Space Charge in Circular Machines

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Overview

Matched 2D Beam Distributions
Space charge on single particle
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Space charge Tuneshift for rms equivalent beams
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Experimental results
Final consideration

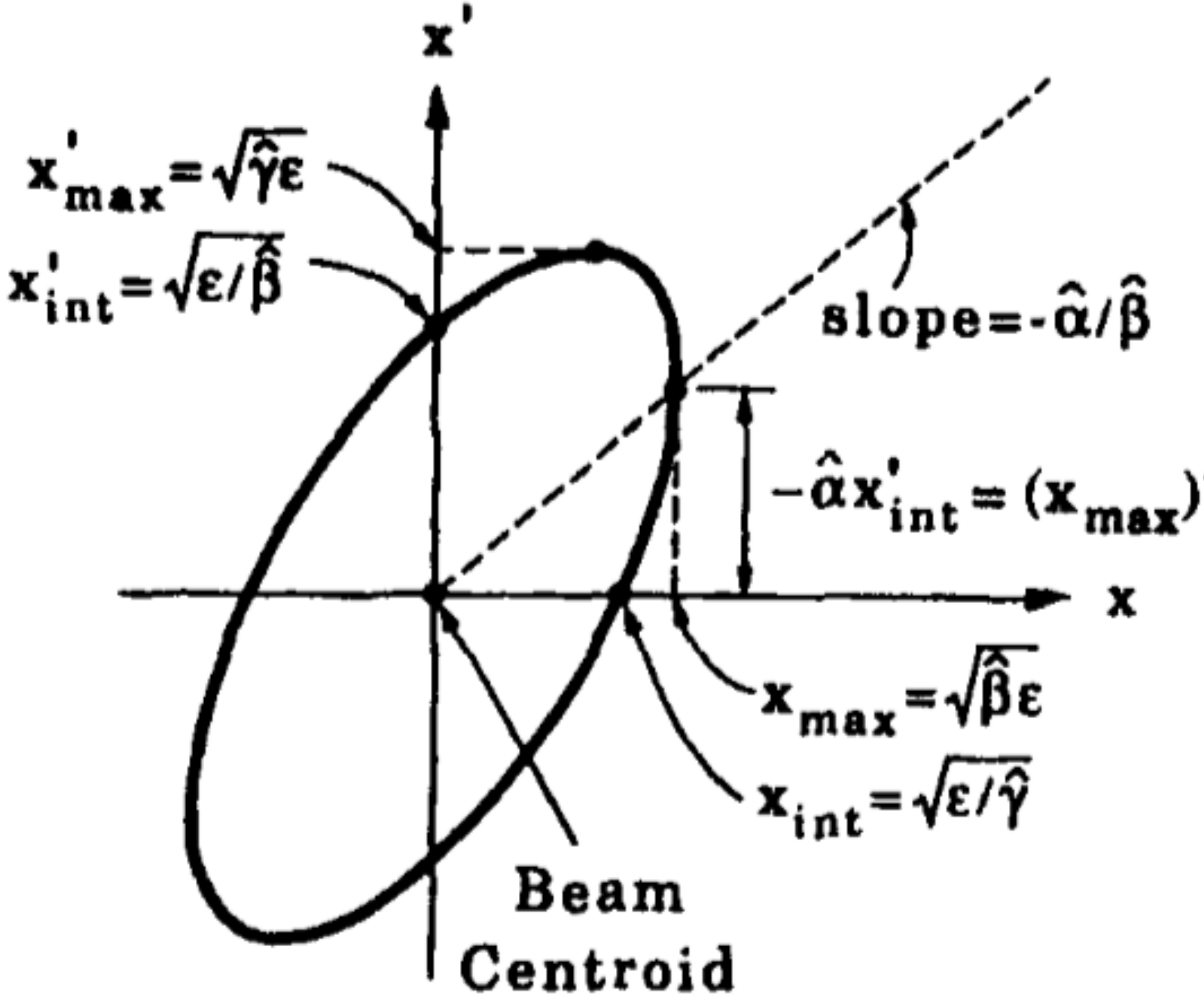
Matched beam at low intensity

With **low intensity** the dynamics at an arbitrary section "s" is determined by the Poincare' section, which for a linear system is described by the Courant-Snyder theory

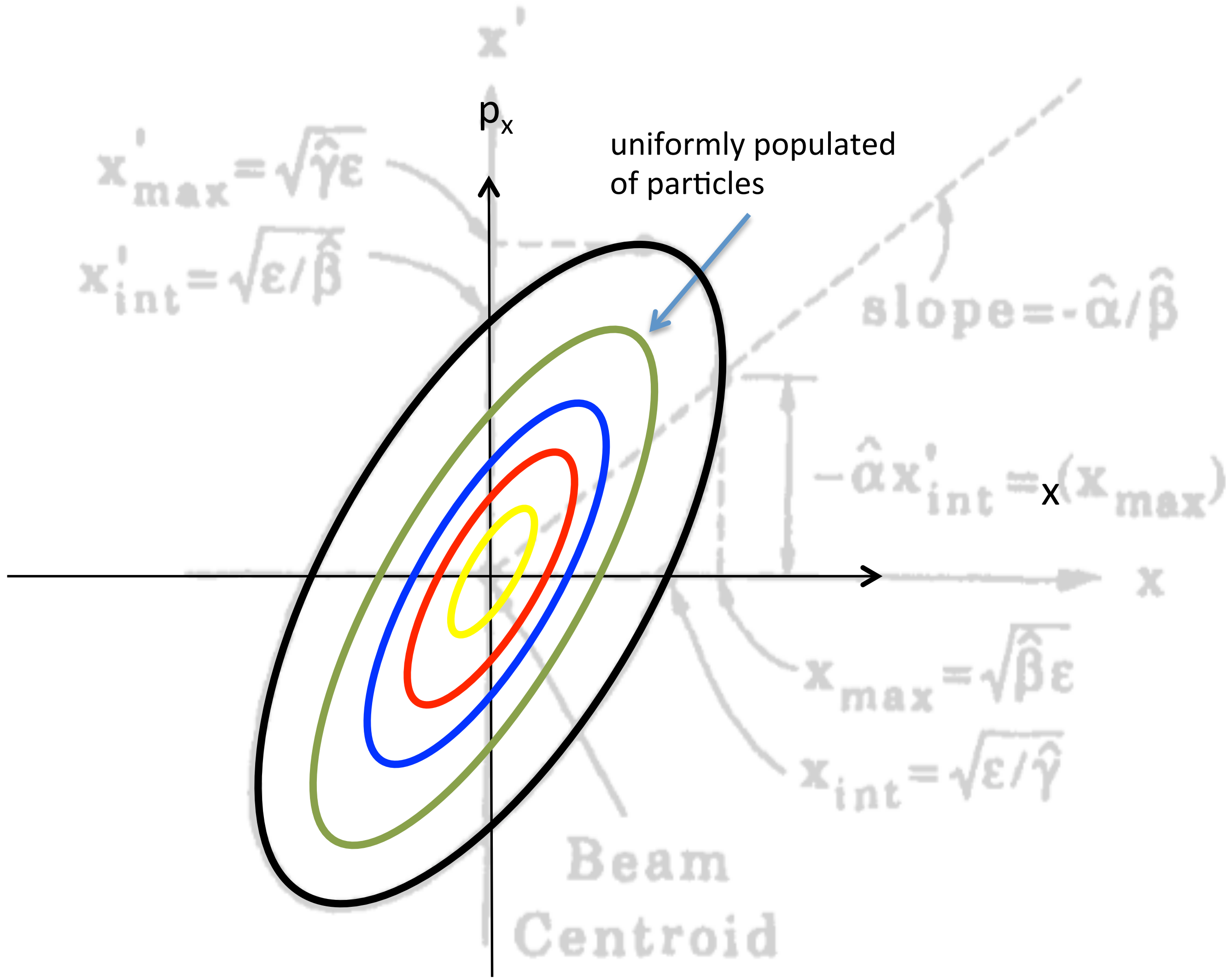
We consider 2D beams matched with the optics:

Matched = "the beam distribution in one section of the machine, will be the same turn after turn"

Orbits in phase space @ "s"



Matched beam



General type of matched distribution

Any type of distribution of the form

$$f(\epsilon_{0x}, \epsilon_{0y})$$

$$\epsilon_{0x} = \gamma_{0x}x^2 + 2\alpha_{0x}xx' + \beta_{0x}x'^2$$
$$\epsilon_{0y} = \gamma_{0y}y^2 + 2\alpha_{0y}yy' + \beta_{0y}y'^2$$

Is matched with the optics at the section "s",
and consequently matched with the optics at any other section

The "0" in the index means "without space charge"

The optical functions are PERIODIC

From sources

Some other consideration has to be used to get down to realistic beam distributions: an energy conservation is here invoked, which basically "incorporates" the physics of the source and with the optics manipulation of the linac.

$$f = f\left(\frac{\epsilon_{0x}}{\mathcal{E}_x} + \frac{\epsilon_{0y}}{\mathcal{E}_y}\right)$$

$\mathcal{E}_x, \mathcal{E}_y$ are "scaling" factors which defines the geometrical extension of the particle distribution in the phase space

Main types of beam distribution

Kapchinsky Vladimirsky

$$f \propto \delta\left(\frac{\epsilon_{0x}}{\mathcal{E}_x} + \frac{\epsilon_{0y}}{\mathcal{E}_y} - 1\right)$$

$$\delta() = \text{Dirac's delta}$$

Waterbag

$$f \propto \Theta\left(1 - \frac{\epsilon_{0x}}{\mathcal{E}_x} - \frac{\epsilon_{0y}}{\mathcal{E}_y}\right)$$

$$\Theta() = \text{Heaviside's function}$$

Gaussian

$$f \propto e^{-\frac{1}{2}\left(\frac{\epsilon_{0x}}{\mathcal{E}_x} + \frac{\epsilon_{0y}}{\mathcal{E}_y}\right)}$$

Beam profile in a 2D matched beam (for low intensity)

Projections

$$f(x,y) = \int f(x,x',y,y')dx'dy'$$

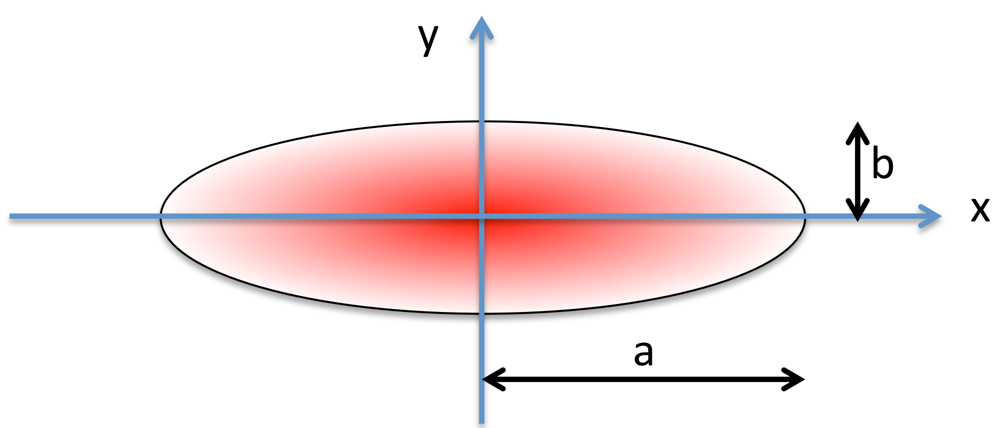
$$f(x,y) = \int f\left(\frac{\epsilon_{0x}(x,x')}{\mathcal{E}_x} + \frac{\epsilon_{0y}(y,y')}{\mathcal{E}_y}\right) dx'dy'$$

Similarly all projections in any plane can be obtained

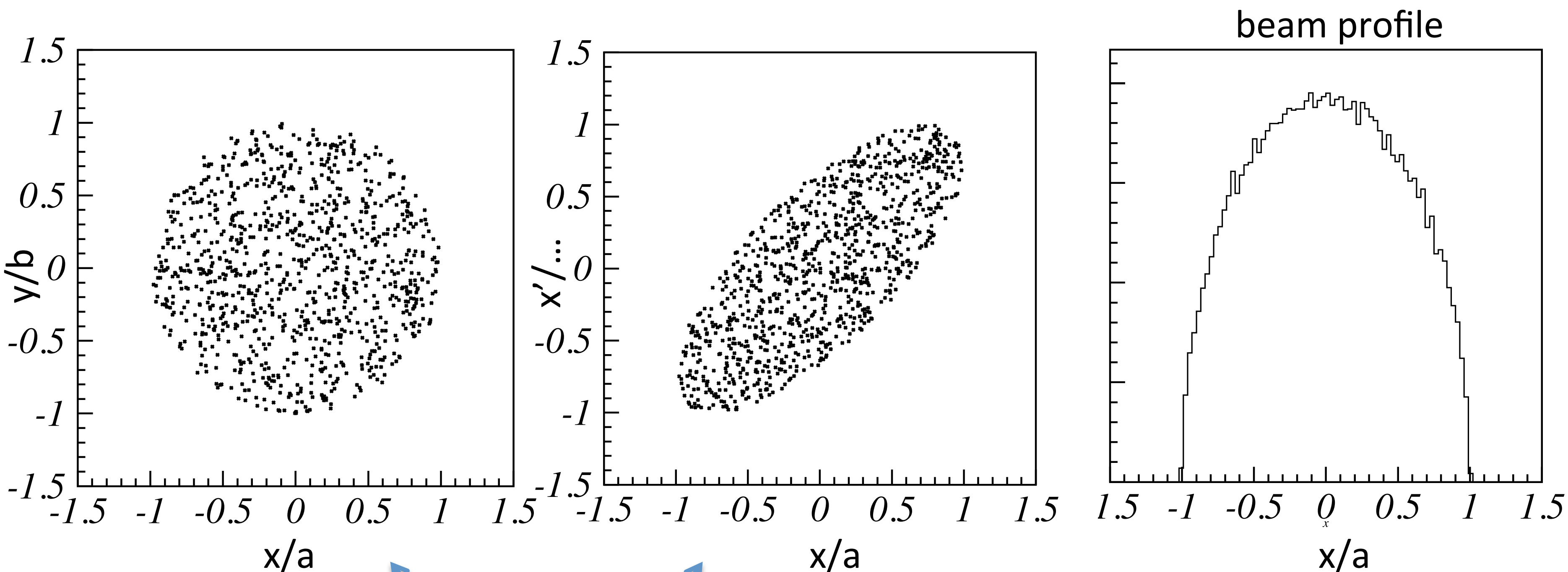
$$f(x,y) = \pi\sqrt{\frac{\mathcal{E}_x\mathcal{E}_y}{\beta_{0x}\beta_{0y}}}\left[F(\infty) - F\left(\frac{x^2}{\beta_{0x}\mathcal{E}_x} + \frac{y^2}{\beta_{0y}\mathcal{E}_y}\right)\right]$$

where $F(t) = \int f(t)dt$ is the primitive of f(t)

we define $a_0 = \sqrt{\beta_{0x}\mathcal{E}_x}$ $b_0 = \sqrt{\beta_{0y}\mathcal{E}_y}$

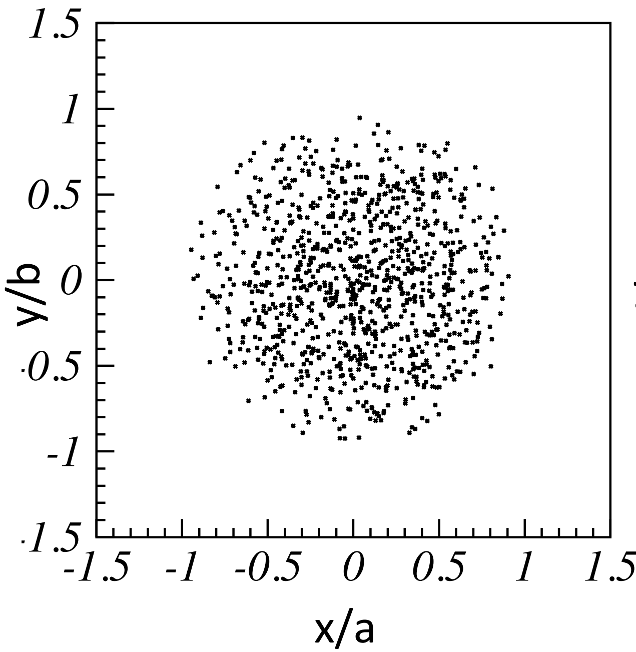


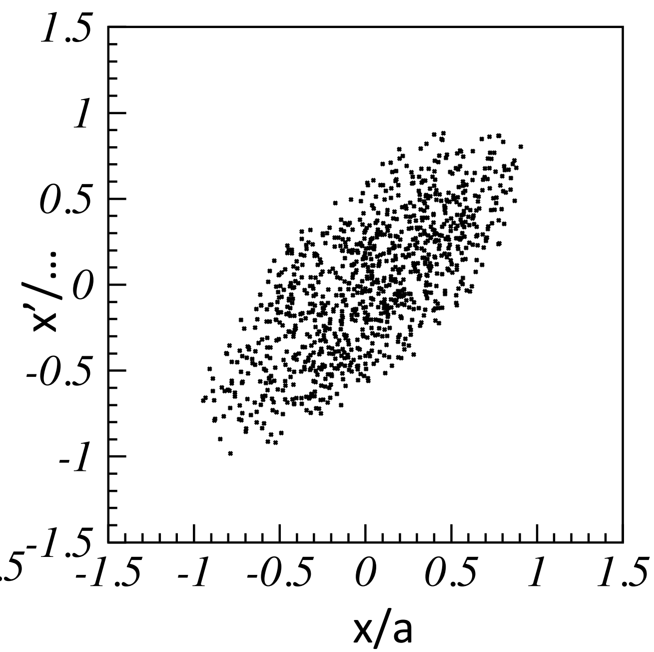
Projections of a KV

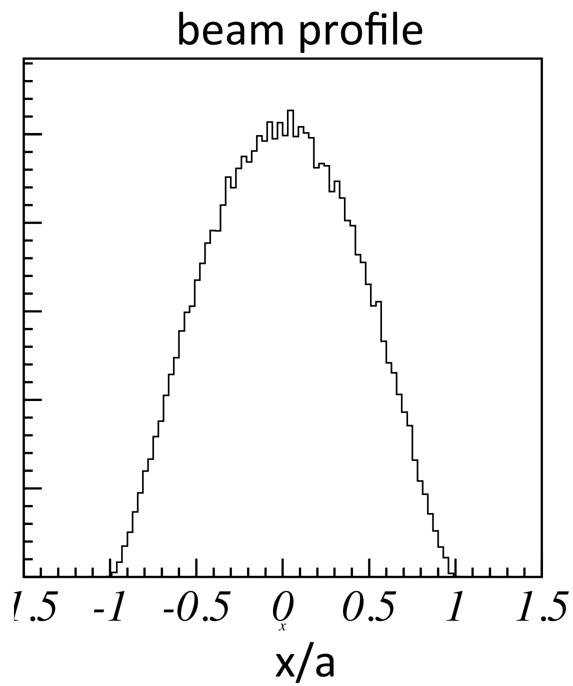


Always uniform in any plane

Projections of a Waterbag (WB)

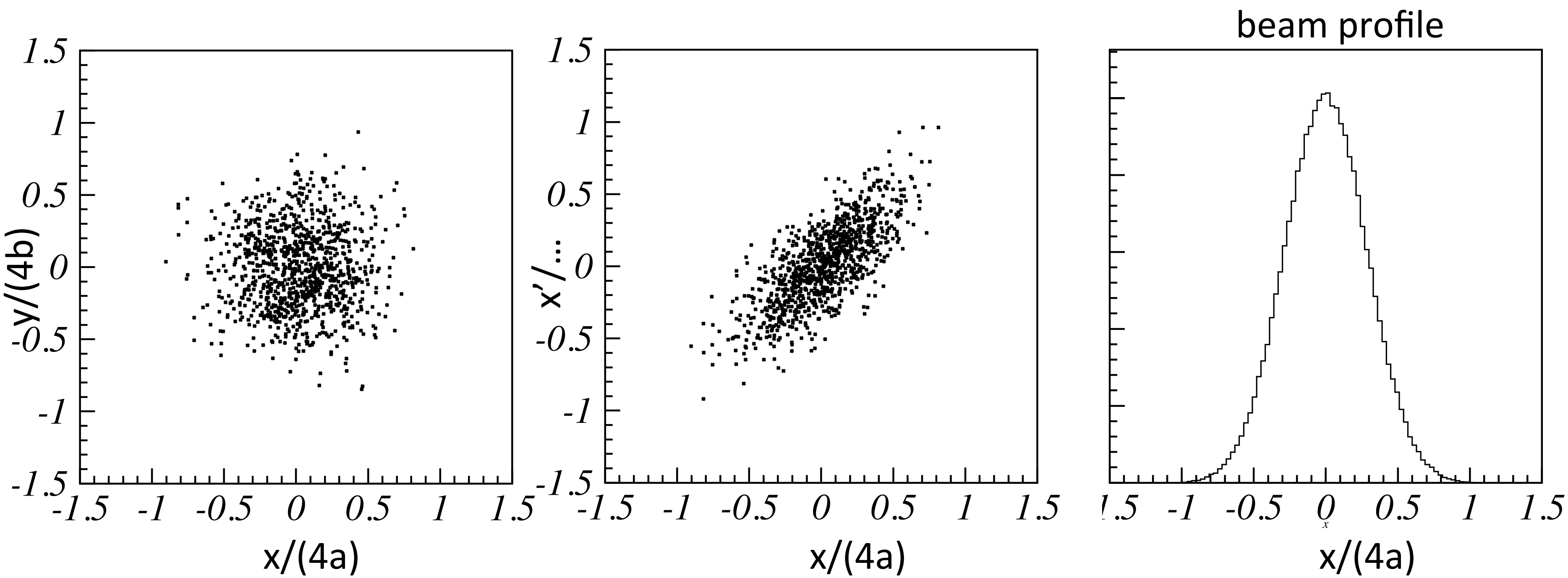






Projections of a Gaussian (G)

Truncated at $$\frac{\epsilon_{0x}}{\mathcal{E}_x} + \frac{\epsilon_{0y}}{\mathcal{E}_y} \leq 16$$



Space charge forces for a **frozen** distribution

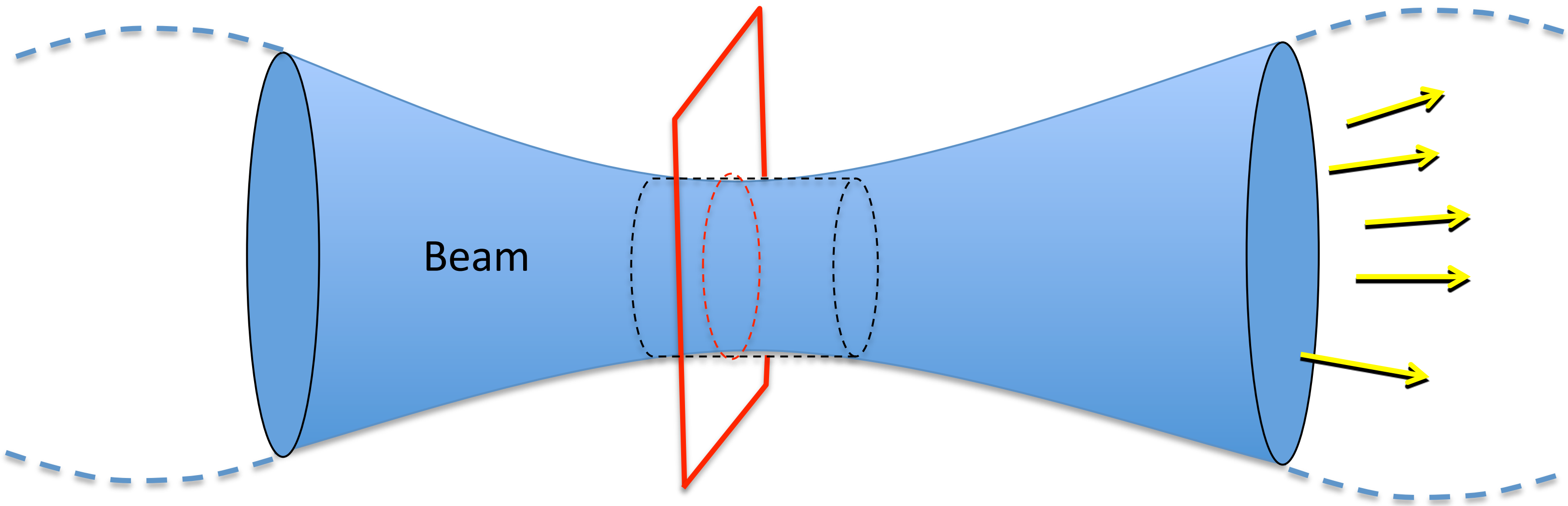
Coulomb electric field

$$\vec{E}(r) = \frac{e}{4\pi\epsilon_0}\sum_i \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3}$$

Approximation

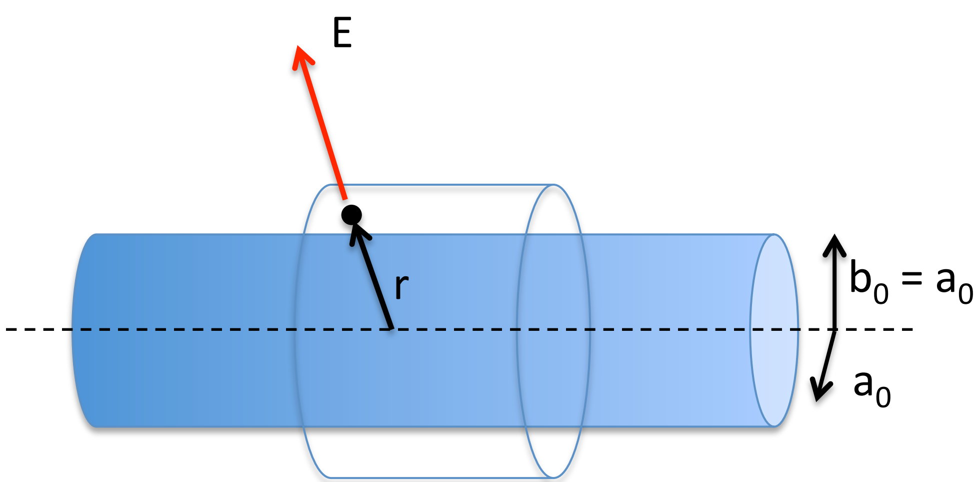
We neglect the longitudinal forces.

