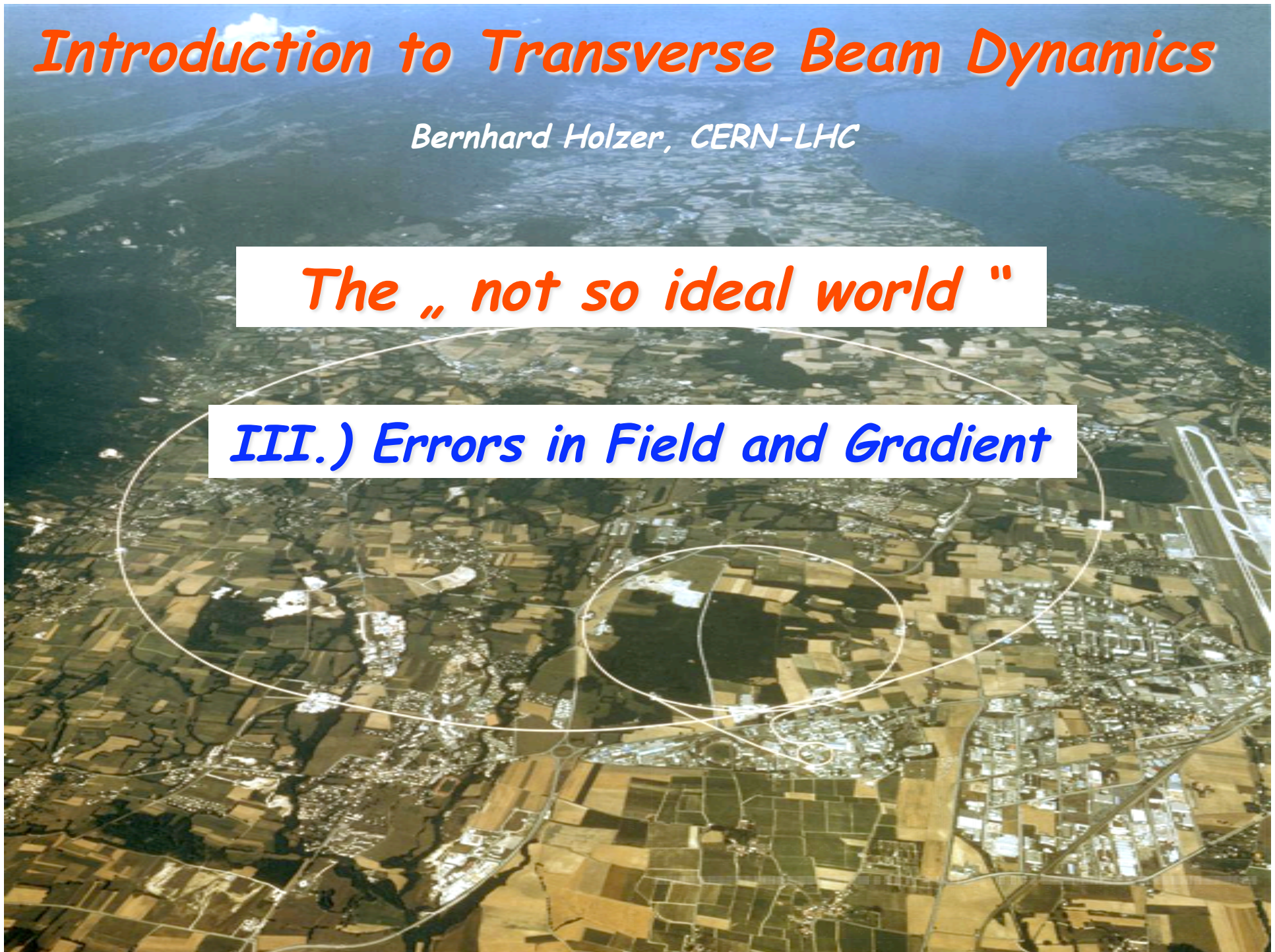


Introduction to Transverse Beam Dynamics

Bernhard Holzer, CERN-LHC

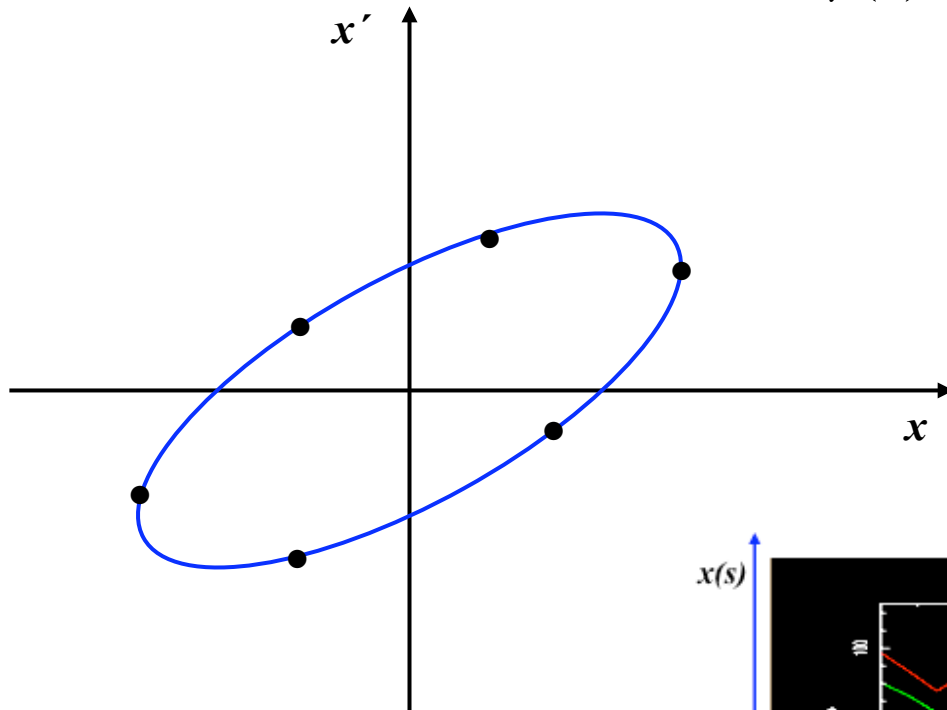
The „ not so ideal world “

III.) Errors in Field and Gradient



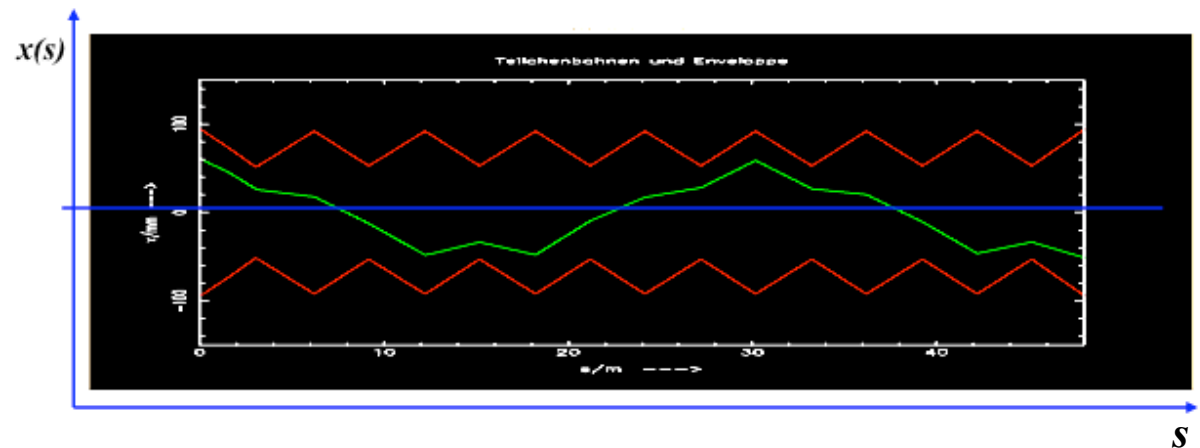
Remember: Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s)$$



Liouville: in reasonable storage rings area in phase space is constant.

$$A = \pi * \varepsilon = \text{const}$$



ε beam emittance = **woozilycity** of the particle ensemble, *intrinsic beam parameter*, cannot be changed by the foc. properties.

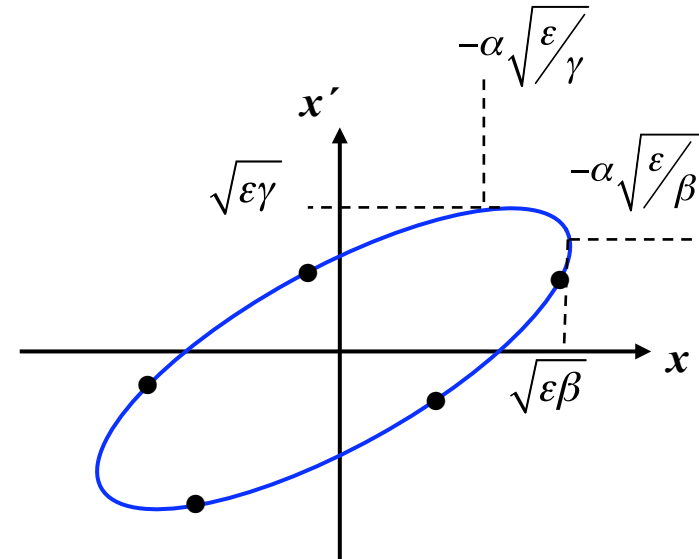
Scientifiquely spoken: area covered in transverse x, x' phase space ... and it is constant !!!

13.) Liouville during Acceleration

$$\varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s)$$

Beam Emittance corresponds to the area covered in the x, x' Phase Space Ellipse

Liouville: Area in phase space is constant.



