(Special) Relativity

With very strong emphasis on electrodynamics and accelerators

Better:

How can we deal with moving charged particles ?

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Reading Material

[1]R.P. Feynman, Feynman lectures on Physics, Vol. 1 + 2, (Basic Books, 2011).

[2]A. Einstein, Zur Elektrodynamik bewegter Körper, Ann. Phys. 17, (1905).

[3]L. Landau, E. Lifschitz, *The Classical Theory of Fields*, Vol2. (Butterworth-Heinemann, 1975)

[4]J. Freund, *Special Relativity*, (World Scientific, 2008).

[5]J.D. Jackson, *Classical Electrodynamics* (Wiley, 1998 ..)

[6]J. Hafele and R. Keating, Science 177, (1972) 166.

Why Special Relativity ?

- We have to deal with moving charges in accelerators
- Electromagnetism and fundamental laws of classical mechanics show inconsistencies
- Ad hoc introduction of Lorentz force
- Applied to moving bodies Maxwell's equations lead to asymmetries [2] not shown in observations of electromagnetic phenomena
- Classical EM-theory not consistent with Quantum theory

Important for beam dynamics and machine design:

- **Longitudinal dynamics (e.g. transition, ...)**
- **Collective effects (e.g. space charge, beam-beam, ...)**
- **Dynamics and luminosity in colliders**
- **Particle lifetime and decay (e.g. μ, π, Z_0, Higgs, ...)**
- **Synchrotron radiation and light sources**
- **...**

We need a formalism to get all that !

OUTLINE

- Principle of Relativity (Newton, Galilei)
  - Motivation, Ideas and Terminology
  - Formalism, Examples
- Principle of Special Relativity (Einstein)
  - Postulates, Formalism and Consequences
  - Four-vectors and applications (Electromagnetism and accelerators)

some slides are for your private study and pleasure and I shall go fast there

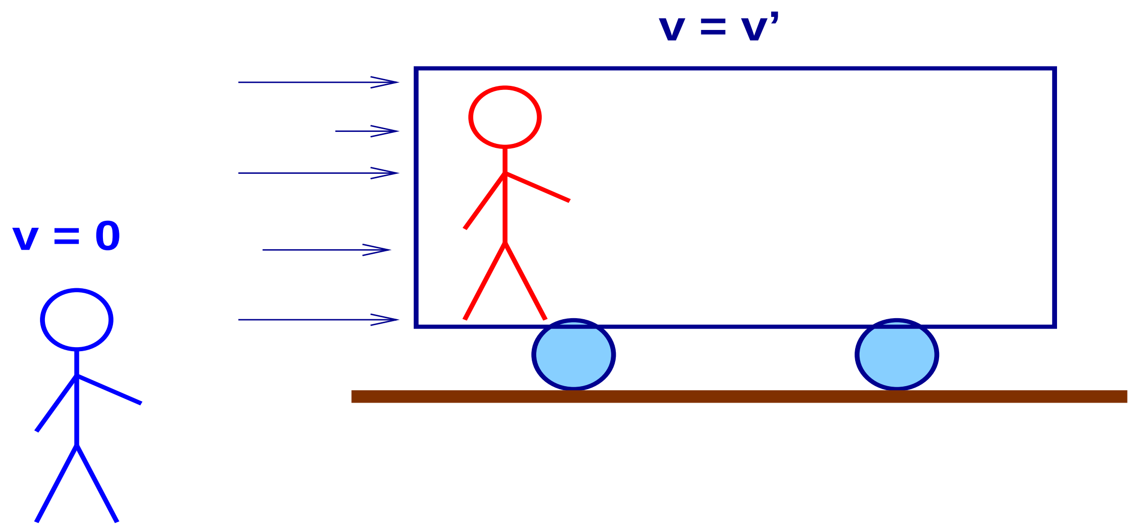
Enjoy yourself ..

Setting the scene (terminology) ..

- To describe an observation and physics laws we use:
  - Space coordinates: $\vec{x} = (x, y, z)$
    (not necessarily Cartesian)
  - Time: t
- What is a "Frame":
  - Where we observe physical phenomena and properties as function of their position $\vec{x}$ and time t.
  - In different frames $\vec{x}$ and t are usually different.
- What is an "Event":
  - Something happening at $\vec{x}$ at time t is an "event", given by four numbers $(x, y, z),\ t$

Example: two frames ...

Assume a frame at rest (S) and another frame (S') moving in x-direction with velocity $\vec{v} = (v', 0, 0)$



- **Passenger performs an experiment and measures the results <u>within</u> his frame**

- **Observer measures the results <u>from</u> the rest frame**

Principles of Relativity (Newton, Galilei)

Definition:

A frame moving at <u>constant velocity</u> is an <u>(Inertial System)</u>

Physical laws are <u>invariant</u> in all inertial systems

<u>invariant</u>:

→ **the mathematical equations keep the <u>same form</u>**

Example:

we would like to have

$Force = m \cdot a$ **and** $Force' = m' \cdot a'$

Relativity: so how to we relate observations ?

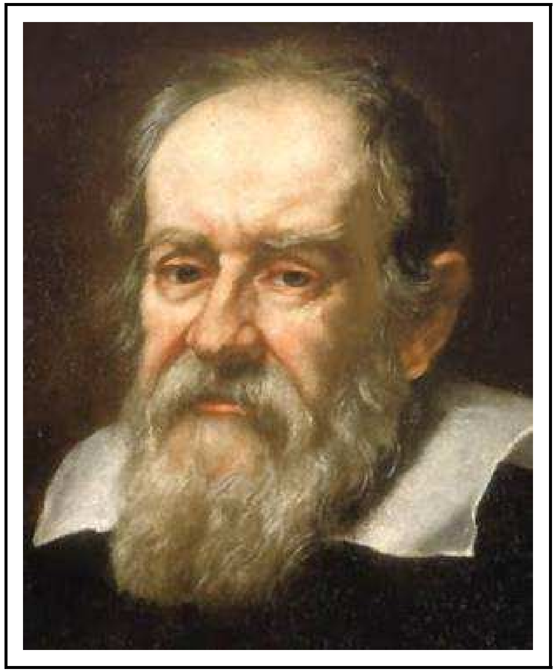
1. We have observed and described an event in rest frame S using coordinates (x, y, z) and time t

2. How can we describe it seen from a moving frame S' using coordinates (x', y', z') and t' ?

3. We need a transformation for: (x, y, z) and t ➡ (x', y', z') and t'.

➡ Then laws should look the same, have the same form

Galilei transformation



$$x' = x - v_x t$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

Galilei transformations relate observations in two frames moving relative to each other (here with constant velocity v_x in x-direction).

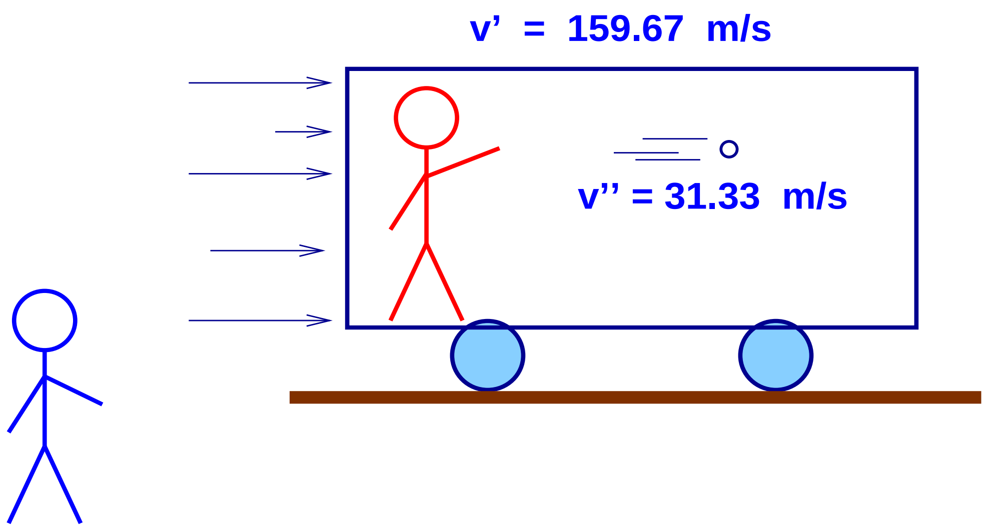
Only the position is changing with time

Frame moves in x-direction with velocity v_x:

- **Space coordinates are changed, time is not changed !**
- **Mass is not transformed/changed !**
- **Space, mass and time are <u>independent</u> quantities**
  - **Absolute space where physics laws are the same**
  - **Absolute time when physics laws are the same**
- **Some examples, plug it in:**

$$\mathrm{m} \cdot \mathrm{a} \quad = \quad \mathrm{m} \cdot \ddot{\mathrm{x}} \quad = \quad \mathrm{m}' \cdot \ddot{\mathrm{x}'} \quad = \quad \mathrm{m}' \cdot \mathrm{a}'$$

$$\mathrm{v}_{\mathrm{x}'} \; = \; \frac{\mathrm{x}'}{\mathrm{dt}} \; = \; \frac{\mathrm{x}}{\mathrm{dt}} \; - \; \mathrm{v}_{\mathrm{x}}$$ **(velocities can be added)**



Fling a ball with 31.33 m/s in a frame moving with 159.67 m/s:

Observed from a non-moving frame: $\mathbf{v}_{tot} = \mathbf{v'} + \mathbf{v''}$

speed of ping-pong ball: $\mathbf{v}_{tot} = \mathbf{191\ m/s}$

Where the trouble starts [2], relative motion of a magnet and a coil:

- If you sit on the coil, you observe:

$$\frac{d\vec{B}}{dt} \rightarrow \vec{\nabla} \times \vec{E} \rightarrow \vec{F} = q \cdot \vec{E} \rightarrow \text{current in coil}$$

- If you sit on the magnet, you observe:

$$\vec{B} = \text{const., moving charge} \rightarrow \vec{F} = q \cdot \vec{v} \times \vec{B} \rightarrow \text{current in coil}$$

<u>Identical results</u>, but seemingly very different mechanisms !

Are the physics laws different ??

Problems with Galilei transformation

Maxwell describes light as waves, wave equation reads:

$$\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right)\Psi \quad = \quad 0$$

With Galilei transformation $x = x' - vt, \;\; y' = y, \;\; z' = z, \;\; t' = t$ **:**

$$\left(\left[1 - \frac{v^2}{c^2}\right]\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{2v}{c^2}\frac{\partial^2}{\partial x \partial t} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi \quad = \quad 0$$

... not quite the same appearance !

Reason: **Waves are required to move in a medium (ether !) which travels along in a fixed reference frame, observed from another frame the speed is different ...**

(try to derive it yourself ..)

Incompatible with experiments:

- Speed of light in vacuum is independent of the motion of the source
- Speed of light in vacuum c is the maximum speed and cannot be exceeded

  c = 299792458.000 m/s

- There is no ether, light is not a wave

Possible options:

1. Maxwell's equations are wrong and should be modified to be invariant with Galilei's relativity (unlikely)
2. Galilean relativity applies to classical mechanics, but not to electromagnetic effects and light has a reference frame (ether). Was defended by many people, sometimes with obscure concepts ...
3. A relativity principle different from Galilei for <u>both</u> classical mechanics and electrodynamics (requires modification of the laws of classical mechanics)

Against all odds and disbelieve of colleagues, Einstein chose the last option ...

Postulates of Special Relativity (Einstein)

All physical laws in inertial frames must have equivalent forms

Speed of light in vacuum c must be the same in all frames

(Implied: energy and momentum conservation)

Need Transformations (not Galileaen) which make ALL physics laws look the same !