Locally the beam can be seen as a "piece" of a coasting beam



transverse section

Infinitely long uniform axi-symmetric cylinder



Longitudinal electric field is zero

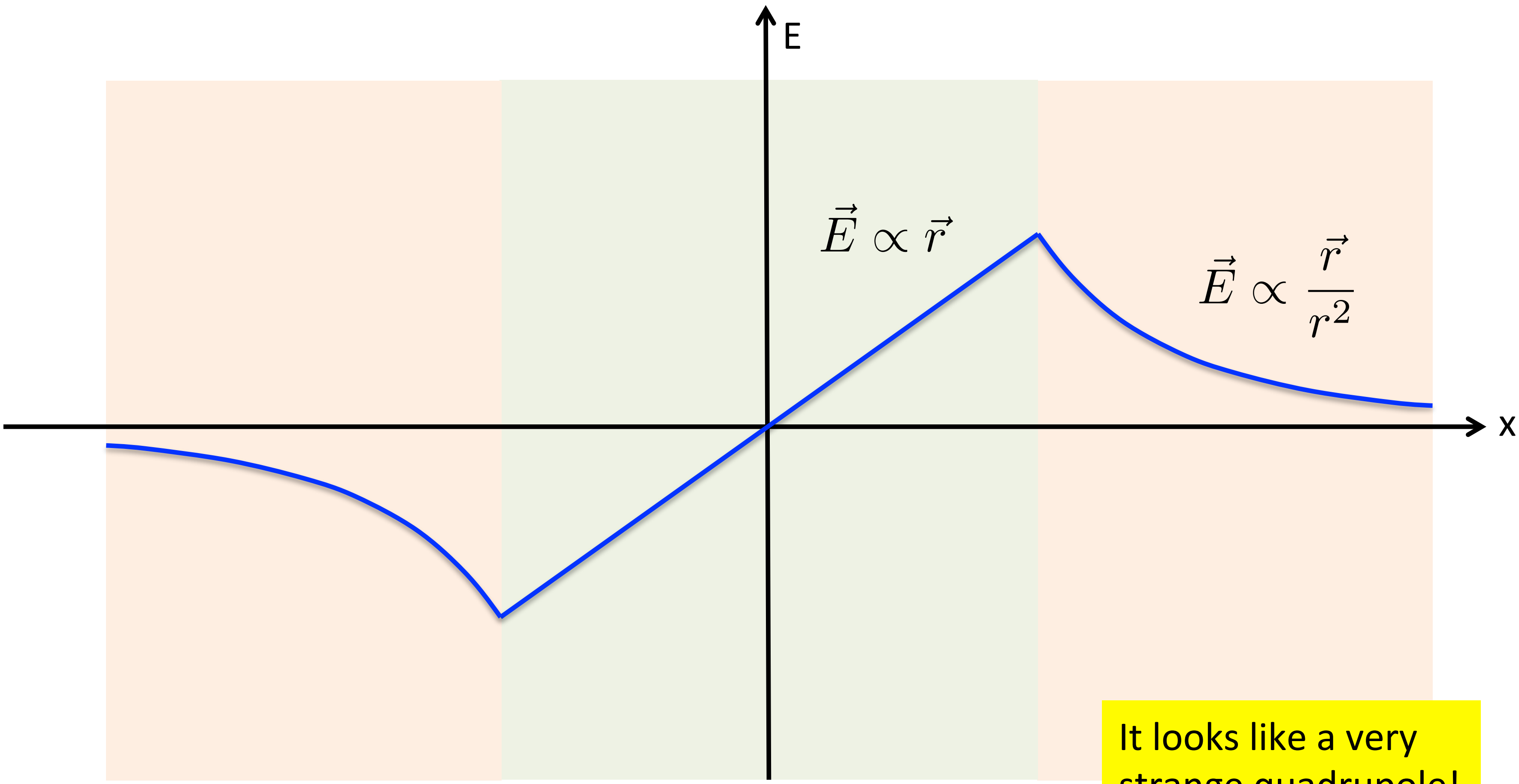
From Gauss law inside

$$E(s) = \frac{\rho(s)}{2\epsilon_0} r$$

Outside the cylinder

$$E(s) = \frac{\rho(s) a_0^2(s)}{2\epsilon_0} \frac{1}{r}$$

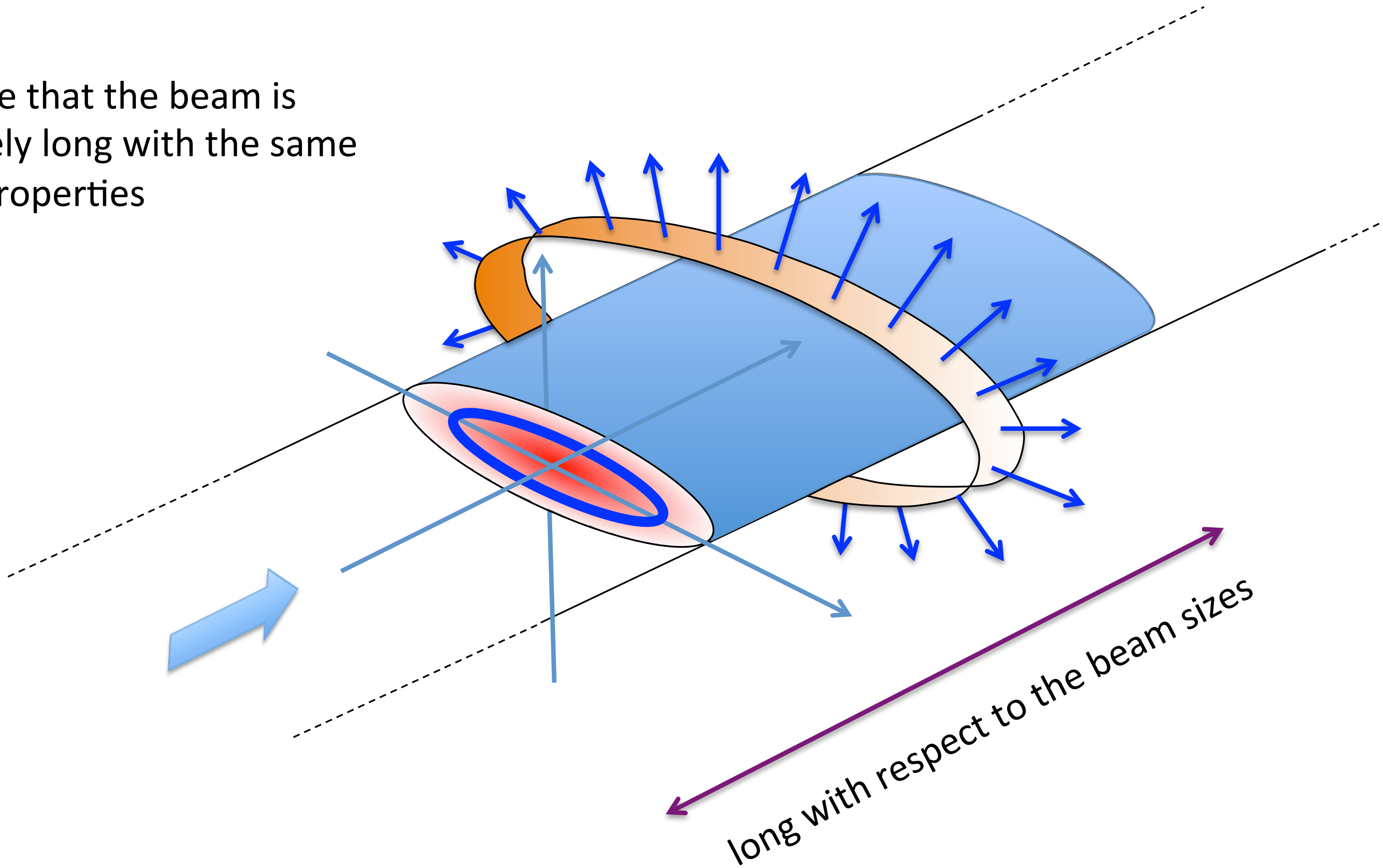
Transverse Electric field: uniform



It looks like a very strange quadrupole!

Ellipsoidal beams

Assume that the beam is infinitely long with the same local properties



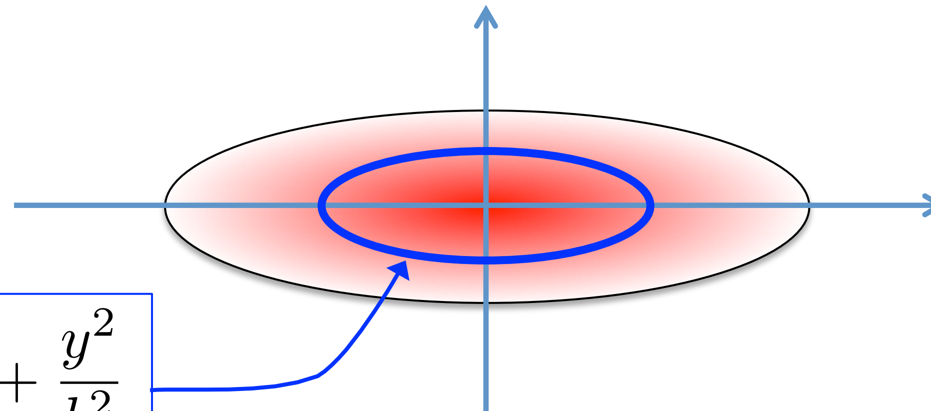
Analytic from for ellipsoidal beams

$$\rho = \lambda n(x,y)$$

$$n(x,y) = \frac{\hat{n}(T)}{\pi a_0 b_0}$$

$$\int_0^\infty \hat{n}(t)dt = 1$$

$$T = \frac{x^2}{a_0^2} + \frac{y^2}{b_0^2}$$



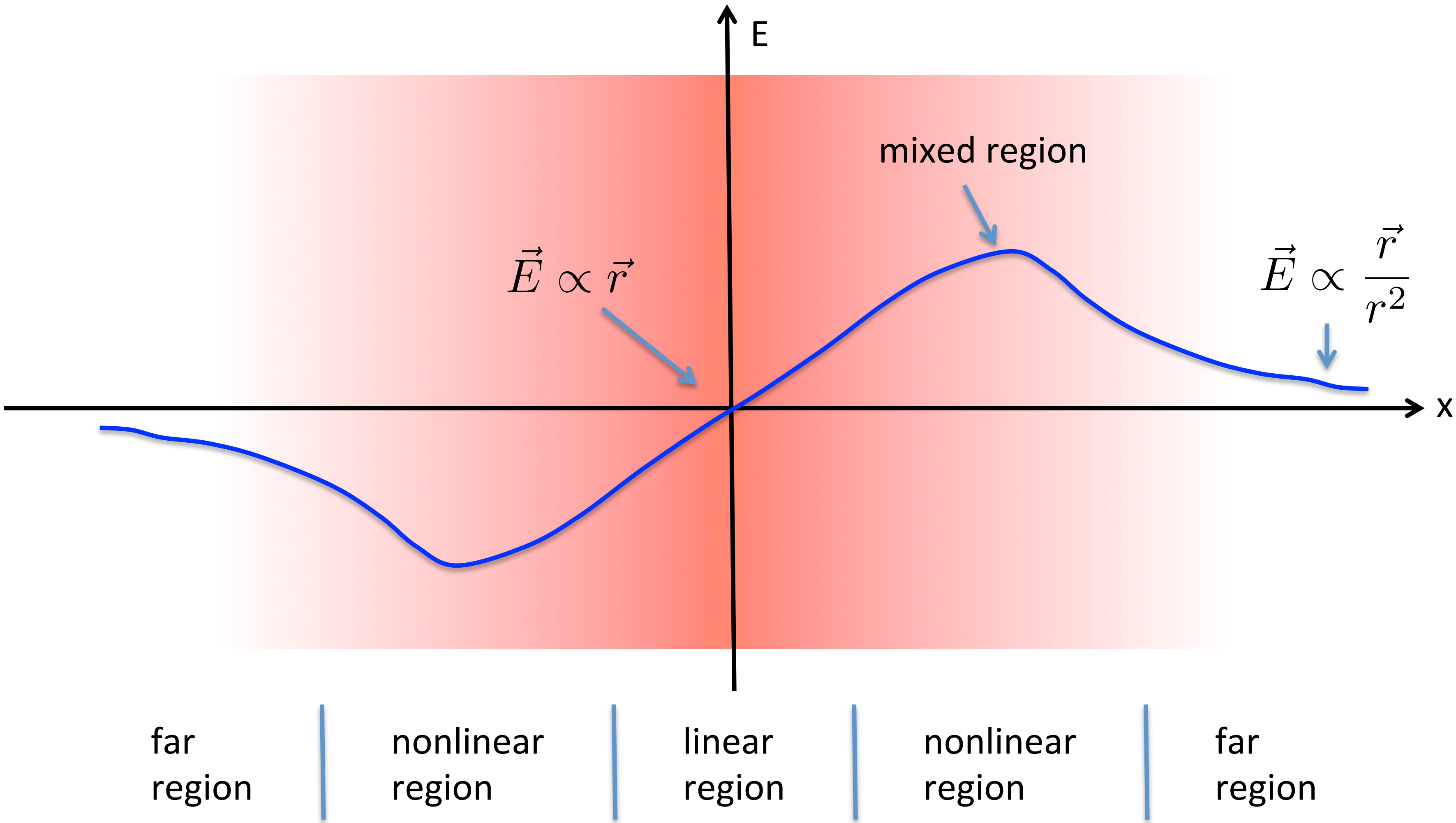
Electric field

$$E_x = \frac{\lambda}{2\pi\epsilon_0} x \int_0^\infty \frac{\hat{n}(\hat{T})}{(a_0^2+t)^{3/2}(b_0^2+t)^{1/2}} dt$$

$$\hat{T} = \frac{x^2}{a_0^2+t} + \frac{y^2}{b_0^2+t}$$

W. Kellog, *Foundation of Potential Theory* (Dover Publications, New York, 1953)

Electric Field



Dynamics in the linear region

In the center of the beam

any ellipsoidal distribution close to the origin always yields

$$E_x = \frac{\lambda \hat{n}(0)}{\pi \epsilon_0} \frac{1}{a_0(a_0 + b_0)} x$$

If a_0, b_0 are frozen then E_x enters in the equation of motion

$$\frac{d^2 x}{ds^2} + k_{0x} x = \frac{e}{m \gamma^3 v_s^2} E_x$$

k_{0x}, k_{0y} s-dependent

This is a Hill equation that includes the effect of the space charge in the center of the beam under the assumption that the beam is coasting, ellipsoidal, and frozen

Dynamics in the linear region

It is convenient to define the quantity

$$K = \frac{eI}{2\pi\epsilon_0 m\gamma^3\beta^3 c^3}$$

(positive)

$$\frac{d^2x}{ds^2} + \left[k_{0x} - \hat{n}(0)\frac{2K}{a_0(a_0+b_0)}\right]x = 0$$

$$\frac{d^2y}{ds^2} + \left[k_{0y} - \hat{n}(0)\frac{2K}{b_0(a_0+b_0)}\right]y = 0$$

KV frozen beams

For a KV beam $\hat{n}(0) = 1$

β_{0x}, α_{0x} Optics without frozen space charge

$$\frac{d^2x}{ds^2} + \left[k_{0x} - \hat{n}(0) \frac{2K}{a_0(a_0 + b_0)} \right] x = 0$$

β_x, α_x Optics with frozen space charge of frozen coasting beams

Effect on dynamics

Given the naked lattice a frozen ellipsoidal coasting beam is created.
This frozen beam has a certain "Emittances" $\mathcal{E}_x, \mathcal{E}_y$

Each particle of the beam evolves according to

$$\frac{d^2x}{ds^2} + \left[k_{0x}(s) - \hat{n}(0) \frac{2K}{a_0(a_0 + b_0)} \right] x = 0$$

The evolution of the particle beam can be predicted (of sizes $a_b(s)$, $b_b(s)$)

Meaning of it all

frozen distribution

$$a_0, b_0$$

naked optics

$$\beta_{0x}, \beta_{0y}$$

particle beam

$$a_0(s), b_0(s)$$

evolve with optics without space charge

$\neq$

effective optics

$$\beta_x, \beta_y$$

a(s), b(s)

evolve with optics with space charge created by frozen beams

Matched beam with space charge (KV)

For a KV beam $\hat{n}(0) = 1$

The frozen beam distribution have to be generated for the optics with space charge !!

$$f\left(\frac{\epsilon_x}{\mathcal{E}_x} + \frac{\epsilon_y}{\mathcal{E}_y}\right)$$

$$\epsilon_x = \beta_x x'^2 + 2\alpha_x x x' + \gamma_x x^2$$
$$\epsilon_y = \beta_y y'^2 + 2\alpha_y y y' + \gamma_y y^2$$

β_{0x}, β_{0y} Optics without frozen space charge

$$\frac{d^2x}{ds^2} + \left[k_{0x}(s) - \frac{2K}{a(a+b)}\right]x = 0$$

β_x, β_y

Optics with frozen space charge of frozen coasting beams

Stationary distribution

a multi-particle system evolves also a type of distribution

Example

during evolution becomes

$$f \propto \Theta\left(1 - \frac{\epsilon_x}{\mathcal{E}_x} - \frac{\epsilon_y}{\mathcal{E}_y}\right) \quad \Longrightarrow \quad f \propto e^{-\frac{1}{2}\left(\frac{\epsilon_x}{\mathcal{E}_x} + \frac{\epsilon_y}{\mathcal{E}_y}\right)}$$

this means that in general

$$f\left(\frac{\epsilon_x}{\mathcal{E}_x} + \frac{\epsilon_y}{\mathcal{E}_y}, s\right)$$

A particle distribution evolves according to the **Vlasov** equation

$$\frac{\partial}{\partial s} f\left(x, x', y, y', s\right) + \frac{\partial}{\partial x} f\left(x, x', y, y', s\right) x' + \frac{\partial}{\partial x'} f\left(x, x', y, y', s\right) x'' + \frac{\partial}{\partial y} f\left(x, x', y, y', s\right) y' + \frac{\partial}{\partial y'} f\left(x, x', y, y', s\right) y'' = 0$$

A stationary distribution satisfies

$$\frac{\partial}{\partial s} f\left(x, x', y, y', s\right) = 0$$

It means: the distribution does not change "type" during evolution

In absence of space charge the matched distribution of the type

$$f\left(\frac{\epsilon_{0x}(x,x',s)}{\mathcal{E}_x}+\frac{\epsilon_{0y}(y,y',s)}{\mathcal{E}_y},s\right)$$

is stationary

It is easy to check that

$$\frac{\partial}{\partial s}f\left(\frac{\epsilon_{0x}(x,x',s)}{\mathcal{E}_x}+\frac{\epsilon_{0y}(y,y',s)}{\mathcal{E}_y},s\right)=0$$

In presence of space charge the only distribution stationary is the KV

$$\delta\left(\frac{\epsilon_x}{\mathcal{E}_x}+\frac{\epsilon_y}{\mathcal{E}_y}\right)$$

$$E_x=\frac{\lambda}{\pi\epsilon_0}\frac{1}{a(a+b)}x$$

linear

$$\frac{\partial}{\partial s}f\left(\frac{\epsilon_x}{\mathcal{E}_x}+\frac{\epsilon_y}{\mathcal{E}_y},s\right)=0$$

therefore delta has no explicit dependence on "s_1"

$$\frac{\partial}{\partial x}f\left(x,x',y,y',s\right)x'+\frac{\partial}{\partial x'}f\left(x,x',y,y',s\right)x''+$$
$$\frac{\partial}{\partial y}f\left(x,x',y,y',s\right)y'+\frac{\partial}{\partial y'}f\left(x,x',y,y',s\right)y''=0$$

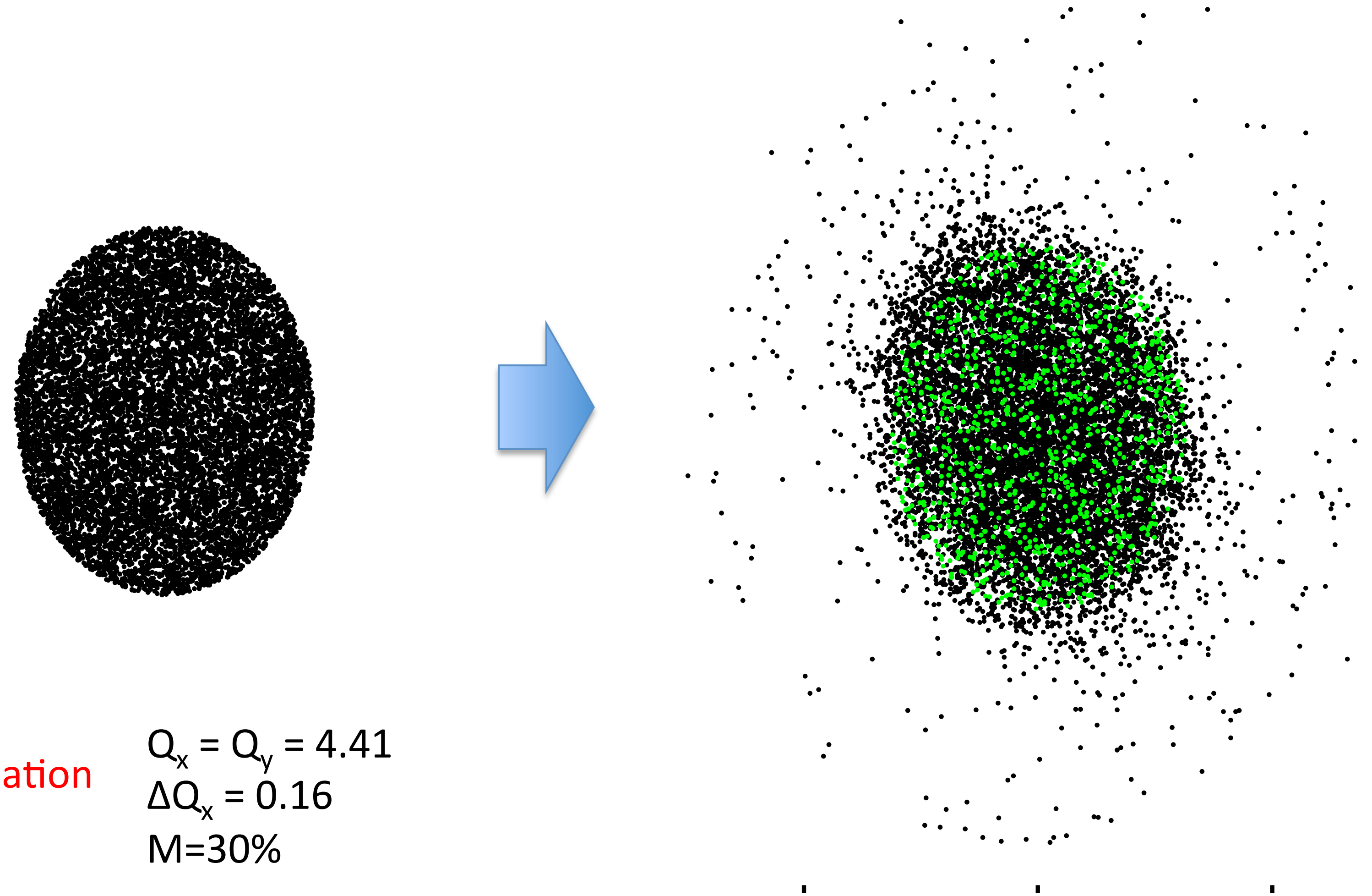
$$\beta_x,\alpha_x.\beta_y,\alpha_y$$

Optics with space charge

$$\frac{d^2x}{ds^2}+\left[k_{0x}(s)-\frac{2K}{a(a+b)}\right]x=0$$

Hill's equation

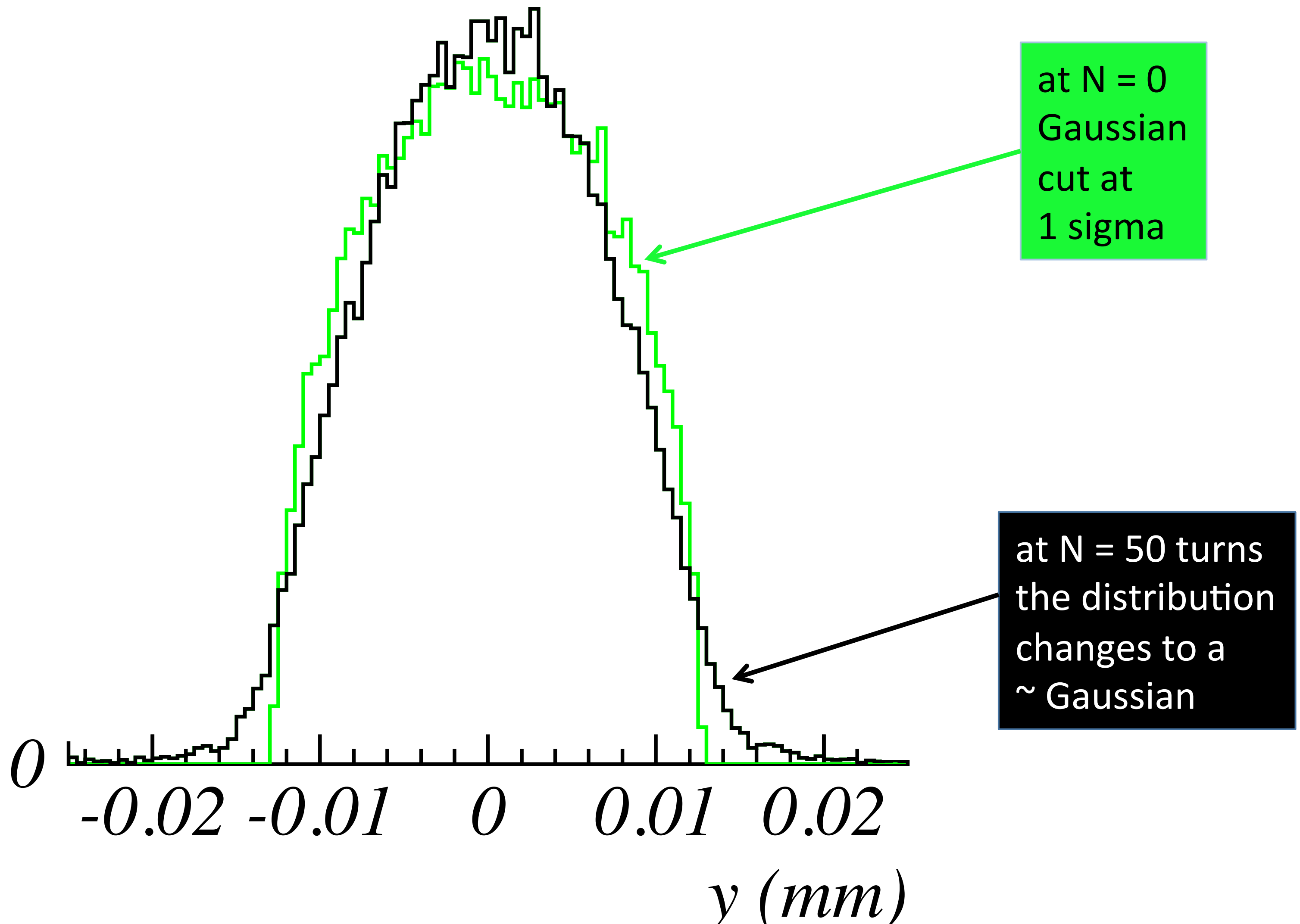
Simulation example of non-stationary beam distribution