But so sorry ... $\varepsilon \neq \text{const} !$

Classical Mechanics:

*phase space = diagram of the two canonical variables
position & momentum*

x p_x

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad ; \quad L = T - V = \text{kin. Energy} - \text{pot. Energy}$$

According to Hamiltonian mechanics:
 phase space diagram relates the variables q and p

$$q = \text{position} = x$$

$$p = \text{momentum} = \gamma m v = mc \gamma \beta_x$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}$$

Liouville's Theorem: $\int p dq = \text{const}$

for convenience (i.e. *because we are lazy bones*) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt} \frac{dt}{ds} = \frac{\beta_x}{\beta} \quad \text{where } \beta_x = v_x / c$$

$$\int p dq = mc \int \gamma \beta_x dx$$

$$\int p dq = mc \gamma \beta \underbrace{\int x' dx}_{\varepsilon}$$

$$\Rightarrow \varepsilon = \int x' dx \propto \frac{1}{\beta \gamma}$$

*the beam emittance
 shrinks during
 acceleration $\varepsilon \sim 1/\gamma$*

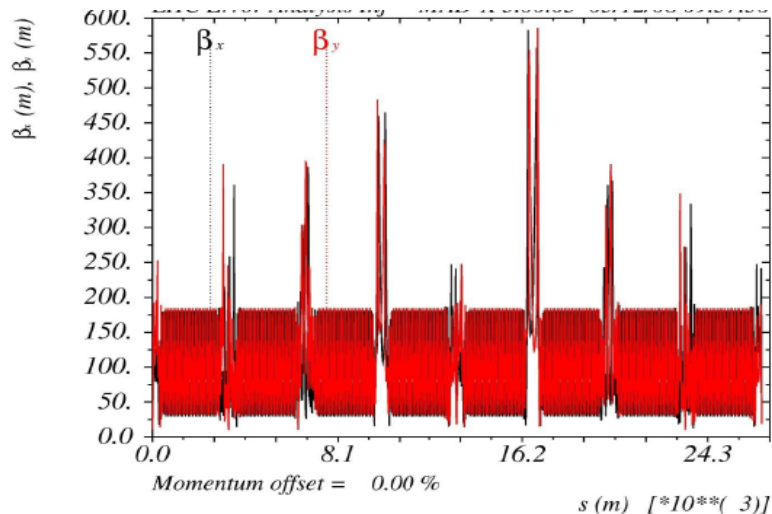
Nota bene:

1.) *A proton machine ... or an electron linac ... needs the highest aperture at injection energy !!!
as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.*

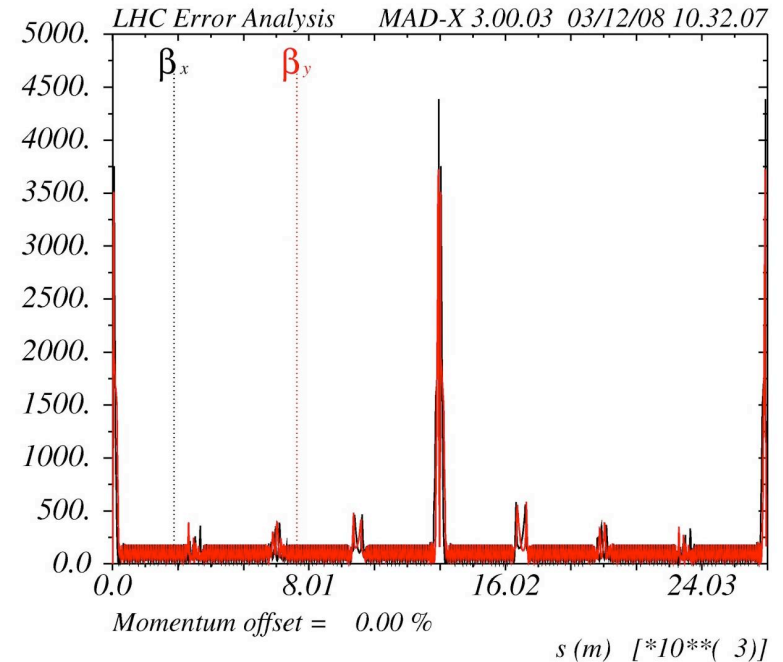
$$\sigma = \sqrt{\epsilon\beta}$$

2.) *At lowest energy the machine will have the major aperture problems,
→ here we have to minimise $\hat{\beta}$*

3.) *we need different beam optics adopted to the energy:
A Mini Beta concept will only be adequate at flat top.*



*LHC injection
optics at 450 GeV*

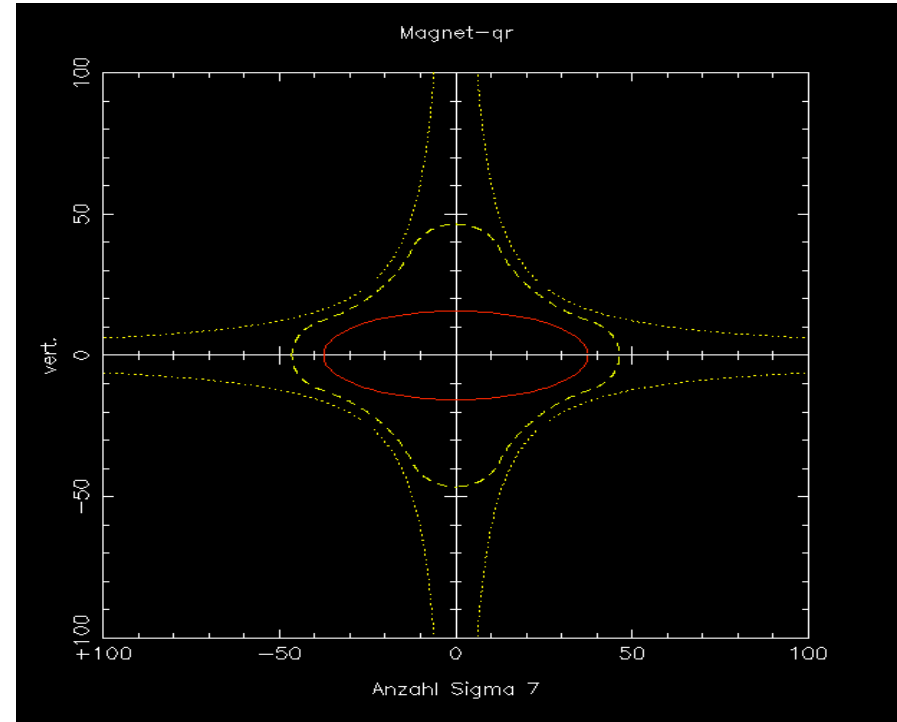
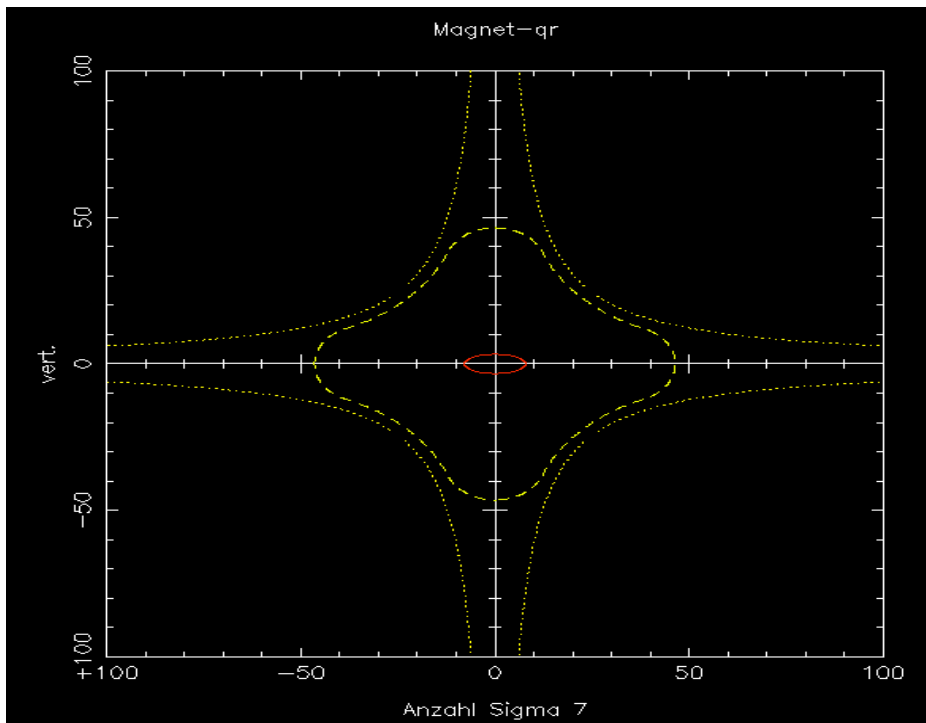


*LHC mini beta
optics at 7000 GeV*

Example: HERA proton ring

*injection energy: 40 GeV $\gamma = 43$
flat top energy: 920 GeV $\gamma = 980$*

*emittance ε (40GeV) = $1.2 * 10^{-7}$
 ε (920GeV) = $5.1 * 10^{-9}$*



7 σ beam envelope at E = 40 GeV

... and at E = 920 GeV

The „ not so ideal world “

14.) The „ $\Delta p / p \neq 0$ “ Problem

ideal accelerator: all particles will see the same accelerating voltage.

$$\rightarrow \Delta p / p = 0$$

„nearly ideal“ accelerator: Cockroft Walton or van de Graaf

$$\Delta p / p \approx 10^{-5}$$



Vivitron, Straßbourg, inner structure of the acc. section



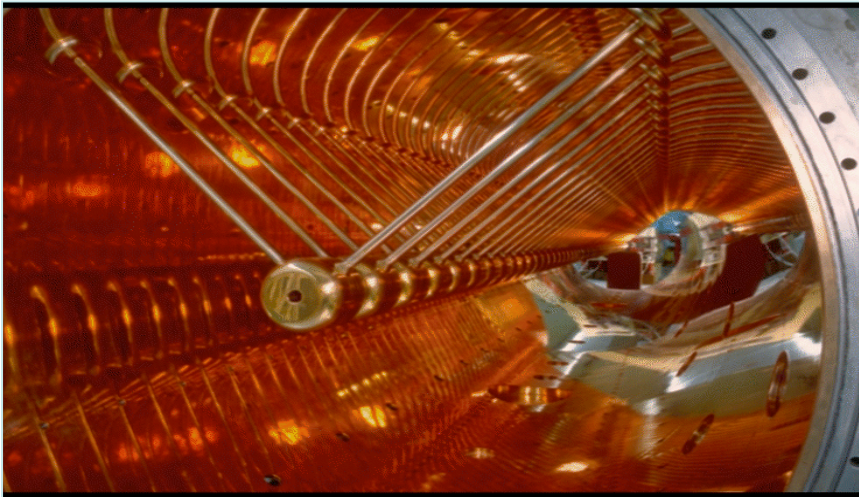
MP Tandem van de Graaf Accelerator at MPI for Nucl. Phys. Heidelberg

RF Acceleration

Energy Gain per „Gap“:

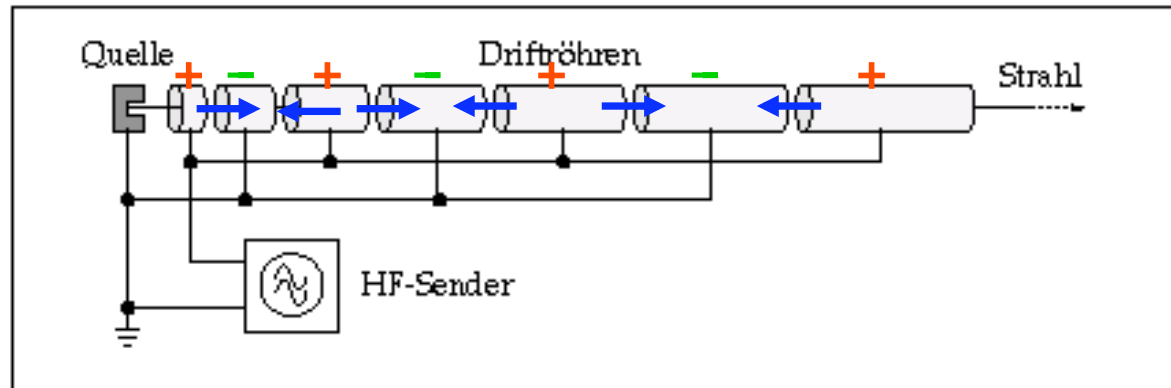
$$W = q U_0 \sin \omega_{RF} t$$

*drift tube structure at a proton linac
(GSI Unilac)*

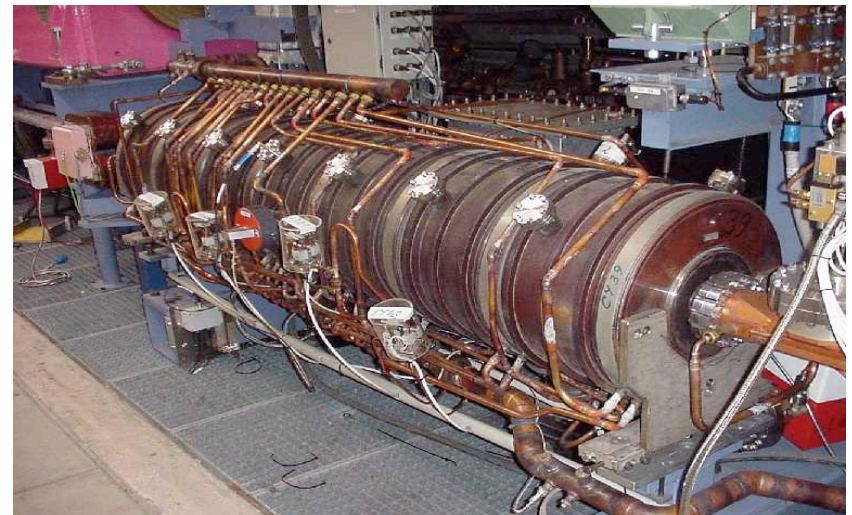


* **RF Acceleration:** multiple application of the same acceleration voltage;
brilliant idea to gain higher energies

1928, Wideroe

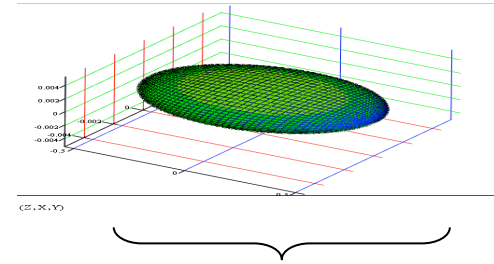


500 MHz cavities in an electron storage ring



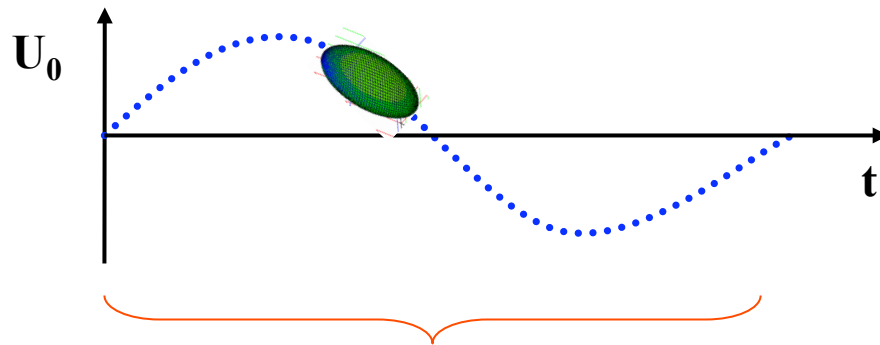
Problem: panta rhei !!!

(Heraklit: 540-480 v. Chr.)



Bunch length of Electrons $\approx 1\text{cm}$

Example: HERA RF:



$\lambda = 60\text{ cm}$

$$\left. \begin{aligned} \nu &= 500\text{ MHz} \\ c &= \lambda \nu \end{aligned} \right\} \lambda = 60\text{ cm}$$

$$\sin(90^\circ) = 1$$

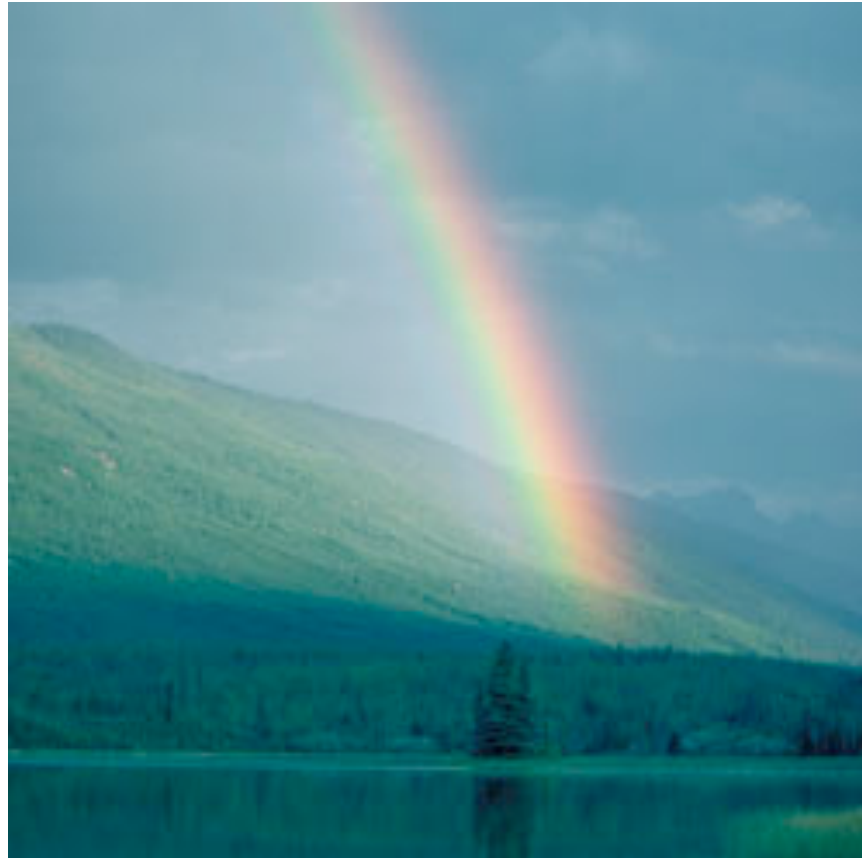
$$\sin(84^\circ) = 0.994$$

$$\frac{\Delta U}{U} = 6.0 \cdot 10^{-3}$$

typical momentum spread of an electron bunch:

$$\frac{\Delta p}{p} \approx 1.0 \cdot 10^{-3}$$

Dispersive and Chromatic Effects: $\Delta p/p \neq 0$



*Are there any Problems ???
Sure there are !!!*

*font colors due to
pedagogical reasons*

16.) Dispersion: trajectories for $\Delta p / p \neq 0$

Question: do you remember last session, page 12 ? ... sure you do

Force acting on the particle

$$F = m \frac{d^2}{dt^2} (x + \rho) - \frac{mv^2}{x + \rho} = e B_y v$$

remember: $x \approx mm$, $\rho \approx m$... \rightarrow develop for small x

$$m \frac{d^2 x}{dt^2} - \frac{mv^2}{\rho} \left(1 - \frac{x}{\rho}\right) = e B_y v$$

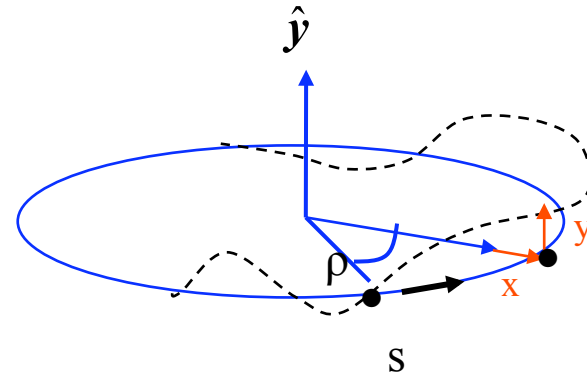
consider only linear fields, and change independent variable: $t \rightarrow s$

$$B_y = B_0 + x \frac{\partial B_y}{\partial x}$$

$$x'' - \frac{1}{\rho} \left(1 - \frac{x}{\rho}\right) = \frac{e B_0}{mv} + \frac{e x g}{mv}$$

$$p = p_0 + \Delta p$$

... but now take a small momentum error into account !!!



Dispersion:

develop for small momentum error

$$\Delta p \ll p_0 \Rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{e B_0}{p_0}}_{-\frac{1}{\rho}} - \frac{\Delta p}{p_0^2} e B_0 + \underbrace{\frac{x e g}{p_0}}_{k * x} - \underbrace{x e g \frac{\Delta p}{p_0^2}}_{\approx 0}$$

$$x'' + \frac{x}{\rho^2} \approx \frac{\Delta p}{p_0} * \underbrace{\frac{(-e B_0)}{p_0}}_{\frac{1}{\rho}} + k * x = \frac{\Delta p}{p_0} * \frac{1}{\rho} + k * x$$

$$x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0} \frac{1}{\rho} \quad \longrightarrow \quad x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0} \frac{1}{\rho}$$

Momentum spread of the beam adds a term on the r.h.s. of the equation of motion.
→ inhomogeneous differential equation.

Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \frac{1}{\rho} \cdot \frac{\Delta p}{p} \end{cases}$$

Normalise with respect to $\Delta p/p$:

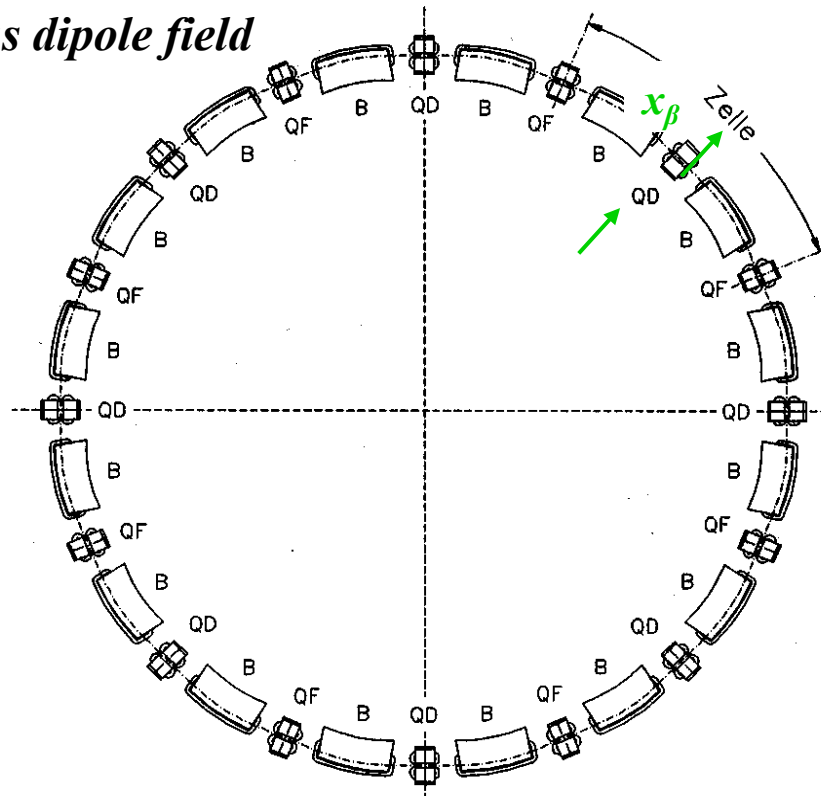
$$D(s) = \frac{x_i(s)}{\frac{\Delta p}{p}}$$

Dispersion function $D(s)$

- * is that **special orbit**, an **ideal particle** would have for $\Delta p/p = 1$
- * the **orbit of any particle** is the **sum** of the well known x_β and the **dispersion**
- * as **$D(s)$ is just another orbit** it will be subject to the focusing properties of the lattice

Dispersion

Example: homogeneous dipole field



valid for $\Delta p/p > 0$

$$: D(s) \cdot \frac{\Delta p}{p}$$

Matrix formalism:

$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x'_0 + D(s) \cdot \frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}$$

$$C = \cos(\sqrt{|k|}s) \quad S = \frac{1}{\sqrt{|k|}} \sin(\sqrt{|k|}s)$$

$$C' = \frac{dC}{ds} \quad S' = \frac{dS}{ds}$$

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$

Example

$$x_\beta = 1 \dots 2 \text{ mm}$$

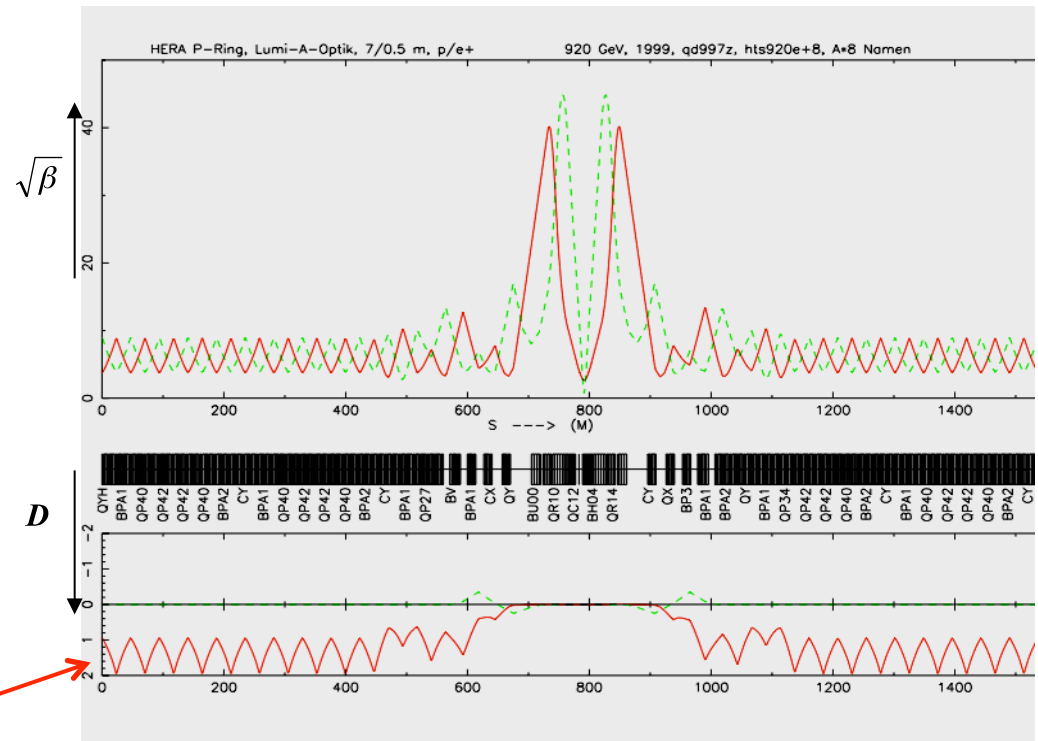
$$D(s) \approx 1 \dots 2 \text{ m}$$

$$\Delta p/p \approx 1 \cdot 10^{-3}$$

Amplitude of Orbit oscillation

contribution due to Dispersion \approx beam size

\rightarrow Dispersion must vanish at the collision point



Calculate D, D' : ... takes a couple of sunny Sunday evenings !

$$D(s) = S(s) \int_{s_0}^{s_1} \frac{1}{\rho} C(\tilde{s}) d\tilde{s} - C(s) \int_{s_0}^{s_1} \frac{1}{\rho} S(\tilde{s}) d\tilde{s}$$

Example: Drift

$$M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$D(s) = S(s) \underbrace{\int_{s_0}^{s_1} \frac{1}{\rho} C(\tilde{s}) d\tilde{s}}_{=0} - C(s) \underbrace{\int_{s_0}^{s_1} \frac{1}{\rho} S(\tilde{s}) d\tilde{s}}_{=0}$$

Example: Dipole

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s) \\ -\sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

$$K = \frac{1}{\rho^2} - \cancel{K}$$

$$s = l_B$$

$$M_{Dipole} = \begin{pmatrix} \cos \frac{l}{\rho} & \rho \sin \frac{l}{\rho} \\ -\frac{1}{\rho} \sin \frac{l}{\rho} & \cos \frac{l}{\rho} \end{pmatrix}$$

→

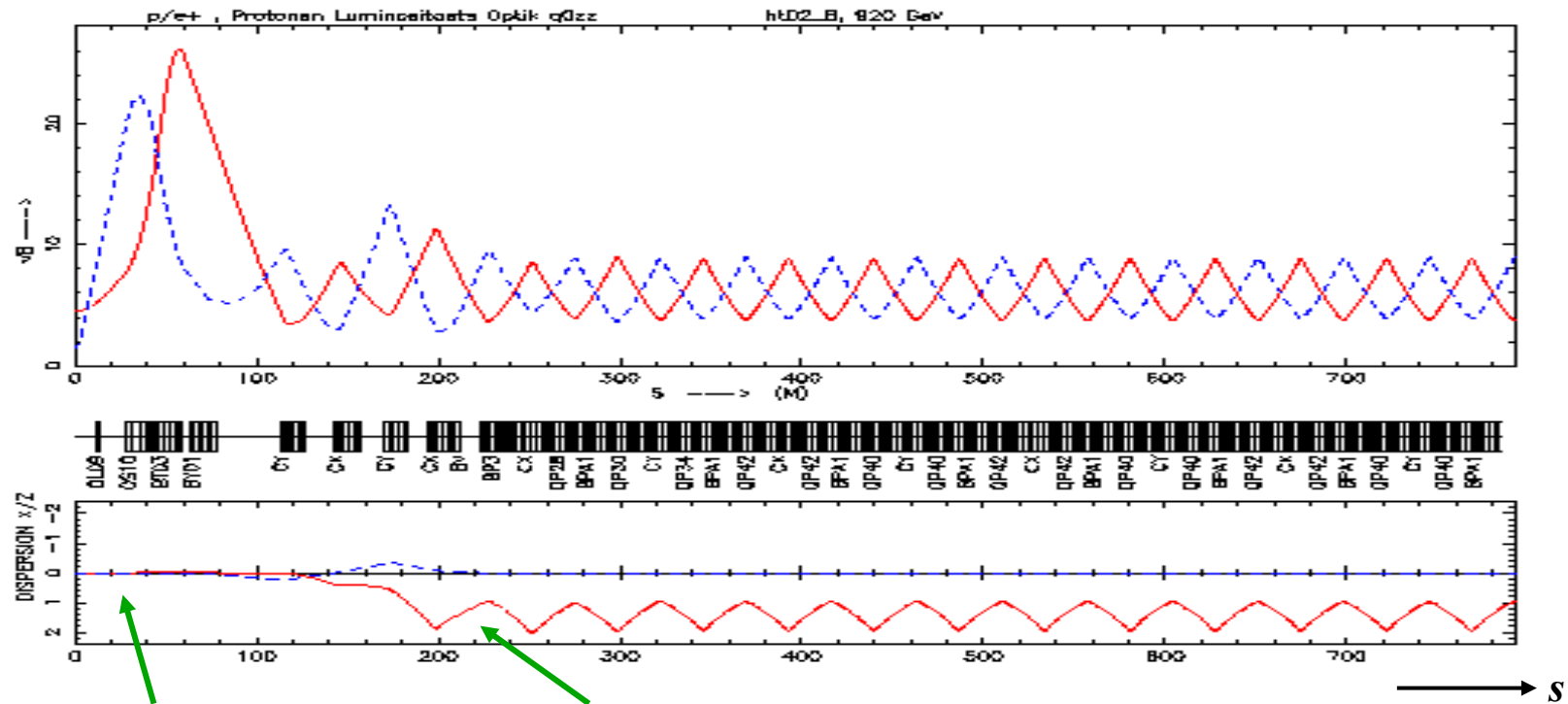
$$D(s) = \rho \cdot \left(1 - \cos \frac{l}{\rho}\right)$$

$$D'(s) = \sin \frac{l}{\rho}$$

Example: Dispersion, calculated by an optics code for a real machine

$$x_D = D(s) \frac{\Delta p}{p}$$

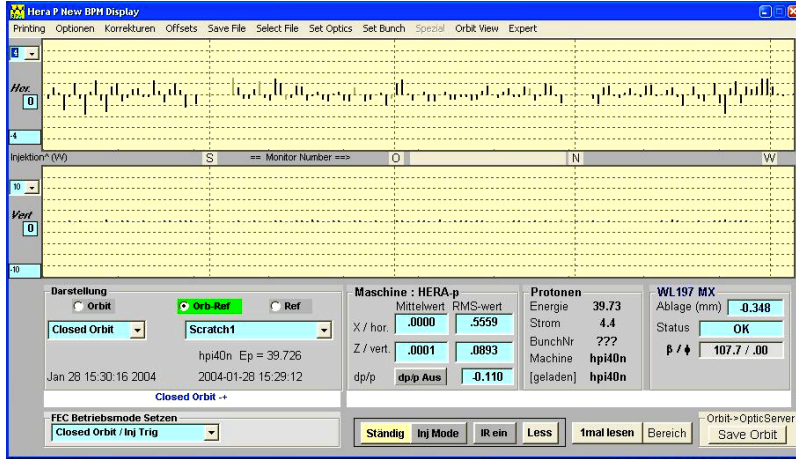
- * *D(s) is created by the dipole magnets*
... and afterwards focused by the quadrupole fields



