CERN

Materials & Properties II:
Thermal & Electrical Characteristics

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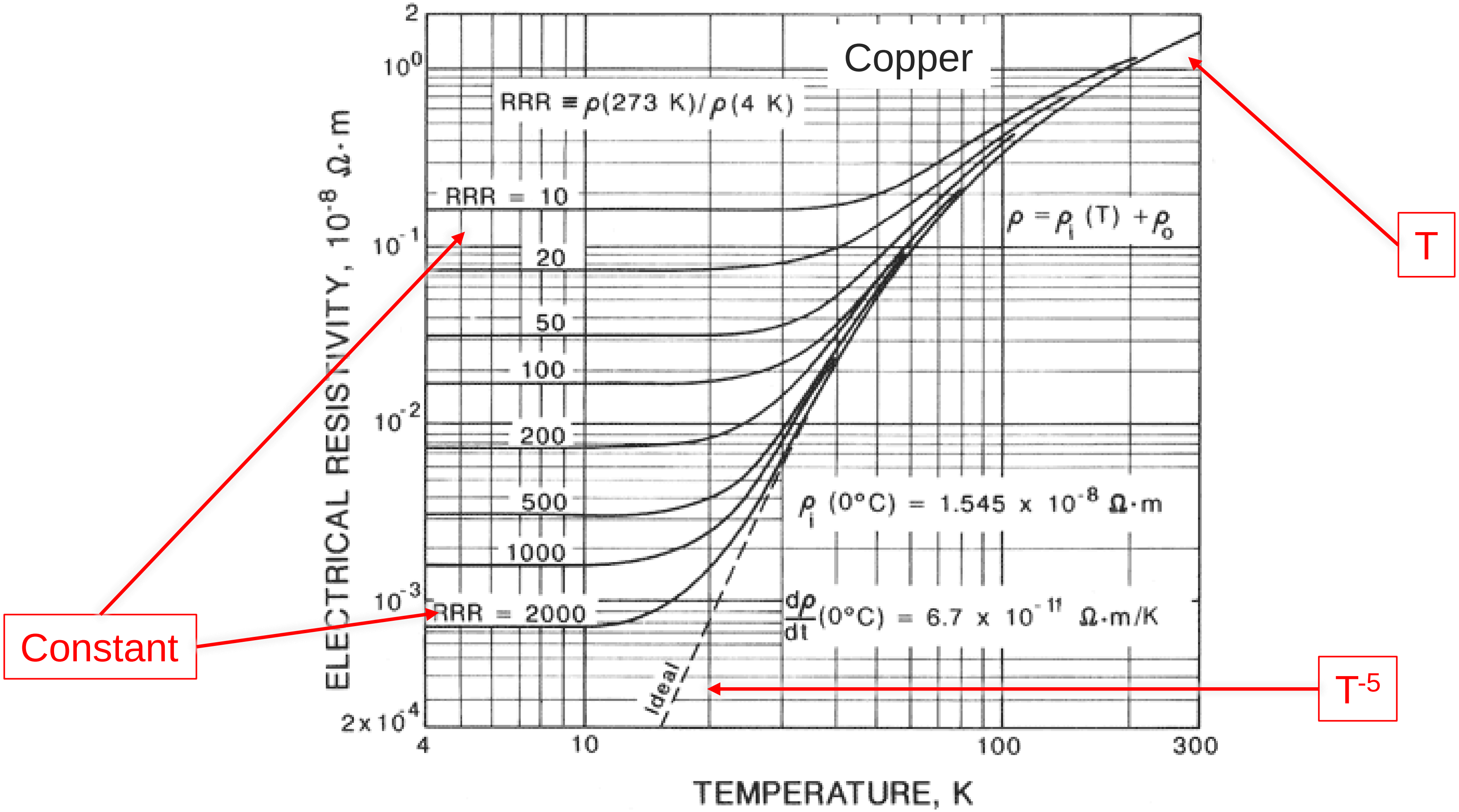
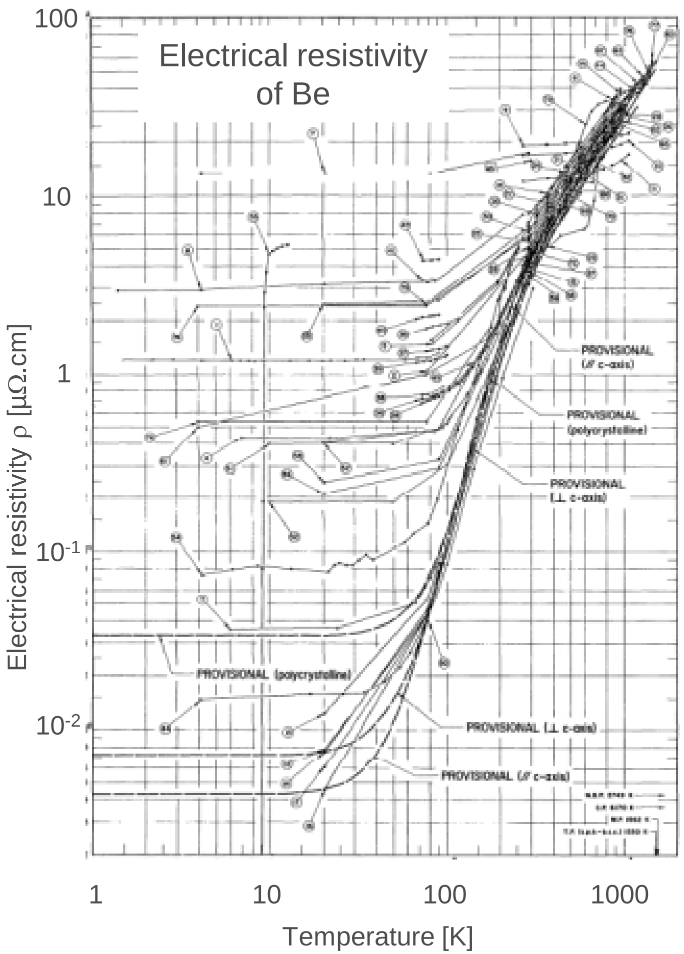
Outline (we will discuss mostly metals)

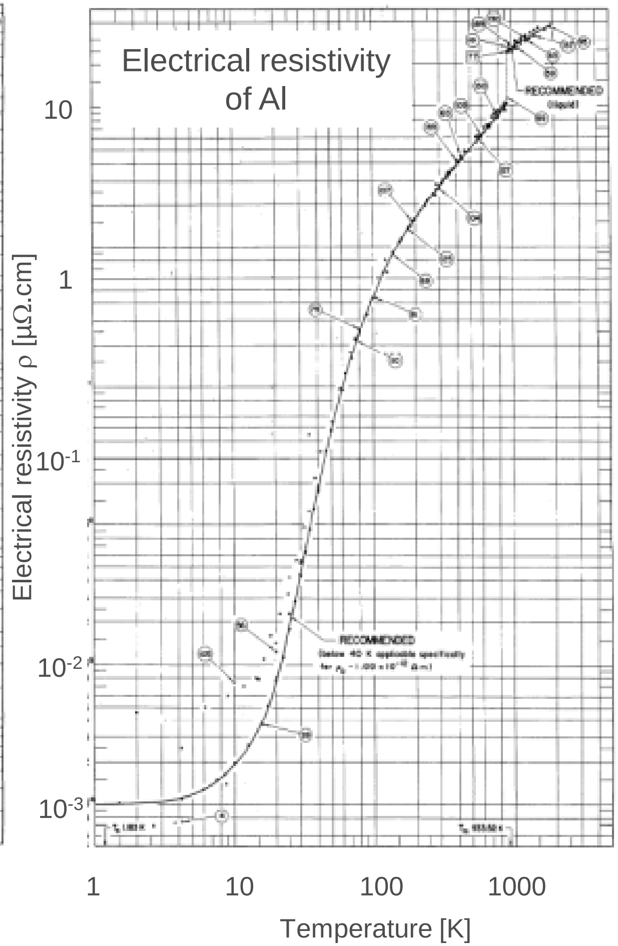
- Electrical properties
  - Electrical conductivity
    - Temperature dependence
    - Limiting factors
  - Surface resistance
    - Relevance for accelerators
    - Heat exchange by radiation (emissivity)
- Thermal properties
  - Thermal conductivity
    - Temperature dependence, electron & phonons
    - Limiting factors

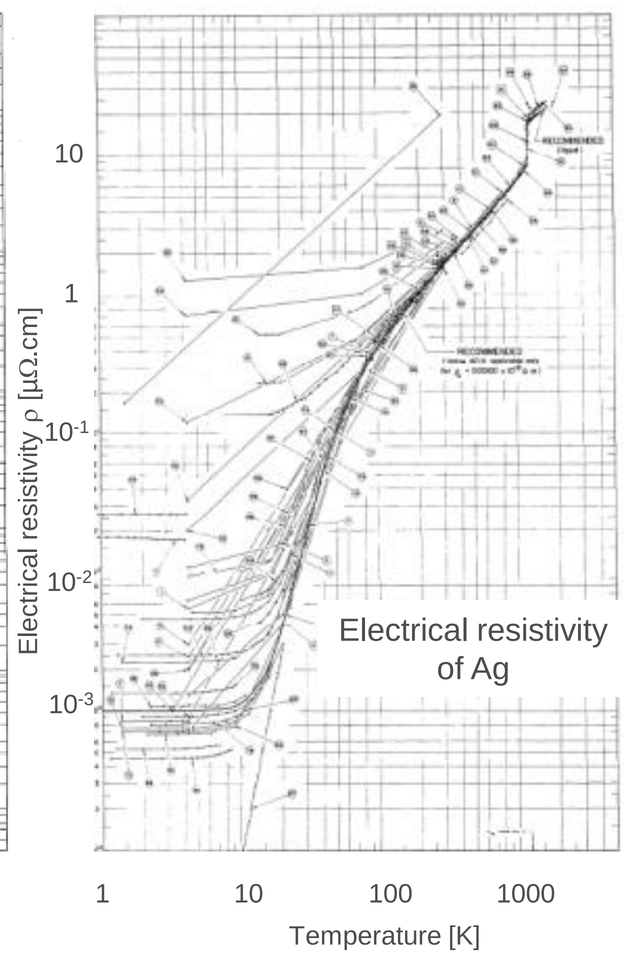
The electrical resistivity of metals changes with temperature



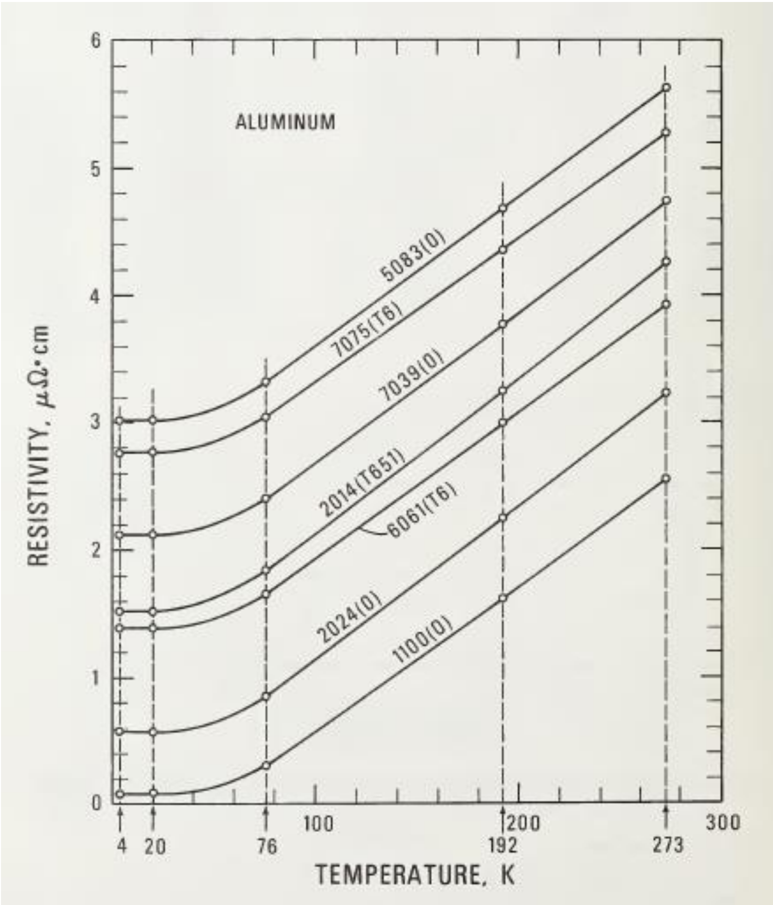
All pure metals…



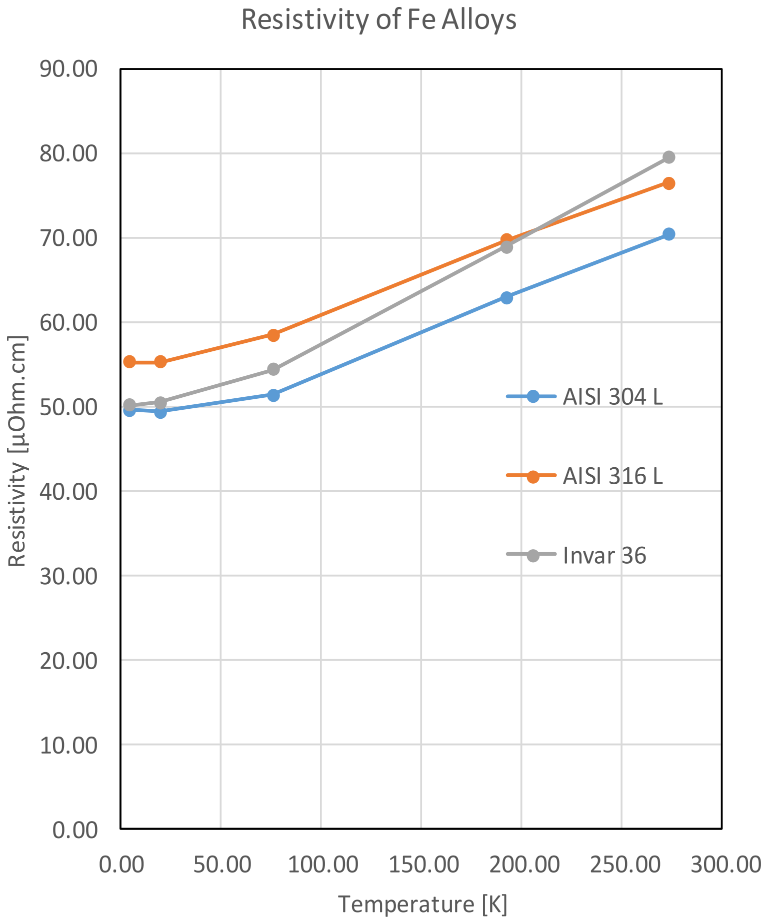




Alloys?