PIC simulation

$Q_x = Q_y = 4.41$
$\Delta Q_x = 0.16$
M=30%

Change of beam profile



How characterize a beam

$$f = f\left(\frac{\epsilon_x}{\mathcal{E}_x} + \frac{\epsilon_y}{\mathcal{E}_y}\right)$$

It means:

1) To know the type of distribution: KV, WB, G
2) To know $\mathcal{E}_x, \mathcal{E}_y$

RMS characterization: moments

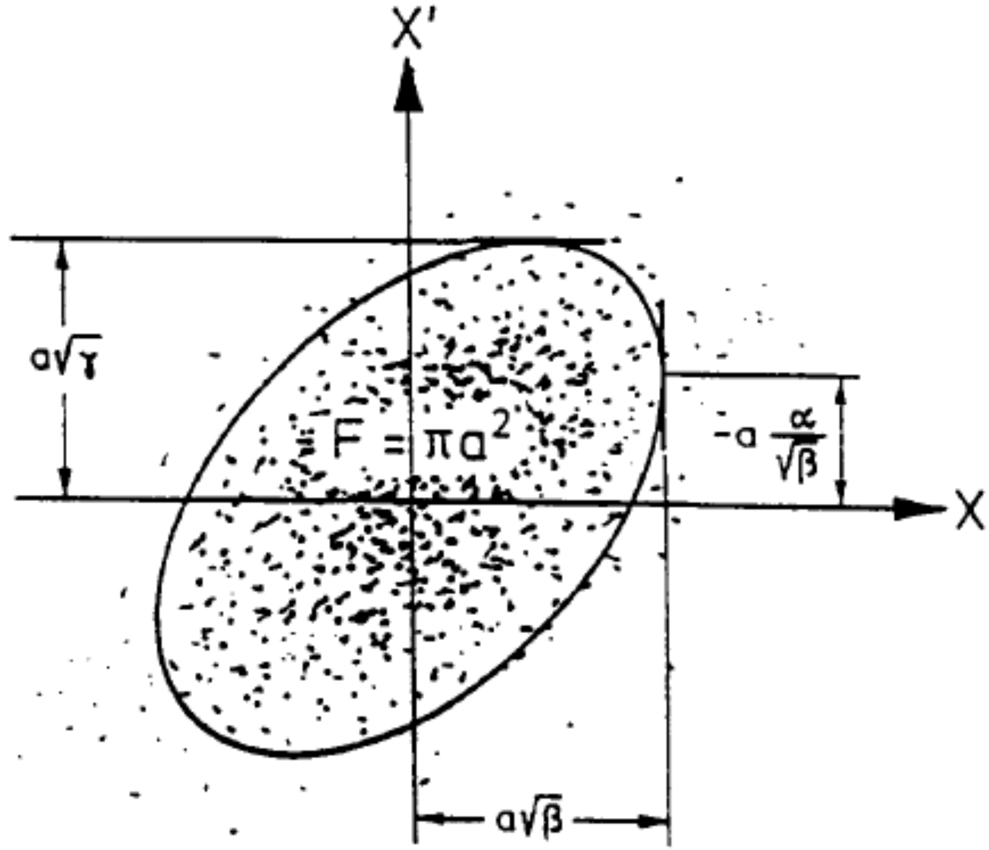
$$\langle x^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i^2$$

$$\langle x'^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i'^2$$

$$\langle xx' \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i x_i'$$

$$\tilde{\epsilon}_x^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2$$

RMS emittance depends
on the beam distribution



RMS quantities and beam distribution

There is a relation between beam distributions and rms quantities

$$a = \sqrt{\beta_x \mathcal{E}_x} \qquad \tilde{x} = \sqrt{\langle x^2 \rangle} \qquad \tilde{\epsilon}_x^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2$$

	$f(x,x',y,y')$	$f(x,y)$	$\frac{a}{\tilde{x}}$	$\frac{\mathcal{E}_x}{\tilde{\epsilon}_x}$
KV	$\frac{1}{\pi^2 \mathcal{E}_x \mathcal{E}_y} \delta\left(\frac{\epsilon_x}{\mathcal{E}_x} + \frac{\epsilon_y}{\mathcal{E}_y} - 1\right)$	$\frac{1}{\pi a b}$	2	4
WB	$\frac{2}{\pi^2 \mathcal{E}_x \mathcal{E}_y} \Theta\left(1 - \frac{\epsilon_x}{\mathcal{E}_x} - \frac{\epsilon_y}{\mathcal{E}_y}\right)$	$\frac{2}{\pi a b}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$	$\sqrt{6}$	6
G	$\frac{1}{4\pi^2 \mathcal{E}_x \mathcal{E}_y} \exp\left[-\frac{1}{2}\left(\frac{\epsilon_x}{\mathcal{E}_x} + \frac{\epsilon_y}{\mathcal{E}_y}\right)\right]$	$\frac{1}{2\pi a b} \exp\left[-\frac{1}{2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\right]$	1	1

Envelope equation

For linear forces, i.e. KV beam $\tilde{\epsilon}_x = const.$ $\tilde{\epsilon}_y = const.$

$$\tilde{x}'' + k_{0x}(s)\tilde{x} - \frac{K}{2(\tilde{x}+\tilde{y})} - \frac{\tilde{\epsilon}_x^2}{\tilde{x}^3} = 0$$

$$\tilde{y}'' + k_{0y}(s)\tilde{y} - \frac{K}{2(\tilde{x}+\tilde{y})} - \frac{\tilde{\epsilon}_y^2}{\tilde{y}^3} = 0$$

RMS equivalent beams

Although other distribution are not self-consistent, beams with the **same rms moments evolve have the same evolution (F. Sacherer)**

it means that the following equation is valid

$$\tilde{x}'' + k_{0x}(s)\tilde{x} - \frac{K}{2(\tilde{x}+\tilde{y})} - \frac{\tilde{\epsilon}_x^2}{\tilde{x}^3} = 0$$

$$\tilde{y}'' + k_{0y}(s)\tilde{y} - \frac{K}{2(\tilde{x}+\tilde{y})} - \frac{\tilde{\epsilon}_y^2}{\tilde{y}^3} = 0$$

but $\tilde{\epsilon}_x, \tilde{\epsilon}_y$ now becomes time dependent !!!

Tuneshift

Naked optics yields the tune

$$Q_{0x} = \frac{1}{2\pi}\int_0^L \frac{1}{\beta_{0x}(s)}ds$$

The space charge is like a distributed gradient.
For moderate intensity, for x ~ y ~ 0 we find

$$\Delta Q_x = \frac{1}{4\pi}\int_0^L \beta_{0x}(s)\hat{E}_x(s)ds = -\frac{1}{4\pi}\int_0^L \beta_{0x}(s)\hat{n}(0)\frac{2K}{a(s)(a(s)+b(s))}ds$$

$$\Delta Q_x = -\frac{1}{4\pi}\int_0^L \frac{\beta_{0x}(s)}{a^2(s)}\hat{n}(0)\frac{2K}{1+\frac{b(s)}{a(s)}}ds$$

$$\Delta Q_x = -\frac{1}{4\pi}\frac{1}{\mathcal{E}_x}\int_0^L \frac{\beta_{0x}(s)}{\beta_x(s)}\hat{n}(0)\frac{2K}{1+\sqrt{\frac{\mathcal{E}_y\beta_y(s)}{\mathcal{E}_x\beta_x(s)}}}ds$$

If space charge is not too large $\quad \dfrac{\beta_{0x}(s)}{\beta_x(s)} \simeq 1$

and

$$\left\langle \frac{1}{1+\sqrt{\frac{\mathcal{E}_y\beta_y(s)}{\mathcal{E}_x\beta_x(s)}}} \right\rangle_s = \frac{1}{1+\sqrt{\frac{\mathcal{E}_y\langle\beta_y\rangle_s}{\mathcal{E}_x\langle\beta_x\rangle_s}}} + O(\langle\Delta\beta_x^2\rangle_s, \langle\Delta\beta_y^2\rangle_s)$$

because beta is periodic

$$Q_x = \frac{R}{\langle \beta_x \rangle_s} + \frac{R}{\langle \beta_x \rangle^3_s} \langle \Delta\beta_x^2 \rangle_s + ...$$

If beta does not oscillate too wild $$Q_x \simeq \frac{R}{\langle \beta_x \rangle_s}$$

$$\Delta Q_x \simeq -\frac{R^2}{Q_x} K \hat{n}(0) \frac{1}{\sqrt{\mathcal{E}_x \langle \beta_x \rangle_s}(\sqrt{\mathcal{E}_x \langle \beta_x \rangle_s} + \sqrt{\mathcal{E}_y \langle \beta_y \rangle_s})}$$

Space charge tuneshift for rms equivalent beams

RMS moment <x^2> of a matched beam is

$$\langle x^2 \rangle(s) = \mathcal{E}_x \beta_x(s) \frac{1}{2} \int_0^\infty T\hat{n}(T) dT$$

and the rms emittance is

$$\tilde{\epsilon}_x = \mathcal{E}_x \frac{1}{2} \int_0^\infty T\hat{n}(T) dT$$

$$\Delta Q_x \simeq -\frac{R^2}{Q_x} K \hat{n}(0) \frac{1}{2} \int_0^\infty T\hat{n}(T) dT \frac{1}{\sqrt{\tilde{\epsilon}_x \langle \beta_x \rangle_s} (\sqrt{\tilde{\epsilon}_x \langle \beta_x \rangle_s} + \sqrt{\tilde{\epsilon}_y \langle \beta_y \rangle_s})}$$

Jackson Laslett

peak tuneshift $$\Delta Q_x \simeq f \Delta \tilde{Q}_x$$

"rms equivalent tuneshift" $$\Delta \tilde{Q}_x \simeq -\frac{R^2}{Q_x} K \frac{1}{\sqrt{\tilde{\epsilon}_x \langle \beta_x \rangle_s} (\sqrt{\tilde{\epsilon}_x \langle \beta_x \rangle_s} + \sqrt{\tilde{\epsilon}_y \langle \beta_y \rangle_s})}$$

distribution factor $$f = \hat{n}(0) \frac{1}{2} \int_0^\infty T \hat{n}(T) dT$$ (f is often called G)

	KV	**WB**	**G**
f	1/4	1/3	1/2

Distribution Summary

from Martin Reiser book

Distribution Function	Definition (Normalized), $f(r_4)$	Ratio of Total Emittance to rms Emittance, $\epsilon_t/\tilde{\epsilon}$	Particle Density in Real Space, $r^2 = x^2 + y^2$
Kapchinsky–Vladimirsky (K–V)	$\frac{1}{2\pi^2 a^3}\delta(r_4 - a)$	4	$\frac{1}{\pi a^2}$
Waterbag (WB)	$\frac{2}{\pi^2 a^4}$	6	$\frac{2}{\pi a^2}\left(1 - \frac{r^2}{a^2}\right)$
Parabolic (PA)	$\frac{6}{\pi^2 a^4}\left(1 - \frac{r_4^2}{a^2}\right)$	8	$\frac{10}{3\pi a^2}\left(1 - 3\frac{r^2}{a^2} + 2\frac{r^3}{a^3}\right)$
Gaussian (GA)	$\frac{1}{4\pi^2\delta^4}\exp\left(-\frac{r_4^2}{2\delta^2}\right)$ $\delta^2 = \overline{x^2}$	$\simeq n^2$ if truncated at $n\delta$, $n \geq 4$	$\frac{1}{2\pi\delta^2}\exp(-\frac{r^2}{2\delta^2})$

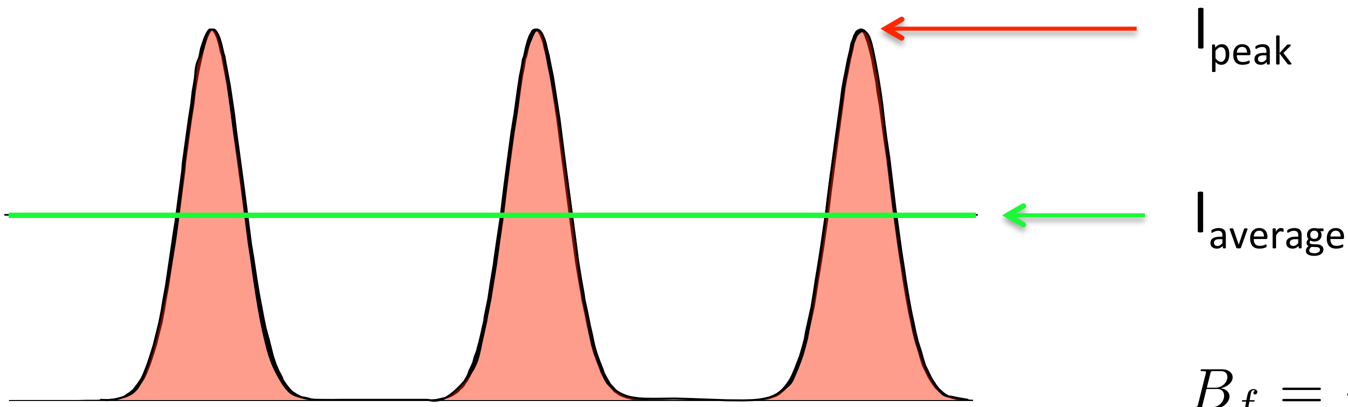
Space Charge limit

General rule $|\Delta Q_{xl}| \leq 0.25$

Maximum longitudinal particle density

$$\lambda = \frac{2\pi\epsilon_0 m A \gamma^3 v^2}{e^2 Z^2} \frac{|\Delta Q_{xl}|}{f} \frac{\tilde{\epsilon}_x}{R} \left(1 + \sqrt{\frac{\langle\beta_y\rangle\tilde{\epsilon}_y}{\langle\beta_x\rangle\tilde{\epsilon}_x}}\right)$$

For bunched beams



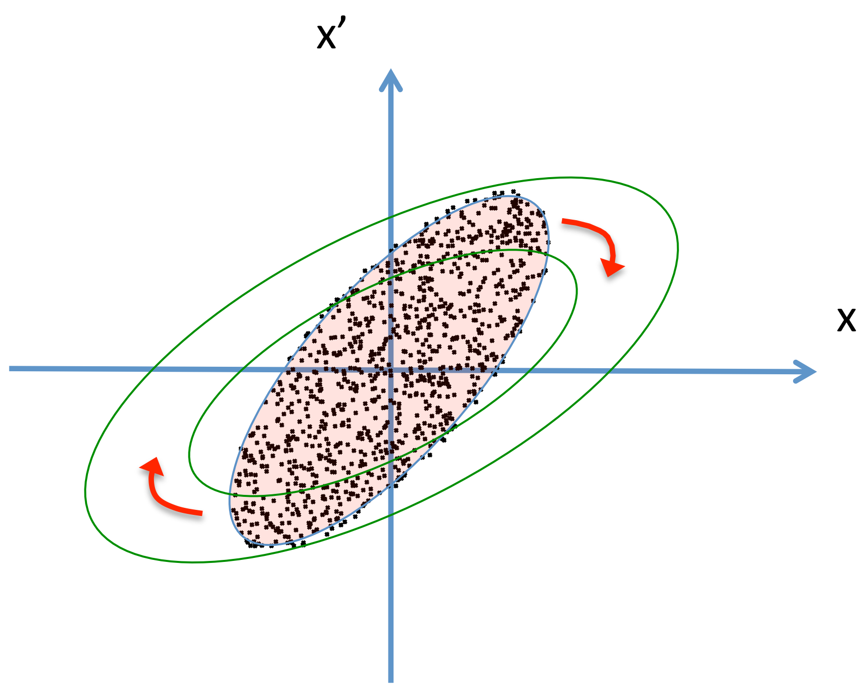
$$B_f = \frac{I_{average}}{I_{peak}}$$

$$N_{tot} = B_f \frac{(2\pi)^2 \epsilon_0 m A \gamma^3 v^2}{e^2 Z^2} \frac{|\Delta Q_{xl}|}{f} \tilde{\epsilon}_x \left(1 + \sqrt{\frac{\langle \beta_y \rangle \tilde{\epsilon}_y}{\langle \beta_x \rangle \tilde{\epsilon}_x}} \right)$$