Coordinates must be transformed differently

Front of a moving light wave in S and S':

$$\text{S}: \quad x^2 + y^2 + z^2 - c^2t^2 = 0$$

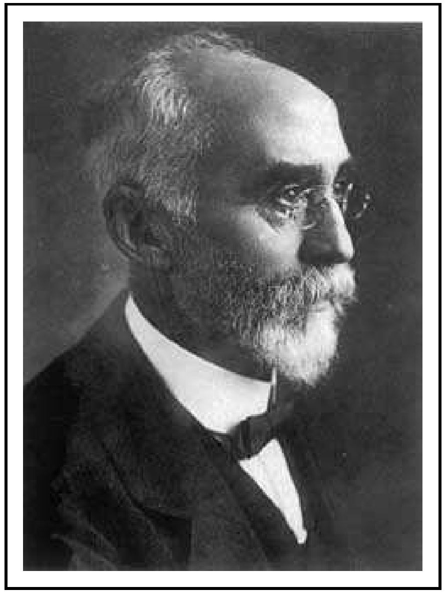
$$\text{S}': \quad x'^2 + y'^2 + z'^2 - c'^2t'^2 = 0$$

Constant speed of light requires $c = c'$

- **To fulfill this condition, time must be changed by transformation as well as space coordinates**
- **Transform** $(x, y, z),\ t \quad \rightarrow \quad (x', y', z'),\ t'$

➡ **After some standard mathematics (e.g. [3, 5]): Lorentz transformation**

Lorentz transformation



$$x' = \frac{x - vt}{\sqrt{(1 - \frac{v^2}{c^2})}} = \gamma \cdot (x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{v \cdot x}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}} = \gamma \cdot (t - \frac{v \cdot x}{c^2})$$

Transformation for constant velocity v along x-axis

Time is now also transformed

Note: for $v \ll c$ it reduces to a Galilei transformation !

Definitions: relativistic factors

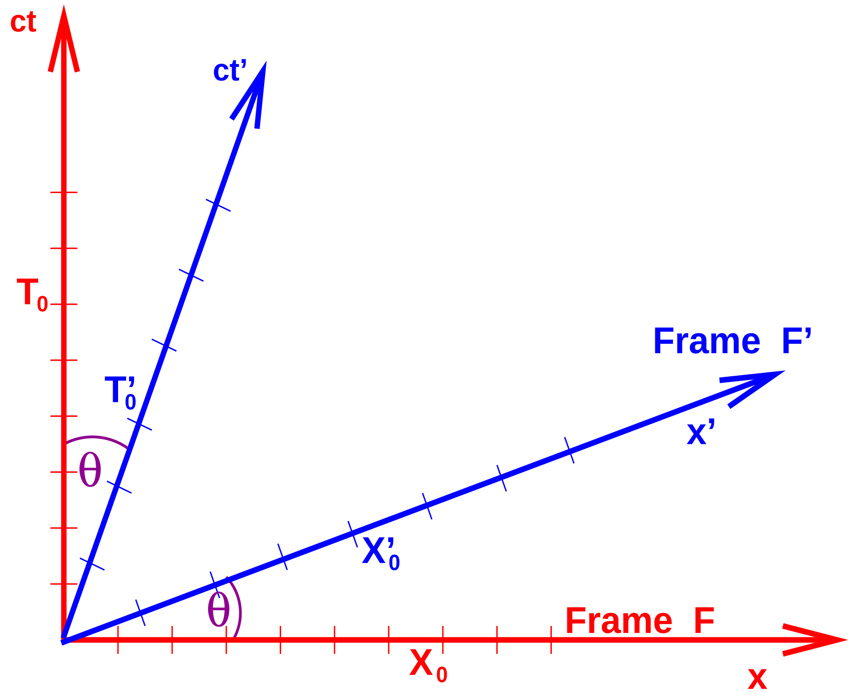
$$\beta_r = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{(1 - \frac{v^2}{c^2})}} = \frac{1}{\sqrt{(1 - \beta_r^2)}}$$

β_r **relativistic speed:** β_r **= [0, 1]**

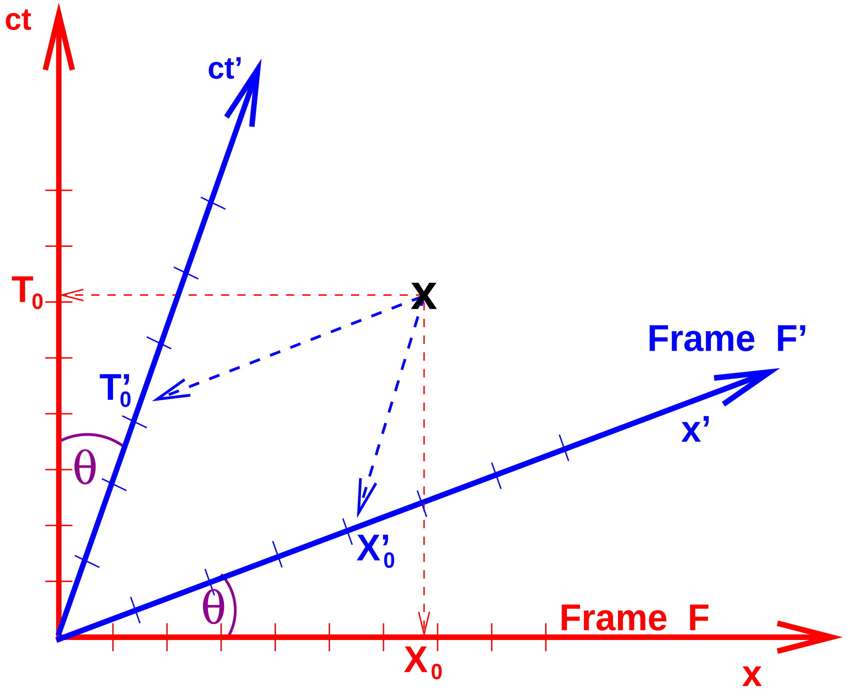
γ **Lorentz factor:** γ **= [1,** ∞**)**

Pictorial Lorentz transformation - Minkowski diagram



- **Rest frame and (skewed) moving frame**
- $\tan(\theta) = \dfrac{v}{c}$

Pictorial Lorentz transformation - Minkowski diagram



- **Event X seen at different time and location in the two frames, projected on axes of F and F'**

Lorentz transformation of velocities

As usual: frame S' moves with constant speed of $\vec{v} = (v, 0, 0)$ relative to frame S

Object inside moving frame moves with $\vec{v}' = (v'_x, v'_y, v'_z)$

What is the velocity $\vec{v} = (v_x, v_y, v_z)$ of the object in the frame S ?

$$v_x = \frac{v'_x + v}{1 + \frac{v'_x v}{c^2}} \qquad v_y = \frac{v'_y}{\gamma(1 + \frac{v'_x v}{c^2})} \qquad v_z = \frac{v'_z}{\gamma(1 + \frac{v'_x v}{c^2})}$$

Addition of velocities

$$v = v_1 + v_2 \quad \longrightarrow \quad v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

→ **Speed of light can never be exceeded by adding velocities !**

Special case: $0.5c + 1.0c = 1.0c$

Consequences of Einstein's interpretation

- Space and time and NOT independent quantities
- There are no absolute time and space, no absolute motion
- Relativistic phenomena (with relevance for accelerators):
  - No speed of moving objects can exceed speed of light
  - (Non-) Simultaneity of events in independent frames
  - Lorentz contraction
  - Time dilation
  - Relativistic Doppler effect
- Formalism with four-vectors introduced (see later)
- Electro dynamics becomes very simple and consistent

- Simultaneity -

(or: what is observed by different observers ..)

Simultaneity between moving frames

Assume two events in frame S at (different) positions x_1 and x_2 happen simultaneously at times $t_1 = t_2$

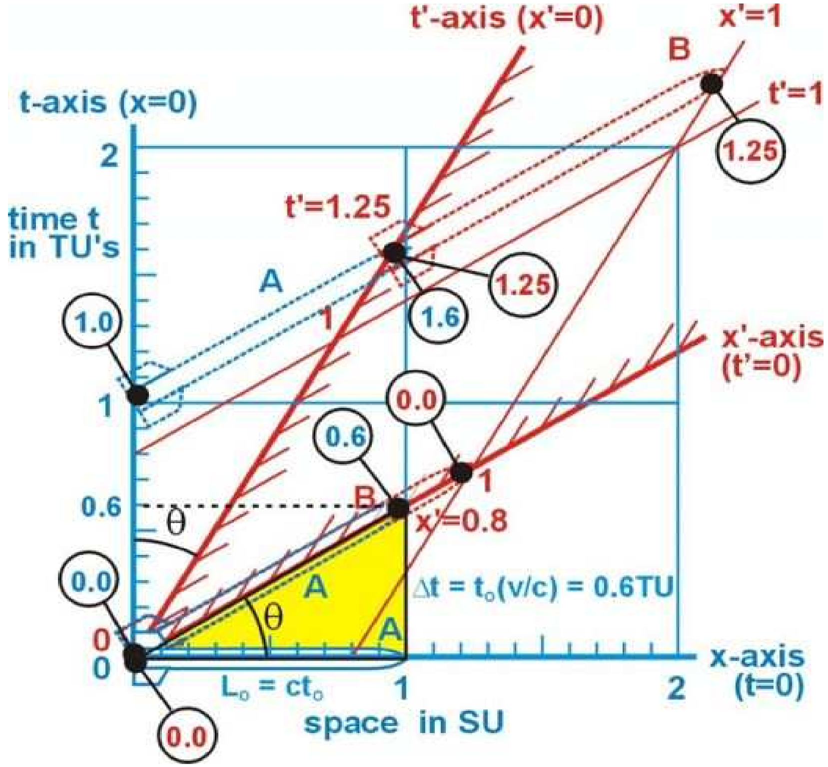
The times t_1' and t_2' in S' we get from:

$$t_1' = \frac{t_1 - \frac{v \cdot x_1}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}} \quad \text{and} \quad t_2' = \frac{t_2 - \frac{v \cdot x_2}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}}$$

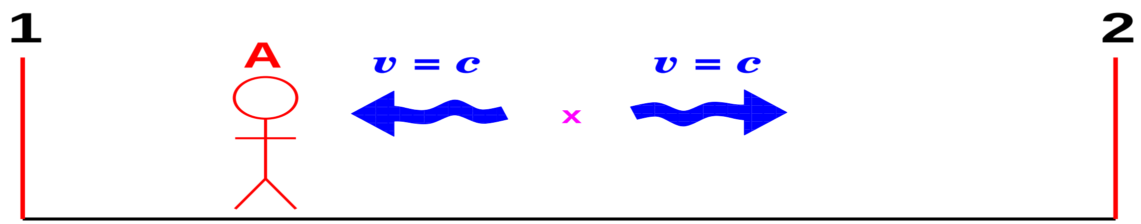
$x_1 \neq x_2$ **in S implies that $t_1' \neq t_2'$ in frame S' !!**

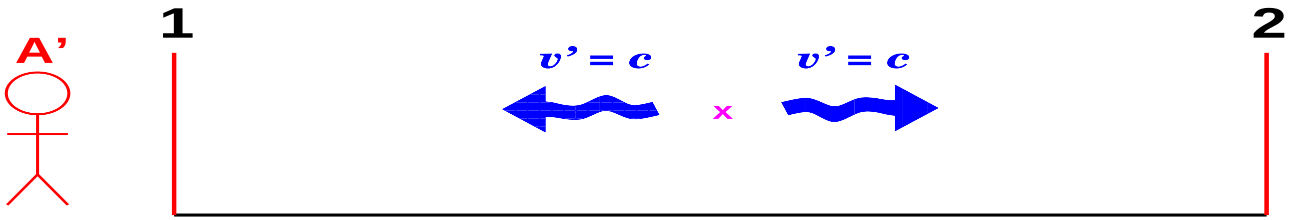
- **Two events simultaneous at (different) positions x_1 and x_2 in S are not simultaneous in S'**

Lack of Simultaneity - explanation:



Simultaneity between moving frames



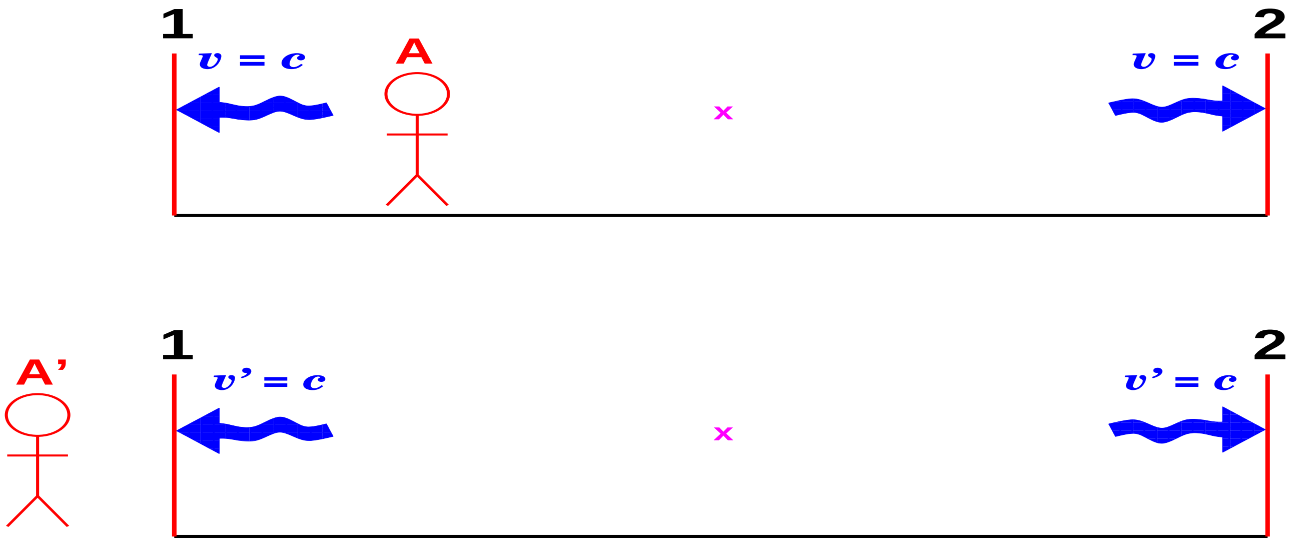


- System with a light source (x) and detectors (1, 2) and
- flashes moving from light source towards detectors

Observer (A) inside this frame

Observer (A') outside

After some time:

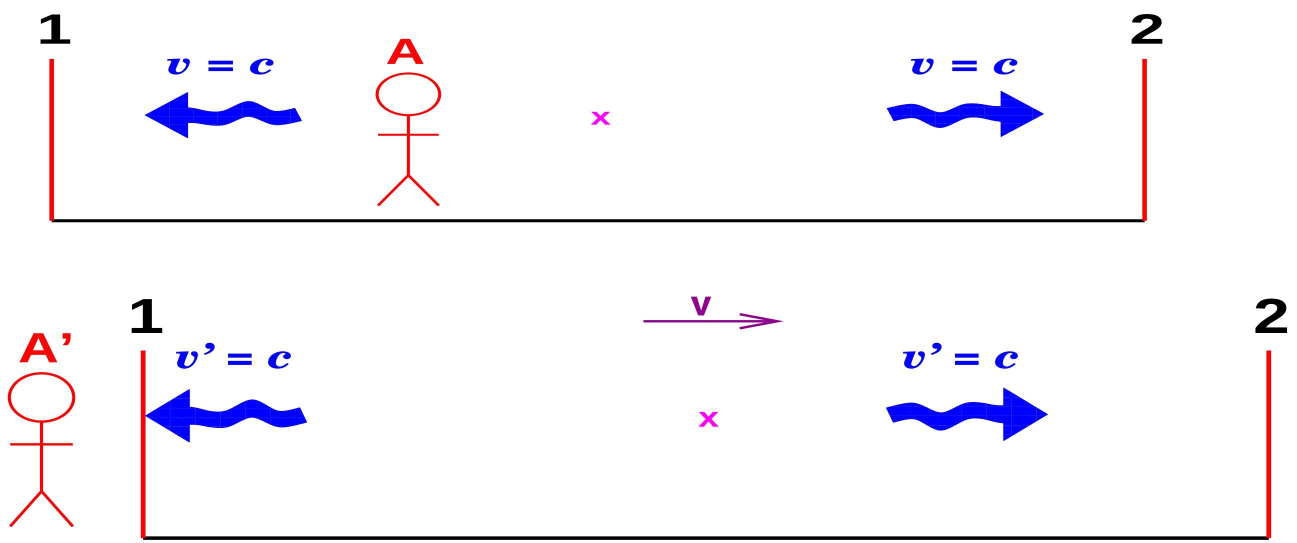


A: both flashes arrive simultaneously at 1 and 2

A': both flashes arrive simultaneously at 1 and 2

What if the frame is moving relative to observer A' ?

Now one frame is moving with speed v:



A: both flashes arrive simultaneously in 1,2

A': flash arrives first in 1, later in 2

A simultaneous event in S is not simultaneous in S'

Why do we care ??

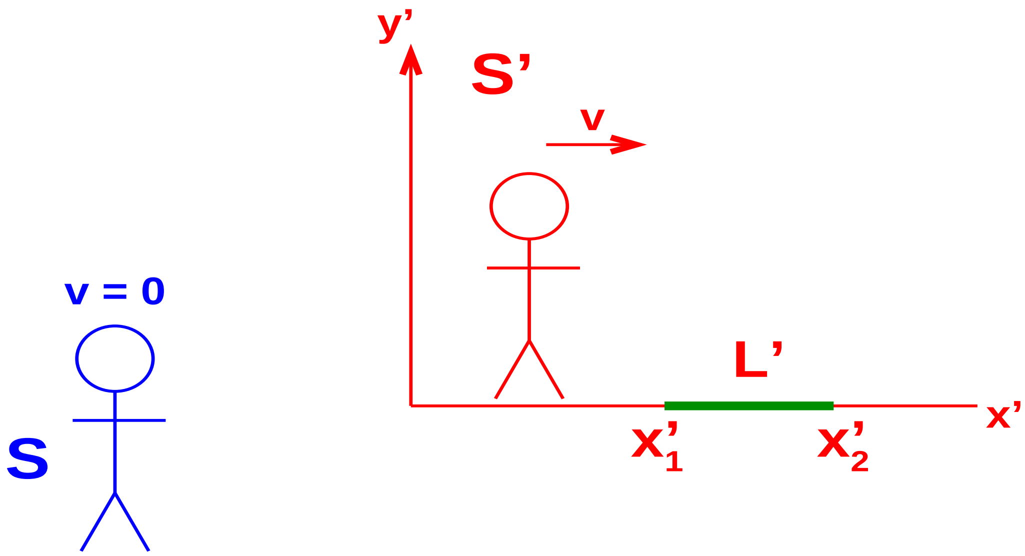
Why care about simultaneity ?

- Simultaneity is not frame independent
- It plays the pivotal role in special relativity
- Almost all paradoxes are explained by that !
- Different observers see a different reality, in particular the sequence of events can change !
  - For $t_1 < t_2$ we may find (not always$^{*)}$!) a frame where $t_1 > t_2$ (concept of before and after depends on the observer)

$^{*)}$ A key to anti-matter - if interested: ask a lecturer at the bar ...

- Lorentz contraction -

How to measure the length of an object ?



Have to measure position of both ends simultaneously !

Length of a rod in S' is $L' = x'_2 - x'_1$, measured simultaneously at a fixed time t' in frame S',

What is the length L measured from S ??

Consequences: length measurement

We have to measure simultaneously (!) the ends of the rod at a fixed time t in frame F, i.e.: $L = x_2 - x_1$

Lorentz transformation of "rod coordinates" into rest frame:

$$x'_1 = \gamma \cdot (x_1 - vt) \quad \text{and} \quad x'_2 = \gamma \cdot (x_2 - vt)$$

$$L' = x'_2 - x'_1 = \gamma \cdot (x_2 - x_1) = \gamma \cdot L$$

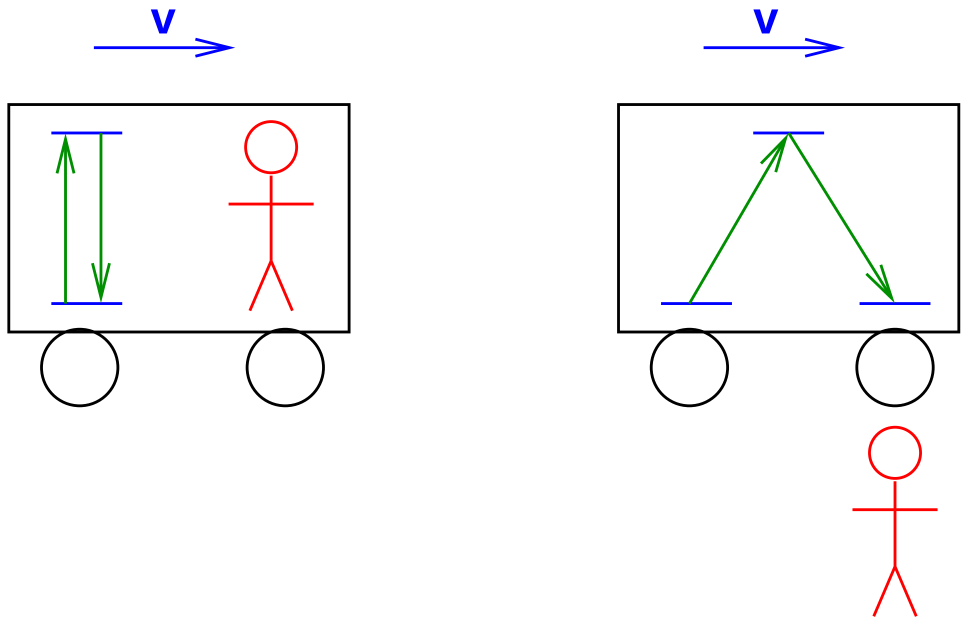
$$\rightarrow \boxed{L = L'/\gamma}$$

In accelerators: bunch length, electromagnetic fields, magnets, ...

- Time dilation -

Time dilation - schematic

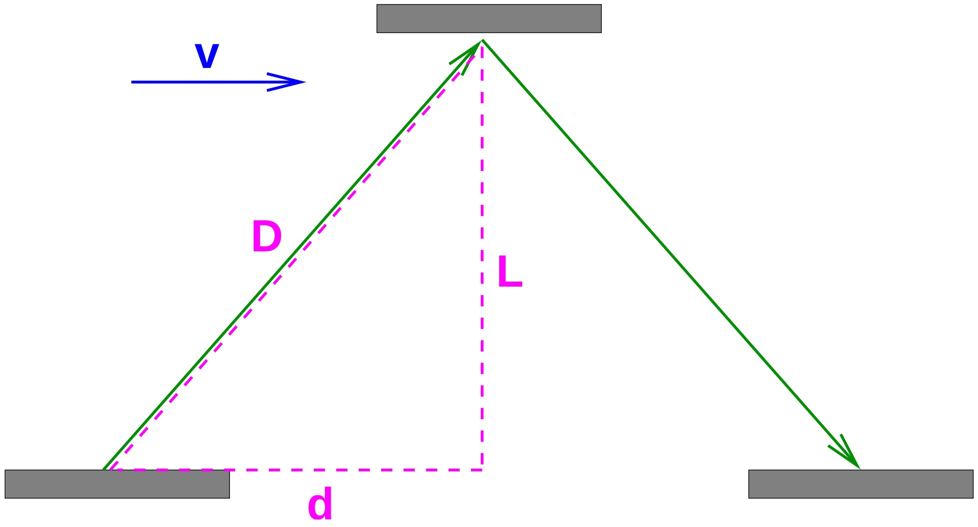
Reflection of light between 2 mirrors seen inside moving frame and from outside



Frame moving with velocity v

Seen from outside the path is longer, but c **must be the same ..**

Time dilation - derivation



In frame S': light travels L in time $\Delta t'$

In frame S: light travels D in time Δt

system moves d in time Δt

$$L = c \cdot \Delta t' \qquad D = c \cdot \Delta t \qquad d = v \cdot \Delta t$$

$$(c \cdot \Delta t)^2 = (c \cdot \Delta t')^2 + (v \cdot \Delta t)^2$$

$$\rightarrow \quad \boxed{\Delta t = \gamma \cdot \Delta t'}$$

Time dilation - the headache

You can derive this two ways:

The car is moving: $\Delta t = \gamma \cdot \Delta t'$

The observer is moving: $\Delta t' = \gamma \cdot \Delta t$

Seems like a contradiction ...