ALUMINUM
RESISTIVITY, μΩ• cm
TEMPERATURE, K
5083(0)
7075(T6)
7039(0)
2014(T651)
6061(T6)
2024(0)
1100(0)
0
1
2
3
4
5
6
4
20
76
100
192
200
273
300



Resistivity of Fe Alloys
Resistivity [μOhm.cm]
Temperature [K]
AISI 304 L
AISI 316 L
Invar 36
0.00
10.00
20.00
30.00
40.00
50.00
60.00
70.00
80.00
90.00
0.00
50.00
100.00
150.00
200.00
250.00
300.00

Some resistivity values (in μΩ.cm) (pure metals)

ELEMENT	77 K	273 K
Li	1.04	8.55
Na	0.8	4.2
K	1.38	6.1
Rb	2.2	11.0
Cs	4.5	18.8
Cu	0.2	1.56
Ag	0.3	1.51
Au	0.5	2.04
Be		2.8
Mg	0.62	3.9
Ca		3.43
Sr	7	23
Ba	17	60
Nb	3.0	15.2
Fe	0.66	8.9
Zn	1.1	5.5
Cd	1.6	6.8
Hg	5.8	Melted
Al	0.3	2.45
Ga	2.75	13.6
In	1.8	8.0
Tl	3.7	15
Sn	2.1	10.6
Pb	4.7	19.0
Bi	35	107
Sb	8	39

Variation of a factor ~70 for pure metals at room temperature

Even alloys have seldom more than a few 100s of μΩcm

We will not discuss semiconductors (or in general effects not due to electron transport)

Definition of electrical resistivity ρ

$$R = \frac{\rho \times length}{section}$$

The electrical resistance of a real object (for example, a cable)

$$\sigma = \frac{1}{\rho}$$

The electrical resistivity is measured in Ohm.m
Its inverse is the conductivity measured in S/m

$$\rho = \frac{m_e v_F}{ne^2 \ell} = \frac{m_e}{ne^2 \tau}$$

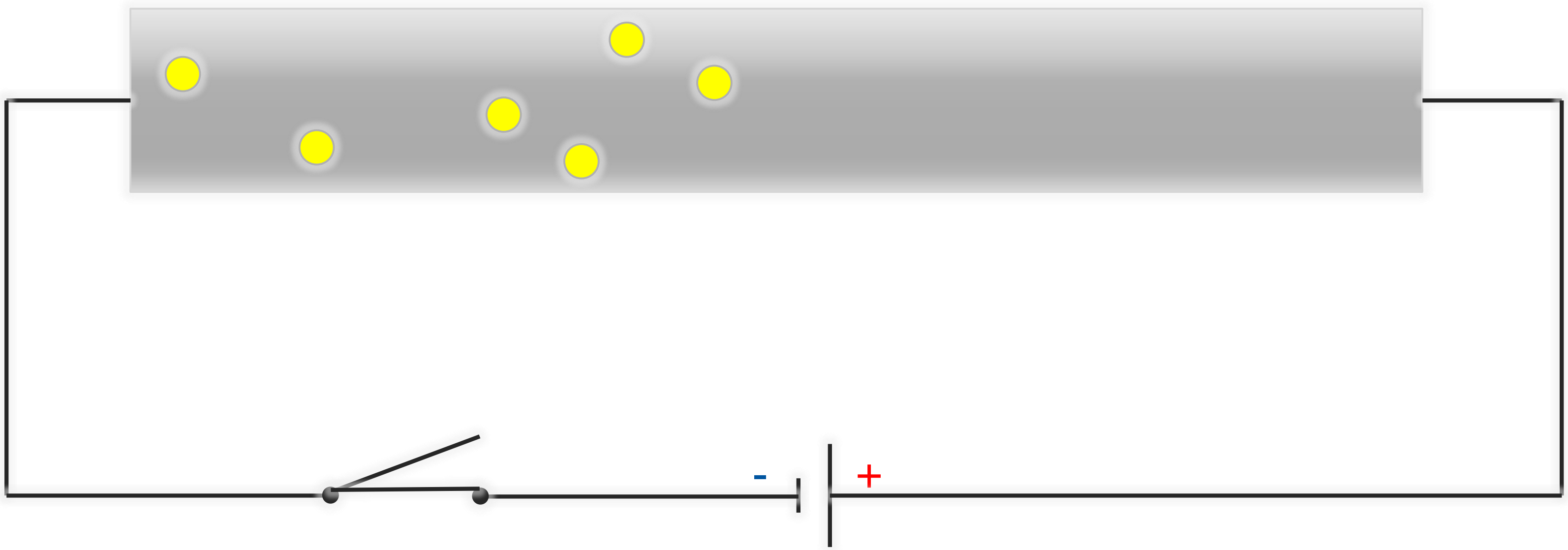
Changes with: temperature, impurities, crystal defects
Electron relaxation time

Electron mean free path

Constant for a given material

Basics (simplified free electron Drude model)

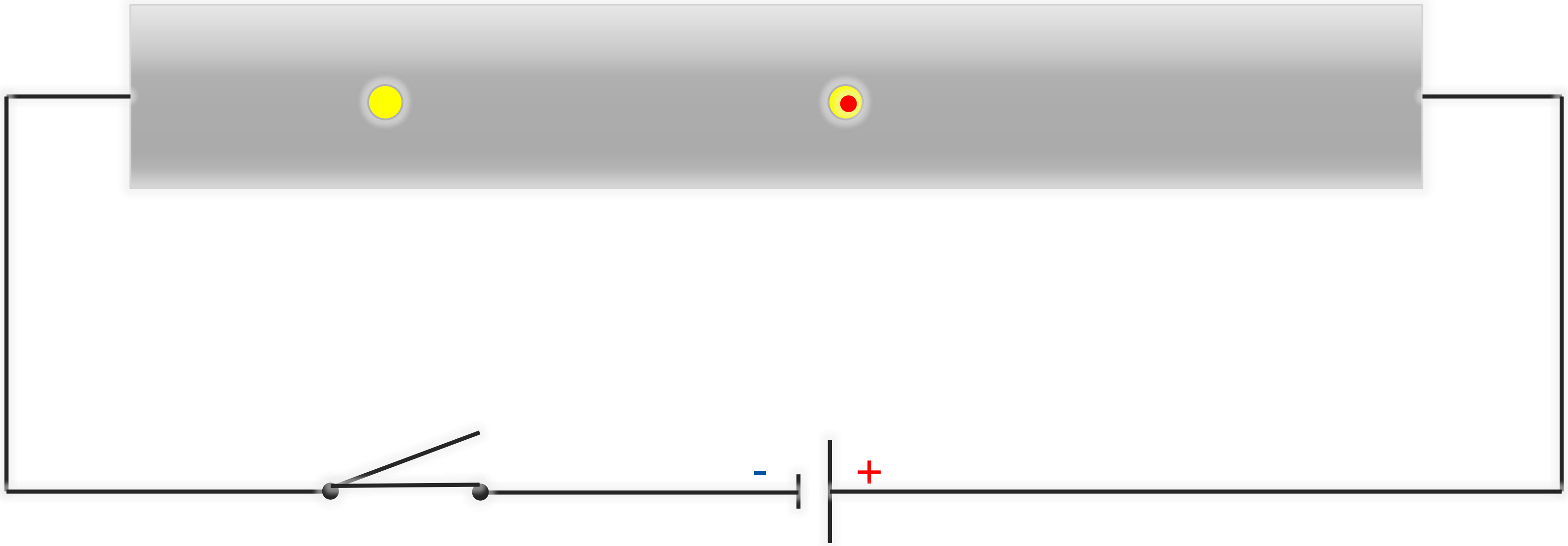
Electrical current = movement of conduction electrons



Defects

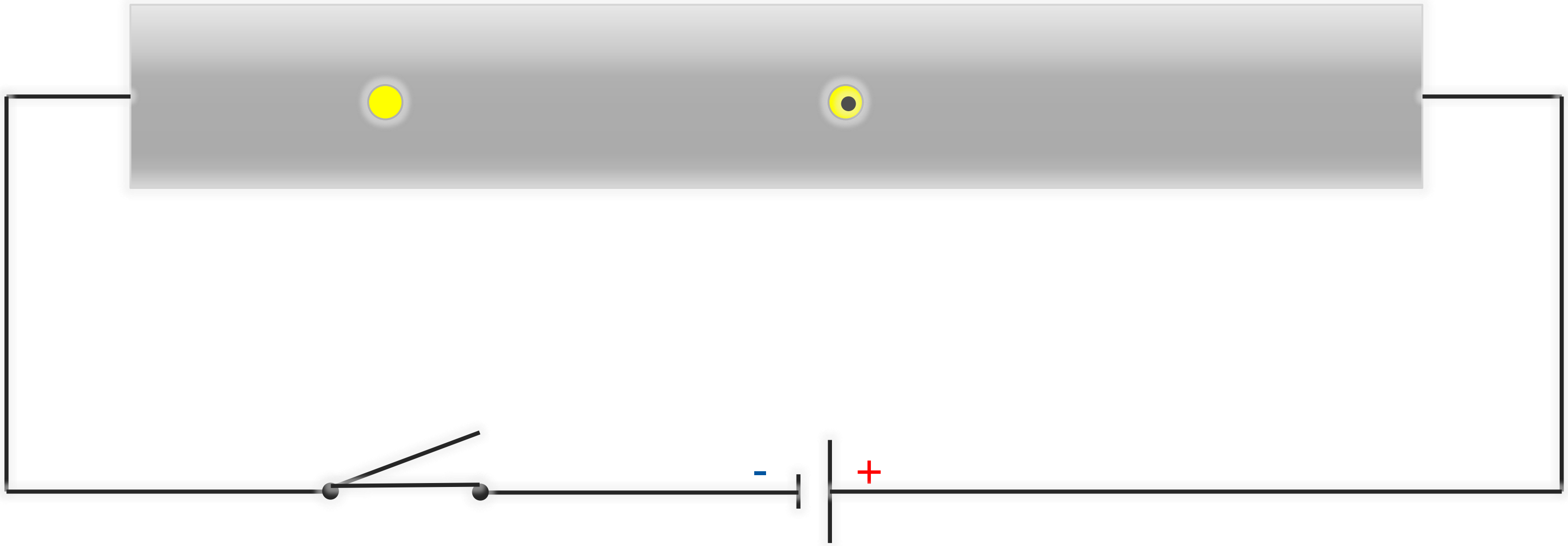
Defects in metals result in electron-defect collisions
They lead to a reduction in mean free path ℓ,
or equivalently in a reduced relaxation time τ.