Oscillation of mismatched beams

Without space charge

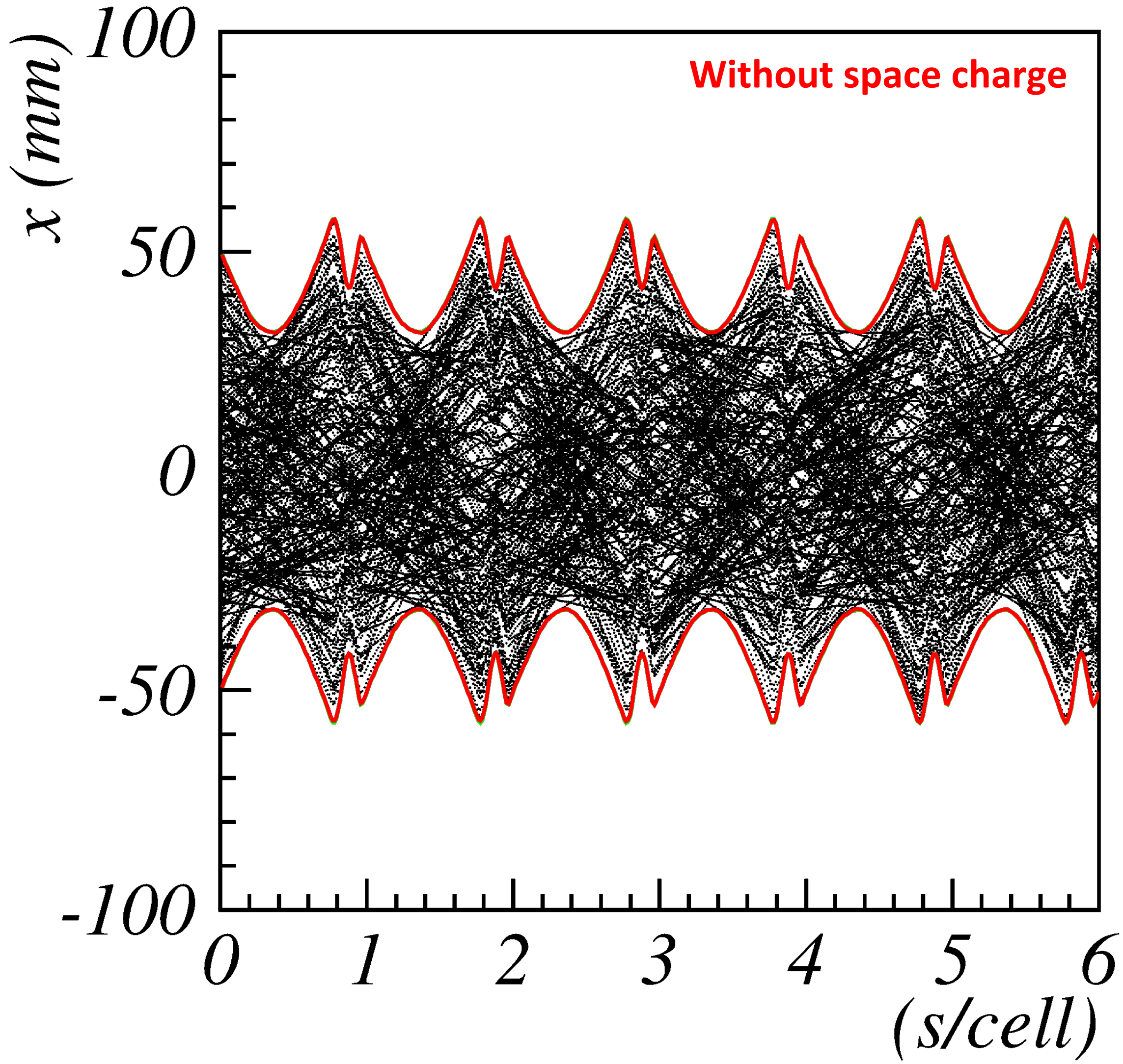
Small oscillation: a mismatched KV



Number of oscillations per turn

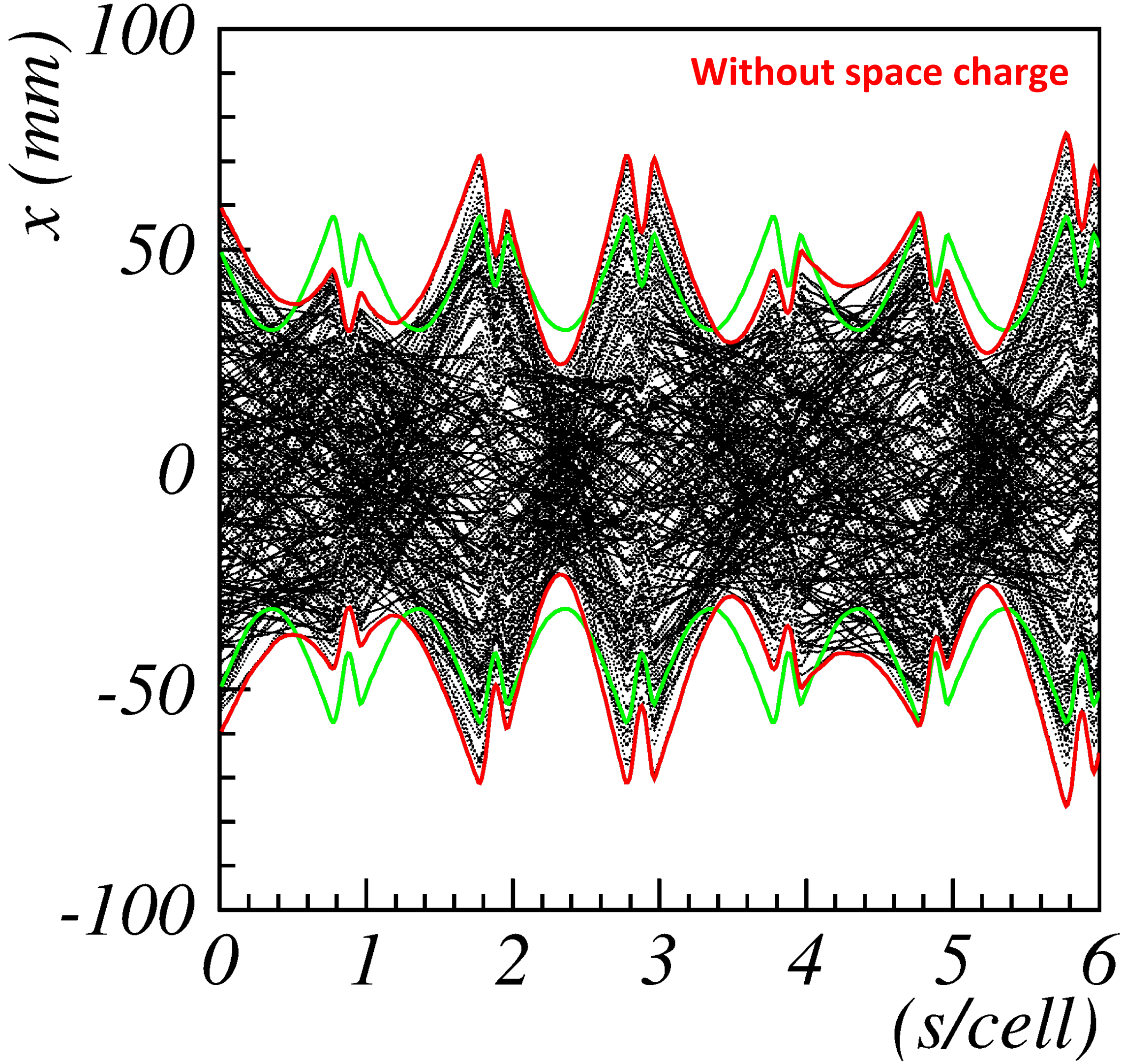
$$2 \times Q_{0x}$$

Matched beam envelope



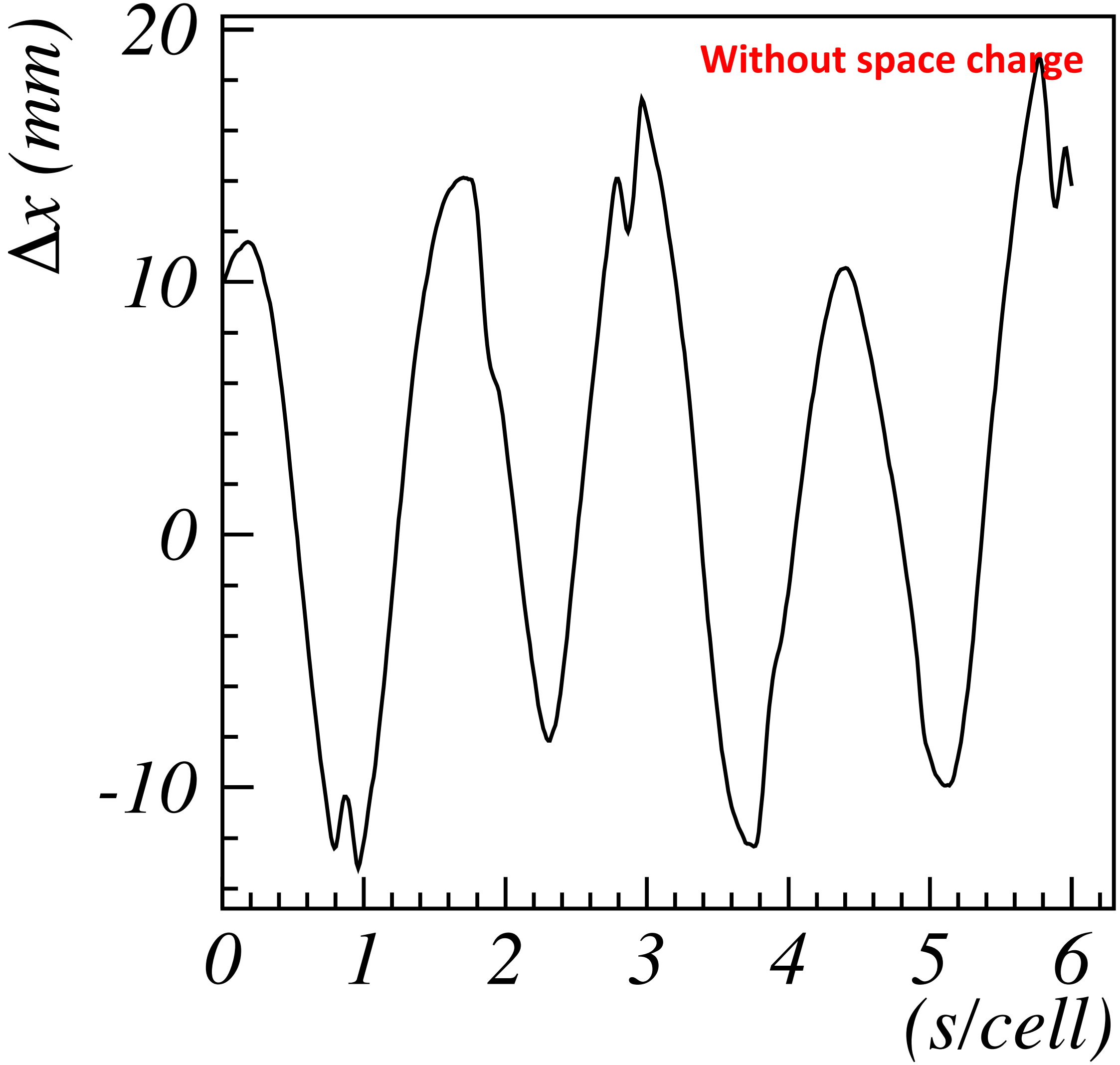
Q_{0x} = 2.145

Mismatched beam



20% mismatch

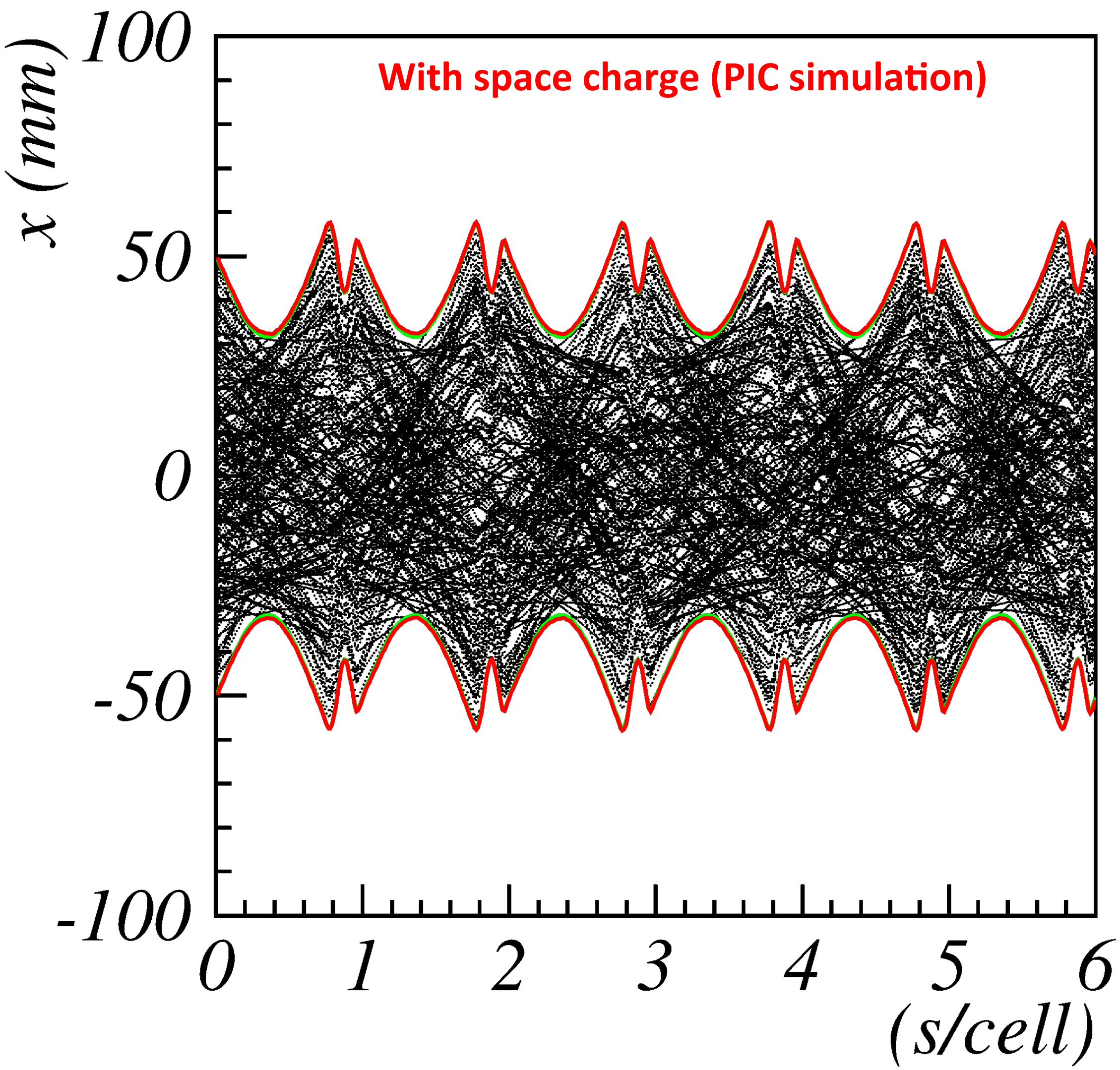
Envelope oscillation

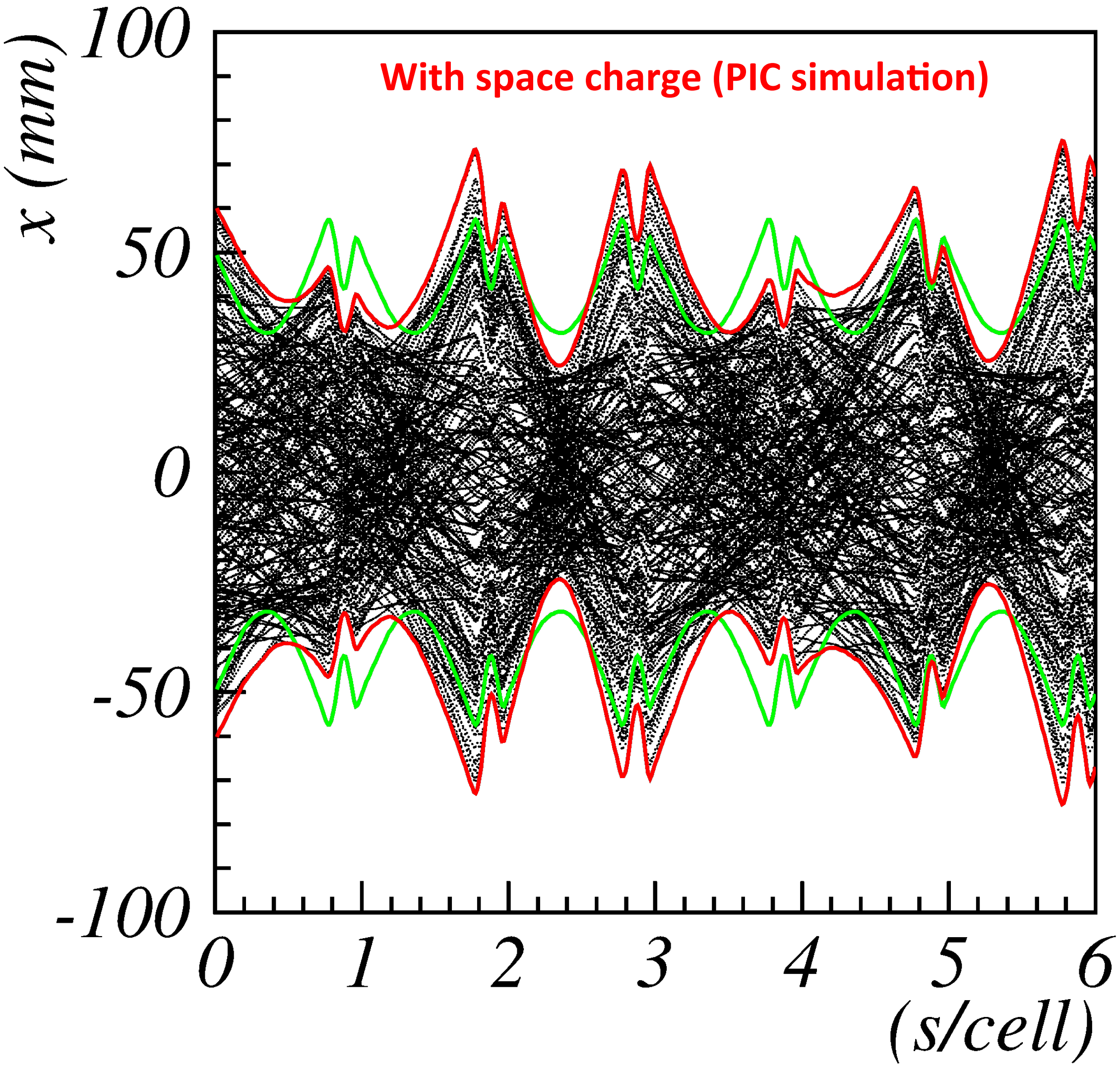


Q_{0x} = 2.145

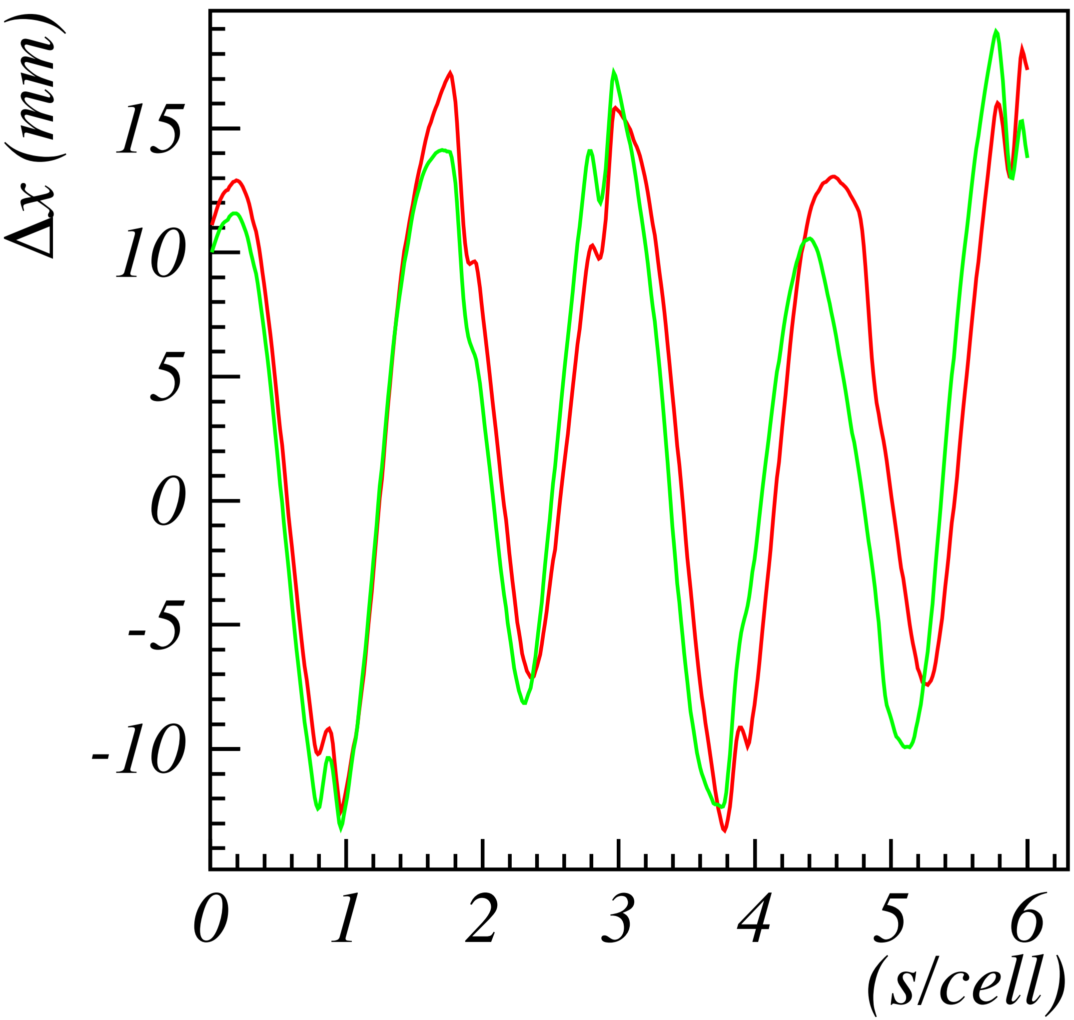
osc. = 4.29

Oscillation of mismatched beams




With space charge (PIC simulation)
x (mm)
100
50
0
-50
-100
0
1
2
3
4
5
6
(s/cell)

20% mismatch



The mismatched envelope oscillates with smaller frequency

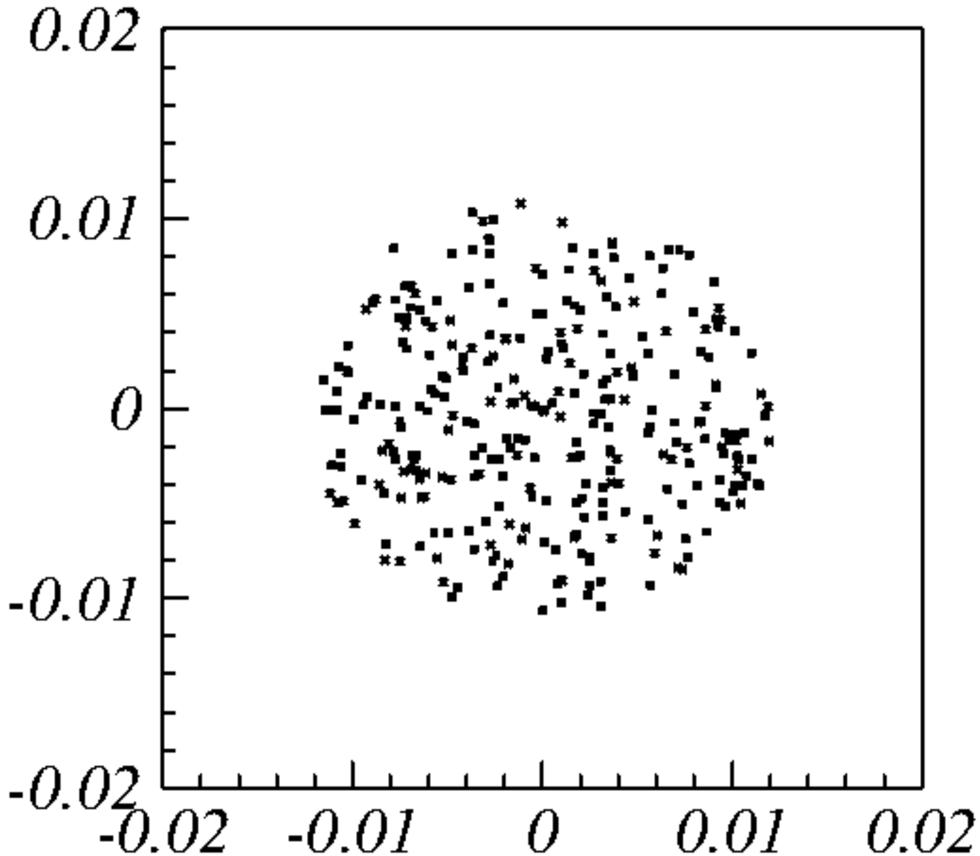
is it

$$2 \times Q_x \quad ?$$

Not only !

Coherent frequencies

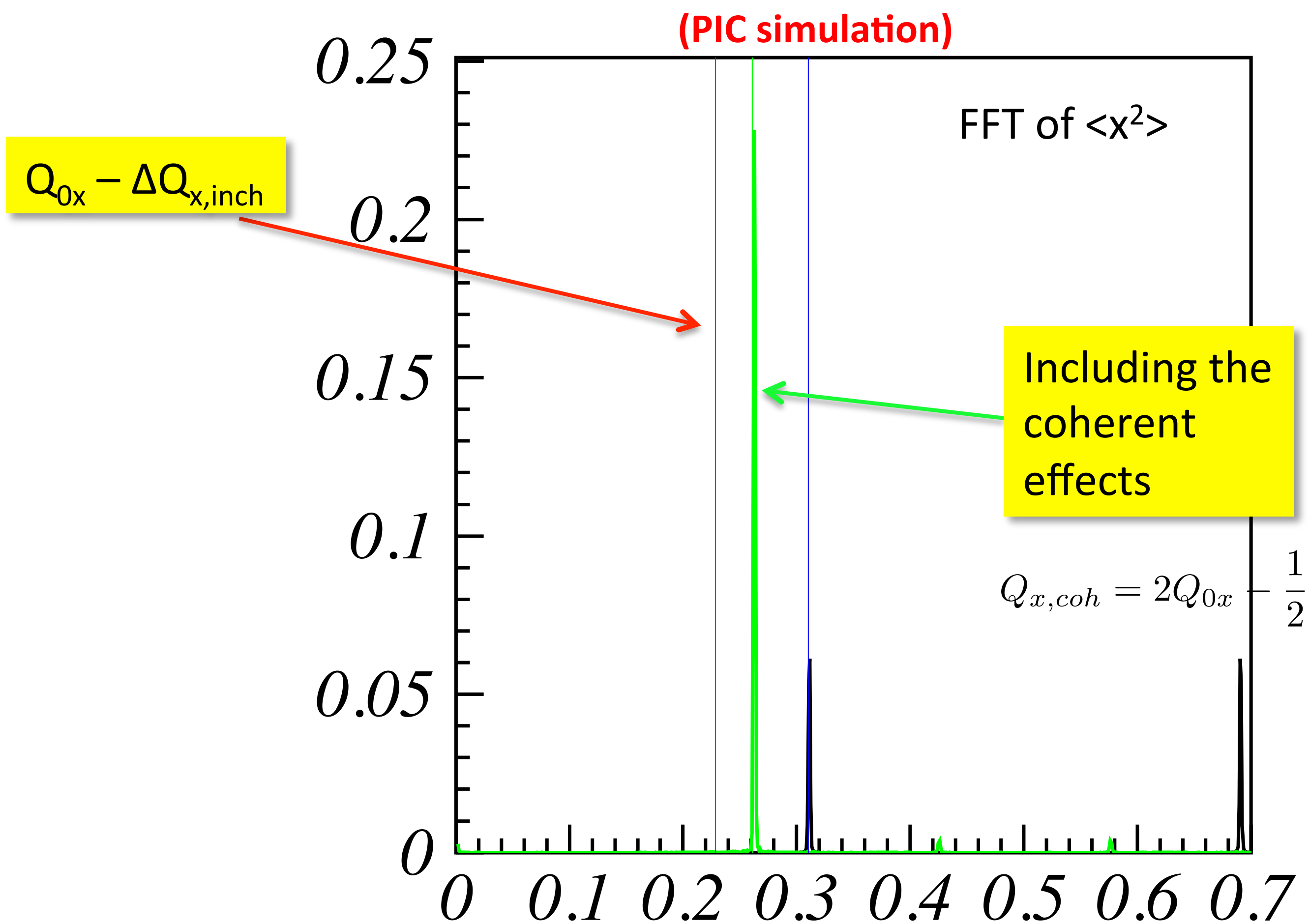
Example of coherent motion driven by an incoherent force (the lattice)
Matched beam kicked with a quadrupolar kick



Coherent frequencies

without space charge
2 x Q_{0x}

Space charge change the collective frequency

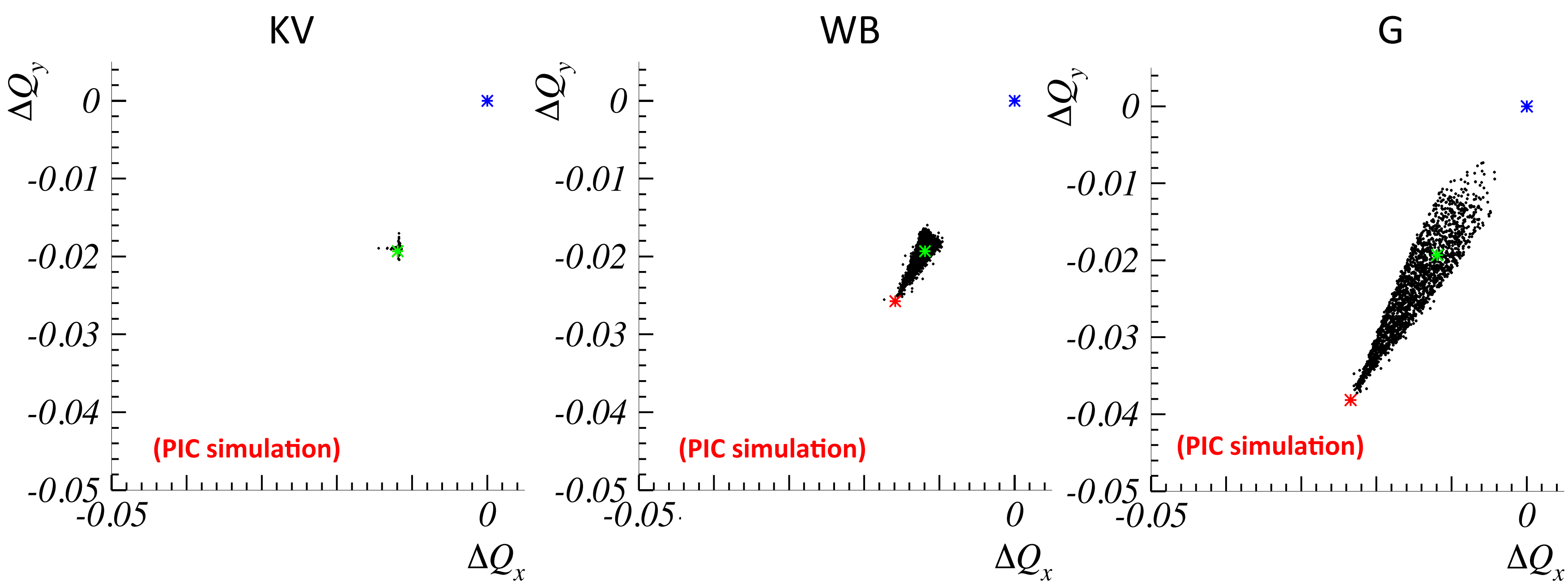


I. Hofmann
Shelter island, 1998
and
Phys. Rev. E **57**, 4713 (1998)

$$Q_{x,coh} = 2Q_{0x} - \frac{1}{2}\left(3 - \frac{\sigma_x}{\sigma_x + \sigma_y}\right)\Delta Q_{x,inch}$$