*Mini Beta Section,
 → no dipoles !!!*

$D(s) \approx 1 \dots 2 \text{ m}$

Dispersion is visible



HERA Standard Orbit

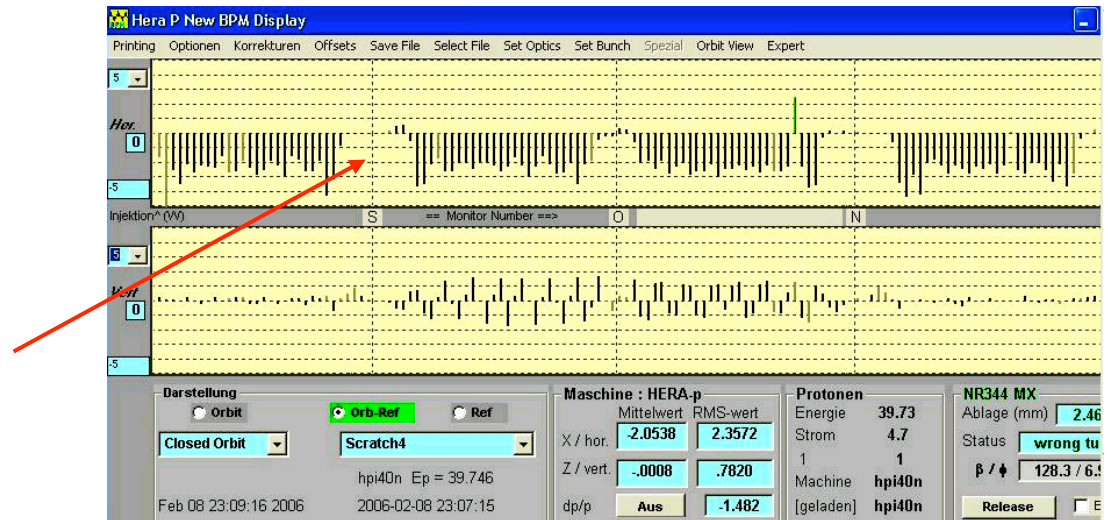
dedicated energy change of the stored beam

→ closed orbit is moved to a dispersions trajectory

$$x_d = D(s) * \frac{\Delta p}{p}$$

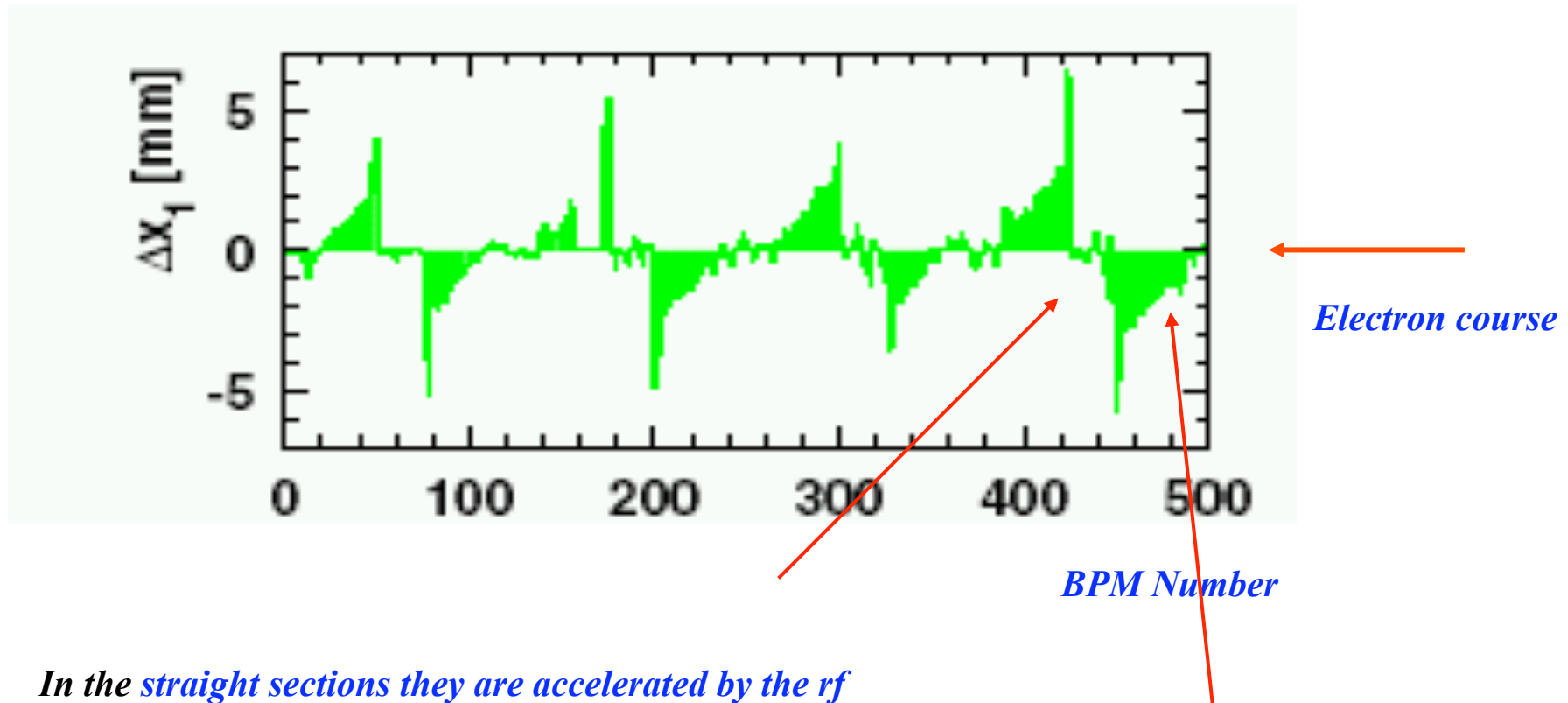
Attention: at the Interaction Points we require $D=D'=0$

HERA Dispersion Orbit



Periodic Dispersion:

„Sawtooth Effect“ at LEP (CERN)



In the straight sections they are accelerated by the rf cavities so much that they „overshoot“ and reach nearly the outer side of the vacuum chamber.

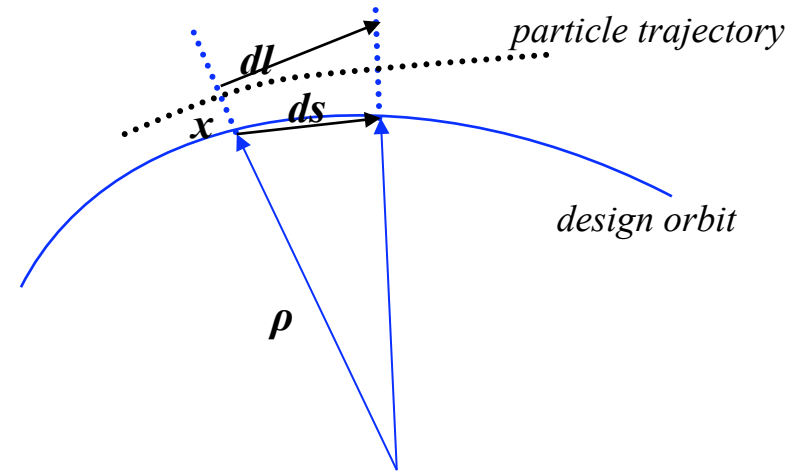
In the arc the electron beam loses so much energy in each octant that the particle are running more and more on a dispersion trajectory.

17.) Momentum Compaction Factor: α_p

particle with a *displacement x* to the design orbit
 → *path length dl* ...

$$\frac{dl}{ds} = \frac{\rho + x}{\rho}$$

$$\rightarrow dl = \left(1 + \frac{x}{\rho(s)} \right) ds$$



circumference of an off-energy closed orbit

$$l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}}{\rho(s)} \right) ds$$

remember:

$$x_{\Delta E}(s) = D(s) \frac{\Delta p}{p}$$

$$\delta l_{\Delta E} = \frac{\Delta p}{p} \oint \left(\frac{D(s)}{\rho(s)} \right) ds$$

** The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.*

Definition:

$$\frac{\delta l_\epsilon}{L} = \alpha_p \frac{\Delta p}{p}$$

$$\rightarrow \alpha_p = \frac{1}{L} \oint \left(\frac{D(s)}{\rho(s)} \right) ds$$

For first estimates assume:

$$\frac{1}{\rho} = \text{const.}$$

$$\int_{\text{dipoles}} D(s) ds \approx l_{\Sigma(\text{dipoles})} \cdot \langle D \rangle_{\text{dipole}}$$

$$\alpha_p = \frac{1}{L} l_{\Sigma(\text{dipoles})} \cdot \langle D \rangle \frac{1}{\rho} = \frac{1}{L} 2\pi\rho \cdot \langle D \rangle \frac{1}{\rho} \rightarrow \alpha_p \approx \frac{2\pi}{L} \langle D \rangle \approx \frac{\langle D \rangle}{R}$$