No, solved by the concept of proper time τ:

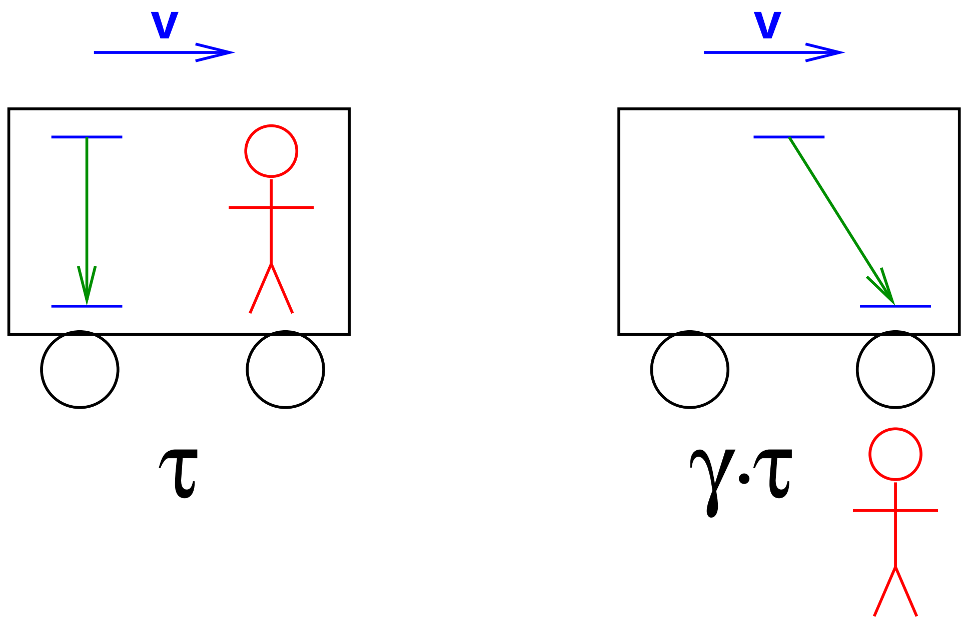
The time measured by the observer at rest relative to the process

Or: The proper time for a given observer is measured by the clock that travels with the observer

$$c^2 \Delta\tau^2 \;=\; c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

Ditto for Lorentz contraction ...

Falling object in a moving car:



Observer in car measures the proper time τ

Proper Length and Proper Time

Time and distances are relative :

- τ is a fundamental time: proper time τ
- The time measured by an observer in its own frame
- From frames moving relative to it, time appears longer

- $\mathcal{L}$ is a fundamental length: proper length $\mathcal{L}$
- The length measured by an observer in its own frame
- From frames moving relative to it, it appears shorter

Example: muon μ

Lifetime of the muon:

In lab frame: $\gamma \cdot \ \tau$ **In frame of muon:** $\tau \ \approx \ 2 \cdot 10^{-6}$ **s**

A clock in the muon frame shows the proper time and the muon decays in $\approx \ 2 \cdot 10^{-6}$ **s, independent of the muon's speed.**

Seen from the lab frame the muon lives γ **times longer**

Example: moving electron

Speed: $v \approx c$

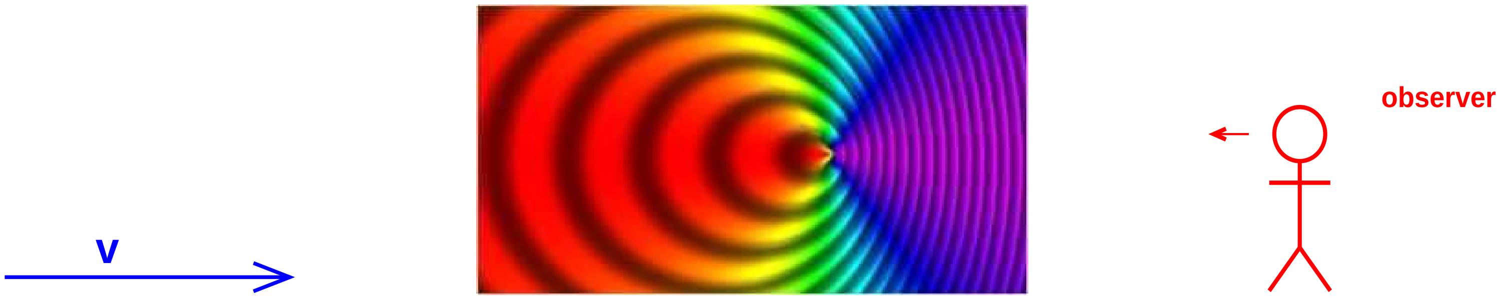
Bunch length:

In lab frame: σ_z **In frame of electron:** $\gamma \cdot \sigma_z$

Length of an object (e.g. magnet, distance between magnets !):

In lab frame: L **In frame of electron:** L/γ

Example: moving light source with speed $\mathbf{v} \approx \mathbf{c}$



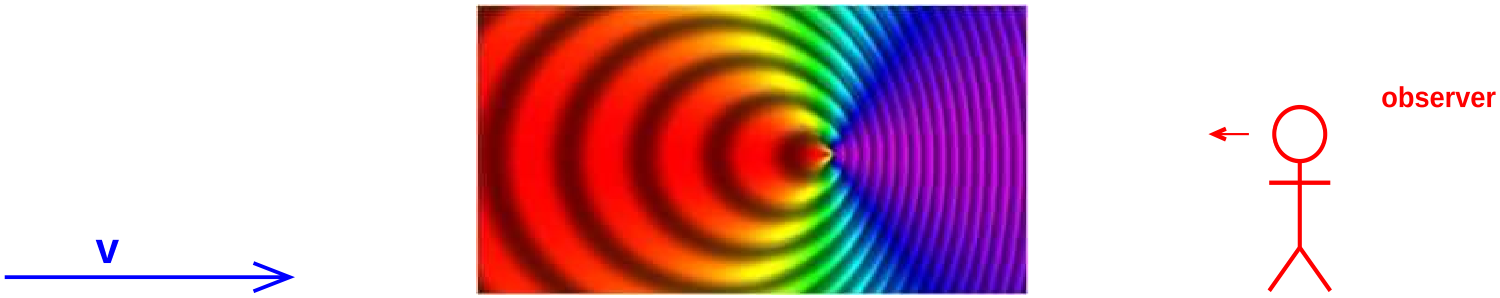
Relativistic Doppler effect:

Unlike sound: no medium of propagation

Observed frequency depends on observation angle θ

➡ **frequency is changed:** $\nu \quad = \quad \nu_0 \cdot \gamma \cdot (1 - \beta_{\mathrm{r}} \cos(\theta))$

Example: moving light source with speed $\mathbf{v} \approx \mathbf{c}$



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$\boxed{\text{Travelling at } v \;\approx\; c \text{ through space can damage your health !}}$

Moving clocks appear to go slower:

Travel by airplane (you age a bit slower compared to ground): tested experimentally with atomic clocks (1971 and 1977)

Assume regular airplane cruising at $\approx$ 900 km/h

On a flight from Montreal to Geneva, the time is slower by 25 - 30 ns (considering only special relativity) !

Not a strong effect, what about other examples ?

Every day example (GPS satellite):

- 20000 km above ground, (unlike popular believe: not on geostationary orbits)
- Orbital speed 14000 km/h (i.e. relative to observer on earth)
- On-board clock accuracy 1 ns
- Relative precision of satellite orbit $\leq 10^{-8}$
- At GPS receiver, for 5 m need clock accuracy $\approx$ 10 ns

Do we correct for relativistic effects ?

Do the math or look it up in the backup slides (and be surprised)..

To make it clear:

Key to understand relativity

- **Lorentz contraction:**
  - It is not the matter that is compressed (was believed before Einstein, some fools still do)
  - It is the space that is modified
- **Time dilation:**
  - It is not the clock that is changed (was believed before Einstein, some people still do)
  - It is the time that is modified

What about the mass m ?

Momentum conservation: $\vec{p} \; = \; \vec{p'}$

To simplify the computation:
Object inside moving frame S' moves with $\vec{u}' \; = \; (0, u'_y, 0)$

We want the expression:

$$\vec{F} \; = \; m \cdot \vec{a} \; = \; m \cdot \frac{d\vec{v}}{dt}$$

in the same form in all frames, transverse momentum must be conserved:

$$p_y = p'_y$$
$$m \; u_y = m' \; u'_y$$
$$m \; u'_y/\gamma = m' \; u'_y$$

implies:

$$\boxed{m = \gamma m'}$$

Relativistic mass

For momentum conservation: <u>mass must also be transformed</u> !

Using the expression for the mass m (using $m' = m_0$):

$$m = m_0 \sqrt{1 - \left(\frac{v}{c}\right)^2} = \gamma \cdot m_0$$

and expand it for small speeds:

$$m \cong m_0 + \frac{1}{2} m_0 v^2 \left(\frac{1}{c^2}\right)$$

and multiplied by c^2:

$$mc^2 \cong m_0 c^2 + \frac{1}{2} m_0 v^2 = m_0 c^2 + T$$

The second term is the kinetic energy T

Relativistic energy

Interpretation:

$$\boxed{E = mc^2 = m_0c^2 + T}$$

- **Total energy** E **is** $E = mc^2$
- **Sum of kinetic energy plus rest energy**
- **Energy of particle at rest is** $E_0 = m_0c^2$

$$\boxed{E = m \cdot c^2 = \gamma m_0 \cdot c^2}$$

using the definition of relativistic mass again: $m = \gamma m_0$

Interpretation of relativistic energy

- For any object, $m \cdot c^2$ is the total energy
- Follows directly from momentum conservations
  - m is the mass (energy) of the object "in motion"
  - m_0 is the mass (energy) of the object "at rest"
- The mass m is not the same in all inertial systems, the **rest mass** m_0 is (prove that ...) !

Relativistic momentum

Classically:

$$p \ = \ m \ v$$

with $m = \gamma m_0$**:**

$$p \ = \ \gamma \cdot m_0 \ v \ = \ \gamma \cdot \beta \cdot c \cdot m_0$$

we re-write:

$$E^2 \ = \ m^2 c^4 \ = \ \gamma^2 m_0^2 c^4 \ = \ (1 + \gamma^2 \beta^2) m_0^2 c^4$$

and finally get:

$$E^2 \ = \ (m_0 c^2)^2 + (pc)^2 \quad \longrightarrow \quad \boxed{\frac{E}{c} \ = \ \sqrt{(m_0 c)^2 \ + \ p^2}}$$

Rather important formula in practice, e.g. accelerators ..

Practical and impractical units

Standard units are not very convenient, easier to use:

$$[\mathbf{E}] = \mathbf{eV} \quad [\mathbf{p}] = \mathbf{eV}/c \quad [\mathbf{m}] = \mathbf{eV}/c^2$$

then: $\boxed{E^2 \;=\; m_0^2 \;+\; p^2}$

Mass of a proton: $m_p = 1.672 \cdot 10^{-27}$ **Kg**
Energy(at rest): $m_p c^2 = 938$ **MeV** $= 0.15$ **nJ**

Ping-pong ball: $m_{pp} = 2.7 \cdot 10^{-3}$ **Kg** **(** $\approx 1.6\ 10^{24}$ **protons)**
Energy(at rest): $m_{pp} c^2 = 1.5 \cdot 10^{27}$ **MeV** $= 2.4 \cdot 10^{14}$ **J**

$\approx$ **750000 times the full LHC beam**

$\approx$ **60 kilotons of TNT**

Masses in accelerators

Recall: the mass of a fast moving particle is increasing like:

$$m = \gamma m_0 = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

When we accelerate:

- **For v $\ll$ c:**
  - **E, m, p, v increase ...**
- **For v $\approx$ c:**
  - **E, m, p increase, but v does (almost) not !**

$$\beta \;=\; \frac{v}{c} \;\approx\; \sqrt{1 - \frac{m_0^2 c^4}{T^2}}$$

→ **Concept of transition (synchrotrons only)**

Collect the formulae: useful kinematic relations

	$\mathbf{cp}$	$\mathbf{T}$	$\mathbf{E}$	γ
$\beta =$	$\frac{1}{\sqrt{(\frac{E_0}{cp})^2+1}}$	$\sqrt{1-\frac{1}{(1+\frac{T}{E_0})^2}}$	$\sqrt{1-(\frac{E_0}{E})^2}$	$\sqrt{1-\gamma^{-2}}$
$\mathbf{cp} =$	cp	$\sqrt{T(2E_0+T)}$	$\sqrt{E^2-E_0^2}$	$E_0\sqrt{\gamma^2-1}$
$E_0 =$	$\frac{cp}{\sqrt{\gamma^2-1}}$	$T/(\gamma-1)$	$\sqrt{E^2-c^2p^2}$	E/γ
$\mathbf{T} =$	$cp\sqrt{\frac{\gamma-1}{\gamma+1}}$	$\mathbf{T}$	$E-E_0$	$E_0(\gamma-1)$
$\gamma =$	$cp/E_0\beta$	$1+T/E_0$	E/E_0	γ

Kinematic relations - logarithmic derivatives

	$\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{dE}{E} = \frac{d\gamma}{\gamma}$
$\frac{d\beta}{\beta} =$	$\frac{d\beta}{\beta}$	$\frac{1}{\gamma^2}\frac{dp}{p}$	$\frac{1}{\gamma(\gamma+1)}\frac{dT}{T}$	$\frac{1}{(\beta\gamma)^2}\frac{d\gamma}{\gamma}$
$\frac{dp}{p} =$	$\gamma^2\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$[\gamma/(\gamma+1)]\frac{dT}{T}$	$\frac{1}{\beta^2}\frac{d\gamma}{\gamma}$
$\frac{dT}{T} =$	$\gamma(\gamma+1)\frac{d\beta}{\beta}$	$(1+\frac{1}{\gamma})\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{\gamma}{(\gamma-1)}\frac{d\gamma}{\gamma}$
$\frac{dE}{E} =$	$(\beta\gamma)^2\frac{d\beta}{\beta}$	$\beta^2\frac{dp}{p}$	$(1-\frac{1}{\gamma})\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$
$\frac{d\gamma}{\gamma} =$	$(\gamma^2-1)\frac{d\beta}{\beta}$	$\frac{dp}{p} - \frac{d\beta}{\beta}$	$(1-\frac{1}{\gamma})\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$

Example LHC (7 TeV): $\frac{\Delta p}{p} \approx 10^{-4}$ **implies:** $\frac{\Delta v}{v} \approx 2 \cdot 10^{-12}$

First summary

- Physics laws the same in all inertial frames ...