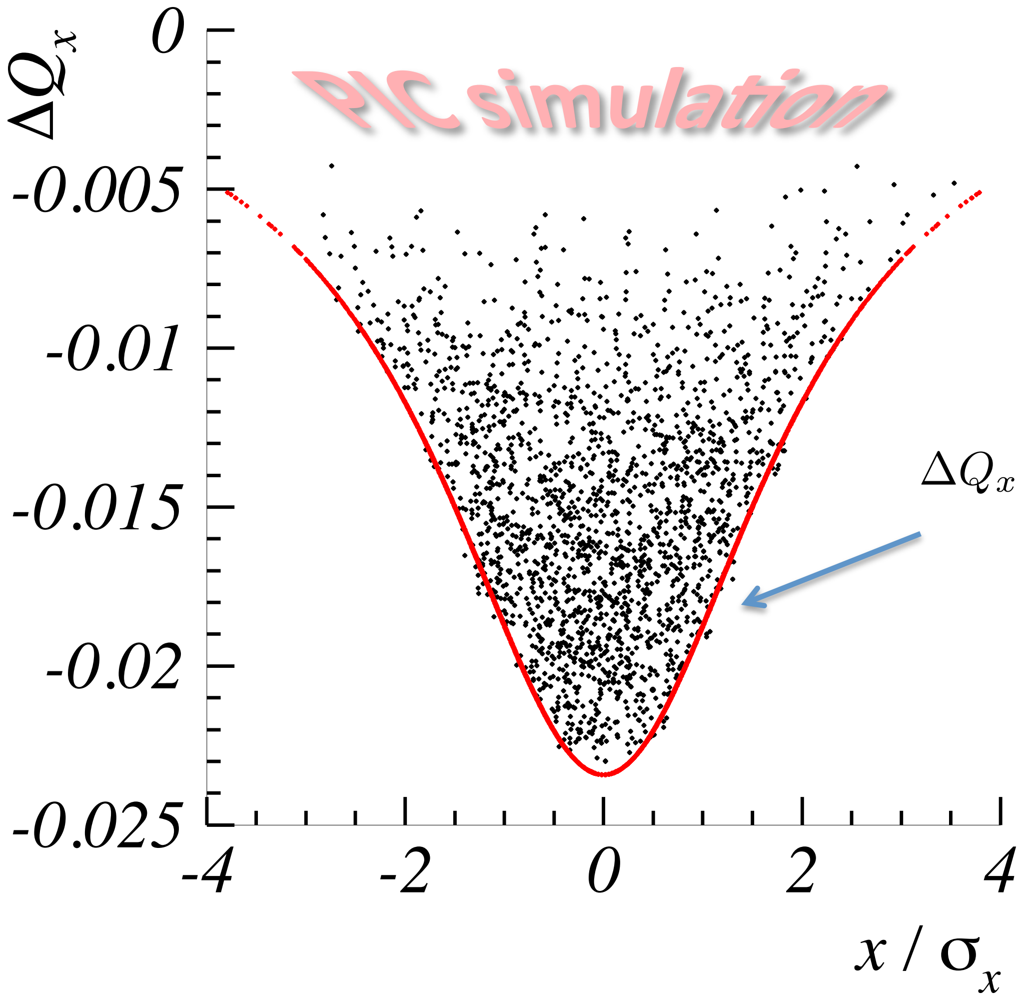
in the non-linear region

Here the tune of each particle becomes amplitude dependent.
and all particles produces a tune-spread



RMS equivalent 2D beams

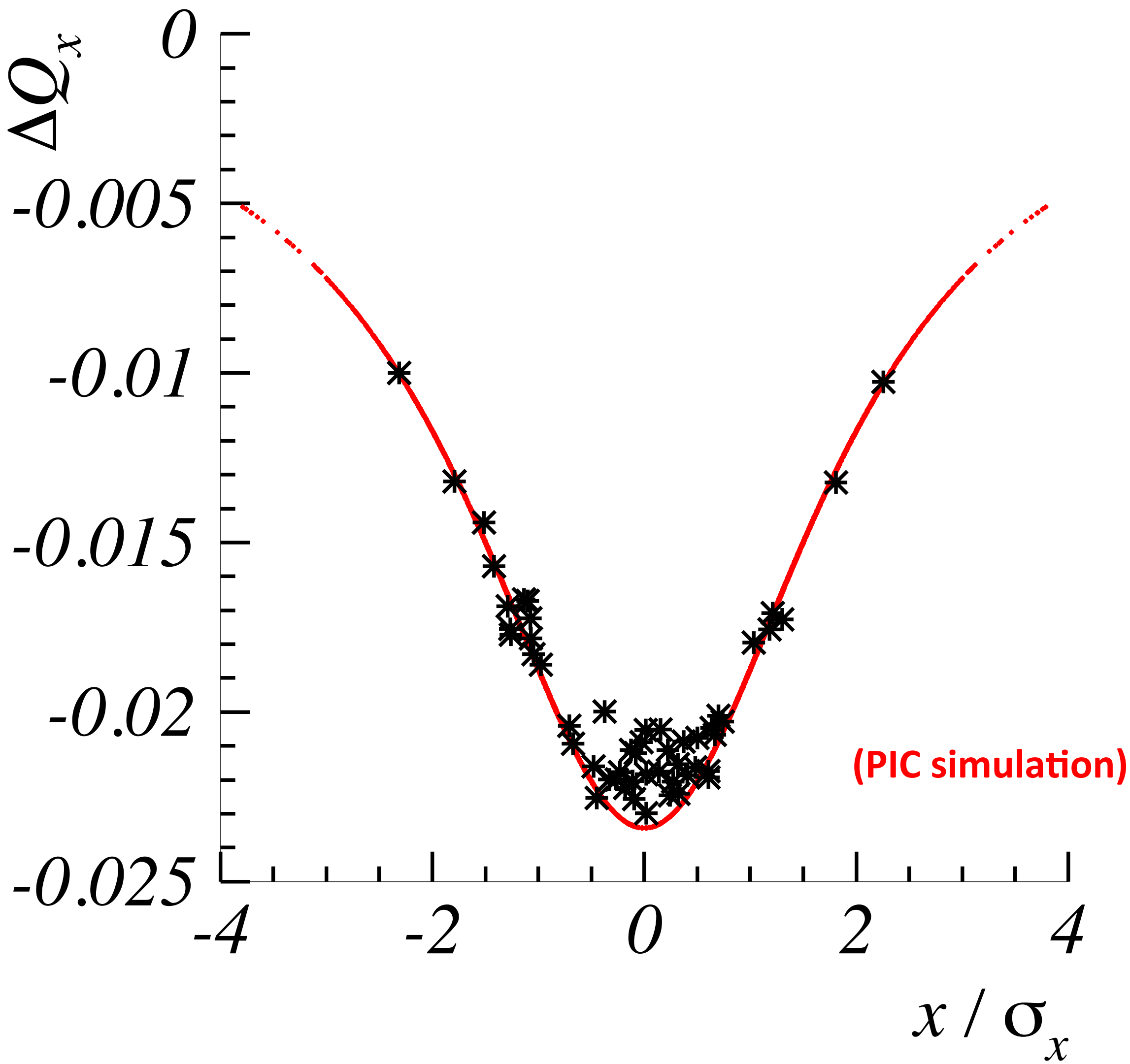
Amplitude dependent detuning



Gaussian

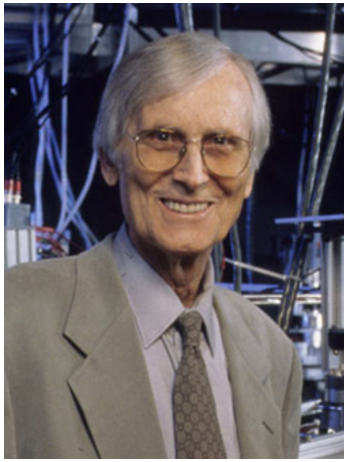
$$\Delta Q_x \simeq \Delta Q_{x,max} \frac{1}{1+\left(\frac{x}{2\sigma_x}\right)^2}$$

It has complicated feature, but for particles with $y \simeq y' \simeq x' \simeq 0$



Mismatched beams
free energy and emittance growth

3 types of nonstationary beams



1) Mismatch in density profile
2) Mismatch in rms radius
3) Beam off center

Free energy = energy of nonstationary state – energy of stationary state

Free energy → emittance growth

Stationary beam

Constant focusing model

$$\tilde{x}'' + k_{0x}\tilde{x} - \frac{K}{2(\tilde{x}+\tilde{y})} - \frac{\tilde{\epsilon}_x^2}{\tilde{x}^3} = 0$$

for round KV beam $\tilde{x} = \tilde{y}$ $\quad a = 2\sqrt{\langle x^2 \rangle}$ $\quad \epsilon_x = 4\tilde{\epsilon}_x$

stationary → $a'' = 0$

$$k_{0x}a - \frac{K}{a} - \frac{\epsilon_x^2}{a^3} = 0$$

Energy budget

Kinetic energy per particle $$E_k = \frac{1}{2}\gamma m(\langle v_x^2\rangle + \langle v_y^2\rangle) = \gamma m v^2 \langle x'^2\rangle$$

but for a stationary beam $\langle x'^2\rangle = k_x\langle x^2\rangle$ ⇒ $E_k = \gamma m v^2 k_x \langle x^2\rangle$

Potential energy per particle $$E_p = \gamma m v^2 k_{0x}\langle x^2\rangle$$

Energy of the electromagnetic field

$$W = \frac{I^2 L}{4\pi\epsilon_0 c^2}\left(\frac{1}{4} + \ln\frac{R_p}{a}\right)\left[\frac{1}{\beta^2} + 1\right]$$

Energy per unit of length of the beam

$$w = \frac{I^2(1-\beta^2)}{16\pi\epsilon_0\beta^2 c^2}\left(1 + 4\ln\frac{R_p}{a}\right)$$

Energy per unit of length per particle

$$\frac{w}{N_L} = \gamma m v^2 \frac{K}{8}\left(1 + 4\ln\frac{R_p}{a}\right)$$

But the perveance is

$$K = (k_{0x} - k_x) 4\langle x^2\rangle$$

Energy per unit of length per particle

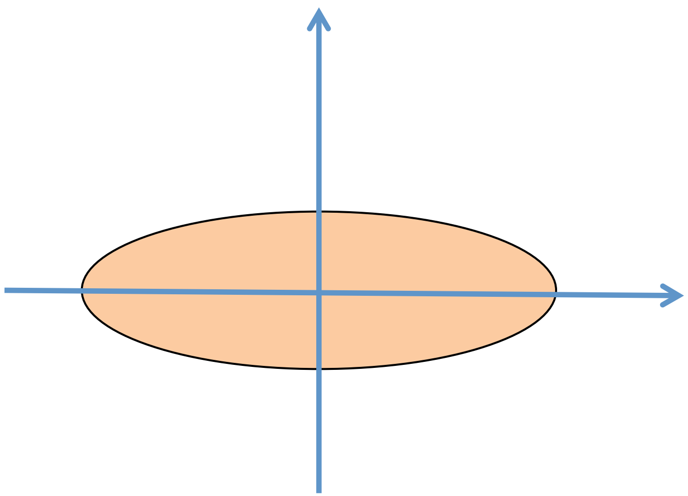
$$E_s = \gamma m v^2 (k_{0x} - k_x)\langle x^2\rangle \left[1 + 4\ln\frac{R_p}{2\tilde{x}}\right]\frac{1}{2}$$

Total energy per particle

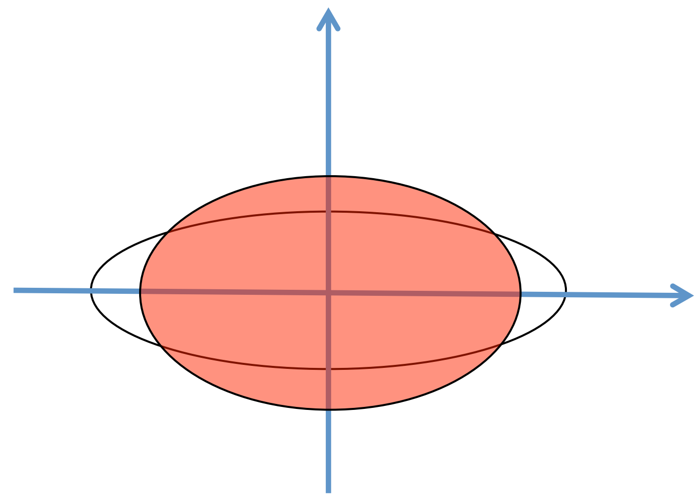
$$E = E_k + E_p + E_s$$

$$E = \frac{1}{4}\gamma m v^2 \left[k_x a^2 + k_{0x} a^2 + \frac{1}{2}[k_{0x} - k_x] a^2 \left(1 + 4 \ln \frac{R_p}{a} \right) \right]$$

Matched beam
total energy per particle E_0



Mismatched beam
total energy per particle E_1



E_1 is higher than E_0 → $\Delta E = E_1 - E_0$ is a “free” energy that can be “thermalized”

to a new stationary state

f = final state I = initial state

$$\frac{\gamma m v^2}{4}\left[k_{fx}a_f^2 + k_{0x}a_f^2 + \frac{1}{2}(k_{0x}^2 - k_{fx}^2)a_f^2\left(1 + 4\ln\frac{R_p}{a_f}\right)^2\right] =$$

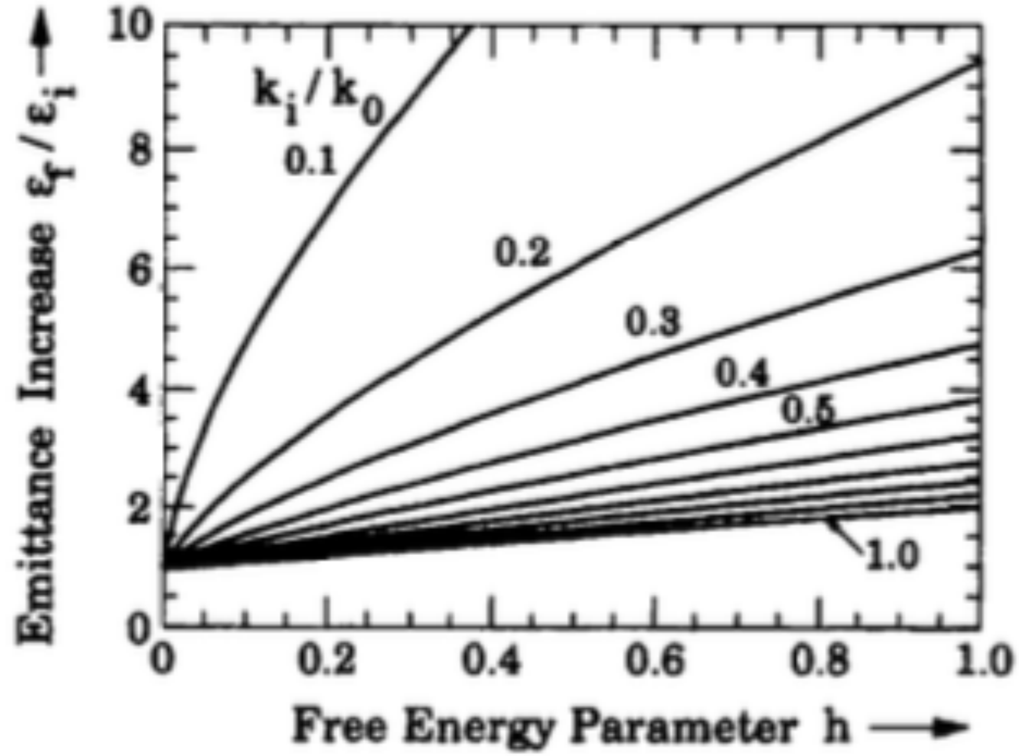
$$\frac{\gamma m v^2}{4}\left[k_{ix}a_i^2 + k_{0x}a_i^2 + \frac{1}{2}(k_{0x}^2 - k_{ix}^2)a_i^2\left(1 + 4\ln\frac{R_p}{a_i}\right)^2\right] + \Delta E$$

rescaling the free energy ΔE to *h* with the formula

$$\Delta E = \frac{1}{2}\gamma m v^2 k_{0x} a_i^2 h$$

$$\frac{a_f}{a_i} \simeq 1 + \frac{h}{1 + k_{ix}/k_x}$$

$$\frac{\epsilon_f}{\epsilon_i} = \frac{a_f}{a_i}\left\{1 + \frac{k_{0x}}{k_{ix}}\left[\frac{a_f^2}{a_i^2} - 1\right]\right\}^{1/2}$$

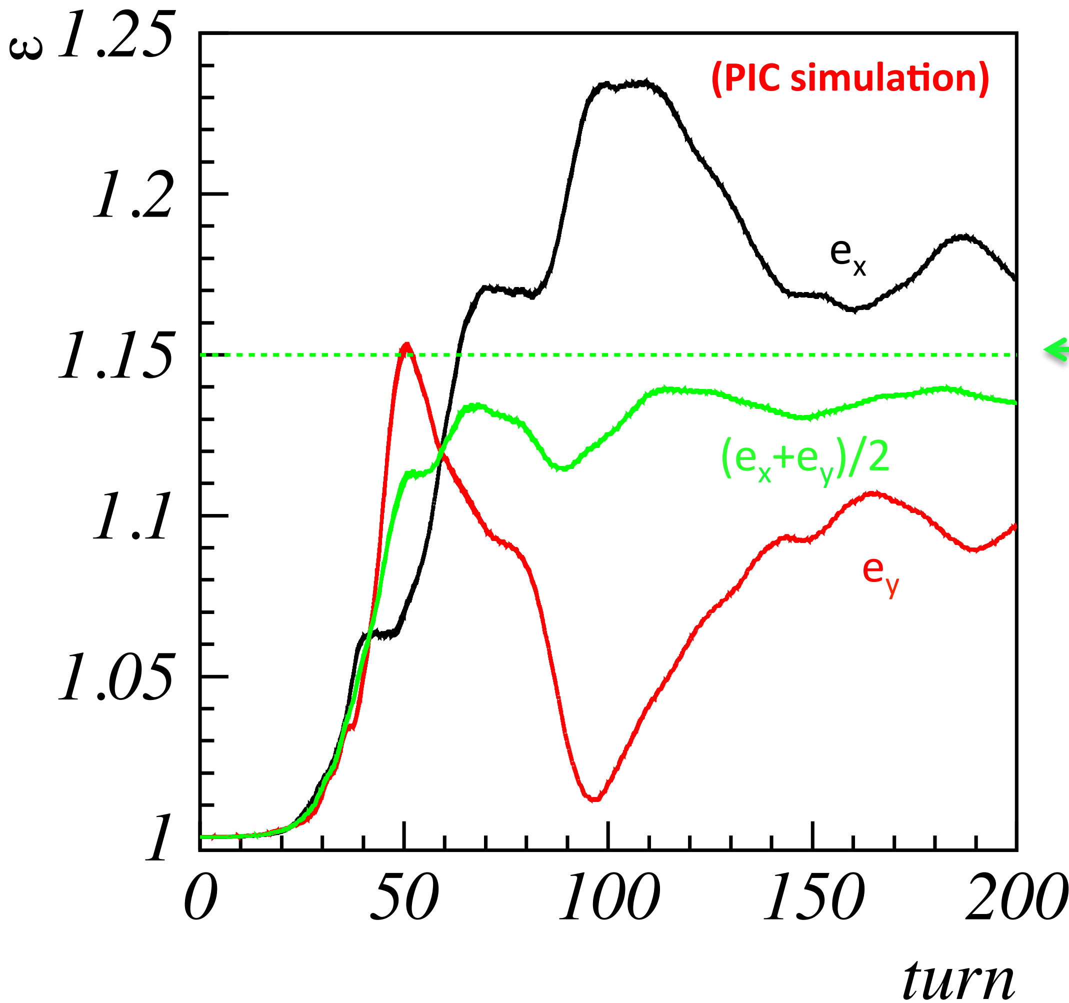


Martin Reiser, JOHN WILEY and Son, Inc, New York 1994

For a mismatched beam

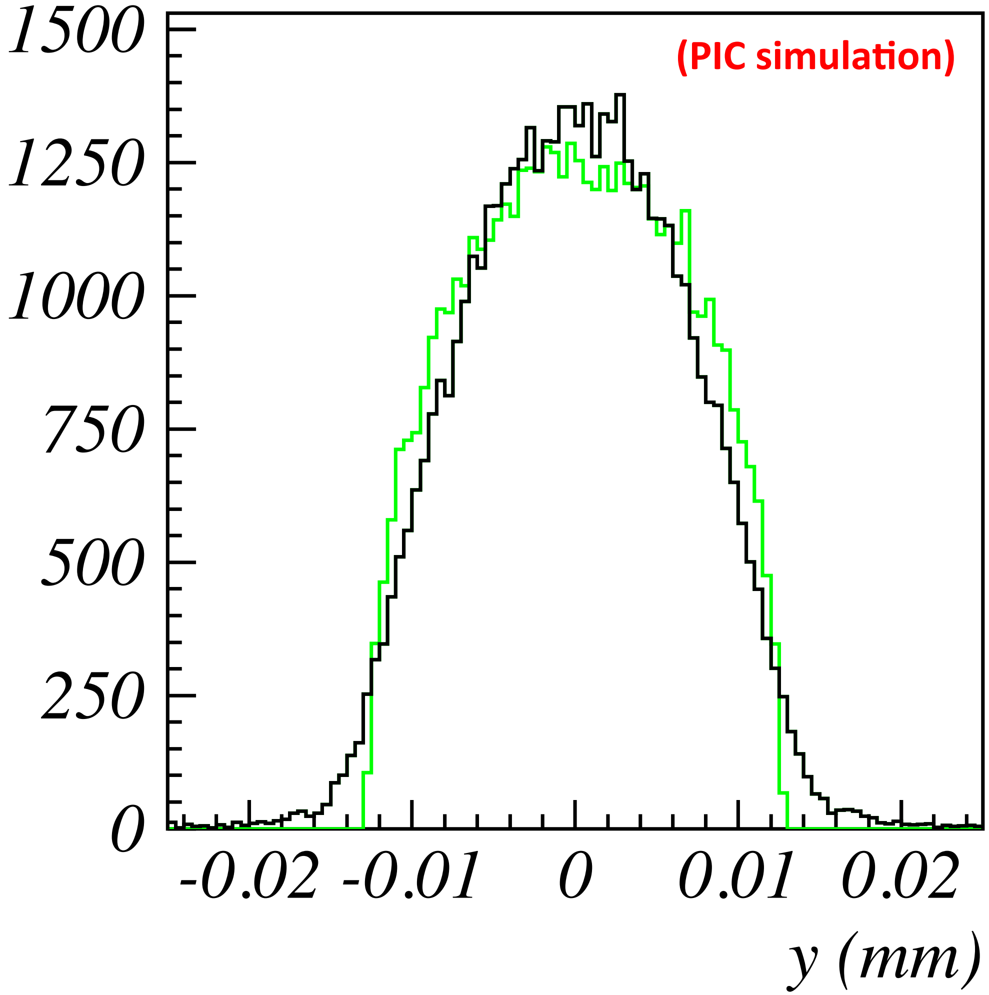
$$h = \frac{1}{2}\frac{k_{ix}}{k_{0x}}\left(\frac{a_i^2}{a_0^2} - 1\right) - \frac{1}{2}\left(1 - \frac{a_0^2}{a_i^2}\right) + \left(1 - \frac{k_{ix}}{k_{0x}}\right)\ln\frac{a_i}{a_0}$$

Simulation example

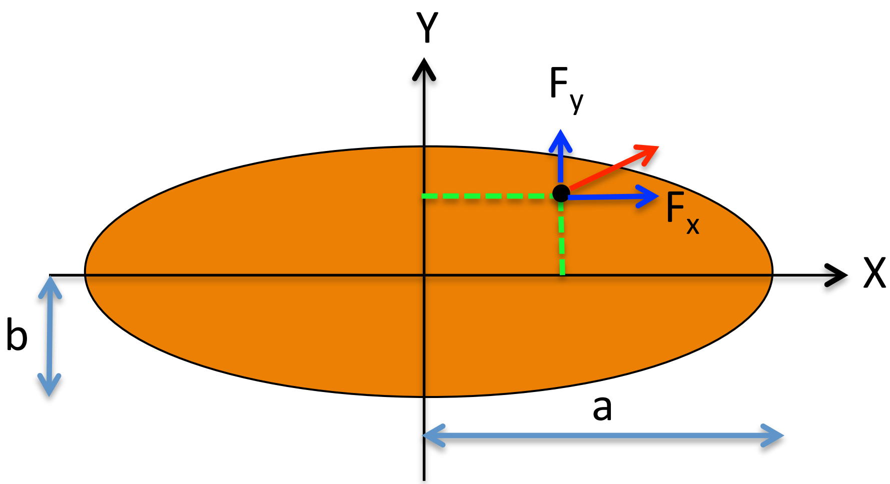


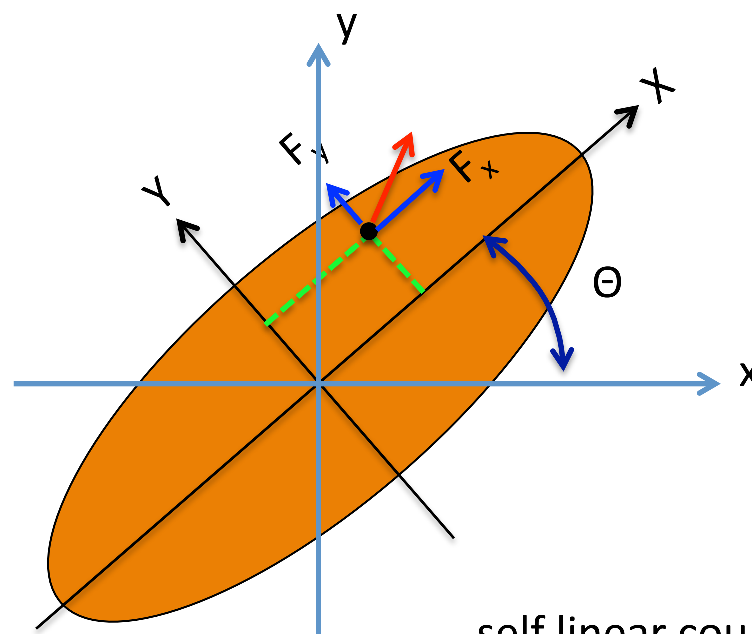
free energy limit

Simulation example



Linear coupling





self linear coupling

$$F_x = \frac{2K}{a(a+b)} X$$

$$F_y = \frac{2K}{b(a+b)} Y$$

$$F_x = \left[\frac{2K}{a(a+b)} \cos^2\theta + \frac{2K}{b(a+b)} \sin^2\theta \right] x + \sin\theta\cos\theta \left[\frac{2K}{a(a+b)} - \frac{2K}{b(a+b)} \right] y$$

$$F_y = \left[\frac{2K}{b(a+b)} \cos^2\theta + \frac{2K}{a(a+b)} \sin^2\theta \right] y + \sin\theta\cos\theta \left[\frac{2K}{a(a+b)} - \frac{2K}{b(a+b)} \right] x$$