They are at the origin of electrical resistivity ρ



Possible defects: phonons

Crystal lattice vibrations: phonons
Temperature dependent



Possible defects: phonons

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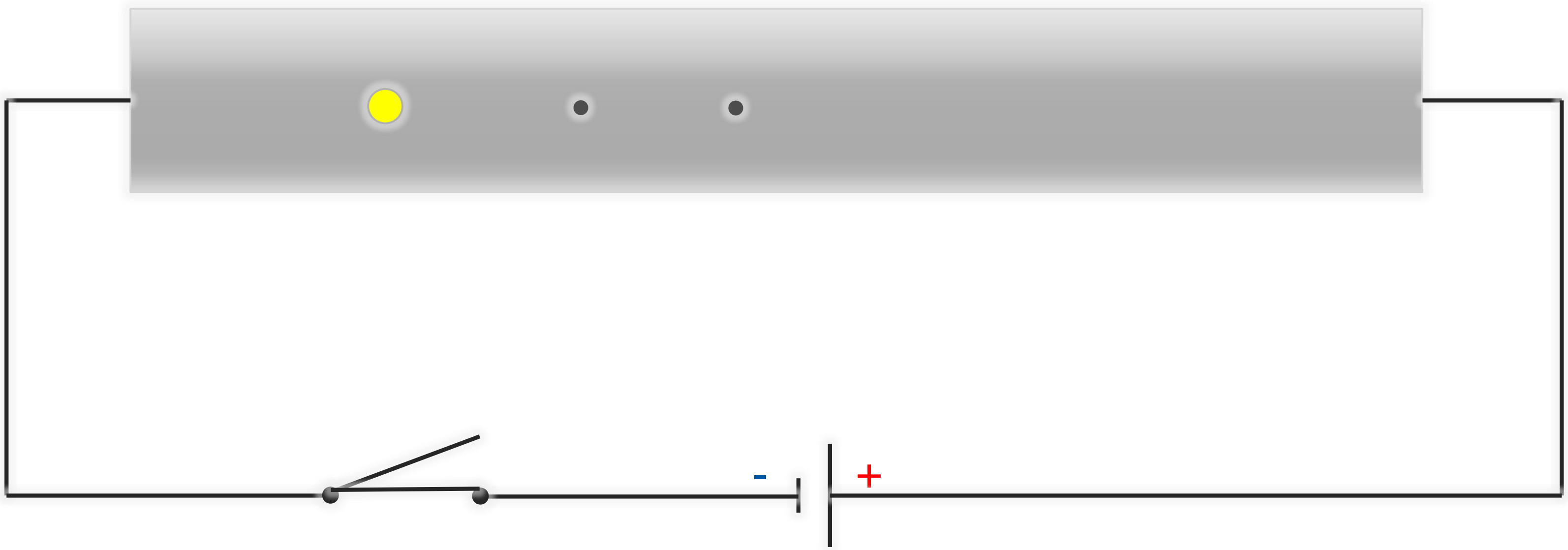
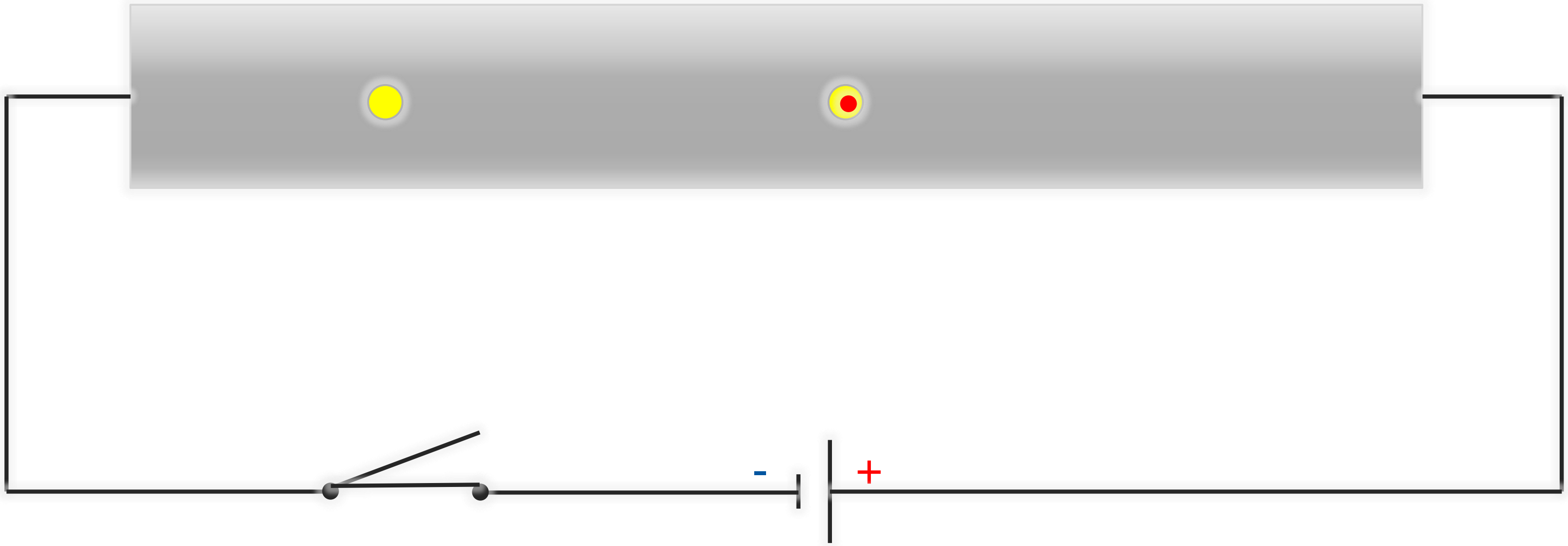


Possible defects: impurities

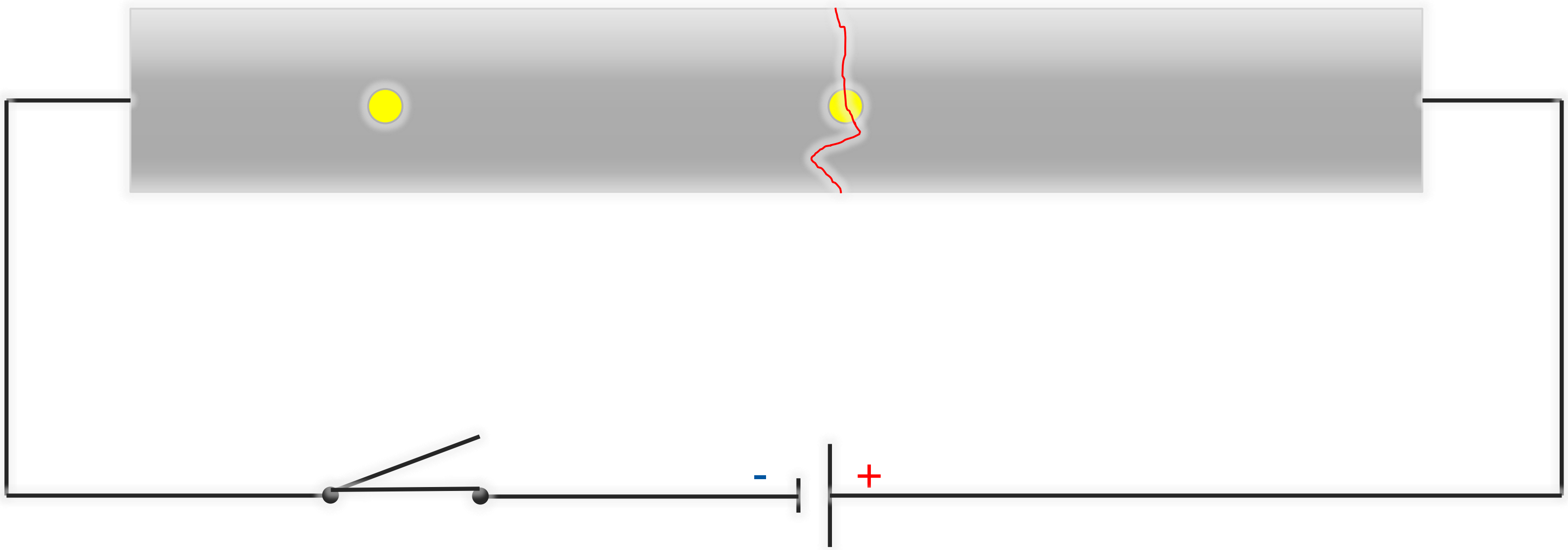
Can be inclusions of foreign atoms, lattice defects, dislocations

Not dependent on temperature



Possible defects: grain boundaries

Grain boundaries, internal or external surfaces
Not dependent on temperature

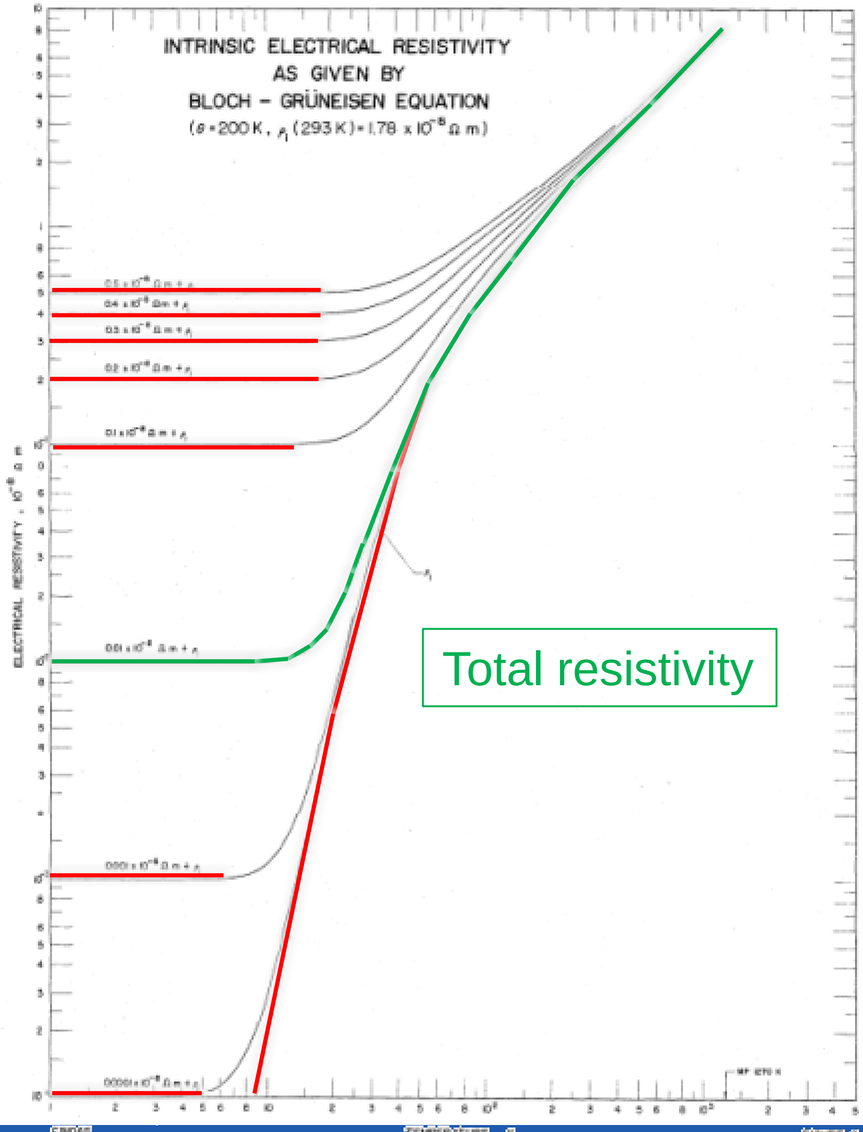


The two components of electrical resistivity

Proportional to:

- Impurity content
- Crystal defects
- Grain boundaries

Does not depend on temperature



Temperature dependent part

It is characteristic of each metal, and can be calculated

Varies of several orders of magnitude between room temperature and “low” temperature

Temperature dependence: Bloch-Grüneisen function

$$\rho_{ph}(T) = \left(\frac{\Theta_d}{T}\right)^{-5} \int_0^{\frac{\Theta_d}{T}} \frac{x^5}{(e^x - 1)(1 - e^{-x})} dx$$

$$\Theta_d = \frac{h v_s}{2\pi k_B}\left(6\pi^2 \frac{N}{V}\right)^{1/3}$$

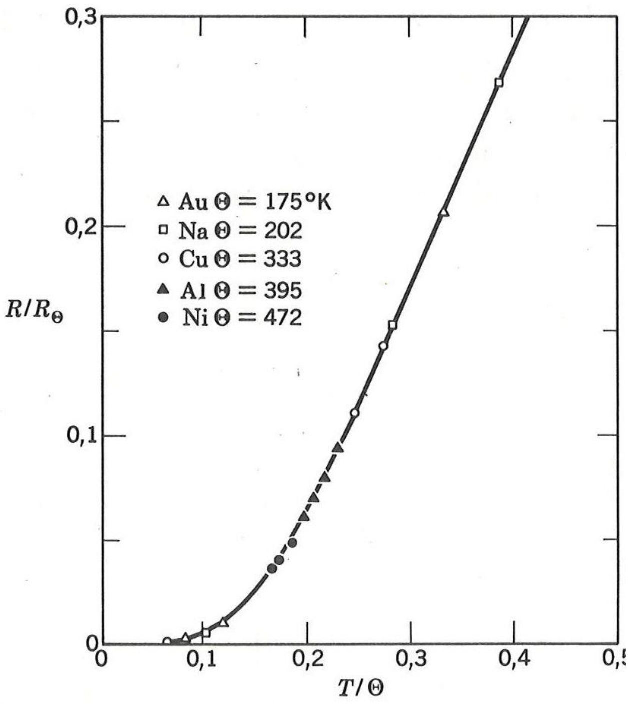
Debye temperature:
~ maximum frequency of crystal lattice vibrations (phonons)

$$\rho \approx T \qquad T >> \Theta_d$$

Given by total number of high-energy phonons proportional ~T

$$\rho \approx T^5 \quad T << \Theta_d$$

Given by total number of phonons at low energy ~T^3 and their scattering efficiency T^2



Low-temperature limits: Matthiessen's rule

$$\rho_{total}(T) = \rho_{phonons}(T) + \rho_{impurities} + \rho_{grain\ boundaries} + \ldots$$

Or in other terms

$$\ell_{total}(T) = \left(\frac{1}{\ell_{phonons}} + \frac{1}{\ell_{impurities}} + \frac{1}{\ell_{grain\ boundaries}} + \ldots \right)^{-1}$$

Every contribution is additive.
Physically, it means that the different sources of scattering for the electrons are independent

Effect of added impurities (copper)

Tabelle 40. Atomare Widerstandserhöhung $\Delta\varrho/a$ von Kupfer durch lösliche Beimengungen [90]. (a = im Kupfer gelöste Beimengung in Atom-%.)

Beimengung	Löslichkeit in Kupfer in Gew.-% bei Raumtemp.	$\Delta\varrho/a$ ($\mu\Omega\cdot$cm) je Atom-%	beobachteter Streubereich von $\Delta\varrho/a$
Ag	0,1	0,6	0,1···0,6
Al	9,4	0,95	0,8···1,1
As	6,5	6,7	6,6···6,8
Au	100	0,55	0,5···0,6
B	0,06	1,4	1,4···2,0
Be	0,2	0,65	0,6···0,7
Ca	<0,01	(0,3)*)	
Cd	<0,5	0,3	0,21···0,31
Co	0,2	6,9	6,0···7,0
Cr	<0,03	4,0	3,8···4,2
Fe	0,10	8,5	8,5···8,6
Ga	20	1,4	1,3···1,5
Ge	11	3,7	3,6···3,75
Hg	?	(1,0)*)	
In	3,0	1,1	1,0···1,2
Ir	1,5	(6,1)*)	
Li	<0,01	(0,7)*)	
Mg	1,0	(0,8)*)	
Mn	24	2,9	2,8···3,0
Ni	100	1,1	1,1···1,15
O	ca. 0,0002	5,3	4,8···5,8
P	0,5	6,7	6,7···6,8
Pb	0,02	3,3	3,0···4,0
Pd	40	0,95	0,9···1,0
Pt	100	2,0	1,9···2,1
Rh	20	(4,4)*)	
S	ca. 0,0003	9,2	8,7···9,7
Sb	2	5,5	5,4···5,6
Se	ca. 0,0004	10,5	10,2···10,8
Si	2	3,1	3,0···3,2
Sn	1,2	3,1	2,8···3,5
Te	ca. 0,0005	8,4	8,4···8,5
Ti	0,4	(16)*)	
U	ca. 0,1	(10)*)	
W	?	(3,8)*)	
Zn	30	0,3	0,28···0,32

*) Geschätzte Werte.