Assume: $v \approx c$

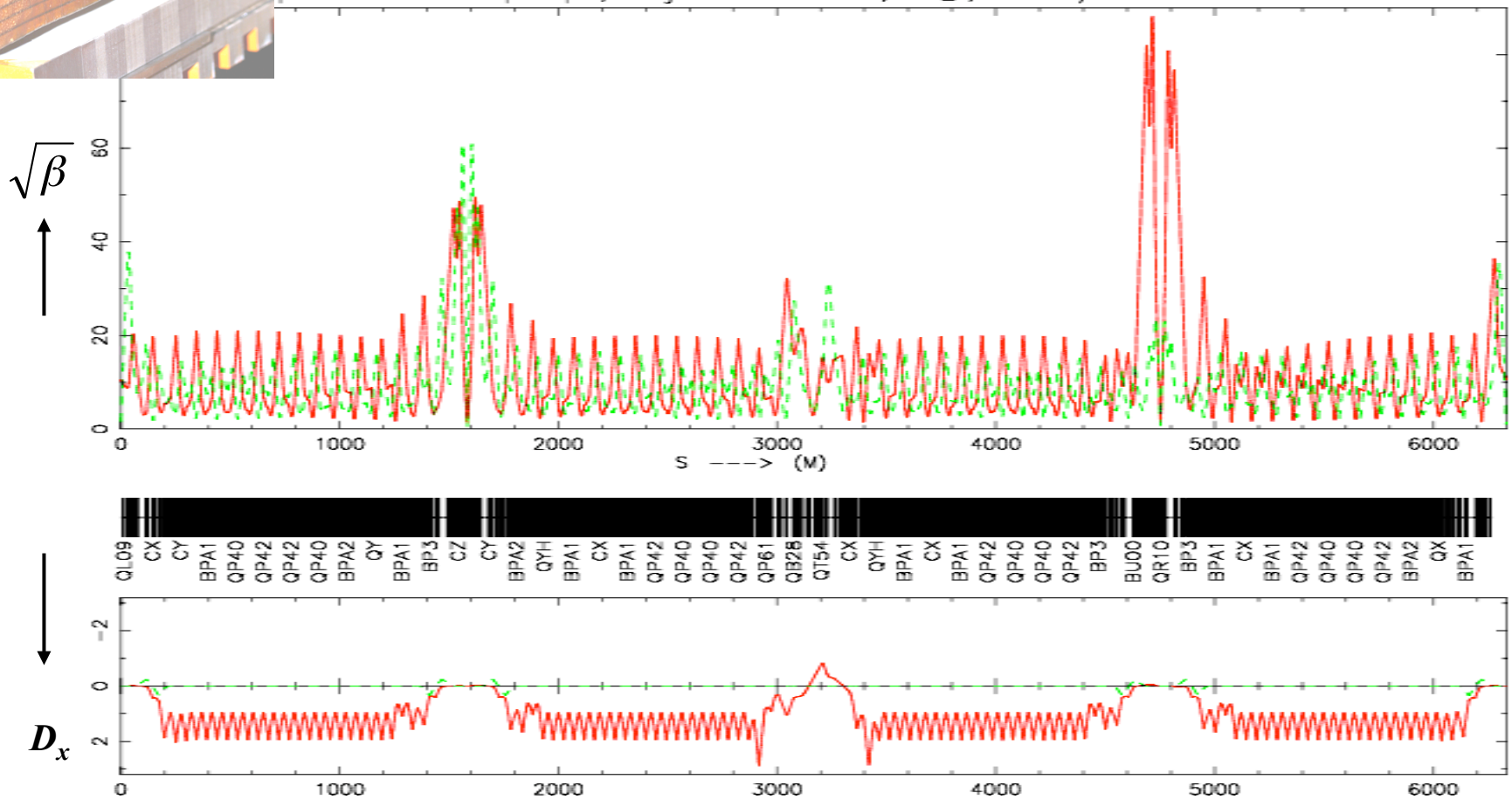
$$\rightarrow \frac{\delta T}{T} = \frac{\delta l_\epsilon}{L} = \alpha_p \frac{\Delta p}{p}$$

α_p combines via the dispersion function the momentum spread with the longitudinal motion of the particle.

18.) Quadrupole Errors



3zz Standard Lumi-Optik Optik, korrigierte Version von 2004, ht02_8, 920 GeV /27.5 GeV



Quadrupole Errors

go back to Lecture I, page 1
single particle trajectory

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M_{QF} * \begin{pmatrix} x \\ x' \end{pmatrix}_1$$

Solution of equation of motion

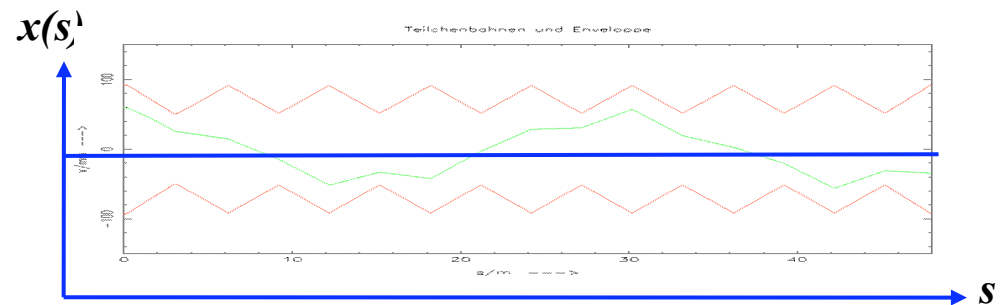
$$x = x_0 \cos(\sqrt{k} l_q) + x'_0 \frac{1}{\sqrt{k}} \sin(\sqrt{k} l_q)$$

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{k} l_q) & \frac{1}{\sqrt{k}} \sin(\sqrt{k} l_q) \\ -\sqrt{k} \sin(\sqrt{k} l_q) & \cos(\sqrt{k} l_q) \end{pmatrix}, \quad M_{thinlens} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

$$M_{turn} = M_{QF} * M_{D1} * M_{QD} * M_{D2} * M_{QF} \dots$$

Definition: phase advance of the particle oscillation per revolution in units of 2π is called **tune**

$$Q = \frac{\psi_{turn}}{2\pi}$$



Matrix in Twiss Form

Transfer Matrix from point „0“ in the lattice to point „s“:

$$M(s) = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}} (\cos\psi_s + \alpha_0 \sin\psi_s) & \sqrt{\beta_s \beta_0} \sin\psi_s \\ \frac{(\alpha_0 - \alpha_s) \cos(\psi_s) - (1 + \alpha_0 \alpha_s) \sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}} (\cos(\psi_s) - \alpha_0 \sin\psi_s) \end{pmatrix}$$



For one complete turn the Twiss parameters have to obey periodic boundary conditions:

$$\beta(s + L) = \beta(s)$$

$$\alpha(s + L) = \alpha(s)$$

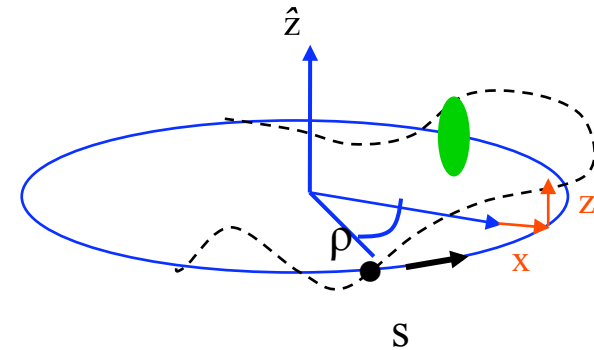
$$\gamma(s + L) = \gamma(s)$$

$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_s & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

Quadrupole Error in the Lattice

optic *perturbation* described by *thin lens quadrupole*

$$M_{dist} = M_{\Delta k} \cdot M_0 = \underbrace{\begin{pmatrix} 1 & 0 \\ \Delta k ds & 1 \end{pmatrix}}_{\text{quad error}} \cdot \underbrace{\begin{pmatrix} \cos\psi_{turn} + \alpha \sin\psi_{turn} & \beta \sin\psi_{turn} \\ -\gamma \sin\psi_{turn} & \cos\psi_{turn} - \alpha \sin\psi_{turn} \end{pmatrix}}_{\text{ideal storage ring}}$$



$$M_{dist} = \begin{pmatrix} \cos\psi_0 + \alpha \sin\psi_0 & \beta \sin\psi_0 \\ \Delta k ds (\cos\psi_0 + \alpha \sin\psi_0) - \gamma \sin\psi_0 & \Delta k ds \beta \sin\psi_0 + \cos\psi_0 - \alpha \sin\psi_0 \end{pmatrix}$$

rule for getting the tune

$$\text{Trace}(M) = 2 \cos\psi = 2 \cos\psi_0 + \Delta k ds \beta \sin\psi_0$$

Quadrupole error \rightarrow Tune Shift

$$\psi = \psi_0 + \Delta\psi \quad \longrightarrow \quad \cos(\psi_0 + \Delta\psi) = \cos\psi_0 + \frac{\Delta k ds \beta \sin\psi_0}{2}$$

remember the old fashioned trigonometric stuff and **assume that the error is small !!!**

$$\underbrace{\cos\psi_0 \cos\Delta\psi}_{\approx 1} - \underbrace{\sin\psi_0 \sin\Delta\psi}_{\approx \Delta\psi} = \cos\psi_0 + \frac{k ds \beta \sin\psi_0}{2}$$

$$\Delta\psi = \frac{k ds \beta}{2}$$

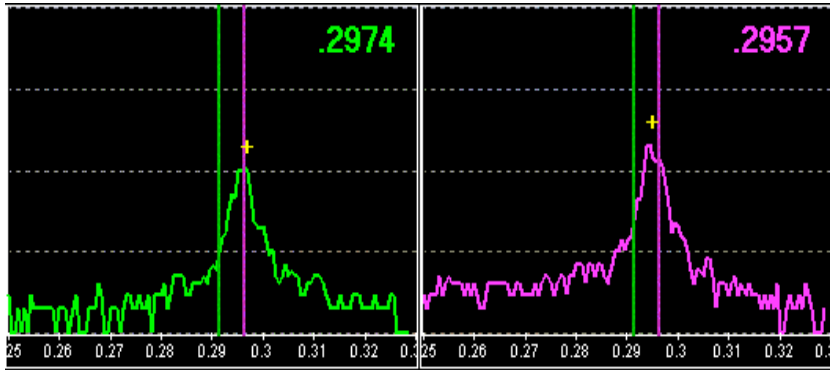
and referring to Q instead of ψ :