Chernin equation

D. Chernin, Part. Accel. 1988, Vol. 24, pp. 29-44

The angle theta depends on the moments $\Sigma_{i,j} \equiv \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle$ v = (x,x'y,y,y')
therefore the envelope equation now should involve all second order moments.

$$\Sigma' = M\Sigma + (M\Sigma)^T$$

One defines

$$M \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\tilde{k}_x & 0 & \tilde{j} & 0 \\ 0 & 0 & 0 & 1 \\ \tilde{j} & 0 & -\tilde{k}_y & 0 \end{pmatrix}$$

$$\tilde{k}_x = k_x - q_{xx}, \tilde{k}_y = k_y - q_{yy}$$
$$\tilde{j} = j + q_{xy}$$

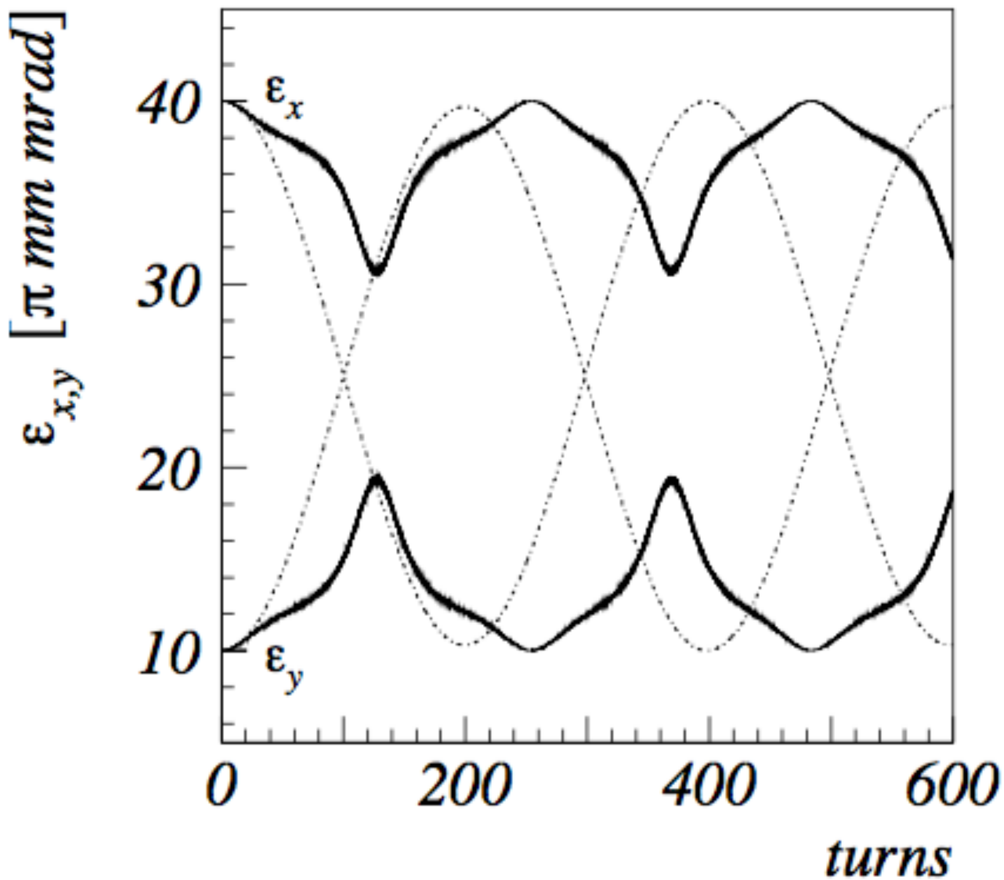
$$q_{xx} = \frac{K}{2} \frac{S_y}{S_0(S_x + S_y)}, \quad q_{yy} = \frac{K}{2} \frac{S_x}{S_0(S_x + S_y)},$$

$$q_{xy} = -\frac{K}{2} \frac{\Sigma_{13}}{S_0(S_x + S_y)},$$

self-coupling due to space charge

$$S_x = \Sigma_{11} + S_0,\ S_y = \Sigma_{33} + S_0,\ S_0 = \sqrt{\Sigma_{11}\Sigma_{33} - \Sigma_{13}^2}$$

Simulation example



Prediction from Chernin equation

G. Franchetti, I. Hofmann, M. Aslaninejad
Phys. Rev. Lett. **94**, 194801 (2005).

Montague resonance

B.W. Montague CERN-68-38

Decomposition of the space charge force for a Gaussian distribution

$$F_x = \frac{q^2 n}{2\pi\epsilon_0\gamma^2}[\frac{1}{\sigma_x(\sigma_x+\sigma_y)}x - \frac{2\sigma_x+\sigma_y}{3!\sigma_x^3(\sigma_x+\sigma_y)^2}x^3 - \frac{1}{2!\sigma_x\sigma_y(\sigma_x+\sigma_y)^2}xy^2 + ...]$$

$$F_y = \frac{q^2 n}{2\pi\epsilon_0\gamma^2}[\frac{1}{\sigma_y(\sigma_y+\sigma_x)}y - \frac{2\sigma_y+\sigma_x}{3!\sigma_y^3(\sigma_y+\sigma_x)^2}x^3 - \frac{1}{2!\sigma_y\sigma_x(\sigma_y+\sigma_x)^2}x^2y + ...]$$

This term is a 4^{th} order force

The stronger harmonics of the space charge force is the "0" order: that means that the average strength of the 4^{th} order component

$$\left\langle \frac{1}{\sigma_x\sigma_y(\sigma_x+\sigma_y)^2} \right\rangle = K_{22}$$ is different from zero

A single particle approach: intuitive argument

$$x'' + \left(\frac{Q_x}{R}\right)^2 x = K_{22}\ xy^2$$
$$y'' + \left(\frac{Q_y}{R}\right)^2 y = K_{22}\ yx^2$$

(this is a "very very" short intuitive version of Montague paper…)

In x plane → $xy^2 \to xcos^2(Q_y/(2\pi R)s)$ the equation (at the beginning of motion)

$$x'' + \left(\frac{Q_x}{R}\right)^2 x = K_{22} x A_y cos^2(\frac{Q_y}{2\pi R}s)$$

the frequency of this term is 2 Q_y

Resonance condition → $nQ_x = \text{frequency of the harmonics}$ → $nQ_x = 2Q_y$

Same argument apply in the other plane, which yields the resonance condition → $n'Q_y = 2Q_x$

Therefore if $2Q_x - 2Q_y = 0$ the "0" order harmonics of the term $K_{22}x^2y^2$ of the space charge potential creates a resonance (Montague resonance)

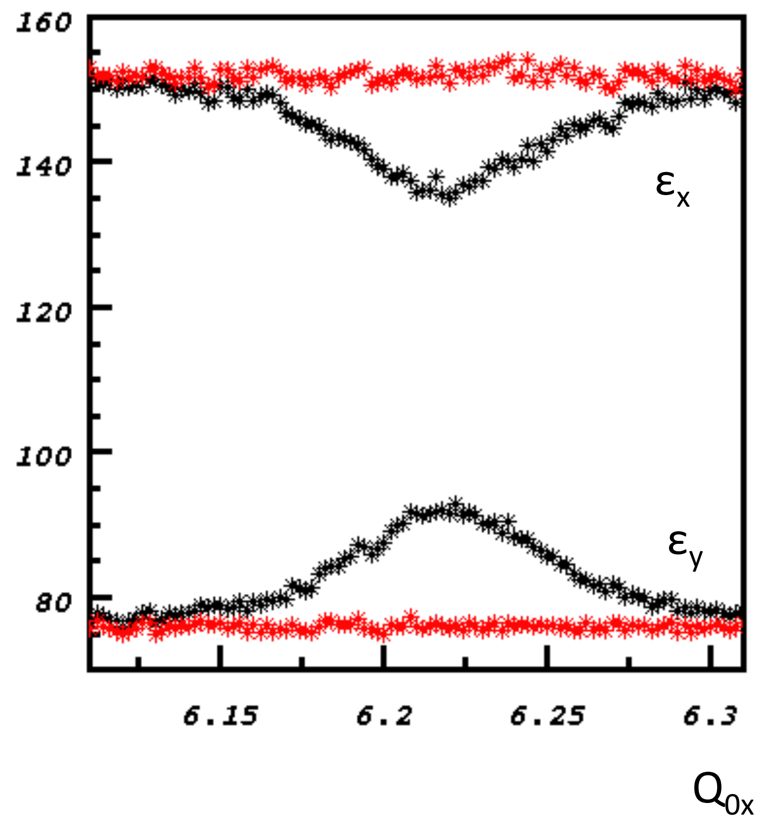
Example

not all space charge, but only K_{22} frozen

$$x'' + \left(\frac{Q_x}{R}\right)^2 x = K_{22}\ xy^2$$

$$y'' + \left(\frac{Q_y}{R}\right)^2 y = K_{22}\ yx^2$$

Q_x = 6.2



The collective part

If emittances change, also K_{22} changes → single particle resonance → collective response

but…

There is also the coherent collective beam response

Ingo's work

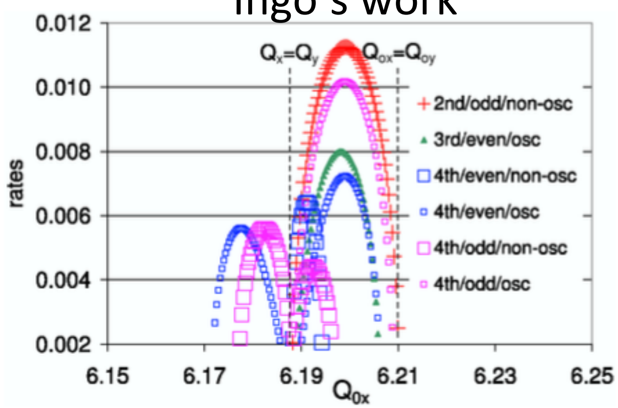


FIG. 6. (Color) Growth rates N_β^{-1} for collective modes of KV-distribution.

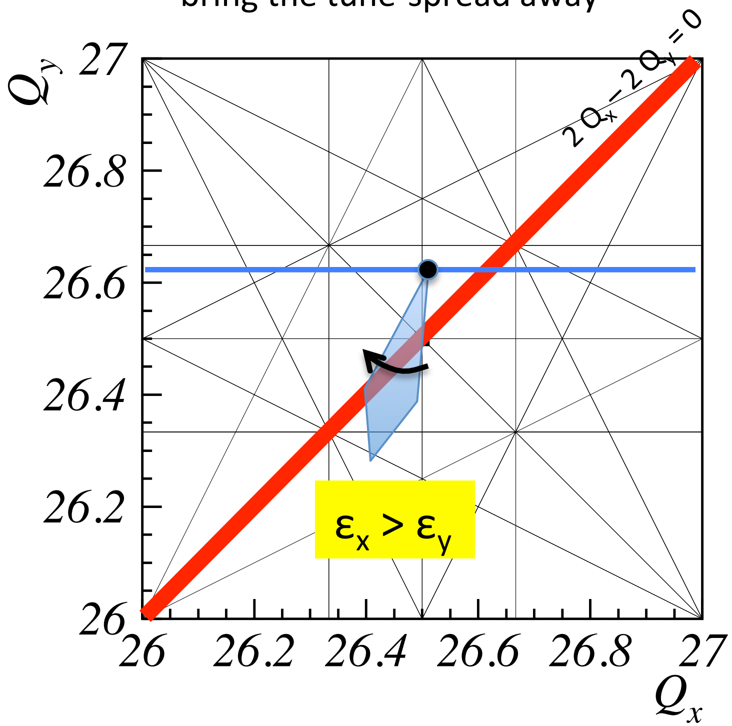
but this theory is for KV

The beam response is complex…

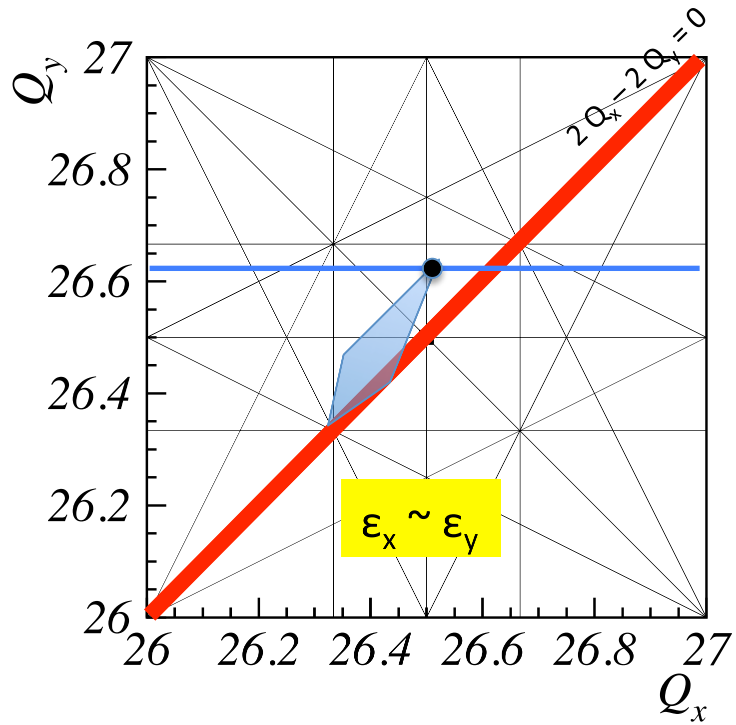
(and if the beam tilts… then we have the self-skew → space charge linear coupling)

Looking at the tunespread

The resonance makes the emittance change, so to bring the tune-spread away



Now the process stops



Simulation example

I.Hofmann & G.Franchetti PRSTAB

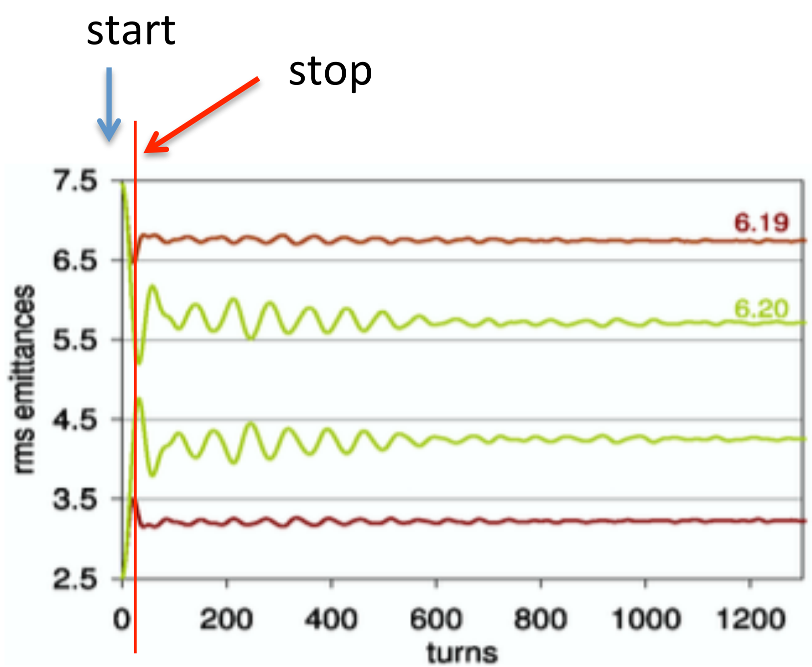


FIG. 1. (Color) Time evolution of rms emittances for $Q_{0,x} =$ 6.19, 6.20.

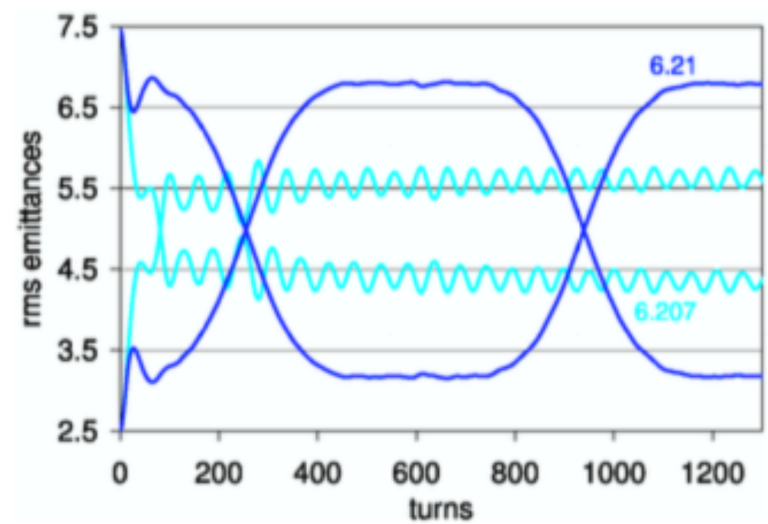


FIG. 2. (Color) Time evolution of rms emittances for $Q_{0,x} =$ 6.207 and $Q_{0,x} = 6.21$.

I.Hofmann & G.Franchetti PRSTAB

Global view of the situation

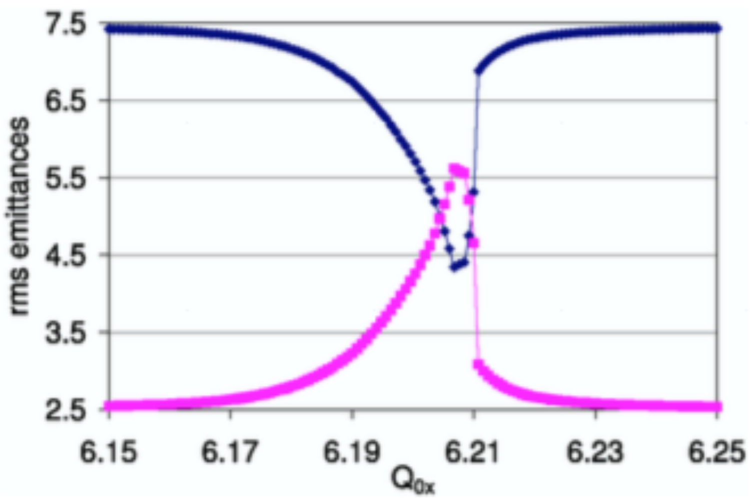


FIG. 3. (Color) Final rms emittances for variable $Q_{0,x}$ and constant focusing.

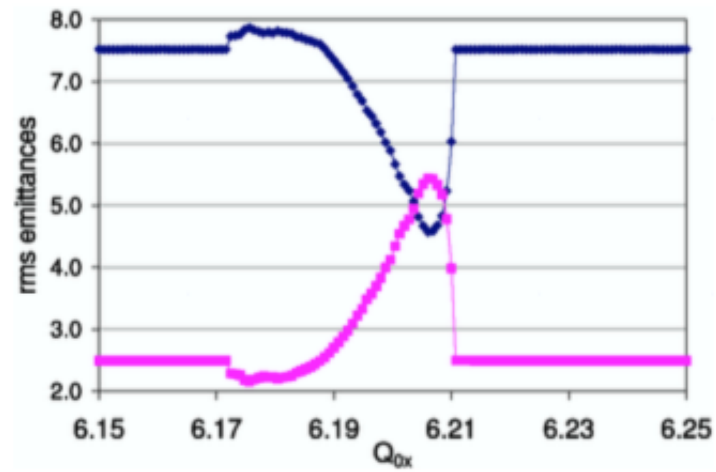
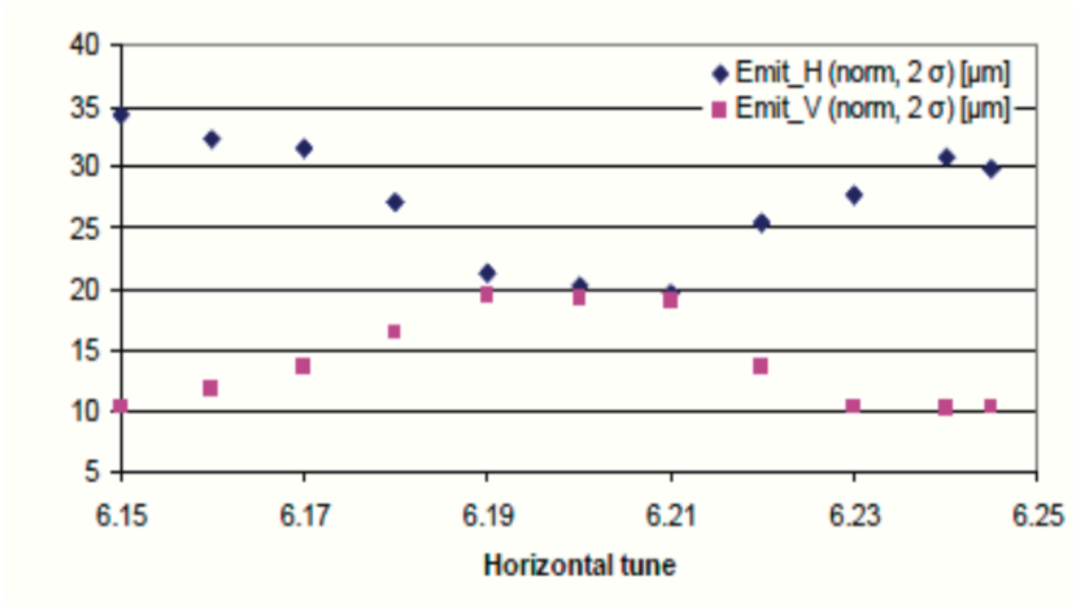
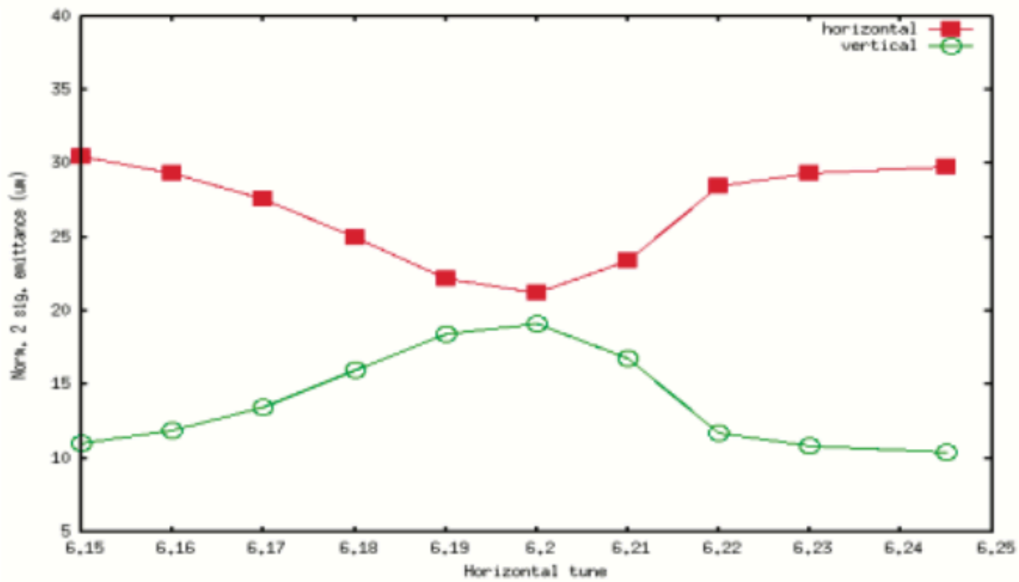


FIG. 5. (Color) Final rms emittances for KV-distribution and constant focusing.

For a KV distribution the 4th order component is not there, but the instability of the coherent modes build it up





CERN-PS experiment on the Montague resonance

R. Cappi, F. Giuliano,
M. Giovannozzi,
I. Hofmann,
M. Martini, E. Metral

3D simulation: this is the last simulation made by Ji Qiang

WEPPR011 Proceedings of IPAC2012, New Orleans, Louisiana, USA

Space charge as incoherent force

$$F_x = \frac{q^2 n}{2\pi\epsilon_0\gamma^2}\left[\frac{1}{\sigma_x(\sigma_x+\sigma_y)}x - \frac{2\sigma_x+\sigma_y}{3!\sigma_x^3(\sigma_x+\sigma_y)^2}x^3 - \frac{1}{2!\sigma_x\sigma_y(\sigma_x+\sigma_y)^2}xy^2 + ...\right]$$

$$F_y = \frac{q^2 n}{2\pi\epsilon_0\gamma^2}\left[\frac{1}{\sigma_y(\sigma_y+\sigma_x)}y - \frac{2\sigma_y+\sigma_x}{3!\sigma_y^3(\sigma_y+\sigma_x)^2}x^3 - \frac{1}{2!\sigma_y\sigma_x(\sigma_y+\sigma_x)^2}x^2 y + ...\right]$$

this is a "octupolar" term

resonance can be excited if the machine tune is at the right distance and the tunespread overlap the resonance

Machida, NIMA 309, 43-59 (1991)

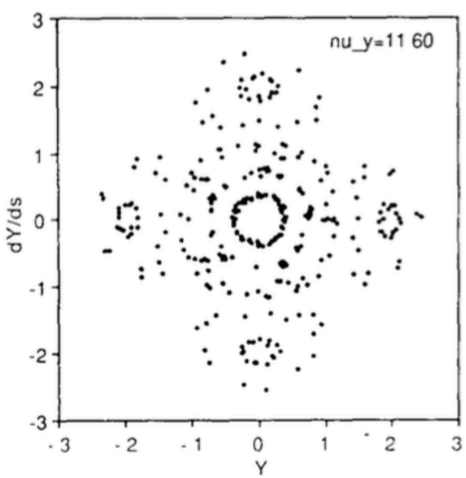


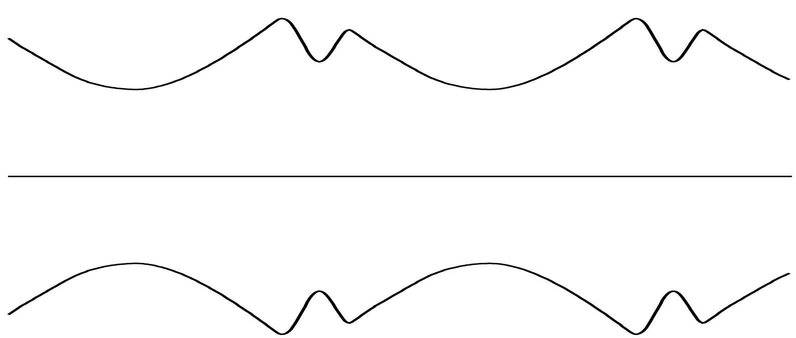
Fig 13. Poincaré map of a test particle of 32 turns for the vertical bare tune: 11.60.