$\rho_{(Cu)}$(300K)=1.65 μΩ.cm

Note: alloys behave as having a very large amount of impurities embedded in the material

An useful quantity: RRR

$$\rho_{total}(300K) = \rho_{phonons}(300K) + \rho_{impurities} + \rho_{grain\ boundaries}$$

$$\rho_{total}(4.2K) = \rho_{impurities} + \rho_{grain\ boundaries}$$

Fixed number

$$RRR = \frac{\rho_{total}(300K)}{\rho_{total}(4.2K)} = \frac{\rho_{phonons}(300K) + \rho_0}{\rho_0}$$

Depends only on "impurities"
Dominant in alloys

$$\rho_0 = \frac{\rho_{phonons}(300K)}{(RRR-1)}$$

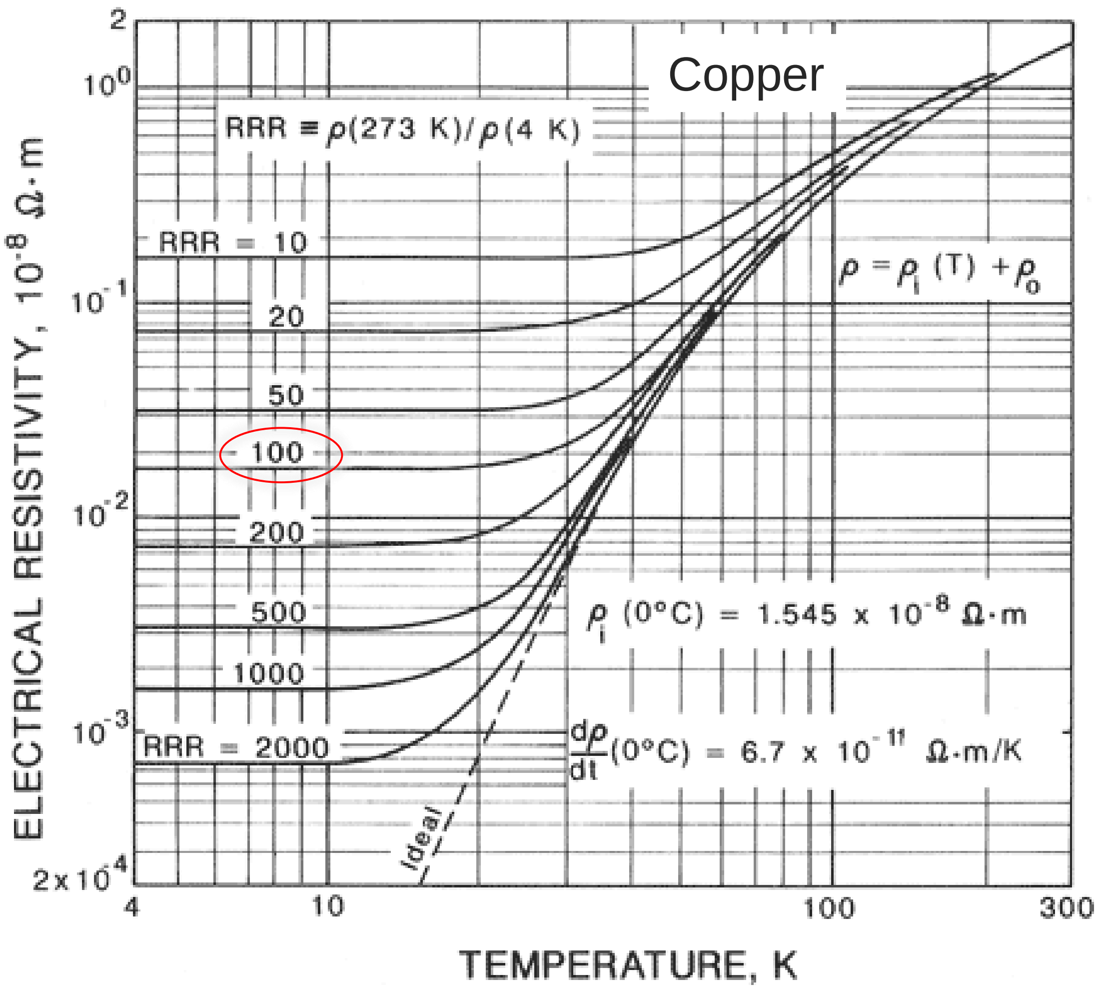
Practical formula

Experimentally, we have a very neat feature remembering that $R = \frac{\rho \times length}{section}$

$$RRR = \frac{R(300K)}{R(4.2K)} = \frac{\rho_{total}(300K)}{\rho_{total}(4.2K)}$$

Independent of the geometry of the sample.

Final example: copper RRR 100



$$RRR = \frac{\rho_{300\,K}}{\rho_{<10\,K}} = 100$$

$\rho_{300K} = \rho_{phonon.} = 1.55\text{x}10^{-8}\ \Omega\text{m}$

$\Rightarrow\ \rho_{10K} = 1.55\text{x}10^{-10}\ \Omega\text{m}$

If this is due only to oxygen:

$\rho_{imp.} = 5.3\ \text{x}10^{-8}\ \Omega\text{m}$ / at% of O

$$\frac{1.55x10^{-10}}{5.3x10^{-8}} = 0.003 \text{ at \% of O}$$

$\Rightarrow$ 30 ppm atomic !

This is Cu-OFE

Estimates of mean free path

$$\rho = \frac{m}{ne^2\tau} = \frac{m_e v_F}{ne^2\ell}$$

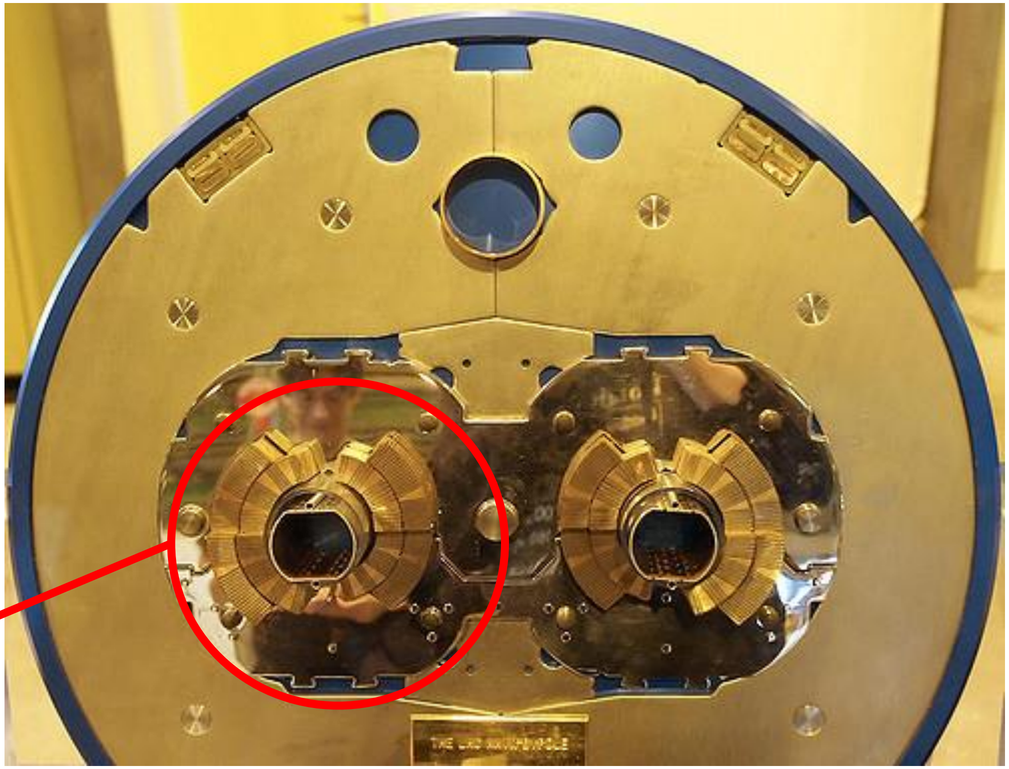
Typical values? Example of Cu at room temperature

- Let's assume one conduction electron per atom.
- ρ = 1.55 x 10^{-8} Ωm.
- *density* = 89400 kg/m^3
- m = 9.11 x 10^{-31} kg, e = 1.6 x 10^{-19} C, A = 63.5, N_A = 6.022 x 10^{23}

Exercise ! Solution:

- $\tau \approx$ 2.5 x 10^{-14} s. Knowing that v_F = 1.6 x 10^6 m/s we have
- $\ell \approx$ 4 x 10^{-8} m at room temperature. It can be x100 ÷ x1000 larger at low temperature

Interlude: LHC



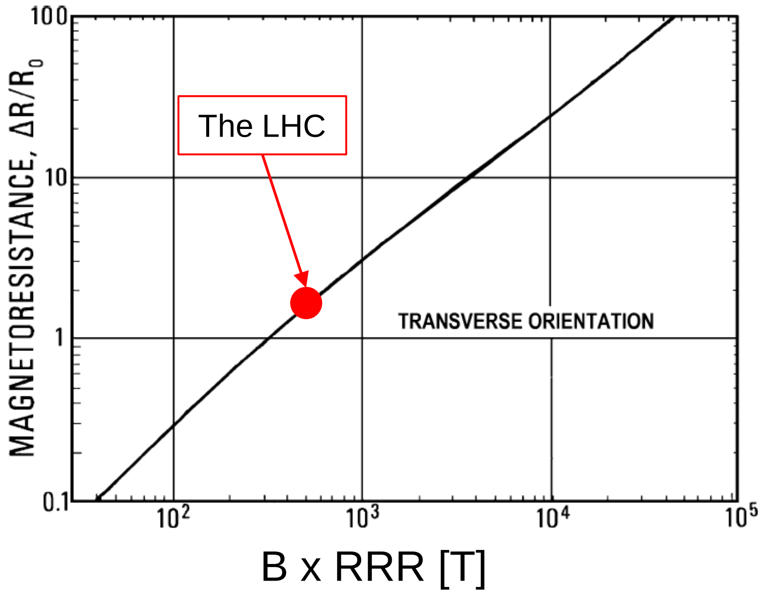
8.33 T dipoles (nominal field) @ 1.9 K

Beam screen operating from 4 K to 20 K
SS + Cu colaminated, RRR ≈ 60

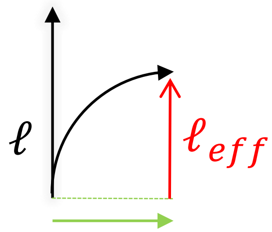
Magnetoresistance

Electron trajectories are bent due to the magnetic field

Cyclotron radius: $r = \dfrac{m v_F}{eB}$



⊗ B-field



$$\ell \longrightarrow \ell/4$$
$$\tau \longrightarrow \tau/4$$

$\ell \quad \ell_{eff} \approx \ell$

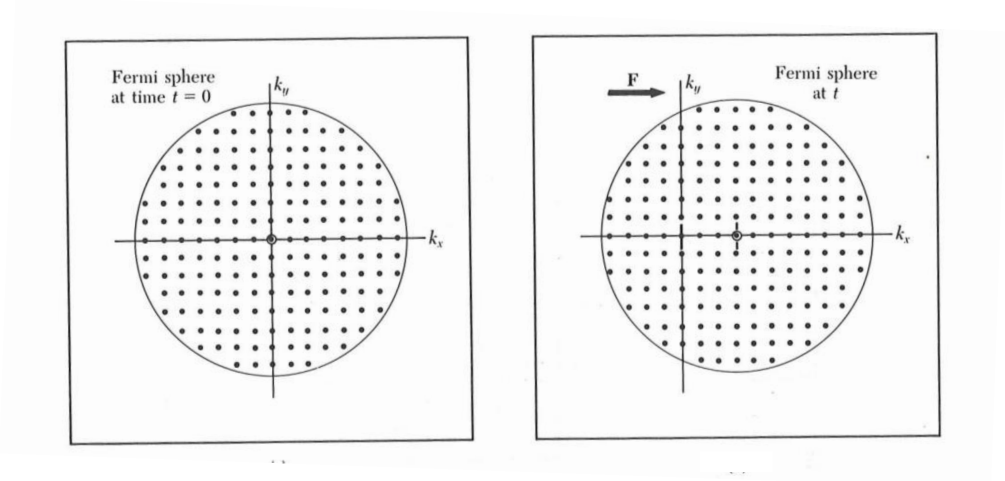
$$\ell_{eff} = r\left[\sin\left(\frac{\ell}{r}\right)\right] = r\left[\sin\left(\frac{\pi}{2}\right)\right]$$

$$\ell_{eff} = r = \ell\,\frac{2}{\pi}$$

$$\ell_{eff} = r\left[\sin\left(\frac{\pi}{8}\right)\right]$$

$$\ell_{eff} \approx \ell\,\frac{\pi}{8}\,\frac{2}{\pi} \approx \ell/4$$

Fermi sphere



- The real picture: the whole Fermi sphere is displaced from equilibrium under the electric field ***E***, the force ***F*** acting on each electron being $-e\boldsymbol{E}$
- This displacement in steady state results in a net momentum per electron $\delta\boldsymbol{k} = \boldsymbol{F}\tau/\hbar$ thus a net speed increment $\delta\boldsymbol{v} = \boldsymbol{F}\tau/m = -e\boldsymbol{E}\tau/m$
- $\boldsymbol{j} = ne\delta\boldsymbol{v} = ne^2\boldsymbol{E}\tau/m$ and from the definition of Ohm's law $\boldsymbol{j} = \sigma\boldsymbol{E}$ we have $\sigma = \frac{ne^2\tau}{m}$