➡ Speed of light in vacuum c is the same in all frames and requires Lorentz transformation

- Moving objects appear shorter
- Moving clocks appear to go slower
- Mass is not independent of motion ($m = \gamma \cdot m_0$) and total energy is $E = m \cdot c^2$
- No absolute space or time: where it happens and when it happens is not independent

➡ Next: how to calculate something and applications ...

Introducing four-vectors

Since space and time are not independent, must reformulate physics taking both into account:

$t, \quad \vec{a} = (x, y, z)$ $\rightarrow$ Replace by one vector including the time

We need two types of four-vectors$^{*)}$ (here position four-vector):

$$X^{\mu} = (ct, x, y, z) \qquad \text{and} \qquad X_{\mu} = (ct, -x, -y, -z)$$

We have a temporal and a spatial part
(time t multiplied by c to get the same units)

General strategy: one $+$ three

$^{*)}$ **Due to "skewed" reference system, for details ask one of the lecturers ..**

Life becomes really simple $\rightarrow$

Lorentz transformation can be written in matrix form:

$$X'^{\mu} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = X^{\mu}$$

$$X'^{\mu} = \Lambda \circ X^{\mu} \qquad\qquad (\Lambda \quad \text{for "Lorentz"})$$

Here for motion in x-direction, but can always rotate into direction of motion

but note:

$$X'_\mu = \begin{pmatrix} ct' \\ -x' \\ -y' \\ -z' \end{pmatrix} = \begin{pmatrix} \gamma & +\gamma\beta & 0 & 0 \\ +\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ -x \\ -y \\ -z \end{pmatrix} = X_\mu$$

This matrix is the inverse of the previous matrix

F.A.Q: Why bother about this $^\mu$ or $_\mu$ stuff ??

Mostly ignored (justifiable), but necessary for an invariant formulation of electrodynamics ...

For many calculations, just blindly follow the rules ➡

Scalar products revisited

Cartesian Scalar Product (Euclidean metric):

$$\vec{x} \cdot \vec{y} \;=\; (x_a, y_a, z_a) \;\cdot\; (x_b, y_b, z_b) = (x_a \cdot x_b \;+ y_a \cdot y_b \;+\; z_a \cdot z_b)$$

Space-time four-vectors like:

$$A^\mu \;=\; (ct_a, x_a, y_a, z_a) \qquad B_\mu \;=\; (ct_b, -x_b, -y_b, -z_b)$$

Four-vector Scalar Product:

$$A^\mu B_\mu = \underbrace{\sum_{\mu=0}^{3} A^\mu B_\mu}_{\text{Einstein convention}} \;= (ct_a \cdot ct_b \;-\; x_a \cdot x_b \;-\; y_a \cdot y_b \;-\; z_a \cdot z_b)$$

For many applications you can use this simplified rule:

$$AB = (ct_a \cdot ct_b \;-\; x_a \cdot x_b \;-\; y_a \cdot y_b \;-\; z_a \cdot z_b)$$

Why bother about four-vectors ?

- **We want invariant laws of physics in different frames**
- **The solution: write the laws of physics in terms of four vectors and use Lorentz transformation**
- **Without proof*): any four-vector (scalar) product $Z^\mu Z_\mu$ has the same value in all inertial frames:**

$$Z^\mu Z_\mu \quad = \quad Z'^\mu Z'_\mu$$

All scalar products of any four-vectors are invariant !

but : $Z^\mu Z^\mu$ and $Z'_\mu Z'_\mu$ are not !!*)

*) **The proof is extremely simple !**

We have important four-vectors:

Coordinates :	$X^\mu = (ct, x, y, z) = (ct, \vec{x})$
Velocities :	$U^\mu = \frac{dX^\mu}{d\tau} \; = \; \gamma(c, \dot{\vec{x}}) = \gamma(c, \vec{u})$
Momenta :	$P^\mu = mU^\mu = m\gamma(c, \vec{u}) = \gamma(mc, \vec{p})$
Force :	$F^\mu = \frac{dP^\mu}{d\tau} = \gamma \frac{d}{d\tau}(mc, \vec{p})$
Wave propagation vector :	$K^\mu = (\frac{\omega}{c}, \vec{k}) \qquad\qquad (\hbar \cdot K^\mu \quad ????)$

Also the <u>Gradient :</u> $\quad \partial^\mu = (\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right)$

<u>ALL</u> four-vectors A^μ transform like:

$$A'^\mu = \Lambda \;\circ\; A^\mu$$

A special invariant

From the velocity four-vector V:

$$U^\mu = \gamma(c, \vec{u})$$

we get the scalar product:

$$U^\mu U_\mu = \gamma^2(c^2 - \vec{u}^2) = c^2 \ !!$$

➡ c **is an invariant, has the same value in all inertial frames**

$$U^\mu U_\mu = U'^\mu U'_\mu = c^2$$

➡ **The invariant of the velocity four-vector** U **is the speed of light** c**, i.e. it is the same in ALL frames !**

Another important invariant

Momentum four-vector P:

$$P^\mu = m_0 U^\mu = m_0 \gamma(c, \vec{u}) = (mc, \vec{p}) \;\; = \;\; (\frac{E}{c}, \vec{p})$$

$$P'^\mu = m_0 U'^\mu = m_0 \gamma(c, \vec{u'}) = (mc, \vec{p'}) \;\; = \;\; (\frac{E'}{c}, \vec{p'})$$

We can get another invariant:

$$P^\mu P_\mu \quad = \quad P'^\mu P'_\mu \quad = \quad m_0^2 c^2$$

Invariant of the four-momentum vector is the mass m_0

➡ **The rest mass is the same in all frames !**

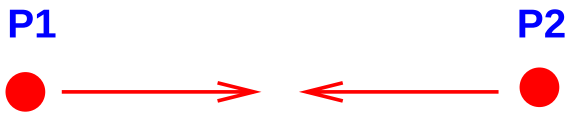
(otherwise we could tell whether we are moving or not !!)

Four vectors

- **Use of four-vectors simplify calculations significantly**
- **Follow the rules and look for invariants**
- **In particular kinematic relationships, e.g.**
  - **Particle decay (find mass of parent particle)**
  - $K'^{\mu} = \Lambda K^{\mu}$ **gives relativistic Doppler shift**
  - **Particle collisions** →

Particle collisions - What is the available energy E_{cm} ?

Collider



Fixed Target

$P_1^\mu = (E, \vec{p}) \qquad P_2^\mu = (E, -\vec{p})$

$P^\mu = P_1^\mu + P_2^\mu = (2E, 0)$

$E_{cm} = \sqrt{P^\mu P_\mu} = 2 \cdot E$

$P_1^\mu = (E, \vec{p}) \qquad P_2^\mu = (m_0, 0)$

$P^\mu = P_1^\mu + P_2^\mu = (E + m_0, \vec{p})$

$E_{cm} = \sqrt{P^\mu P_\mu} = \sqrt{2 m_0 E}$

Works for as many particle as you like : $P^\mu = P_1^\mu + P_2^\mu + P_3^\mu + \ldots$

Works for any configuration, also for particle decay ...

Examples:

collision	E beam energy	E_{cm} (collider)	E_{cm} (fixed target)
pp	315 (GeV)	630 (GeV)	24.3 (GeV)
pp	6500 (GeV)	13000 (GeV)	110.4 (GeV)
pp	90 (PeV)$^{*)}$	180 (PeV)	13000 (GeV)
e+e−	100 (GeV)	200 (GeV)	0.320 (GeV)

$^{*)}$ VFC: for $B_{NC} \approx 3$ T → $C \approx 480\,000$ km (Jupiter $\approx 450\,000$ km)
(although cosmic ray particles can have MUCH higher energies,
Oh-My-God particle had more than 10^{20} eV, $\gamma \approx 10^{11}$ ..)

Relativity and electrodynamics

- Back to the start: electrodynamics and Maxwell equations
- Life made easy with four-vectors ..
- Strategy: one $+$ three

Write potentials and currents as four-vectors:

$$\Phi,\ \vec{A} \quad \Rightarrow \quad A^{\mu} \ = \ (\frac{\Phi}{c},\ \vec{A})$$

$$\rho,\ \vec{j} \quad \Rightarrow \quad J^{\mu} \ = \ (\rho \cdot c,\ \vec{j})$$

What about the transformation of current and potentials ?

Transform the four-current like:

$$\begin{pmatrix} \rho' c \\ j'_x \\ j'_y \\ j'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho c \\ j_x \\ j_y \\ j_z \end{pmatrix}$$

It transforms via: $J'^{\mu} = \Lambda\ J^{\mu}$ **(always the same** Λ**)**

Ditto for: $A'^{\mu} = \Lambda\ A^{\mu}$ **(always the same** Λ**)**

Note: $\partial_\mu J^\mu = \dfrac{\partial \rho}{\partial t} + \vec{\nabla}\vec{j} = 0$ **(charge conservation)**

Electromagnetic fields using potentials:

Magnetic field: $\vec{B} = \nabla \times \vec{A}$

e.g. the x-component:

$$B_x = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

Electric field: $\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$

e.g. for the x-component:

$$E_x = -\frac{\partial A_0}{\partial x} - \frac{\partial A_1}{\partial t} = -\frac{\partial A_t}{\partial x} - \frac{\partial A_x}{\partial t}$$

➡ **after getting all combinations ..**

Electromagnetic fields described by field-tensor $F^{\mu\nu}$:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & \frac{-E_x}{c} & \frac{-E_y}{c} & \frac{-E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

It transforms via: $F'^{\mu\nu} = \Lambda \; F^{\mu\nu} \; \Lambda^T$ **(same Λ as before)**

(Warning: There are different ways to write the field-tensor $F^{\mu\nu}$, I use the convention from [1, 3, 5])

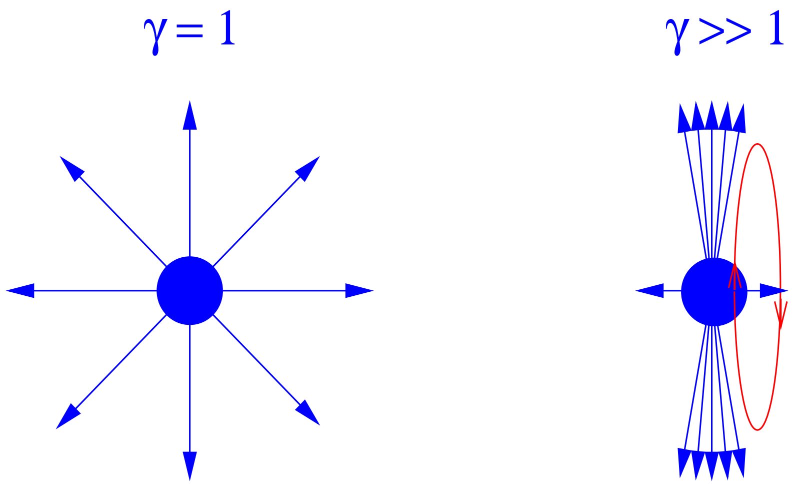
Transformation of fields into a moving frame (x-direction):

Use Lorentz transformation of $F^{\mu\nu}$ and write for components:

$$
\begin{aligned}
E'_x &= E_x & B'_x &= B_x \\
E'_y &= \gamma(E_y - v \cdot B_z) & B'_y &= \gamma(B_y + \tfrac{v}{c^2} \cdot E_z) \\
E'_z &= \gamma(E_z + v \cdot B_y) & B'_z &= \gamma(B_z - \tfrac{v}{c^2} \cdot E_y)
\end{aligned}
$$

Fields perpendicular to movement are transformed

Example Coulomb field: **(a charge moving with constant speed)**



- In rest frame purely electrostatic forces
- In moving frame $\vec{E}$ transformed and $\vec{B}$ appears

How do the fields look like ?