Space charge structure resonances

$$F_x = \frac{q^2 n}{2\pi\epsilon_0\gamma^2}\left[\frac{1}{\sigma_x(\sigma_x+\sigma_y)}x - \frac{2\sigma_x+\sigma_y}{3!\sigma_x^3(\sigma_x+\sigma_y)^2}x^3 - \frac{1}{2!\sigma_x\sigma_y(\sigma_x+\sigma_y)^2}xy^2 + ...\right]$$

$$F_y = \frac{q^2 n}{2\pi\epsilon_0\gamma^2}\left[\frac{1}{\sigma_y(\sigma_y+\sigma_x)}y - \frac{2\sigma_y+\sigma_x}{3!\sigma_y^3(\sigma_y+\sigma_x)^2}x^3 - \frac{1}{2!\sigma_y\sigma_x(\sigma_y+\sigma_x)^2}x^2y + ...\right]$$

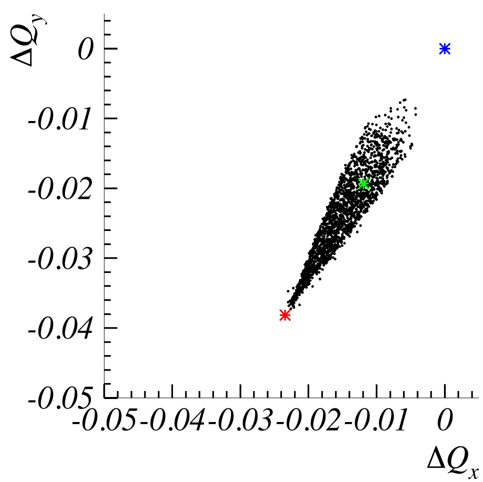
All orders of the space charge force may excite incoherent resonances due to envelope dependence on "s" or due to oscillations

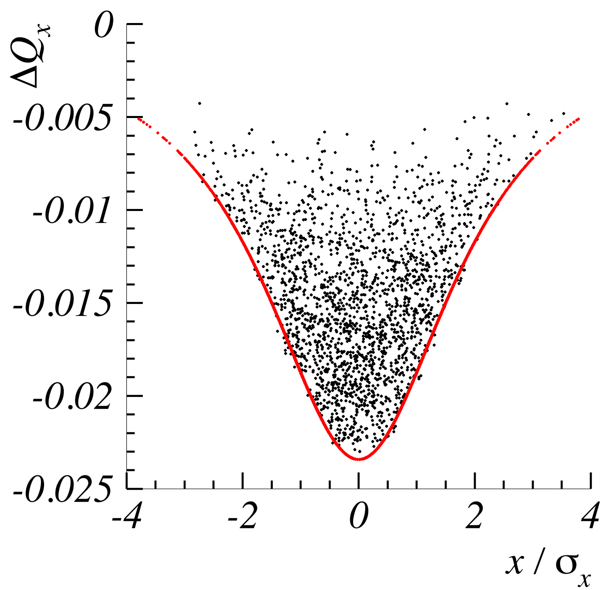


Spectral analysis reveals the harmonics excited

Space charge as source of amplitude dependent detuning

(PIC simulation)

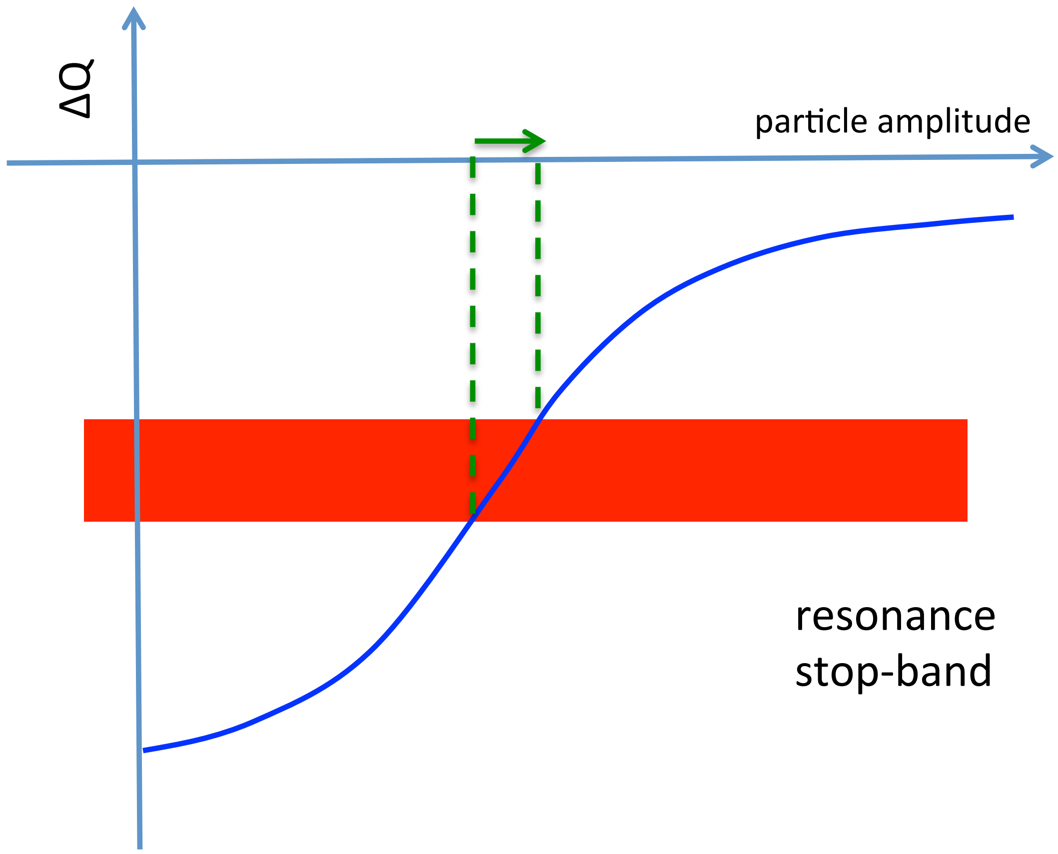




Stabilizing effect on resonant phenomena

Space charge and machine resonances

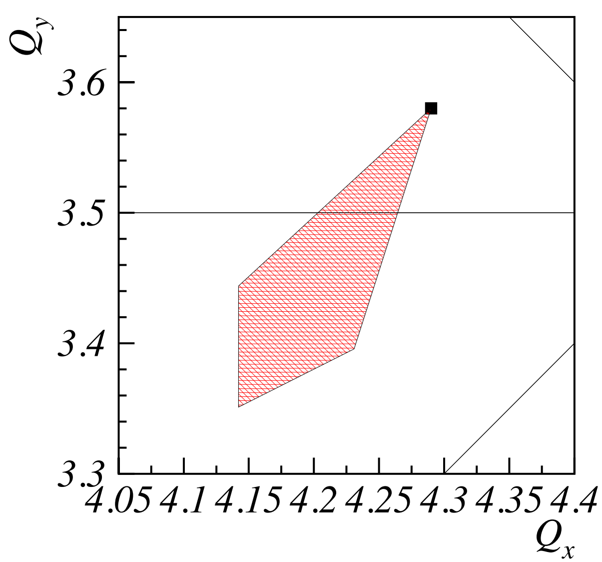
Space charge may have a stabilizing effect

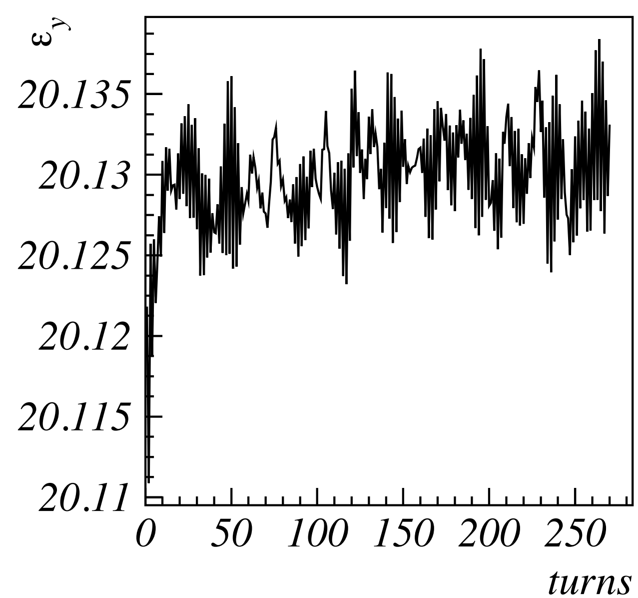


Example with half integer

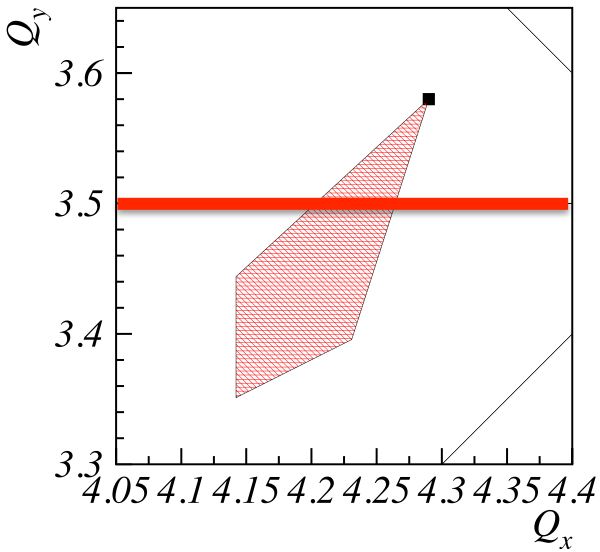
Beam coasting: ΔQ_x = -0.14, ΔQ_y = -0.22

half integer not excited

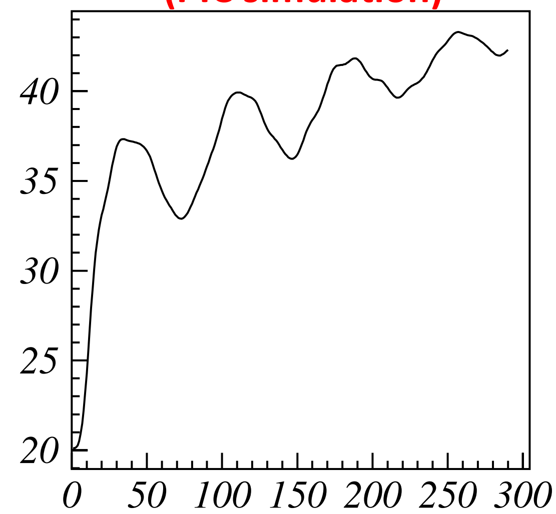


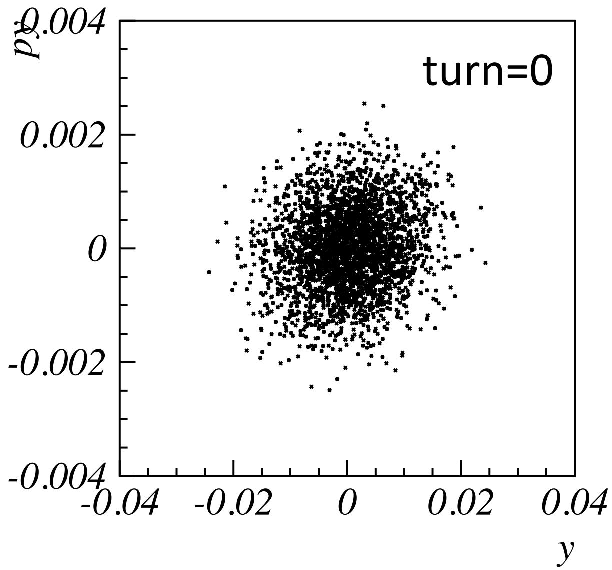


half integer is excited

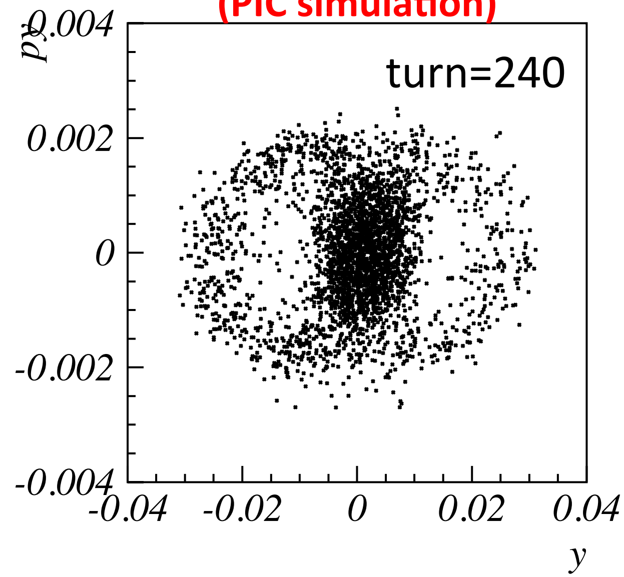


(PIC simulation)





(PIC simulation)

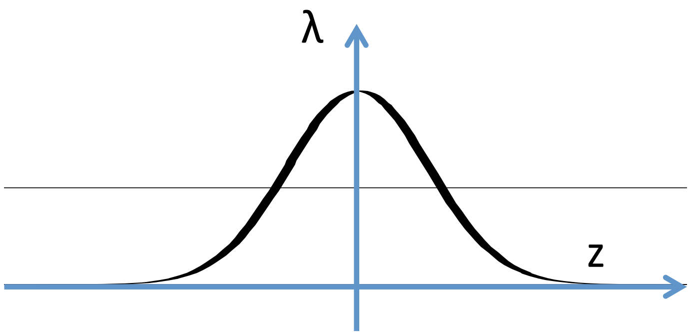


The longitudinal envelope equation

There is an equivalent to the KV distribution → D. Neuffer distribution

longitudinal
space charge
force

$$F_z = A\frac{d\lambda}{dz}$$



longitudinal equation of motion

$$z'' + k_{0z}z = A\frac{d\lambda}{dz}$$

if λ is parabolic

$$\lambda(z) = \lambda_0 \left(1 - \frac{z^2}{z_m^2}\right) = \frac{3}{4} N \frac{1}{z_m} \left(1 - \frac{z^2}{z_m^2}\right)$$

Longitudinal space charge force is linear (like for KV)

$$F_z = -\frac{3N}{2} \frac{1}{z_m^3} z$$

$$z'' + k_{0z} z + A \frac{3N}{2} \frac{1}{z_m^3} z = 0 \qquad k_z = k_{0z} + A \frac{3N}{2} \frac{1}{z_m^3}$$

D. Neuffer (IPAC 1979 proceeding) showed that

$$f(H) = C \sqrt{2(H_{max} - k_z z^2 - z'^2)}$$

is stationary and self-consistent, and generates a parabolic distribution

Longitudinal envelope equation

$$z_m'' + k_{z0}^2 z_m - \frac{K_L}{z_m^2} - \frac{\epsilon_L^2}{z_m^3} = 0$$

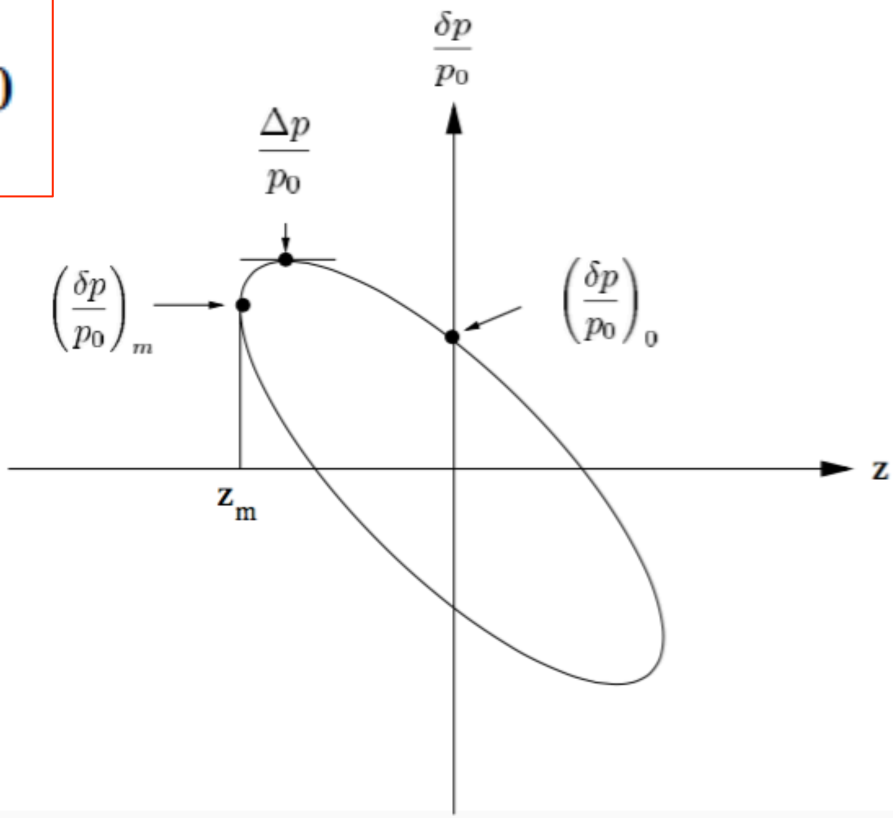
$$k_{z0}^2 = eZVh\eta/(2\pi R^2 \gamma \beta^2 Amc^2)$$

$$K_L = -3gN(Z^2/A)r_p\eta/(2\beta^2\gamma^3)$$

$$\epsilon_L = |\eta| z_m (\delta p/p_0)_0$$

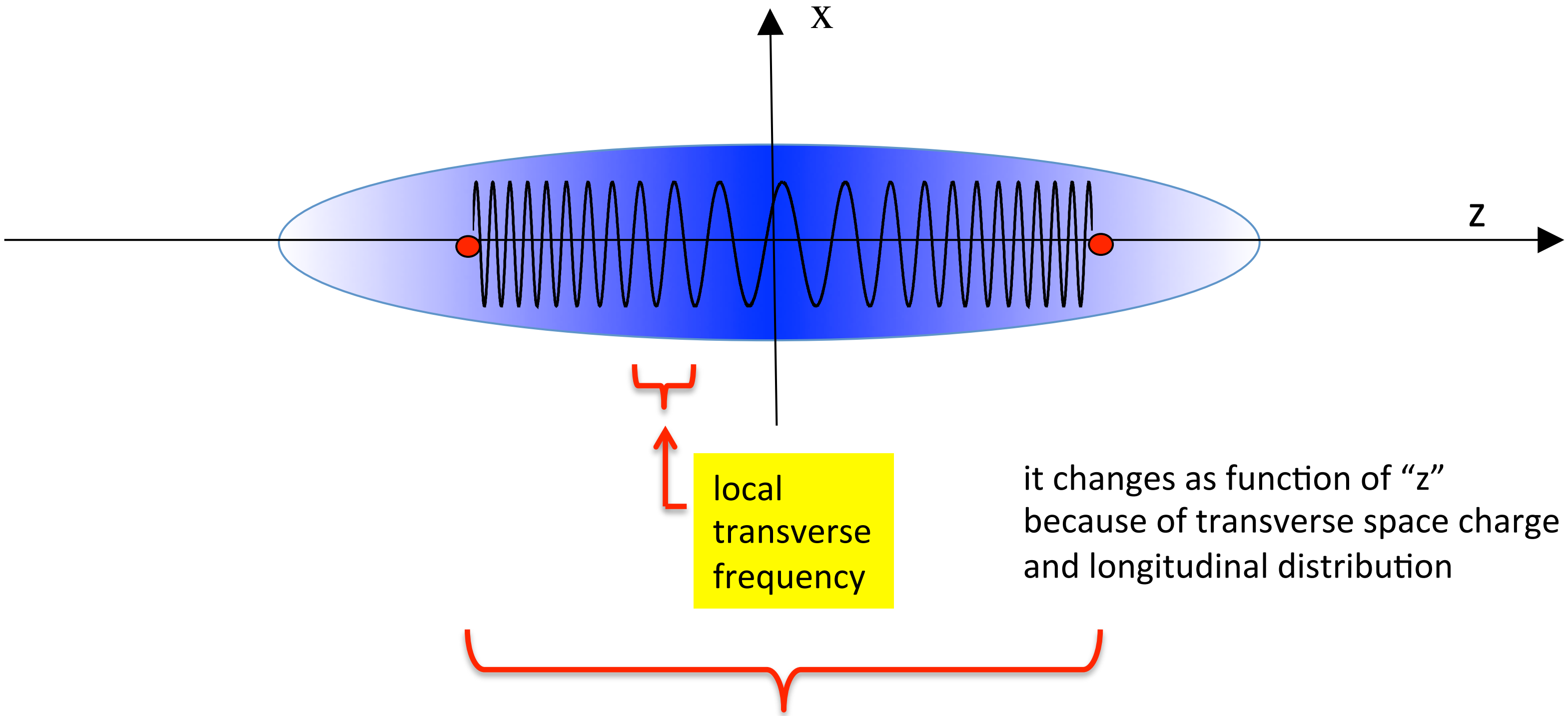
$$\eta = 1/\gamma_t^2 - 1/\gamma^2$$

$$g = 0.5 + 2\ln(R_p/R_b)$$



$$z' = -\eta \delta p/p_0$$

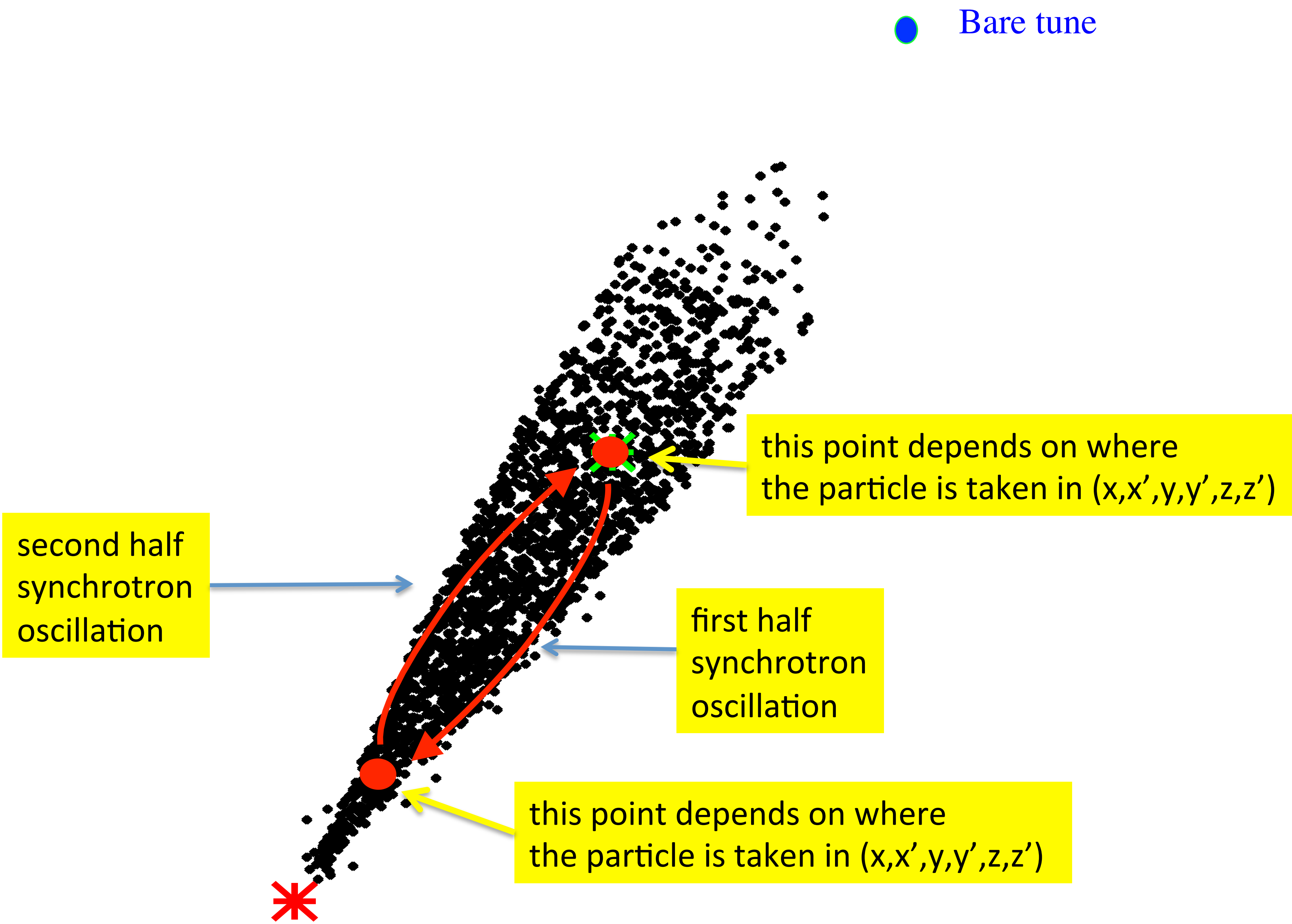
Transverse-Longitudinal coupling via space charge