The speed of conduction electrons

- Fermi velocity $\boldsymbol{v_F}$ = 1.6 x 10^6 m/s
- $\delta\boldsymbol{v} = \boldsymbol{j}/ne$ thus $\delta\boldsymbol{v} = \frac{\sigma\boldsymbol{E}}{ne} = \frac{e\tau\boldsymbol{E}}{m}$

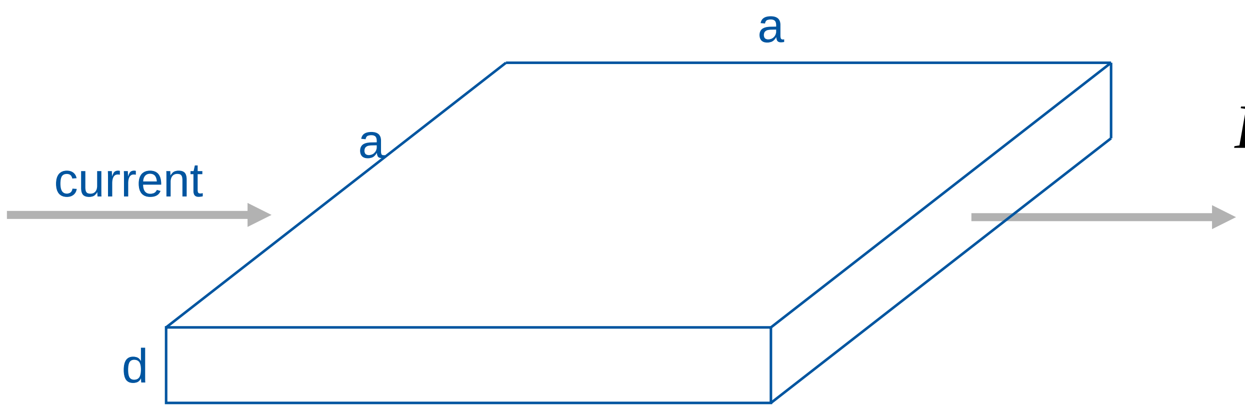
➢ As an order-of-magnitude, in a common conductor, we may have a potential drop of ~1V over ~1m

- $\boldsymbol{E} = \frac{V}{d}$ ≈1 V/m and as a consequence $\delta\boldsymbol{v}$ ≈4 x 10^{-3} m/s
- The drift velocity of the conduction electrons is orders of magnitude smaller than the Fermi velocity

➢ (Repeat the same exercise with 1 A of current, in a copper conductor of 1 cm^2 cross section)

Square resistance and surface resistance

Consider a square sheet of metal and calculate its resistance to a transverse current flow:

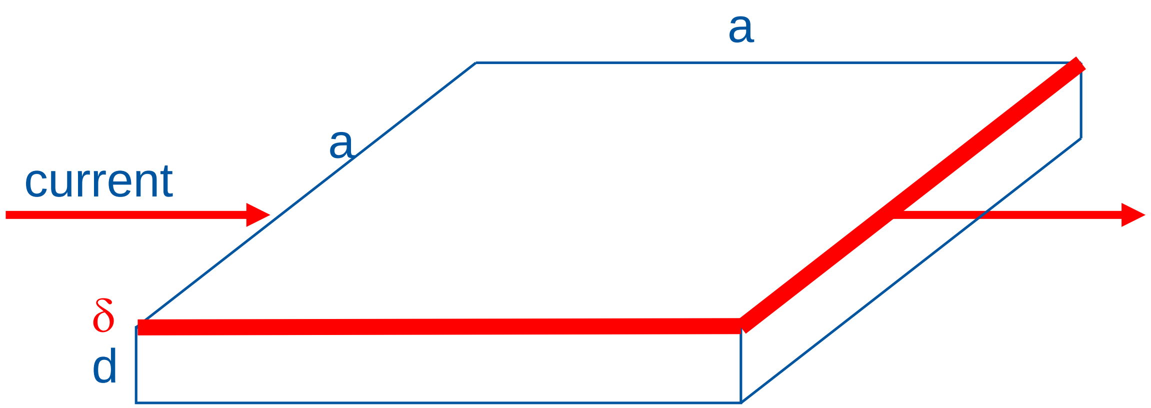


$$R = \frac{\rho\, \cancel{a}}{d\, \cancel{a}} = \frac{\rho}{d}$$

This is the so-called square resistance often indicated as $R_{\blacksquare}$

Square resistance and surface resistance

And now imagine that instead of DC we have RF, and the RF current is confined in a skin depth: $\delta = \sqrt{\dfrac{2\rho}{\mu_0 \omega}}$



$$R = \frac{\rho \cancel{a}}{d \cancel{a}} = \frac{\rho}{d}$$

$$R_s = \frac{\rho}{\delta} = \sqrt{\frac{\rho \mu_0 \omega}{2}}$$

This is a (simplified) definition of surface resistance R_s

(We will discuss this in more details at the tutorials)

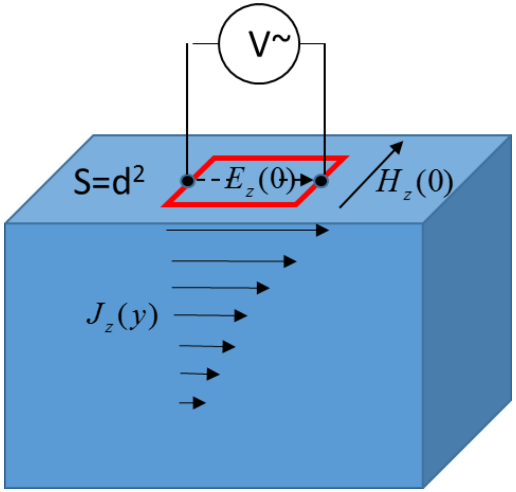
Surface impedance in normal metals

- The Surface Impedance Z_s is a complex number defined at the interface between two media.
- The real part R_s contains all information about power losses (per unit surface)

$$\overline{P} = \frac{\frac{1}{2} R_s I^2}{d^2} = \frac{1}{2} R_s \mathrm{H}_0^2$$

- The imaginary part X_s contains all information about the field penetration in the material

$$\delta = \frac{2}{\mu_0 \omega} X_s$$



- For copper (ρ = 1.75x10^{-8} µΩ.cm) at 350 MHz:
- $R_s = X_s = 5$ mΩ and δ = 3.5 µm

Why the surface resistance (impedance)?

- It is used for all interactions between E.M. fields and materials

- In RF cavities: quality factor $Q_0 = \dfrac{\Gamma}{R_s}$

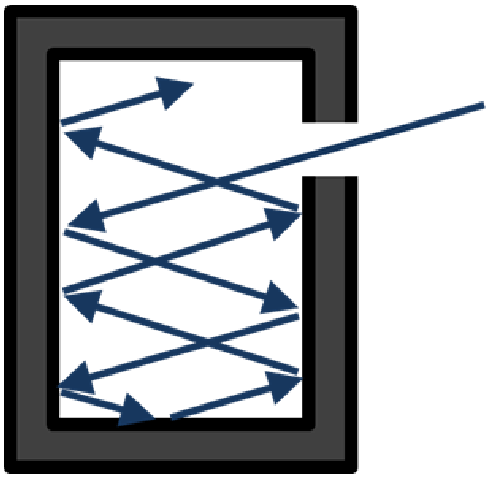
- In beam dynamics (more at the tutorials):

  - Longitudinal impedance and power dissipation from wakes is $P_{loss} = MI_b^2 \,\mathrm{Re}\left|Z_s^{eff}\right|$ where Z_s^{eff} is a summation of $\left(2\pi R/2\pi b\right) Z_s$ over the bunch frequency spectrum

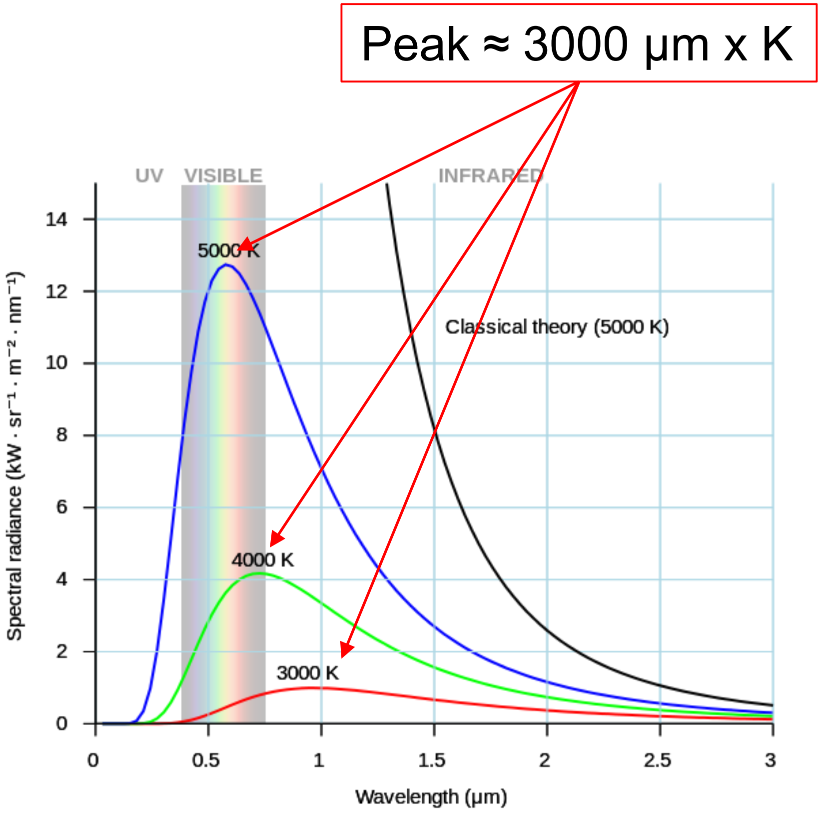
  - Transverse impedance: $Z_T = \dfrac{2\pi R\, c}{\pi b^3\, \omega} Z_s$

From RF to infrared: the blackbody

Thermal exchanges by radiation are mediated by EM waves in the infrared regime.



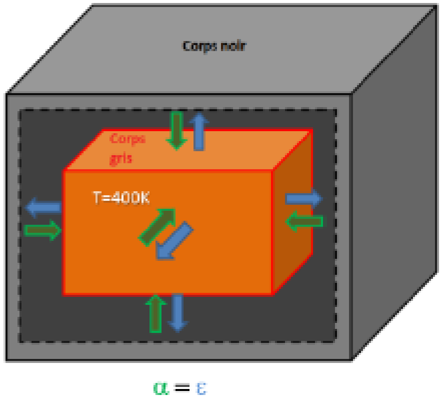
Schematization of a blackbody



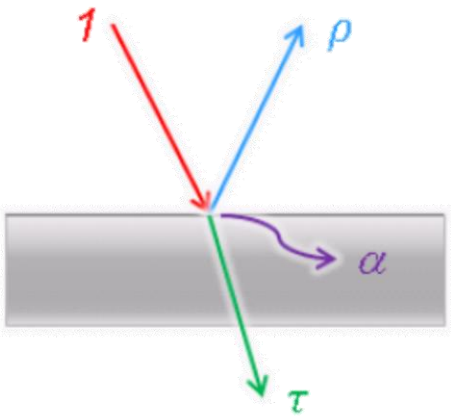
Blackbody radiation

- A blackbody is an idealized perfectly emitting and absorbing body (a cavity with a tiny hole)
- Stefan-Boltzmann law of radiated power density:

$$\frac{P}{A} = \sigma T^4 \qquad \sigma \approx 5.67 \times 10^{-8}\ \mathrm{W/(m^2K^4)}$$



- At thermal equilibrium: $\varepsilon = \alpha$
- ε is the emissivity (blackbody=1)



- A "grey" body will obey: $1 = r + \alpha\ (+t)$
- Thus for a grey body: $\frac{P}{A} = \varepsilon\sigma T^4$

From RF to infrared in metals

- Thermal exchanges by radiation are mediated by EM waves in the infrared regime.