Needed to compute e.g. radiation of a moving charge, wake fields, ...

For the static charge we have the Coulomb potential (see lecture on Electrodynamics) and $\vec{A} = 0$

Transformation into the new frame (moving in x-direction) with our transformation of four-potentials:

$$\frac{\Phi'}{c} = \gamma(\frac{\Phi}{c} - A_x) = \gamma\frac{\Phi}{c}$$

$$A'_x = \gamma\left(A_x - \frac{v\Phi}{c^2}\right) = -\gamma\frac{v}{c^2}\Phi = -\frac{v}{c^2}\Phi'$$

i.e. all we need to know is Φ' $\rightarrow$

$$\Phi'(\vec{r}) \ = \ \gamma\Phi(\vec{r}) \ = \ \gamma \cdot \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{|\vec{r} - \vec{r}_q|}$$

After transformation of coordinates, e.g. $x \ = \ \gamma(x' - vt')$
The resulting potentials can be used to compute the fields.

Watch out !!

We have to take care of causality:
The field observed at a position $\vec{r}$ at time t was caused at an <u>earlier</u> time $t_r < t$ at the location $\vec{r}_0(t_r)$

$$\Phi(\vec{r},t) = \frac{qc}{|\vec{R}|c - \vec{R}\vec{v}} \qquad \vec{A}(\vec{r},t) = \frac{q\vec{v}}{|\vec{R}|c - \vec{R}\vec{v}}$$

The potentials $\Phi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ depend on the state at <u>retarted</u> time t_r, not t
$\vec{v}$ is the velocity at time t_r and $\vec{R} = \vec{r} - \vec{r}_0(t_r)$ relates the retarted position to the observation point.

Q: Can we also write a <u>Four-Maxwell</u> ?

Re-write Maxwell's equations using four-vectors and $F^{\mu\nu}$:

$$\nabla\vec{E} = \frac{\rho}{\epsilon_0} \qquad \textbf{and} \qquad \nabla\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J} \qquad \xrightarrow{1+3}$$

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 J^\nu \quad \text{(Inhomogeneous Maxwell equation)}}$$

$$\nabla\vec{B} = 0 \qquad \textbf{and} \qquad \nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \qquad \xrightarrow{1+3}$$

$$\boxed{\partial_\gamma F^{\mu\nu} + \partial_\mu F^{\nu\lambda} + \partial_\nu F^{\lambda\mu} = 0 \quad \text{(Homogeneous Maxwell equation)}}$$

We have Maxwell's equation in a very compact form,
transformation between moving systems very easy

How to use all that stuff ??? **Look at first equation:**

$$\partial_\mu F^{\mu\nu} \quad = \quad \mu_0 J^\nu$$

Written explicitly (Einstein convention, sum over μ):

$$\partial_\mu F^{\mu\nu} \; = \; \sum_{\mu=0}^{3} \partial_\mu F^{\mu\nu} \; = \; \partial_0 F^{0\nu} \; + \; \partial_1 F^{1\nu} \; + \; \partial_2 F^{2\nu} \; + \; \partial_3 F^{3\nu} \quad = \quad \mu_0 J^\nu$$

Choose e.g. $\nu = 0$ and replace $F^{\mu\nu}$ by corresponding elements:

$$\begin{array}{lllllll} \partial_0 F^{00} & + \partial_1 F^{10} & + \partial_2 F^{20} & + \partial_3 F^{30} & = & \mu_0 J^0 & \\ 0 & + \partial_x \dfrac{E_x}{c} & + \partial_y \dfrac{E_y}{c} & + \partial_z \dfrac{E_z}{c} & = & \mu_0 J^0 & = \quad \mu_0 \; c\rho \end{array}$$

This corresponds exactly to:

$$\vec{\nabla} \cdot \; \vec{E} \quad = \quad \frac{\rho}{\epsilon_0} \qquad\qquad (c^2 \; = \; \epsilon_0 \mu_0)$$

→ **For** $\nu = 1,2,3$ **you get Ampere's law**

For example in the x-plane (ν **= 1) and the S frame:**

$$\partial_y B_z - \partial_z B_y - \partial_t \frac{E_x}{c} = \mu_0 J^x$$

after transforming ∂^γ **and** $F^{\mu\nu}$ **to the S' frame:**

$$\partial'_y B'_z - \partial'_z B'_y - \partial'_t \frac{E'_x}{c} = \mu_0 J'^x$$

Now Maxwell's equation have the identical form in S and S'

(In matter: can be re-written with $\vec{D}$ **and** $\vec{H}$ **using "magnetization tensor")**

Finally: since $$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

$$\partial_\gamma F^{\mu\nu} + \partial_\mu F^{\nu\lambda} + \partial_\nu F^{\lambda\mu} = 0$$

We can re-write them two-in-one in a new form:

$$\boxed{\frac{\partial^2 A^\mu}{\partial x_\nu \partial x^\nu} = \mu_0 J^\mu}$$

This contains all four Maxwell's equations, and the only one which stays the same in all frames !!

There are <u>no</u> separate electric and magnetic fields, just a frame dependent manifestation of a single electromagnetic field

Quite obvious in Quantum ElectroDynamics !

What about forces ??

Start with the (four-)force as the time derivative of the four-momentum:

$$\mathcal{F}_L^{\mu} \quad = \quad \frac{\partial P^{\mu}}{\partial \tau}$$

We get the four-vector for the Lorentz force, with the well known expression in the second part:

$$\mathcal{F}_L^{\mu} \; = \; \gamma q (\tfrac{\vec{E} \cdot \vec{u}}{c}, \; \vec{E} \; + \; \vec{u} \times \vec{B}) \quad = \quad q \cdot F^{\mu\nu} U_{\nu}$$

Quote Einstein (1905):

For a charge moving in an electromagnetic field, the force experienced by the charge is equal to the electric force, transformed into the rest frame of the charge

There is no mystic, velocity dependent coupling between a charge and a magnetic field !

It is just a consequence of two reference frames

An important consequence - remember:

$$E'_x = E_x \qquad B'_x = B_x$$
$$E'_y = \gamma(E_y - v \cdot B_z) \qquad B'_y = \gamma(B_y + \frac{v}{c^2} \cdot E_z)$$
$$E'_z = \gamma(E_z + v \cdot B_y) \qquad B'_z = \gamma(B_z - \frac{v}{c^2} \cdot E_y)$$

Assuming that $\vec{B}' = 0$, we get for the transverse forces:

$$\boxed{\vec{F}_{mag} = -\beta^2 \cdot \vec{F}_{el}}$$

For β = 1, Electric and Magnetic forces cancel, plenty of consequences, e.g. Space Charge

Most important for stability of beams (so watch out for $\beta \ll 1$) !

No more inconsistencies:

Mechanisms are the same, physics laws are the same:

- **Formulated in an invariant form and transformed with Lorentz transformation**
- **Different reference frames are expected to result in different observations**

→ **In an accelerator we have always at least two reference frames**

Where are we ?

- We have to deal with moving charges in accelerators
- Electromagnetism and fundamental laws of classical mechanics show inconsistencies
- Ad hoc introduction of Lorentz force
- Applied to moving bodies Maxwell's equations lead to asymmetries [2] not shown in observations of electromagnetic phenomena
- Classical EM-theory not consistent with Quantum theory

Summary I (things to understand)

- Special Relativity is very simple, few basic principles
  - Physics laws are the same in all inertial systems
  - Speed of light in vacuum the same in all inertial systems
- Everyday phenomena lose their meaning (do not ask what is "real"):
  - Only union of space and time preserve an independent reality: space-time
  - Electric and magnetic fields do not exist !
    - Just different aspects of a single electromagnetic field
    - Its manifestation, i.e. division into electric $\vec{E}$ and magnetic $\vec{B}$ components, depends on the chosen reference frame

Summary II (accelerators - things to remember)

Write everything as four-vectors, blindly follow the rules and you get it all easily, in particular transformation of fields etc.

- **Relativistic effects in accelerators (used in later lectures)**
  - **Lorentz contraction and Time dilation (e.g. FEL, ..)**
  - **Relativistic Doppler effect (e.g. FEL, ..)**
  - **Invariants !**
  - **Relativistic mass effects and dynamics**
  - **New interpretation of electric and magnetic fields, in particular "Lorentz force"**
- **If you do not take relativity into account, you are sunk ...**

Interesting, but not treated here:

- Principles of Special Relativity apply to inertial (non-accelerated) systems
- Is it conceivable that the principle applies to accelerated systems ?
- Introduces General Relativity, with consequences:
  - Space and time are dynamical entities:

    → space and time change in the presence of matter
  - Explanation of gravity (sort of ..)
  - Black holes, worm holes, Gravitational Waves, ...
  - Time depends on gravitational potential, different at different heights (Airplanes, GPS !)

A last word ...

- If you do not yet have enough or are bored, look up some of the popular paradoxes (entertaining but mostly irrelevant for accelerators):
  - Ladder-garage paradox (*)
  - Twin paradox (**)
  - Bug - Rivet paradox (**)
  - J. Bell's rocket-rope paradox (***)
  - ...

- BACKUP SLIDES -

Personal comments:

Special relativity is very simple - but not intuitive, may violate common sense ...

We have to rely on the deductive procedure (and accept the results)

In any kind of theory the main difficulty is to formulate a problem mathematically.

A rudimentary knowledge of high school mathematics suffices to solve the resulting equations in this theory.

→ Derivations and proofs are avoided when they do not serve a didactical purpose (see e.g. [2, 4, 5])...

But no compromise on correctness, not oversimplified !

Small history

- 1678 (Römer, Huygens): Speed of light c is finite ($c \approx 3 \cdot 10^8$ m/s)
- 1630-1687 (Galilei,Newton): Principles of Relativity
- 1863 (Maxwell): Electromagnetic theory, light are waves moving through static ether with speed c
- 1887 (Michelson, Morley): Speed c independent of direction, $\rightarrow$ no ether
- 1892 (Lorentz, FitzGerald, Poincaré): Lorentz transformations, Lorentz contraction
- 1897 (Larmor): Time dilation
- 1905 (Einstein): Principles of Special Relativity
- 1907 (Einstein, Minkowski): Concepts of Spacetime

Relativistic Principles

- **Relativity in (classical) inertial systems:**
  - Classical relativity
  - Newton, Galilei
- **Relativity in (all) inertial systems:**
  - Special relativity
  - Lorentz, Einstein, Minkowski
- **Relativity in accelerated systems:**
  - General relativity
  - Einstein

Lorentz contraction

- In moving frame an object has always the same length (it is invariant, our principle !)
- From stationary frame moving objects appear contracted by a factor γ (Lorentz contraction)
- Why do we care ?
- Turn the argument around: assume length of a proton bunch appears always at 0.1 m in laboratory frame (e.g. in the RF bucket), what is the length in its own (moving) frame ?
  - At 5 GeV ($\gamma \approx 5.3$) $\rightarrow$ L' = 0.53 m
  - At 450 GeV ($\gamma \approx 480$) $\rightarrow$ L' = 48.0 m