$$\psi = 2\pi Q$$

$$\Delta Q = \int_{s_0}^{s_0+l} \frac{\Delta k(s) \beta(s) ds}{4\pi}$$

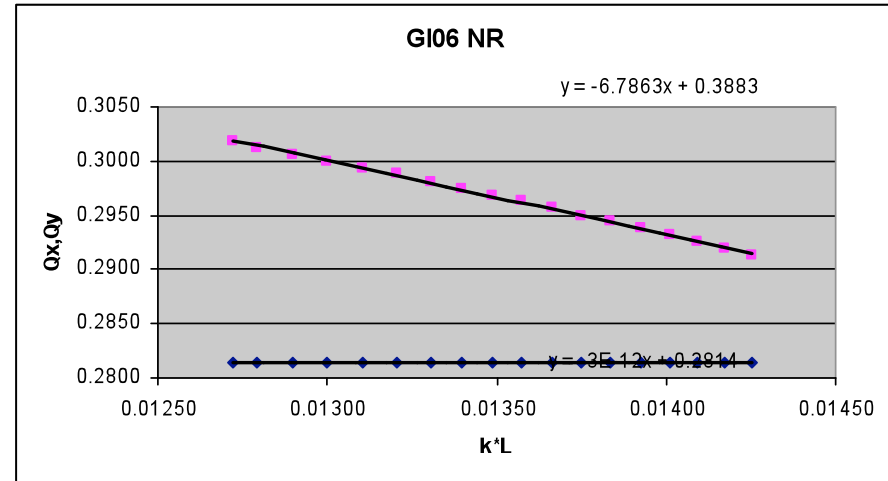
- ! the tune shift is **proportional to the β -function** at the quadrupole
- !! field quality, power supply tolerances etc are **much tighter at places where β is large**
- !!! mini beta quads: $\beta \approx 1900$ m
arc quads: $\beta \approx 80$ m
- !!!! β is a measure for the sensitivity of the beam

a quadrupol error leads to a shift of the tune:



$$\Delta Q = \int_{s_0}^{s_0+l} \frac{\Delta k \beta(s)}{4\pi} ds \approx \frac{\Delta k l_{quad} \bar{\beta}}{4\pi}$$

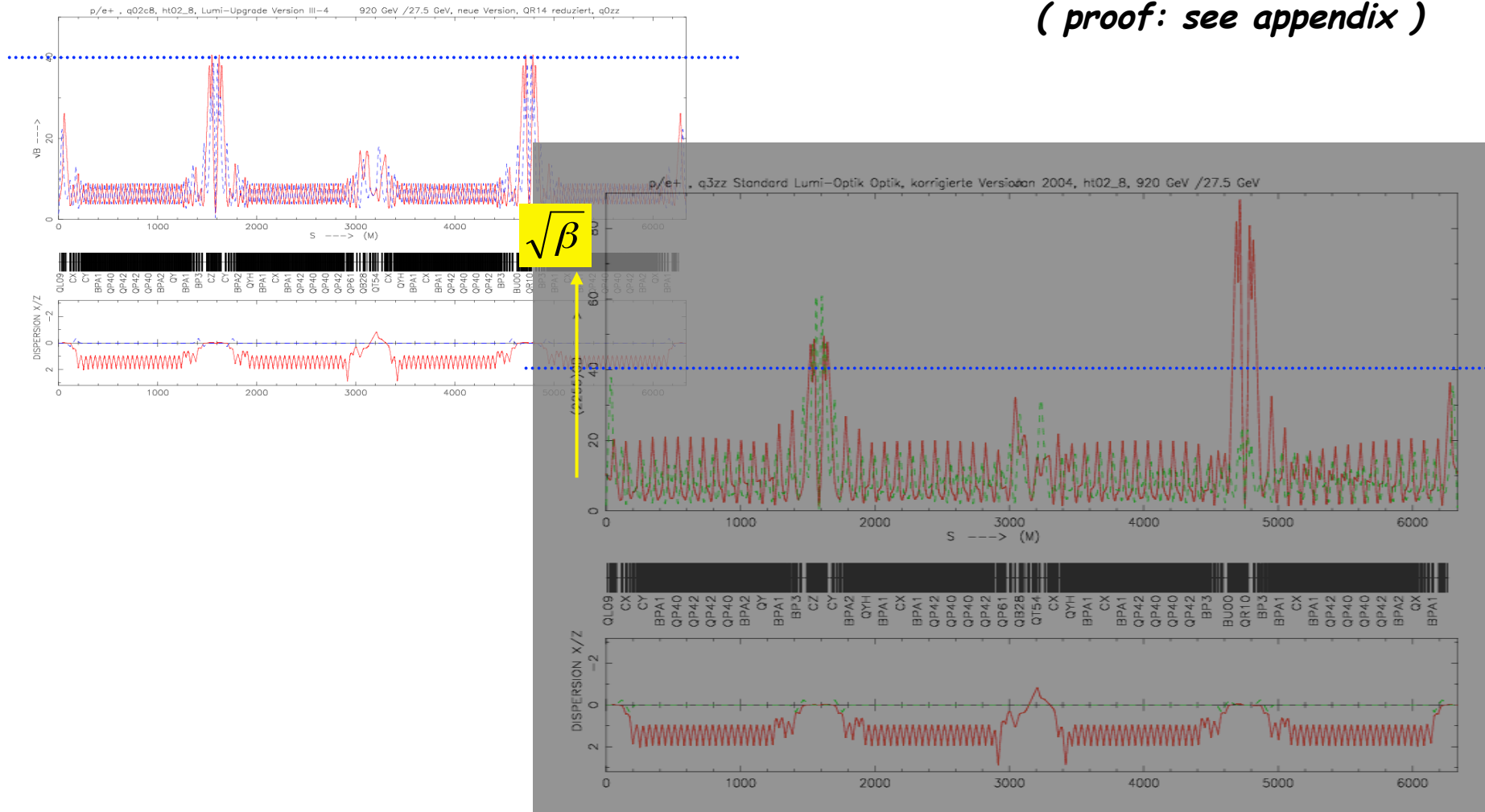
*Example: measurement of β in a storage ring:
tune spectrum*



Quadrupole error: Beta Beat

$$\Delta\beta(s_0) = \frac{\beta_0}{2 \sin 2\pi Q} \int_{s_1}^{s_1+l} \beta(s_1) \Delta K \cos(2|\psi_{s_1} - \psi_{s_0}| - 2\pi Q) ds$$

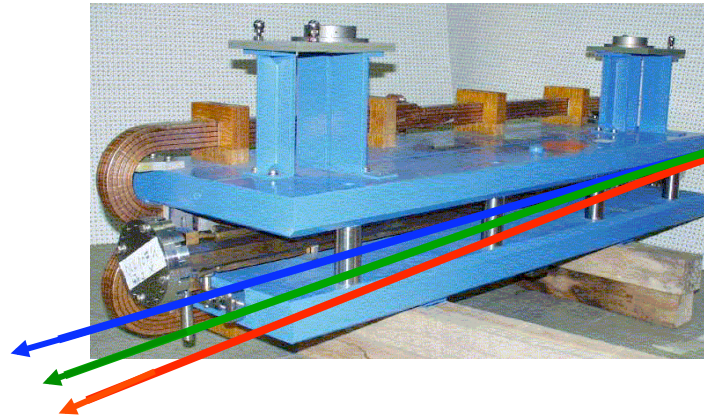
(proof: see appendix)



19.) Chromaticity: A Quadrupole Error for $\Delta p/p \neq 0$

Influence of external fields on the beam: *prop. to magn. field & prop. zu $1/p$*

dipole magnet $\alpha = \frac{\int B dl}{p/e}$



$$x_D(s) = D(s) \frac{\Delta p}{p}$$

focusing lens $k = \frac{g}{p/e}$

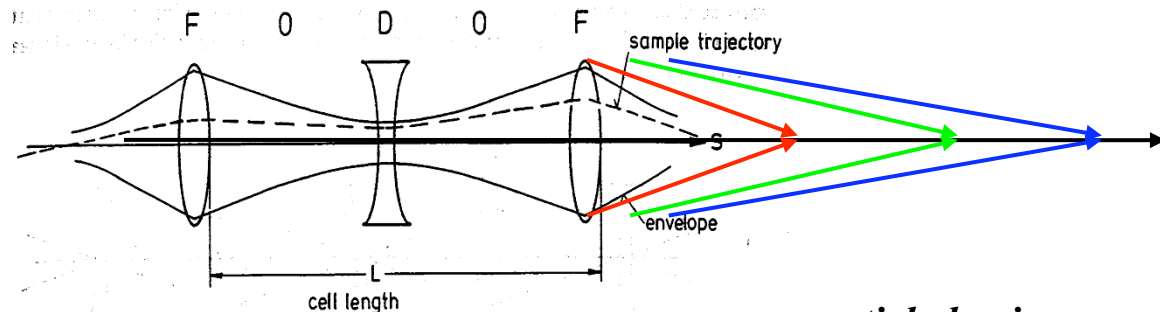


Figure 29: FODO cell

particle having ...
to high energy
to low energy
ideal energy

Chromaticity: Q'

$$k = \frac{g}{\frac{p}{e}} \qquad p = p_0 + \Delta p$$

in case of a momentum spread:

$$k = \frac{eg}{p_0 + \Delta p} \approx \frac{e}{p_0} \left(1 - \frac{\Delta p}{p_0}\right) g = k_0 + \Delta k$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

... which acts like a quadrupole error in the machine and leads to a tune spread:

$$\Delta Q = -\frac{1}{4\pi} \frac{\Delta p}{p_0} k_0 \beta(s) ds$$

definition of chromaticity:

$$\Delta Q = Q' \frac{\Delta p}{p} \quad ; \quad Q' = -\frac{1}{4\pi} \oint k(s) \beta(s) ds$$

Where is the Problem ?