- At 300 K, $\lambda_{peak} \approx 10\ \mu m$ of wavelength -> $\approx 10^{13}$ Hz or $\tau_{RF} \approx 10^{-13}$ s

- The theory of normal skin effect is usually applied for: $\omega_{RF}\tau < 1$
- But it can be applied also for: $\omega_{RF}\tau > 1$
- In the latter case it means: $\tau > \tau_{RF}$

- For metals at moderate T we can then use the standard skin effect theory to calculate emissivity

Emissivity of metals

- From: $1 = \rho + \alpha \;\Rightarrow\; \varepsilon = 1 - r$
- Thus we can calculate emissivity from reflectivity:

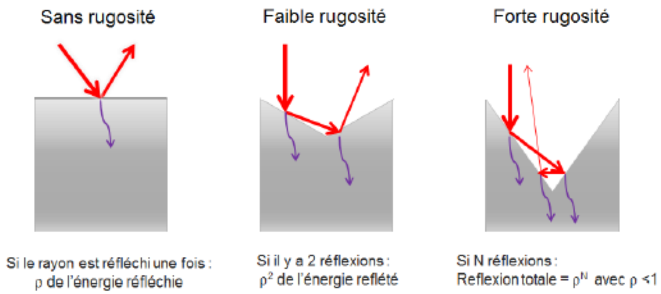
$$\varepsilon = 1 - r \approx 4\frac{R_s}{R_{vacuum}}$$

$$R_s = \sqrt{\frac{\rho\mu_0\omega}{2}} \approx \Omega$$

$$R_{vacuum} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7\Omega$$

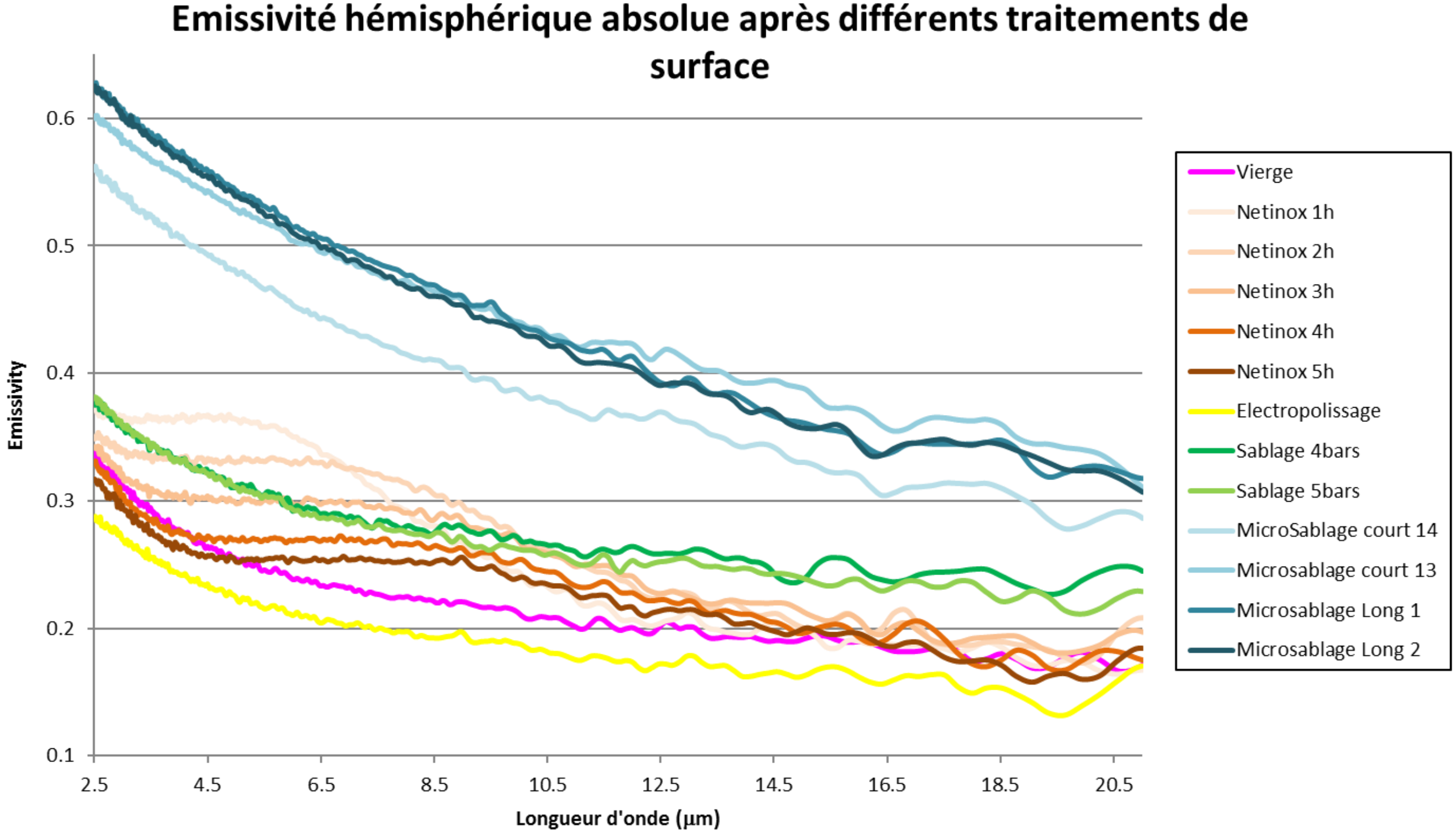
- The emissivity of metals is small
- The emissivity of metals depends on resistivity
- Thus, the emissivity of metals depends on temperature and on frequency

Practical case: 316 LN

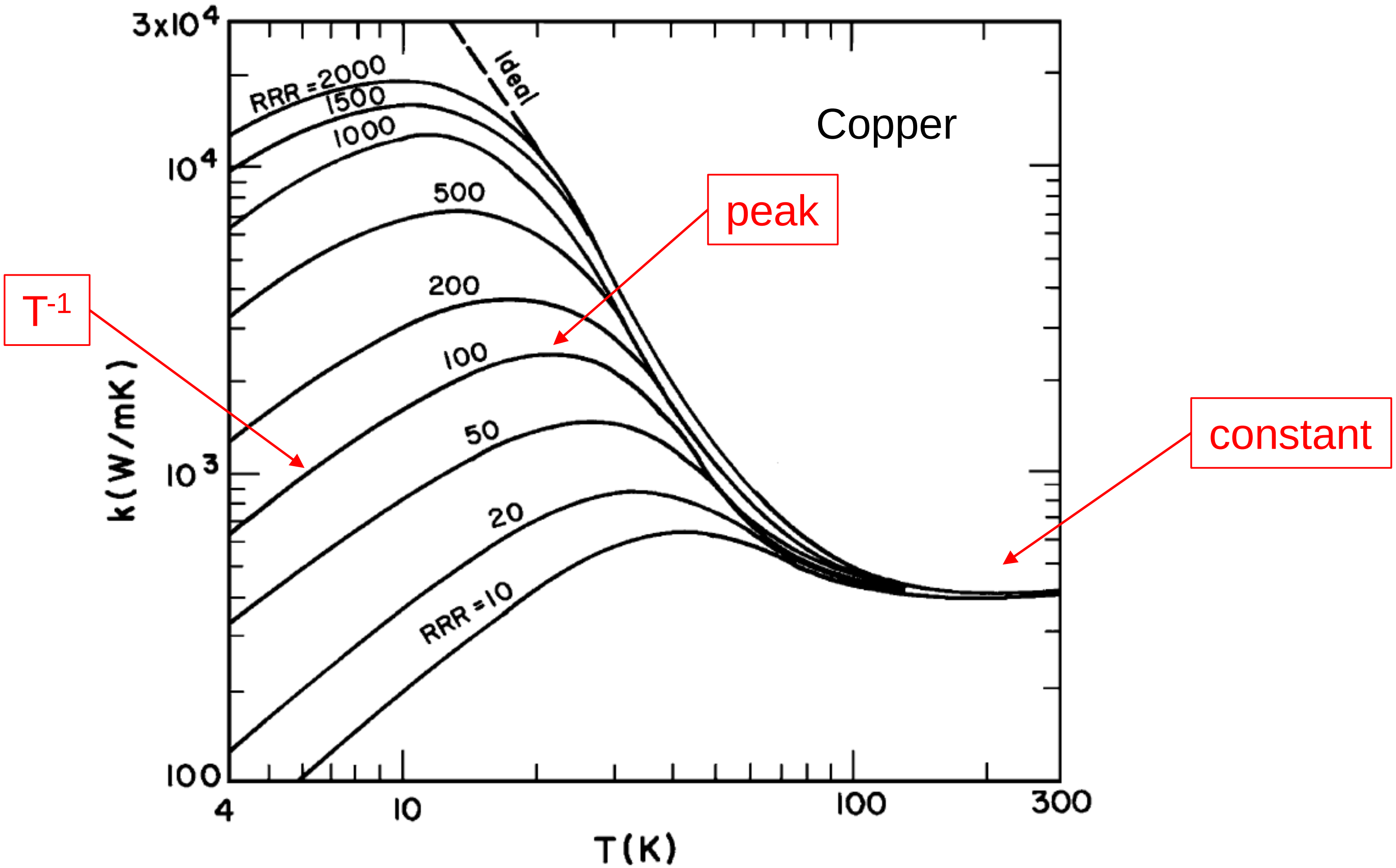


$$\rho_t = \prod \rho_i$$

ρ_i : *Réflectance*
ρ_t : *Réflectance totale*

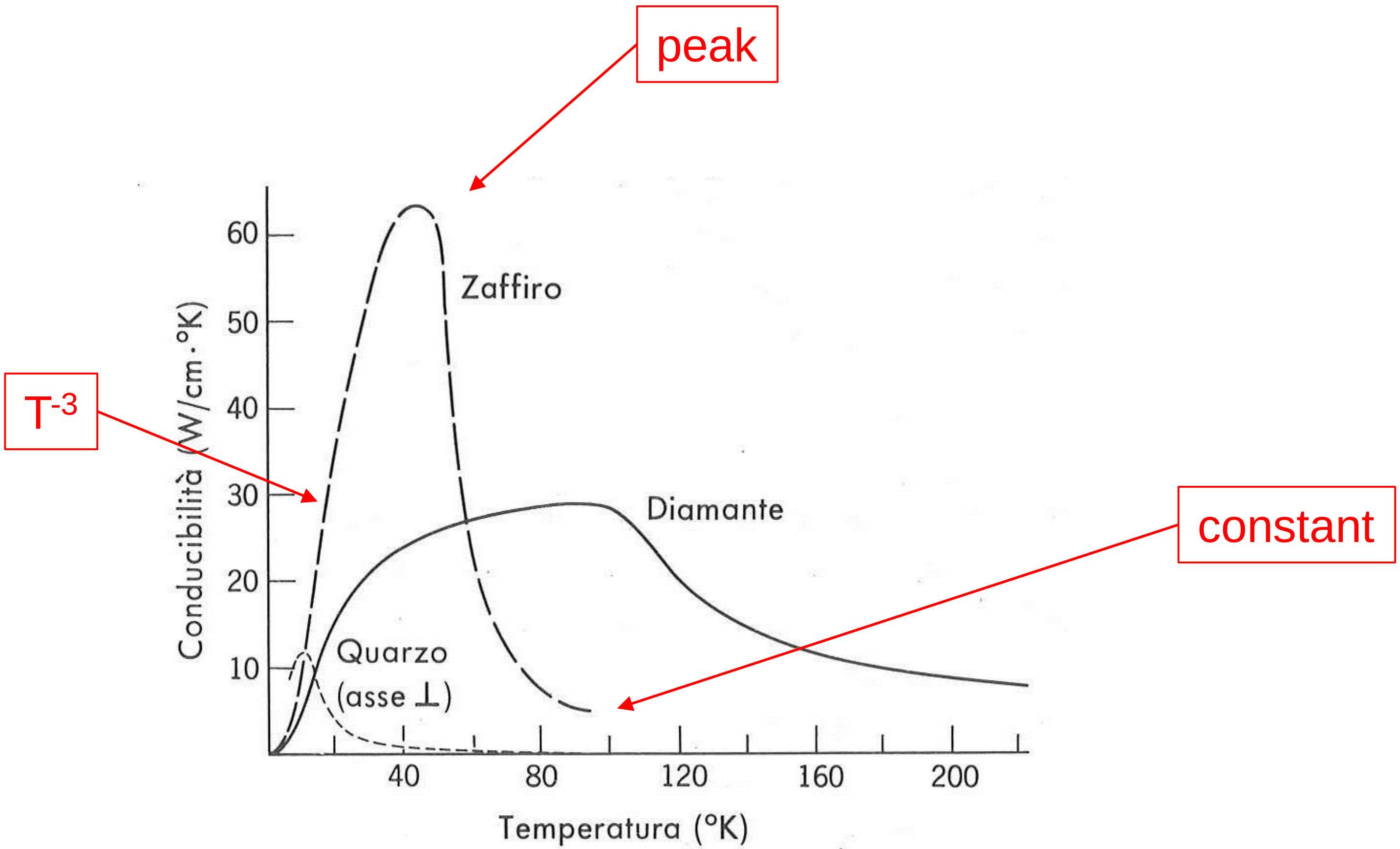


Thermal conductivity of metals



Thermal conductivity: insulators

Determined by phonons (lattice vibrations). Phonons behave like a “gas”



Thermal conductivity: insulators

Thermal conductivity K_{ph} from heat capacity C_{ph} (as in thermodynamics of gases)

$$K_{ph} = \frac{1}{3} C_{ph} v_s \ell = \frac{1}{3} C_{ph} v_s^2 \tau$$

$$C_{ph} = 3Nk_B T = 3RT \qquad T >> \Theta_d$$

$$C_{ph} = \frac{12\pi^4}{5} N k_B \left(\frac{T}{\Theta_d}\right)^3 \qquad T << \Theta_d$$

$$\tau = \frac{1}{T} \qquad T >> \Theta_d$$

$$\tau = const. \qquad T << \Theta_d$$

$$K_{ph} \approx const \qquad T >> \Theta_d$$

$$K_{ph} \approx \left(\frac{T}{\Theta_d}\right)^3 \qquad T << \Theta_d$$

for ultra-pure crystals

$$K_{peak} = \frac{1}{3} C_{ph} v_s \varnothing$$

$\varnothing$ = max dimension of specimen

Thermal conductivity: metals

Determined by both electrons and phonons.