Relations to remember

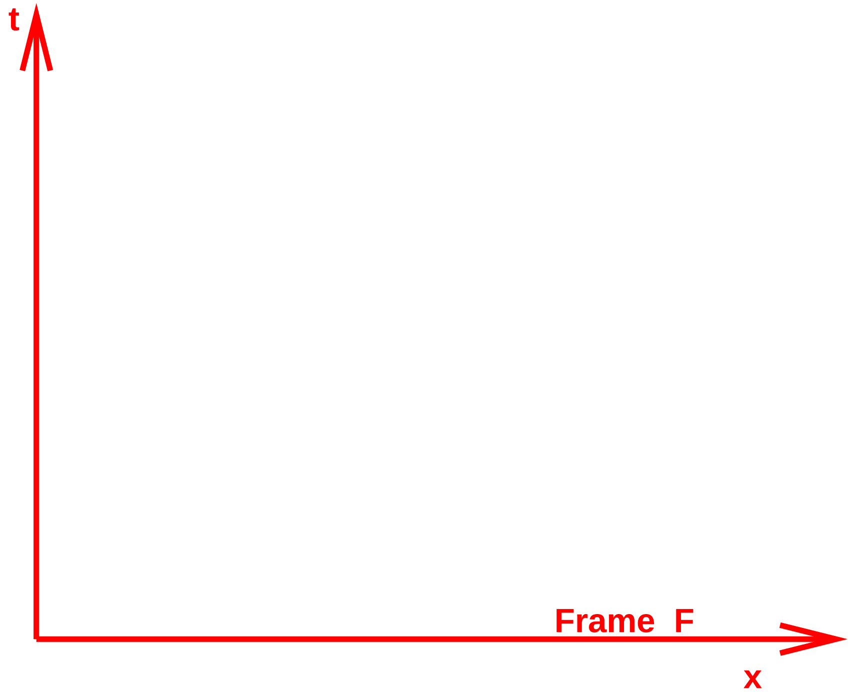
Note:

$$E = mc^2 = \gamma \cdot m_0 c^2 \quad \rightarrow \quad \boxed{E = \gamma m_0}$$

$$p = m_0 \gamma v = \gamma m_0 \cdot \beta c \quad \rightarrow \quad \boxed{p = \gamma m_0 \cdot \beta}$$

$$T = m_0(\gamma - 1) \cdot c^2 \quad \rightarrow \quad \boxed{T = \gamma m_0 - m_0}$$

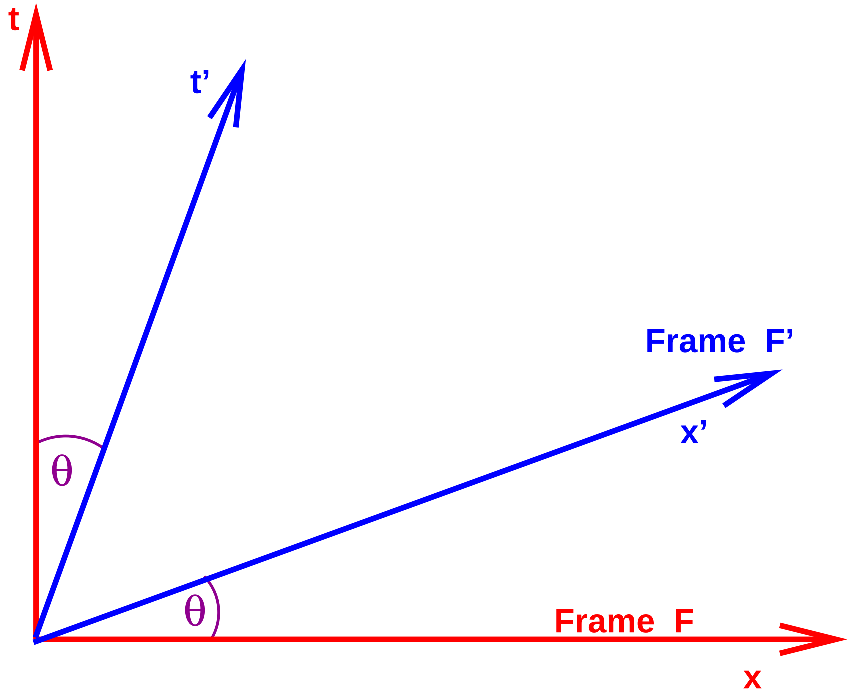
Lorentz transformation - schematic



- Rest frame (x only, difficult to draw many dimensions)

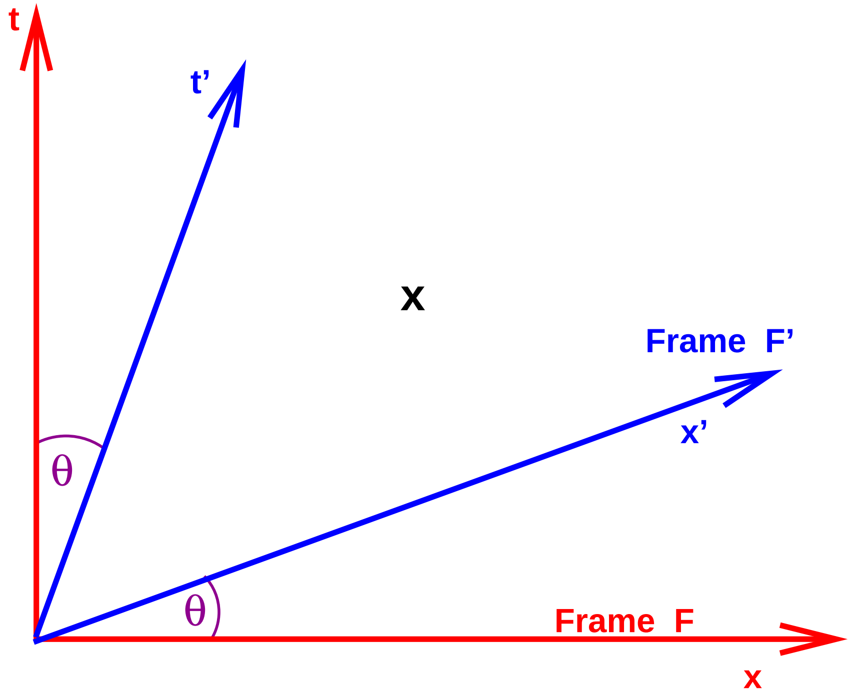
  y and z coordinates are not changed (transformed)

Lorentz transformation - schematic



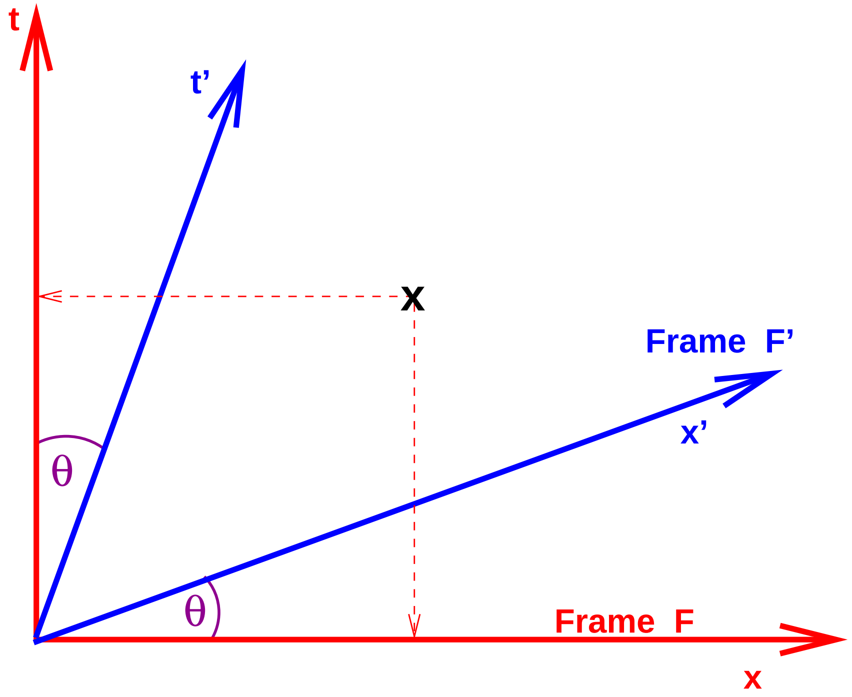
- **Rest frame and moving frame**
- **tan**$(\theta) = \frac{v}{c}$

Lorentz transformation - schematic



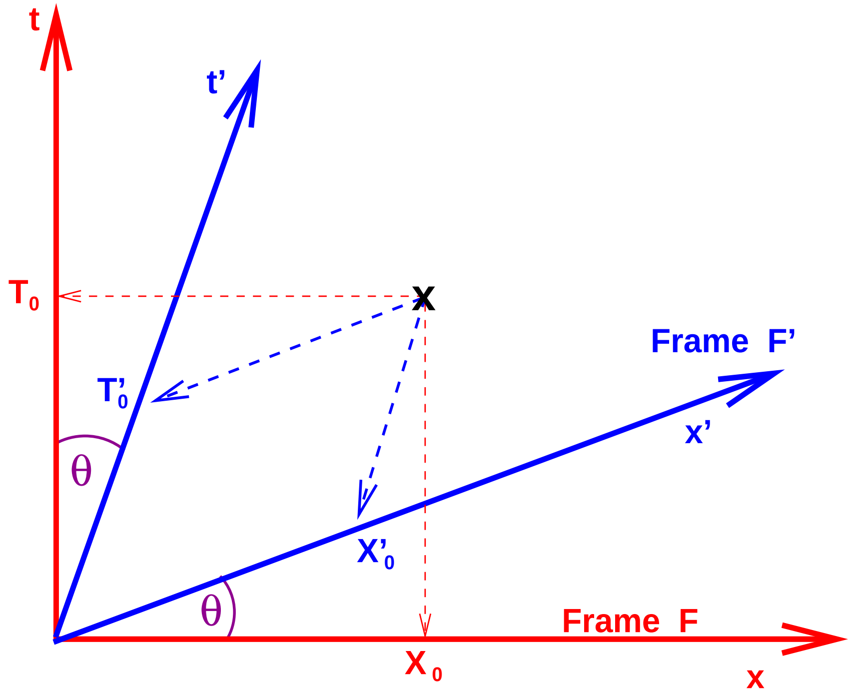
- An event X

Lorentz transformation - schematic



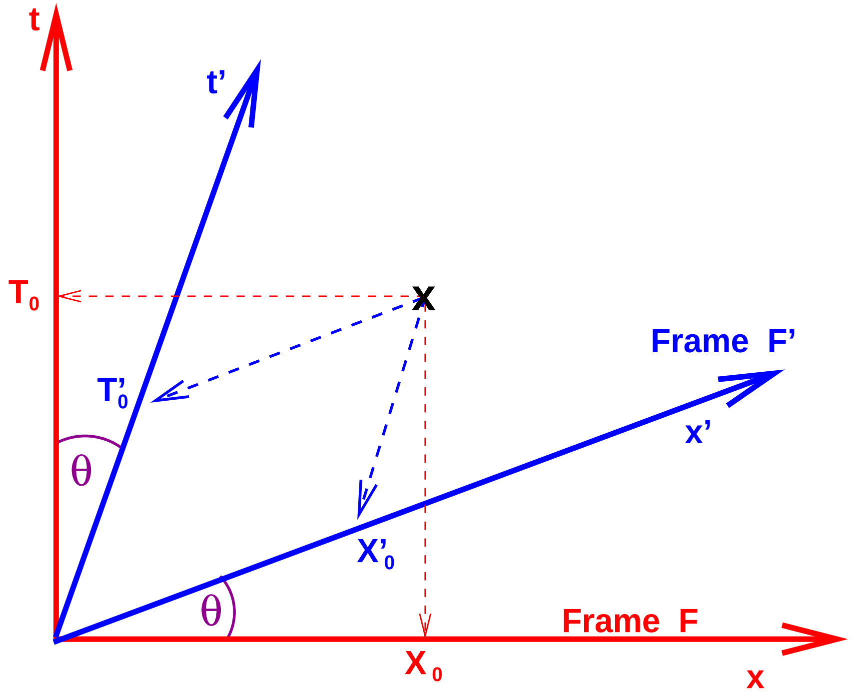
- **Event X as seen from rest frame, projected on F-axes**