Resume':

quadrupole error: tune shift

$$\Delta Q \approx \int_{s_0}^{s_0+l} \frac{\Delta k(s) \beta(s)}{4\pi} ds \approx \frac{\Delta k(s) l_{quad} \bar{\beta}}{4\pi}$$

beta beat

$$\Delta\beta(s_0) = \frac{\beta_0}{2 \sin 2\pi Q} \int_{s_1}^{s_1+l} \beta(s_1) \Delta k \cos(2(\psi_{s_1} - \psi_{s_0}) - 2\pi Q) ds$$

chromaticity

$$\Delta Q = Q' \frac{\Delta p}{p}$$

$$Q' = -\frac{1}{4\pi} \oint k(s) \beta(s) ds$$

momentum compaction

$$\frac{\delta l_\varepsilon}{L} = \alpha_p \frac{\Delta p}{p}$$

$$\alpha_p \approx \frac{2\pi}{L} \langle \mathbf{D} \rangle \approx \frac{\langle \mathbf{D} \rangle}{R}$$

beta function in a symmetric drift

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

Appendix I:

Dispersion: Solution of the inhomogeneous equation of motion

Ansatz:
$$D(s) = S(s) \int_{s_0}^{s_1} \frac{1}{\rho} C(\tilde{s}) d\tilde{s} - C(s) \int_{s_0}^{s_1} \frac{1}{\rho} S(\tilde{s}) d\tilde{s}$$

$$D'(s) = S' * \int \frac{1}{\rho} C dt + S \cancel{\frac{1}{\rho} C} - C' * \int \frac{1}{\rho} S dt - C \cancel{\frac{1}{\rho} S}$$

$$D'(s) = S' * \int \frac{C}{\rho} dt - C' * \int \frac{S}{\rho} dt$$

$$\begin{aligned} D''(s) &= S'' * \int \frac{C}{\rho} d\tilde{s} + S' \frac{C}{\rho} - C'' * \int \frac{S}{\rho} d\tilde{s} - C' \frac{S}{\rho} \\ &= S'' * \int \frac{C}{\rho} d\tilde{s} - C'' * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho} \underbrace{(CS' - S C')} \\ &= \det M = 1 \end{aligned}$$

remember: for $C(s)$ and $S(s)$ to be independent solutions the Wronski determinant has to meet the condition

$$W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} \neq 0$$

and as it is independent of the variable „s“ $\frac{dW}{ds} = \frac{d}{ds}(CS' - SC') = CS'' - SC'' = -K(CS - SC) = 0$

we get for the initial conditions that we had chosen ... $C_0 = 1, C'_0 = 0$
 $S_0 = 0, S'_0 = 1$ } $W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} = 1$

$$D'' = S'' * \int \frac{C}{\rho} d\tilde{s} - C'' * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$


remember: S & C are solutions of the homog. equation of motion:

$$S'' + K * S = 0$$

$$C'' + K * C = 0$$

$$D'' = -K * S * \int \frac{C}{\rho} d\tilde{s} + K * C * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

$$D'' = -K * \left\{ S \int \frac{C}{\rho} d\tilde{s} + C \int \frac{S}{\rho} d\tilde{s} \right\} + \frac{1}{\rho}$$


 $=D(s)$

$$D'' = -K * D + \frac{1}{\rho}$$

... or

$$\underline{\underline{D'' + K * D = \frac{1}{\rho}}}$$

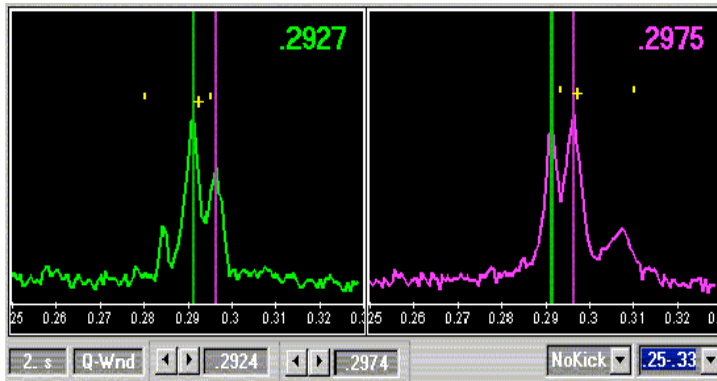
qed

Appendix II:

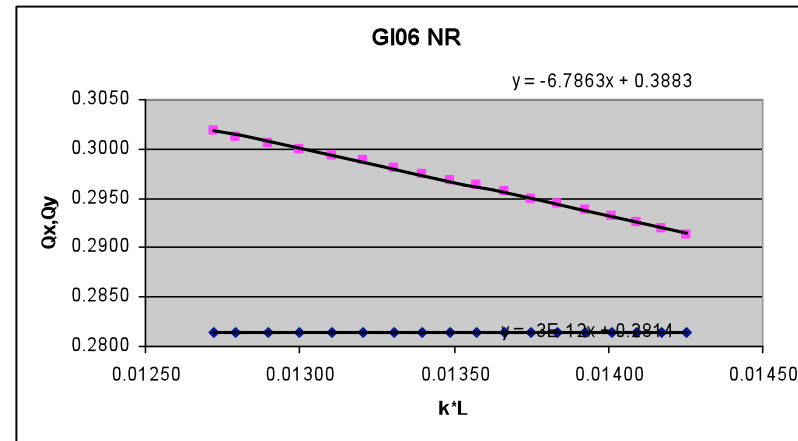
Quadrupole Error and Beta Function

a change of quadrupole strength in a synchrotron leads to tune shift:

$$\Delta Q \approx \int_{s_0}^{s_0+l} \frac{\Delta k(s) \beta(s)}{4\pi} ds \approx \frac{\Delta k(s) * l_{quad} * \bar{\beta}}{4\pi}$$



tune spectrum ...



tune shift as a function of a gradient change

*But we should expect an error in the β -function as well ...
... shouldn't we ???*

Quadrupole Errors and Beta Function

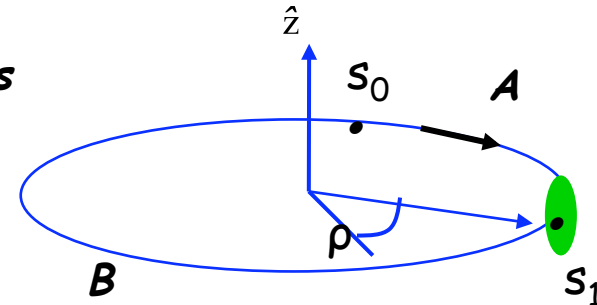
a quadrupole error will not only influence the oscillation frequency ... „tune“
... but also the amplitude ... „beta function“

split the ring into 2 parts, described by two matrices
A and B

$$M_{turn} = B * A$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$



matrix of a quad error
between A and B

$$M_{dist} = \begin{pmatrix} m_{11}^* & m_{12}^* \\ m_{21}^* & m_{22}^* \end{pmatrix} = B \begin{pmatrix} 1 & 0 \\ -\Delta k ds & 1 \end{pmatrix} A$$

$$M_{dist} = B \begin{pmatrix} a_{11} & a_{12} \\ -\Delta k ds a_{11} + a_{12} & -\Delta k ds a_{12} + a_{22} \end{pmatrix}$$

$$M_{dist} = \begin{pmatrix} \sim & b_{11} a_{12} + b_{12} (-\Delta k ds a_{12} + a_{22}) \\ \sim & \sim \end{pmatrix}$$

the beta function is usually obtained via the matrix element „m12“, which is in Twiss form for the undistorted case

$$m_{12} = \beta_0 \sin 2\pi Q$$

and including the error:

$$m_{12}^* = \underbrace{b_{11}a_{12} + b_{12}a_{22} - b_{12}a_{12}}_{m_{12}} \Delta k ds$$

$$m_{12} = \beta_0 \sin 2\pi Q$$

$$(1) \quad m_{12}^* = \beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds$$

As M^* is still a matrix for one complete turn we still can express the element m_{12} in twiss form:

$$(2) \quad m_{12}^* = (\beta_0 + d\beta)^* \sin 2\pi(Q + dQ)$$

Equalising (1) and (2) and assuming a small error

$$\beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds = (\beta_0 + d\beta)^* \sin 2\pi(Q + dQ)$$

$$\beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds = (\beta_0 + d\beta)^* \sin 2\pi Q \underbrace{\cos 2\pi dQ}_{\approx 1} + \cos 2\pi Q \underbrace{\sin 2\pi dQ}_{\approx 2\pi dQ}$$

$$\cancel{\beta_0 \sin 2\pi Q} - a_{12} b_{12} \Delta k ds = \cancel{\beta_0 \sin 2\pi Q} + \beta_0 2\pi dQ \cos 2\pi Q + d\beta_0 \sin 2\pi Q + \cancel{d\beta_0 2\pi dQ \cos 2\pi Q}$$

ignoring second order terms

$$- a_{12} b_{12} \Delta k ds = \beta_0 2\pi dQ \cos 2\pi Q + d\beta_0 \sin 2\pi Q$$

remember: tune shift dQ due to quadrupole error: $dQ = \frac{\Delta k \beta_1 ds}{4\pi}$
 (index „1“ refers to location of the error)

$$- a_{12} b_{12} \Delta k ds = \frac{\beta_0 \Delta k \beta_1 ds}{2} \cos 2\pi Q + d\beta_0 \sin 2\pi Q$$

solve for $d\beta$

$$d\beta_0 = \frac{-1}{2 \sin 2\pi Q} \{ 2 a_{12} b_{12} + \beta_0 \beta_1 \cos 2\pi Q \} \Delta k ds$$

express the matrix elements a_{12} , b_{12} in Twiss form

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}} (\cos \psi_s + \alpha_0 \sin \psi_s) & \sqrt{\beta_s \beta_0} \sin \psi_s \\ \frac{(\alpha_0 - \alpha_s) \cos \psi_s - (1 + \alpha_0 \alpha_s) \sin \psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}} (\cos \psi_s - \alpha_s \sin \psi_s) \end{pmatrix}$$

$$d\beta_0 = \frac{-1}{2 \sin 2\pi Q} \{2a_{12}b_{12} + \beta_0\beta_1 \cos 2\pi Q\} \Delta k ds$$

$$a_{12} = \sqrt{\beta_0\beta_1} \sin \Delta\psi_{0 \rightarrow 1}$$

$$b_{12} = \sqrt{\beta_1\beta_0} \sin(2\pi Q - \Delta\psi_{0 \rightarrow 1})$$

$$d\beta_0 = \frac{-\beta_0\beta_1}{2 \sin 2\pi Q} \{2 \sin \Delta\psi_{12} \sin(2\pi Q - \Delta\psi_{12}) + \cos 2\pi Q\} \Delta k ds$$

... after some TLC transformations ... = $\cos(2\Delta\psi_{01} - 2\pi Q)$

$$\Delta\beta(s_0) = \frac{-\beta_0}{2 \sin 2\pi Q} \int_{s_1}^{s_1+l} \beta(s_1) \Delta k \cos(2(\psi_{s_1} - \psi_{s_0}) - 2\pi Q) ds$$

Nota bene: ! the beta beat is proportional to the strength of the error Δk

!! and to the β function at the place of the error ,

!!! and to the β function at the observation point,
(... remember orbit distortion !!!)

!!!! there is a resonance denominator