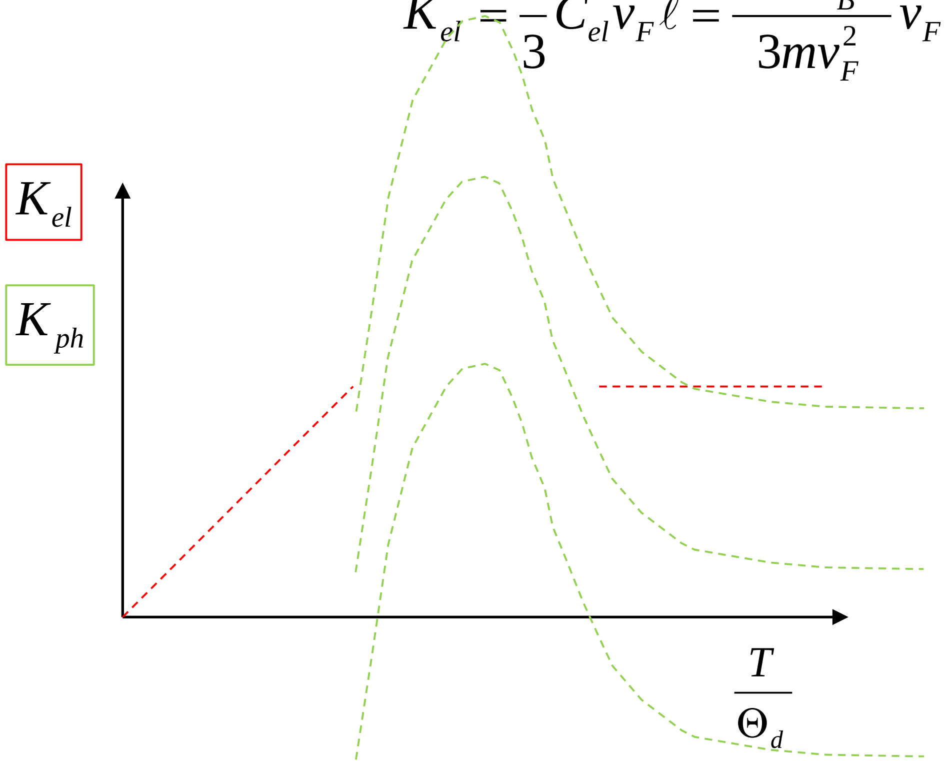
Thermal conductivity K_{el} from heat capacity C_{el}

$$K_{el} = \frac{1}{3} C_{el} v_F \ell = \frac{\pi^2 n k_B^2 T}{3 m v_F^2} v_F \ell = \frac{\pi^2 n k_B^2 T}{3m} \tau$$

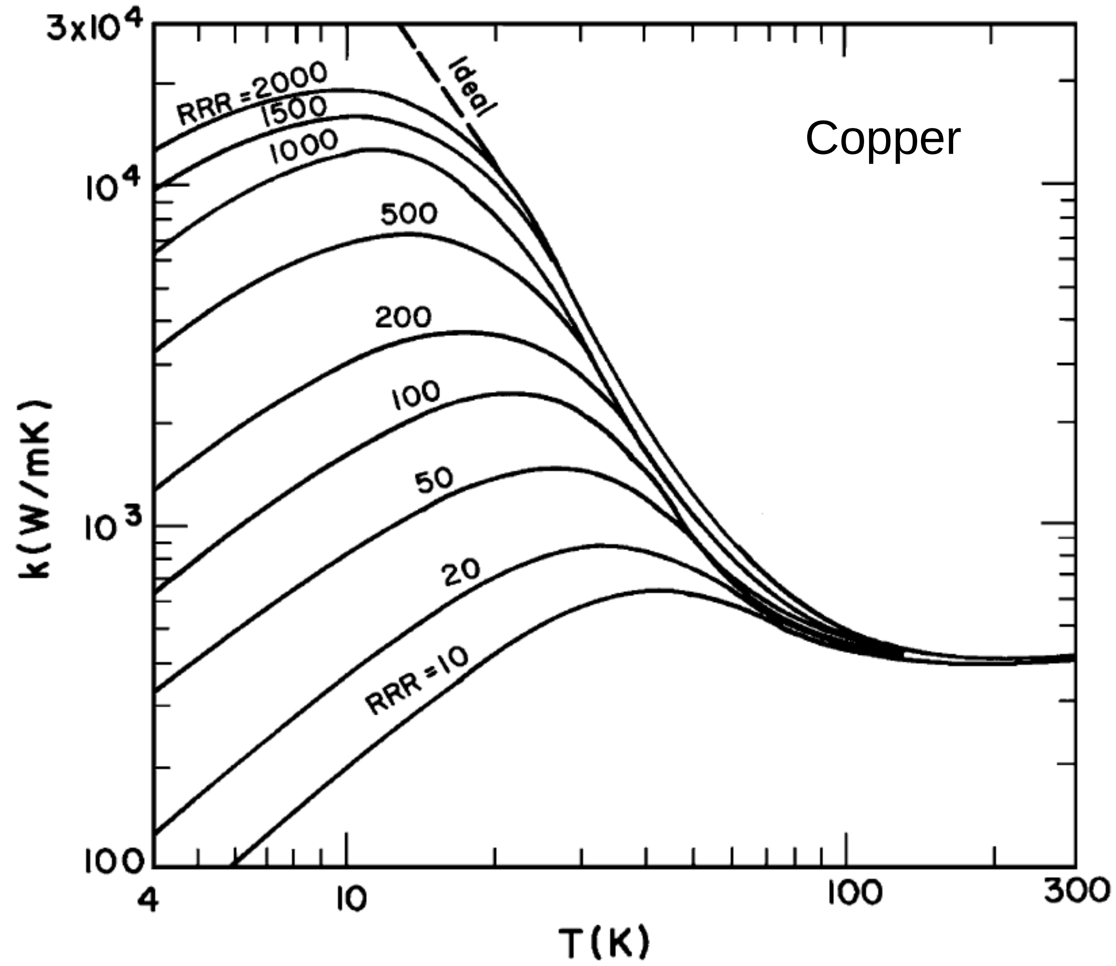
$$\tau = \frac{1}{T} \qquad T >> \Theta_d$$

$$\tau = const. \qquad T << \Theta_d$$

impurities



Thermal conductivity of metals: total



Wiedemann-Franz

Proportionality between thermal conductivity and electrical conductivity

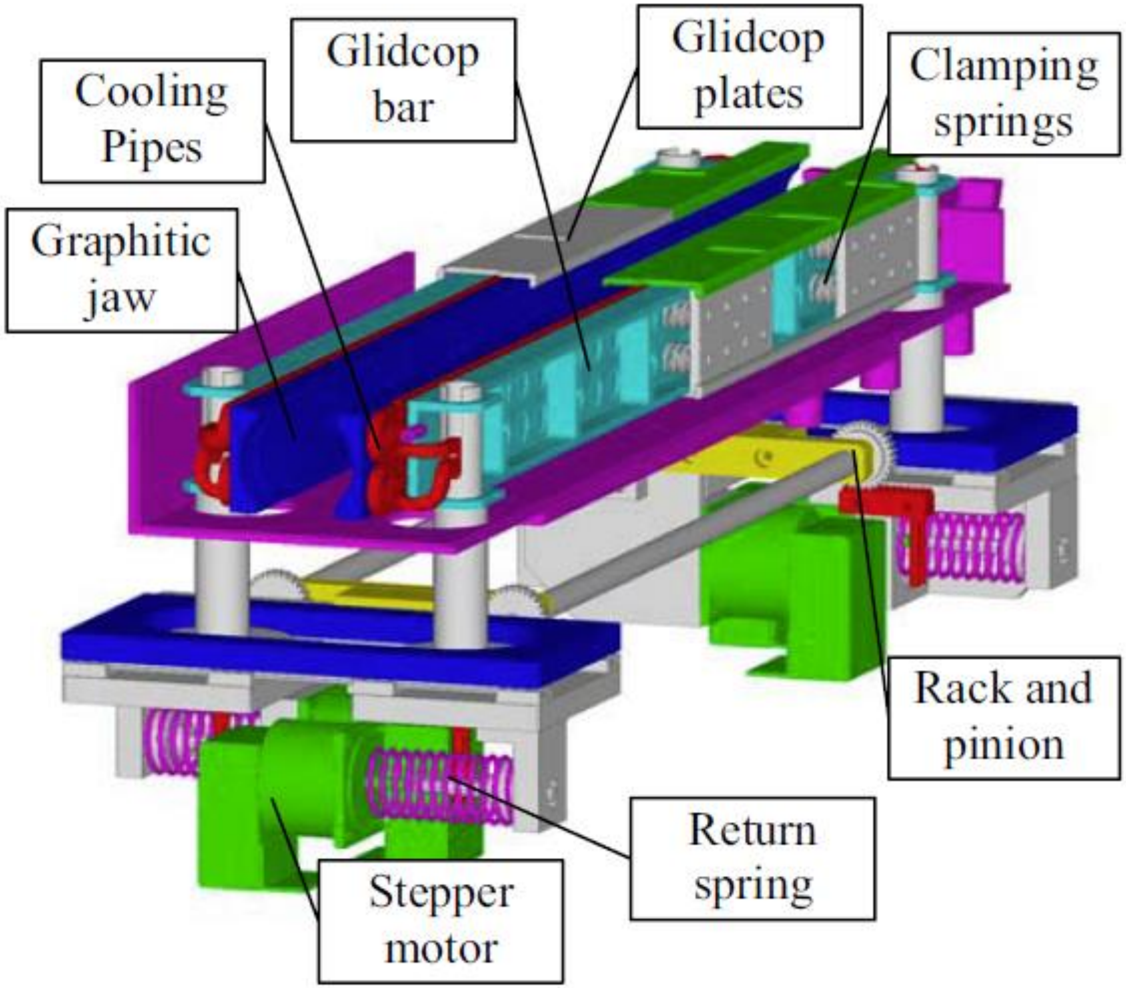
$$\frac{K_{el}}{\sigma} = \frac{\pi^2}{3}\left(\frac{k_B}{e}\right)^2 T = L\,T \qquad T > \Theta_d$$

$L = 2.45\text{x}10^{-8}$ WΩK^{-2}
(*Lorentz number*)

Useful for simple estimations, if one or the other quantity are known

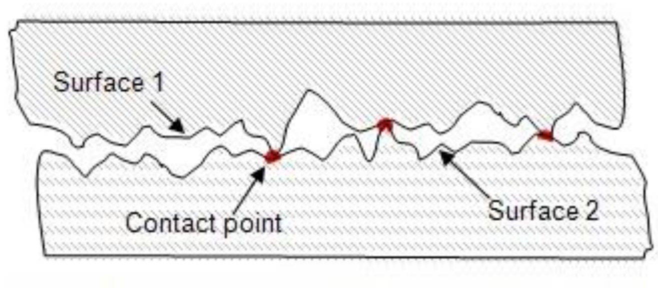
Useful also (very very approximately) to estimate contact resistances

The LHC collimator



Contact resistance (both electrical and thermal)

- Complicated… and no time left ☹



Contact area: $A \sim P^n \quad n \approx O(1)$

n depends on:

Plastic deformation
Elastic deformation

Roughness “height” and “shape”

- Contacts depend also on oxidation, material(s) properties, temperature…

Example for electric contacts:

- Theoretically:
  - $R \propto P^{-1/3}$ in elastic regime
  - $R \propto P^{-1/2}$ in plastic regime
- Experimentally:
  - $R \propto P^{-1 \div -1/2}$ (same as for thermal contacts)

References

- Charles Kittel, “Introduction to solid state physics”
- Ashcroft & Mermin, “Solid State Physics”
- S. W. Van Sciver, “Helium Cryogenics”
- M. Hein, “HTS thin films at μ-wave frequencies”
- J.A. Stratton, “Electromagnetic Theory”
- Touloukian & DeWitt, “Thermophysical Properties of Matter”

The end. Questions?

Plane waves in vacuum

Plane wave solution of Maxwell's equations in vacuum:

$$E = E_0\, \mathrm{e}^{i(kz-\omega t)} \qquad H = H_0\, \mathrm{e}^{i(kz-\omega t)} \qquad H = E_0 \frac{k}{\omega \mu_0} \mathrm{e}^{i(kz-\omega t)}$$

Where (in vacuum): $$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} = \omega\sqrt{\varepsilon_0 \mu_0} = \sqrt{\frac{\varepsilon_0}{\mu_0}}$$

So that: $$E = E_0\, \mathrm{e}^{i(kz-\omega t)} \qquad H = E_0 \sqrt{\frac{\varepsilon_0}{\mu_0}}\, \mathrm{e}^{i(kz-\omega t)}$$

The ratio $Z = \frac{|\mathbf{E}|}{|\mathbf{H}|} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7\Omega$ is often called impedance of the free space and the above equations are valid in a continuous medium

Plane waves in normal metals

More generally, in metals:

$$k^2 = \omega^2 \varepsilon\mu + i\omega\sigma\mu \qquad Z = \frac{|\mathbf{E}|}{|\mathbf{H}|} = \sqrt{\frac{\mu}{\varepsilon}} = \frac{\omega\mu}{k}$$

This results from taking the full Maxwell's equations, plus a supplementary equation which relates locally current density and field:

$$\mathbf{J}(\vec{x},t) = \sigma\,\mathbf{E}(\vec{x},t) \qquad \sigma_0 = \frac{ne^2\ell}{m_e v_F} = \frac{ne^2\tau}{m_e}$$

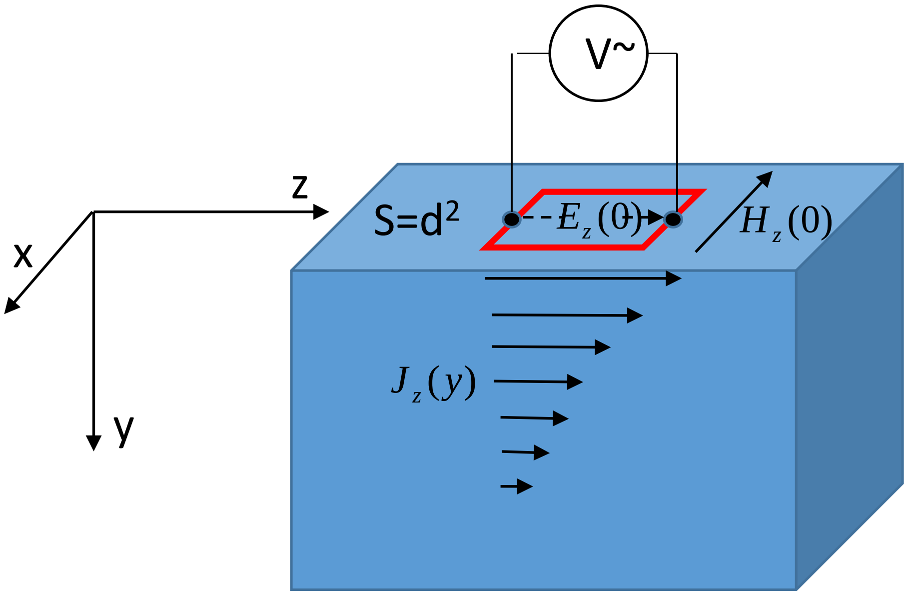
In metals $\omega\sigma\mu >> \omega^2\varepsilon\mu \Rightarrow k^2 = i\omega\sigma\mu \qquad k = \sqrt{i\omega\sigma\mu} = \alpha + i\beta$

and the wave equations become:

$$E = E_0\,\mathrm{e}^{i(kz-\omega t)} = E_0\,\mathrm{e}^{i(\alpha z-\omega t)}\mathrm{e}^{-\beta z}$$

With $\delta = \dfrac{1}{\beta} = \sqrt{\dfrac{\sigma\omega\mu}{2}}$ is the damping coefficient of the wave inside a metal, and δ is also called the field penetration depth.