Lorentz transformation - schematic



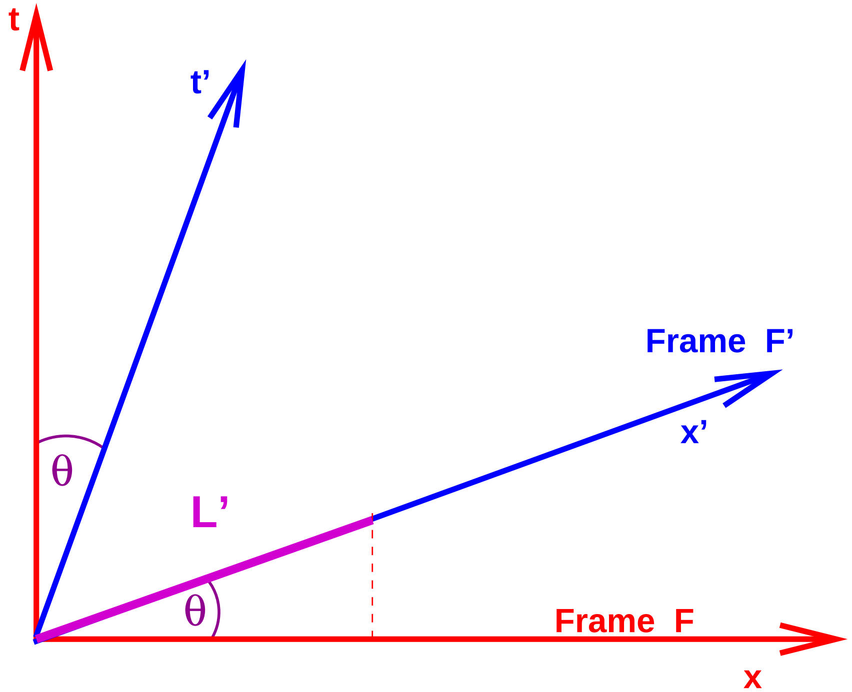
- **Event X seen at different time and location in the two frames, projected on axes of F and F'**

Lorentz transformation - schematic



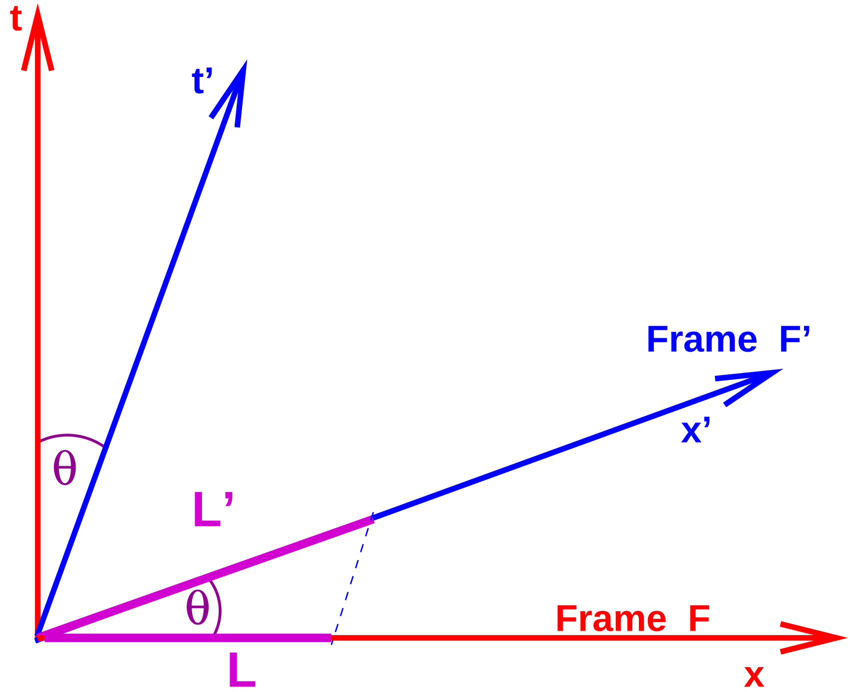
- **Q: How would a Galilei-transformation look like ??**

Lorentz contraction - schematic



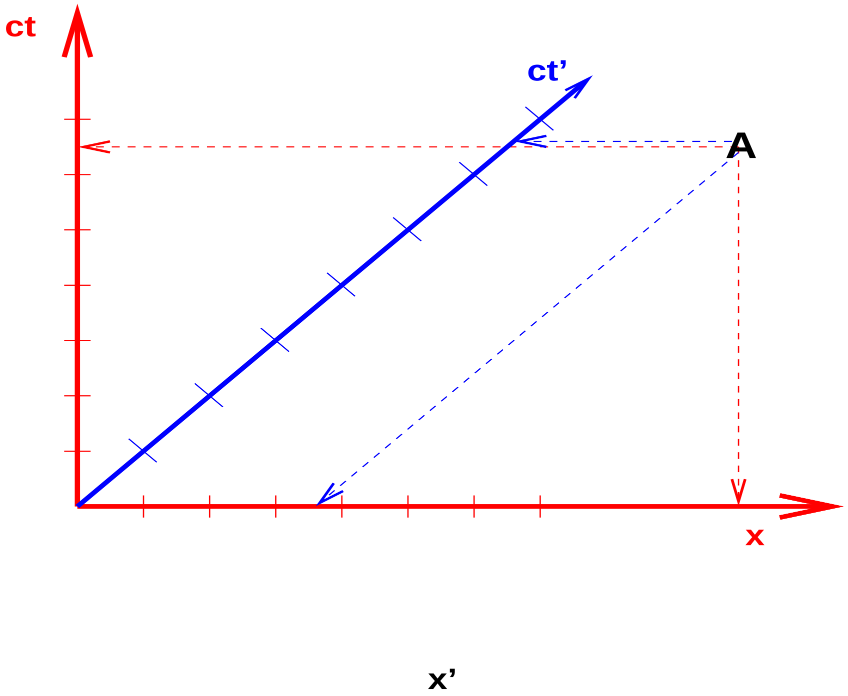
- **Length L' as measured in moving frame**

Lorentz contraction - schematic



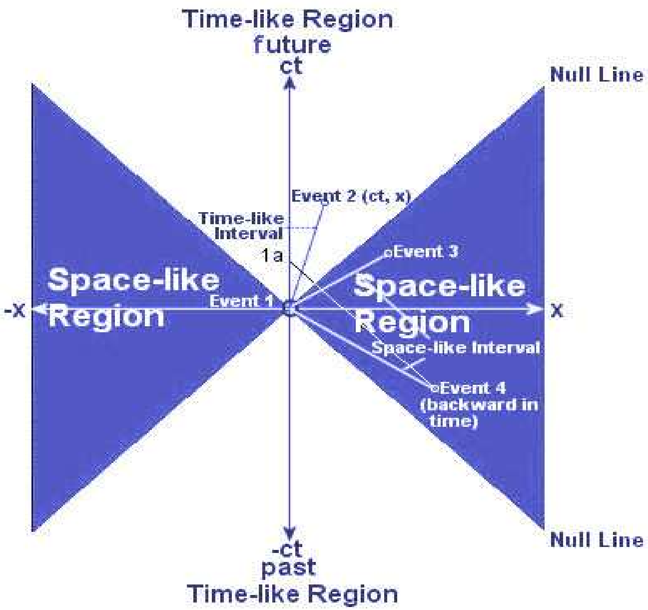
- From moving frame: L appears shorter in rest frame
- Length is maximum in frame (F') where object is at rest

Galilei transformation - schematic



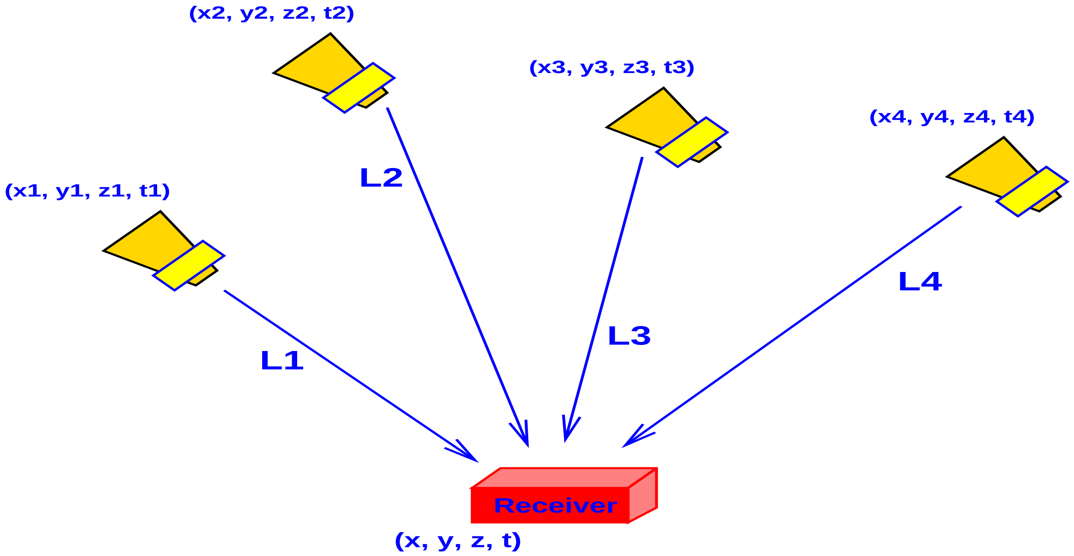
- **Rest frame and Galilei transformation ...**

Time-like and Space-like events



- **Event 1 can communicate with event 2**
- **Event 1 cannot communicate with event 3, would require to travel faster than the speed of light**

GPS principle ...



$$
\begin{aligned}
L_1 &= c(t - t_1) = \sqrt{((x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} \\
L_2 &= c(t - t_2) = \sqrt{((x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2} \\
L_3 &= c(t - t_3) = \sqrt{((x - x_3)^2 + (y - y_3)^2 + (z - z_3)^2} \\
L_4 &= c(t - t_4) = \sqrt{((x - x_4)^2 + (y - y_4)^2 + (z - z_4)^2}
\end{aligned}
$$

$t_1, t_2, t_3, t_4,$ **need relativistic correction !**

4 equations and 4 variables ➡ (x, y, z, t) **of the receiver !**

Kinematic relations

	cp	**T**	**E**	γ
$\beta =$	$\frac{1}{\sqrt{(\frac{E_0}{cp})^2+1}}$	$\sqrt{1-\frac{1}{(1+\frac{T}{E_0})^2}}$	$\sqrt{1-(\frac{E_0}{E})^2}$	$\sqrt{1-\gamma^{-2}}$
cp =	cp	$\sqrt{T(2E_0+T)}$	$\sqrt{E^2-E_0^2}$	$E_0\sqrt{\gamma^2-1}$
$E_0 =$	$\frac{cp}{\sqrt{\gamma^2-1}}$	$T/(\gamma-1)$	$\sqrt{E^2-c^2p^2}$	E/γ
T =	$cp\sqrt{\frac{\gamma-1}{\gamma+1}}$	**T**	$E-E_0$	$E_0(\gamma-1)$
$\gamma =$	$cp/E_0\beta$	$1+T/E_0$	E/E_0	γ

Kinematic relations

Example: CERN Booster

At injection: T = 50 MeV

➡ **E = 0.988 GeV, p = 0.311 GeV/c**

➡ γ **= 1.0533,** β **= 0.314**

At extraction: T = 1.4 GeV

➡ **E = 2.338 GeV, p = 2.141 GeV/c**

➡ γ **= 2.4925,** β **= 0.916**

Gravitational time dilation

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{2GM}{Rc^2}}$$
$$\frac{d\tau}{dt} \approx 1 - \frac{GM}{Rc^2}$$
$$\Delta\tau = \frac{GM}{c^2} \cdot \left(\frac{1}{R_{earth}} - \frac{1}{R_{gps}} \right)$$

With:

$R_{earth} = 6357000$ **m,** $R_{gps} = 26541000$ **m**
$G = 6.674 \cdot 10^{-11} \ \frac{N \cdot m^2}{kg^2}$ $M = 5.974 \cdot 10^{24}$ **kg**

We have:

$$\Delta\tau \approx 5.3 \cdot 10^{-10}$$

Do the math:

Orbital speed 14000 km/h $\approx$ 3.9 km/s

➡ $\beta \approx 1.3 \cdot 10^{-5}$, $\gamma \approx 1.000000000084$

Small, but accumulates 7 μs during one day compared to reference time on earth !

After one day: your position wrong by $\approx$ 2 km !!

(including general relativity error is 10 km per day, for the interested: backup slide, coffee break or after dinner discussions)

➡ **Countermeasures:**

(1) Minimum 4 satellites (avoid reference time on earth)

(2) Detune data transmission frequency from 1.023 MHz to 1.022999999543 MHz prior to launch