Surface impedance



$$Z_s = \frac{V}{I}$$

$$V = dE_z(0)\ ;\ I = d\int_0^{+\infty} J_z(y)dy$$

$$\delta = \frac{1}{J_z(0)}\int_0^{+\infty} J_z(y)dy$$

$$\boxed{Z_s = R_s + iX_s = \frac{E_z(0)}{\int_0^{+\infty} J_z(y)dy}} = \frac{E_z(0)}{\delta J_z(0)} = \boxed{\frac{E_z(0)}{H_x(0)}}$$

$$\overline{P_{tot}(t)} = \frac{1}{2}R_s I^2 = \frac{1}{2}R_s d^2 H_x^2 \qquad \overline{P}/S = \boxed{P_{rf} = \frac{1}{2}R_s H_{rf}^2} = \frac{1}{2}R_s\left(\frac{B_{rf}}{\mu_o}\right)^2$$

Normal metals in the local limit

$$J_z(y) = J_z(0)e^{-\frac{y}{\delta}} \qquad \delta = \sqrt{\frac{2}{\omega\mu_o\sigma_n}} \qquad \left(\vec{J}(t) = \vec{J}(0)e^{i\omega t}\right)$$

$$Z_n = R_n + iX_n = \frac{E_z(0)}{\delta J_z(0)} = \frac{1}{\delta\sigma_n}(1+i) = \sqrt{\frac{\mu_o\omega}{2\sigma_n}}(1+i)$$

$$\boxed{R_n = X_n = \frac{1}{\delta\sigma_n} = \frac{\rho_n}{\delta} = \sqrt{\frac{\mu_o\omega}{2\sigma_n}} = \sqrt{\frac{\mu_o\omega}{2}\rho_n}} \qquad (R_n \propto \sqrt{\omega})$$

Limits for conductivity and skin effect

$$\delta = \sqrt{\frac{2\rho}{\omega\mu_0}} \qquad \ell \sim \frac{1}{\rho}$$

1. Normal skin effect if: $\ell << \delta$ e.g.: high temperature, low frequency

2. Anomalous skin effect if: $\ell >> \delta$ e.g.: low temperature, high frequency

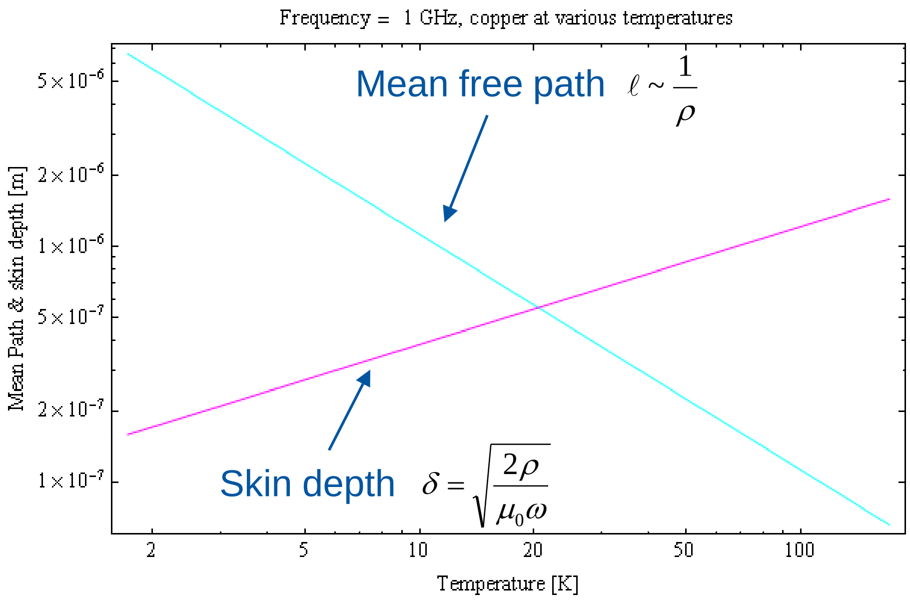
Note: 1 & 2 valid under the implicit assumption $\omega\tau << 1$

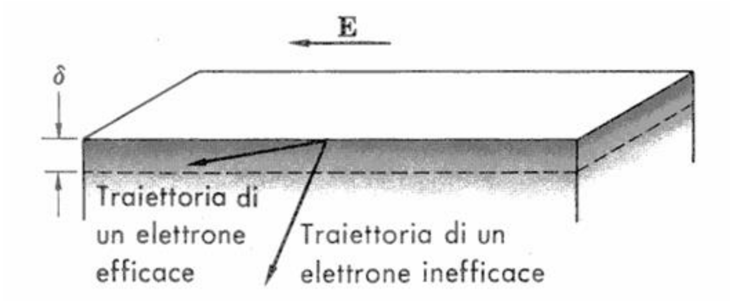
1 & 2 can also be rewritten (in advanced theory) as:

$$\frac{\ell}{\delta} << \left(1 + \omega^2\tau^2\right)^{3/4}$$

It derives that 1 can be true for $\omega\tau << 1$ and also for $\omega\tau >> 1$

Mean free path and skin depth



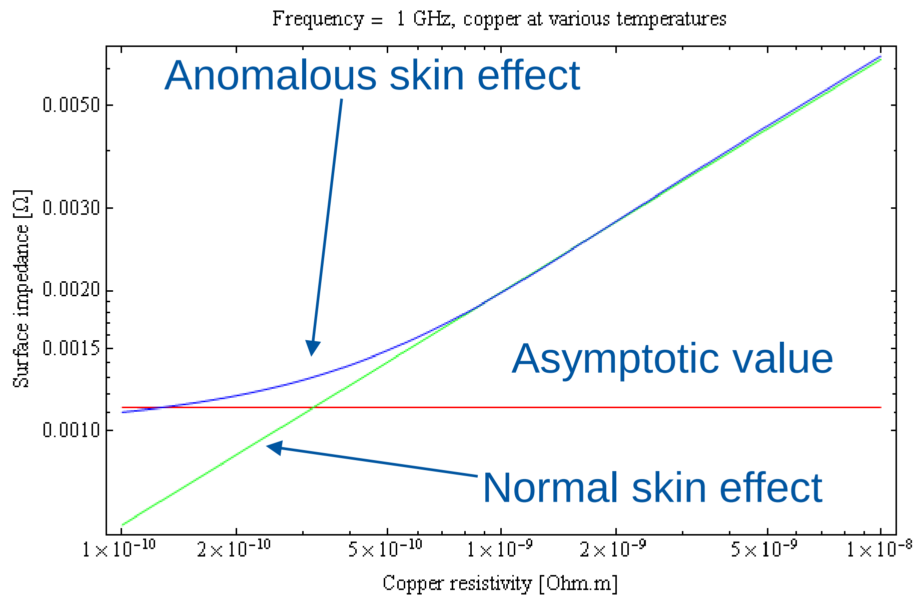


$$n_{eff} \approx n_0 \frac{\delta}{\ell}$$

$$\sigma_0 = \frac{ne^2\ell}{m_e v_F} \Rightarrow \sigma_{eff} \approx \frac{ne^2\delta}{m_e v_F}$$

$$\delta = \sqrt{\frac{2}{\sigma\omega\mu_0}} \Rightarrow \delta \xrightarrow[\tau\to\infty]{} const.$$

Anomalous skin effect

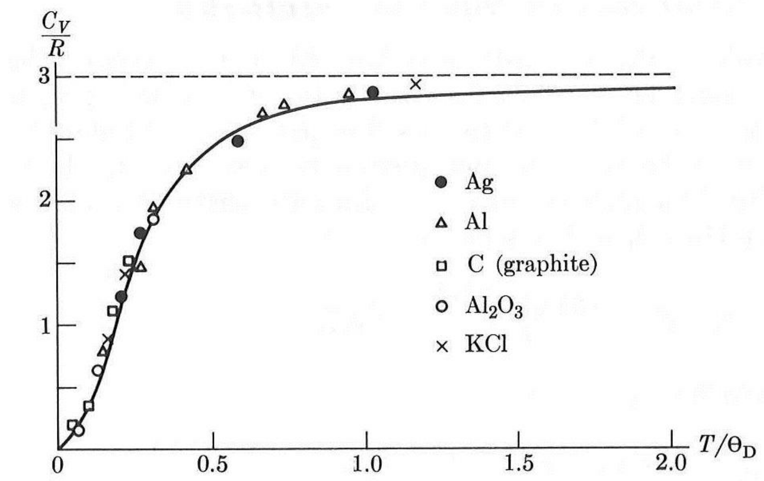


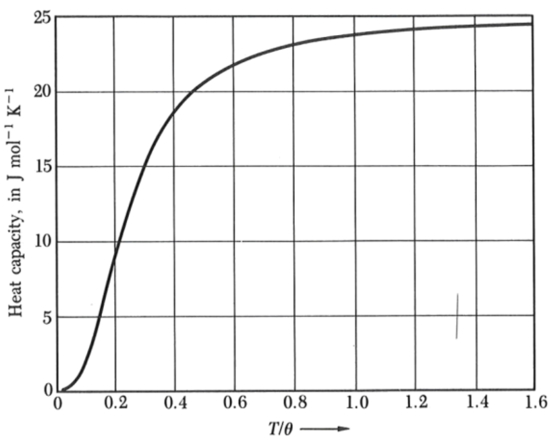
Understood by Pippard, Proc. Roy. Soc. A191 (1947) 370
Exact calculations Reuter, Sondheimer, Proc. Roy. Soc. A195 (1948) 336

Debye temperatures

Element	Θ_D (K)
Al	428
Au	165
Cd	209
Cr	630
Cu	343
Fe	470
Ga	320
Hf	252
Hg	71.9
In	108
Nb	275
Ni	450
Pb	105
Sn	200
Ti	420
V	380
Zn	327

Heat capacity of solids: Dulong-Petit law





Low-temperature heat capacity of phonon gas

(simplified plot in 2D)

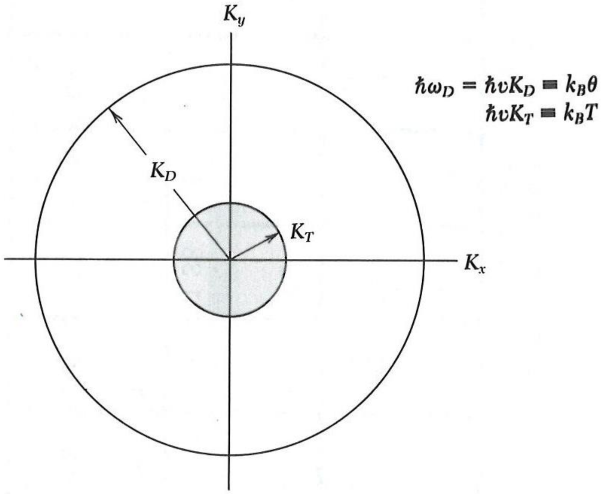


Figure 10 To obtain a qualitative explanation of the Debye T^3 law, we suppose that all phonon modes of wavevector less than K_T have the classical thermal energy k_BT and that modes between K_T and the Debye cutoff K_D are not excited at all. Of the $3N$ possible modes, the fraction excited is $(K_T/K_D)^3 = (T/\theta)^3$, because this is the ratio of the volume of the inner sphere to the outer sphere. The energy is $U \approx k_BT \cdot 3N(T/\theta)^3$, and the heat capacity is $C_V = \partial U/\partial T \approx 12Nk_B(T/\theta)^3$.

Phonon spectrum and Debye temperature

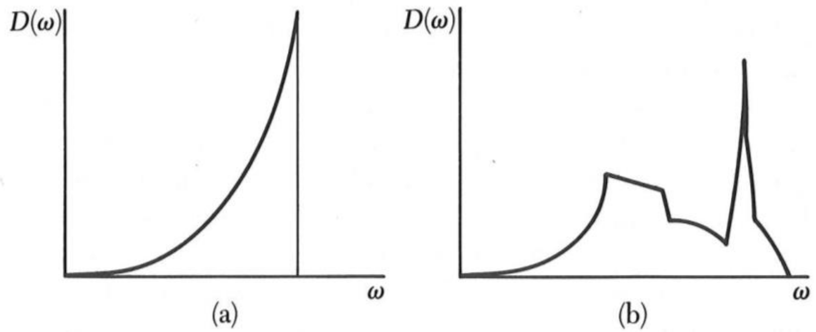


Figure 14 Density of states as a function of frequency for (a) the Debye solid and (b) an actual crystal structure. The spectrum for the crystal starts as ω^2 for small ω, but discontinuities develop at singular points.

Density of states $D(\omega)$:
How many elemental oscillators of frequency ω

Assuming constant speed of sound