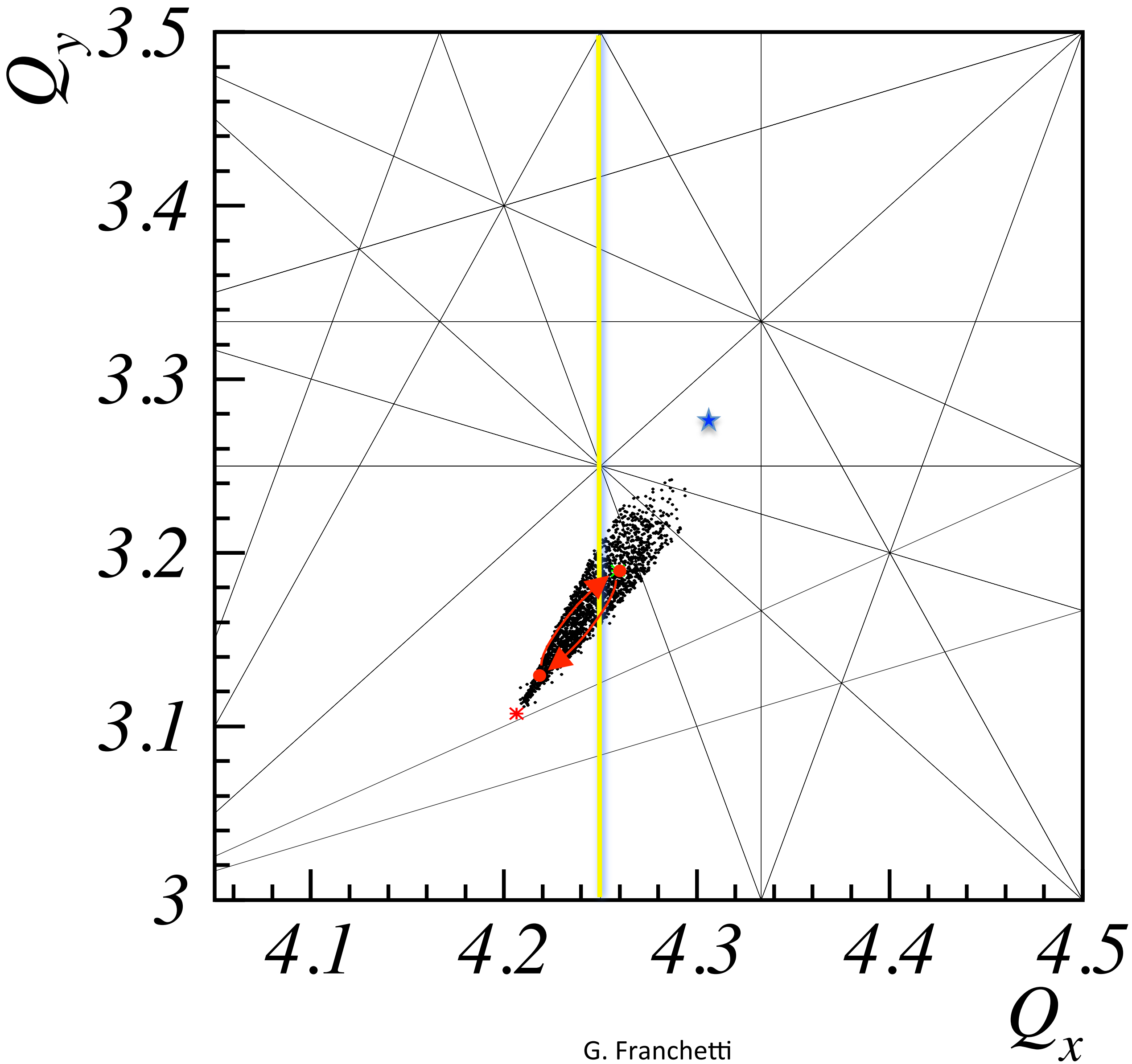
local transverse frequency

it changes as function of "z" because of transverse space charge and longitudinal distribution

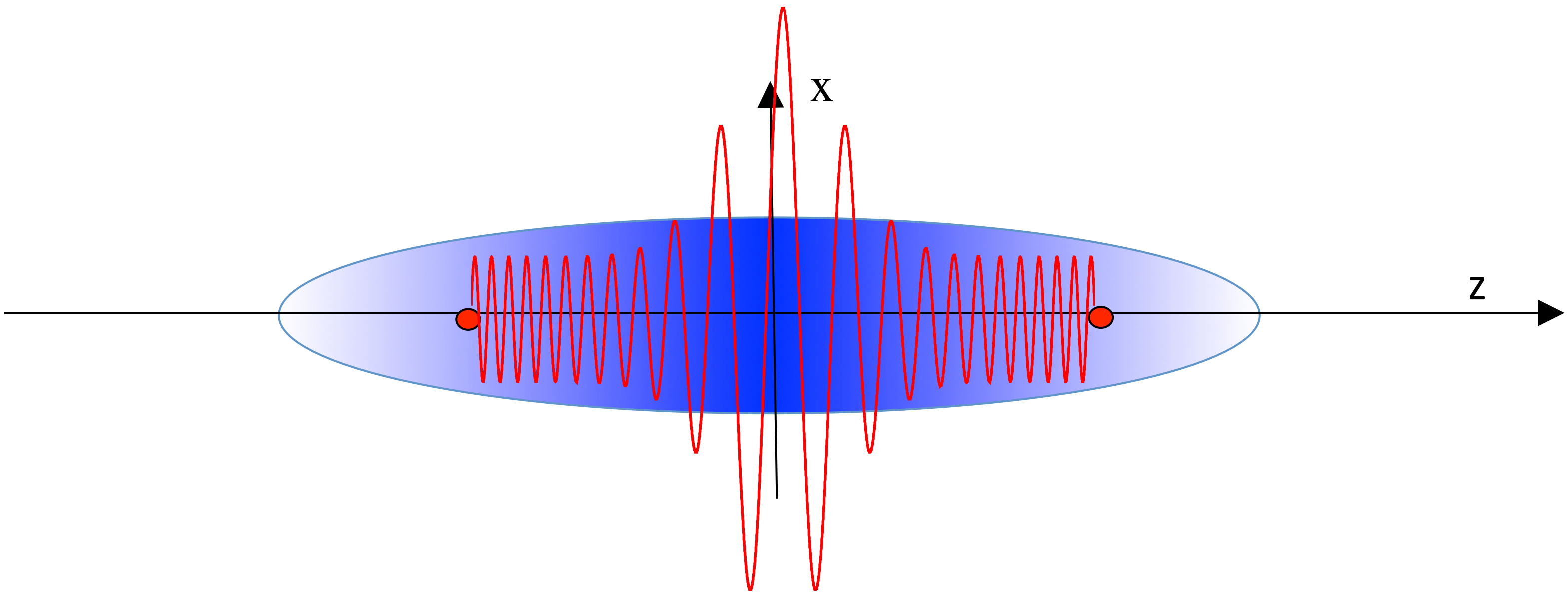
longitudinal synchrotron amplitude


Bare tune
this point depends on where
the particle is taken in (x,x',y,y',z,z')
second half
synchrotron
oscillation
first half
synchrotron
oscillation
this point depends on where
the particle is taken in (x,x',y,y',z,z')

Periodic crossing of a resonance

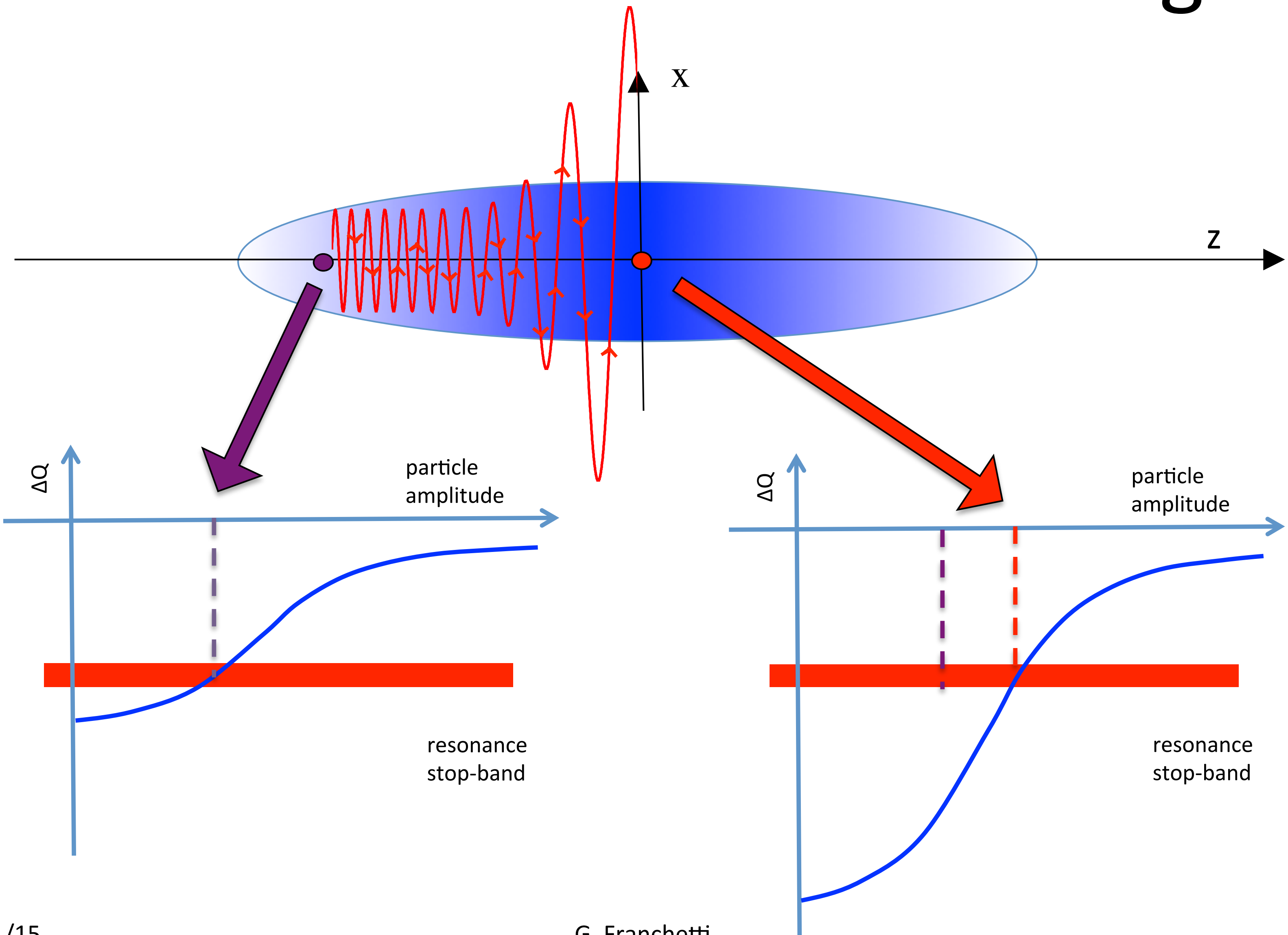


Periodic resonance crossing

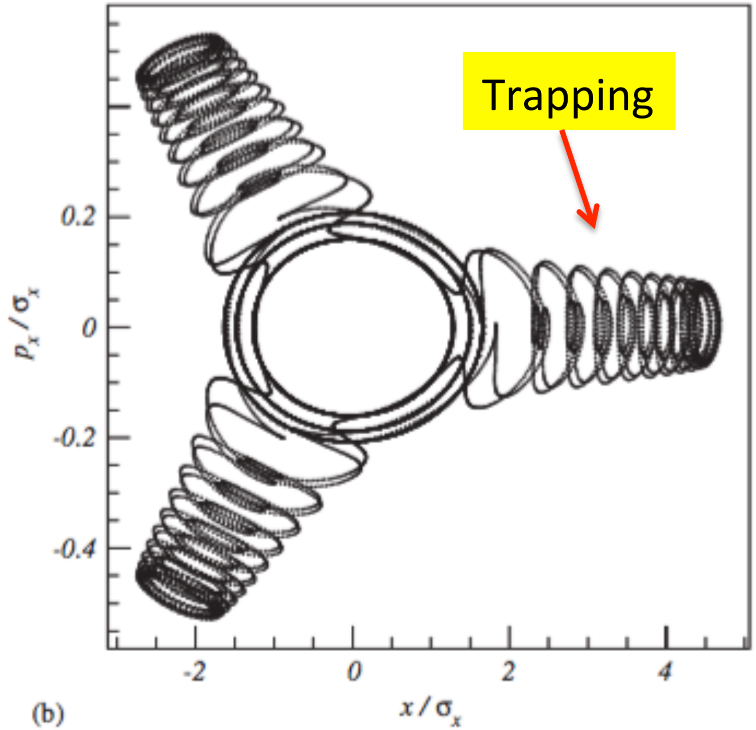


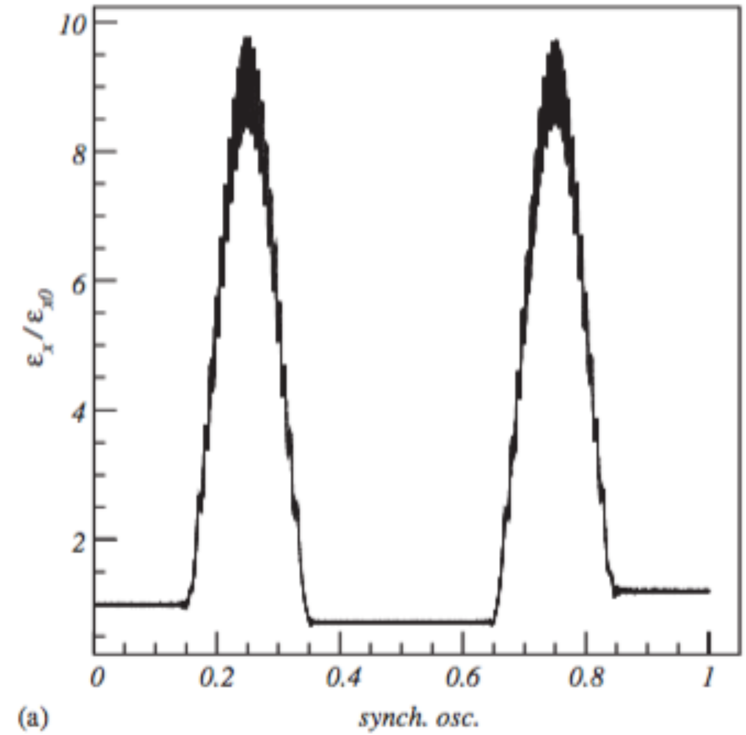
if crossing is adiabatic (Trapping)

Adiabatic resonance crossing



Numerical example



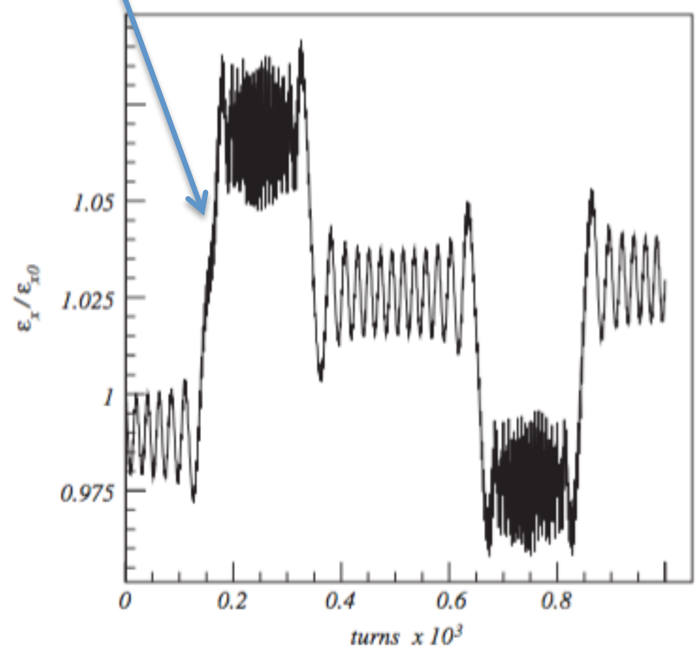


G. Franchetti, I. Hofmann, NIMA

Here the space charge is “frozen”

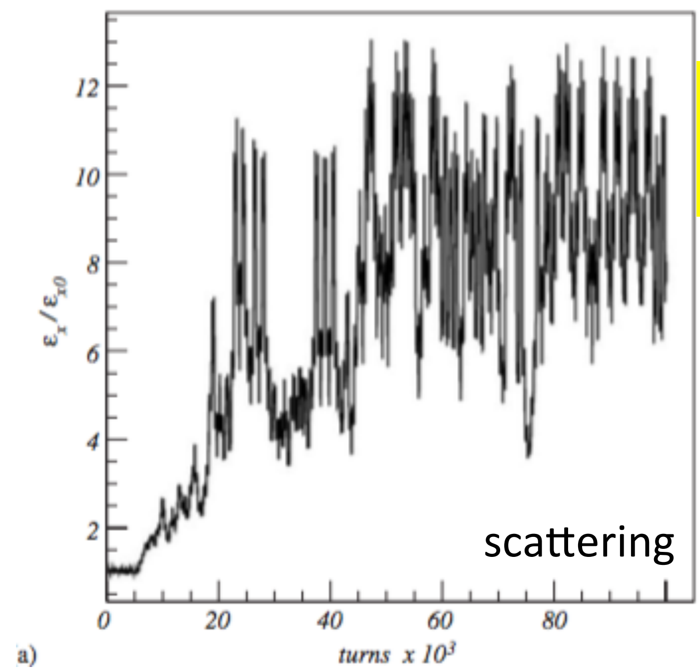
Non adiabatic resonance crossing

kick given by the "resonance"



Multiple resonance crossing
4 x the number of synchrotron oscillations

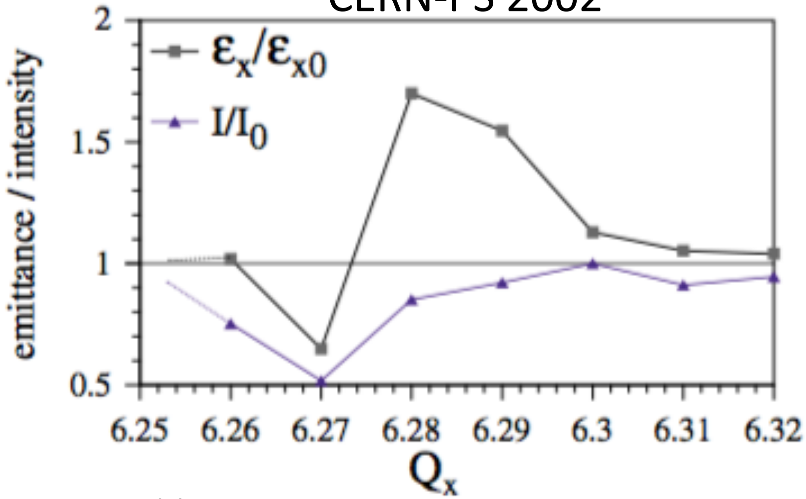
very slow diffusion to large amplitudes till largest island



G. Franchetti, I. Hofmann, NIMA

Experimental results

CERN-PS 2002



R. Cappi
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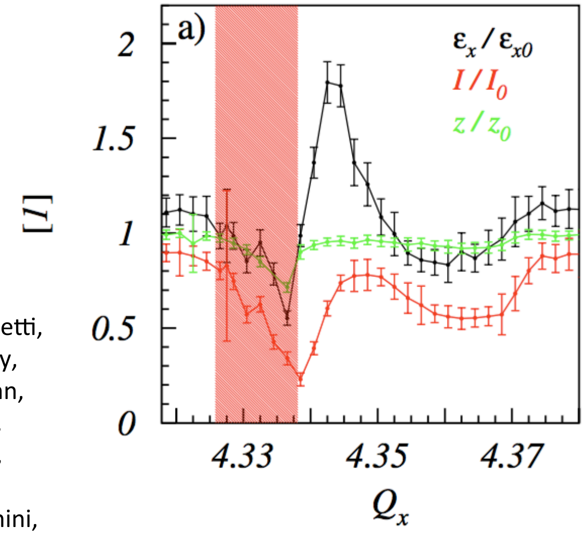
M. Giovannozzi
I. Hofmann
M. Martini
E. Metral

PRSTAB 2003

GSI 2010



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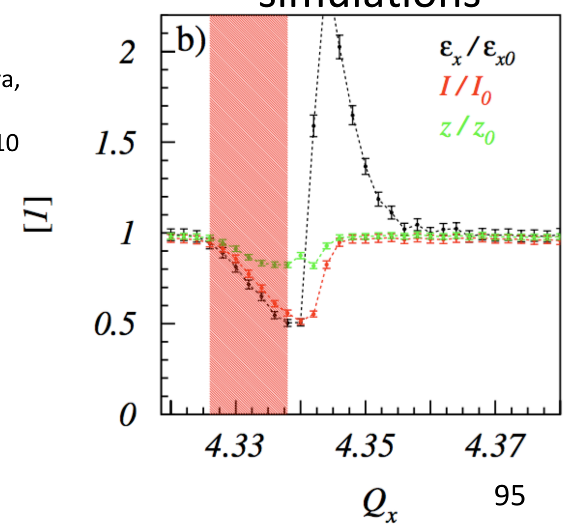
O. Chorniy,
I. Hofmann,
W. Bayer,
F. Becker,
P. Forck,
T. Giacomini,
M. Kirk,
T. Mohite,
C. Omet,
A. Parfenova,
P. Schü¨tt

PRSTAB 2010

simulations



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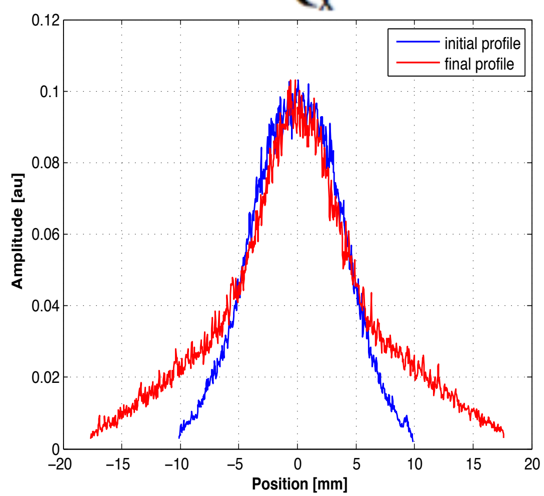
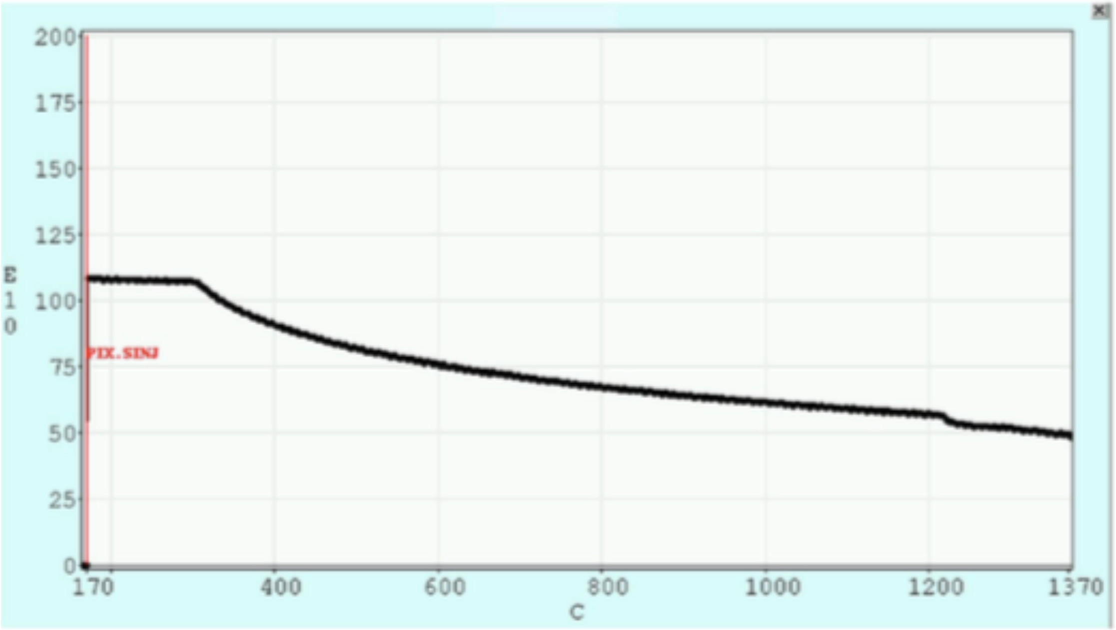
A. Huschuaer
F. Schmidt
R. Wasef

CERN-PS
2013



Very slow beam loss: over 1 second

R. Cappi, G. Franchetti, M. Giovannozzi, I. Hofmann, M. Martini, E. Metral , PRSTAB 2003



CERN-PS 2002

Control / compensation of lattice resonances is a cure, but the effect of space charge on resonance theory (hence compensation) is a subject still under development

Final Considerations

Space charge effects are a mixed of incoherent and coherent effect

The incoherent effects are measured by the incoherent tuneshift
The coherent effects by the “coherent tuneshift”

Most of space charge effects are fully nonlinear, which makes extremely difficult to predict beam evolution.

Simulation codes are necessary to make predictions, but without a physical understanding it is very difficult decide whether to believe to the code predictions

Experiment performed to verify the space charge effects are necessary, but it is not easy to validate a specific mechanism (due to the limits of the observables from real life diagnostics)

Final Considerations

Interplay of space charge in bunches with machine nonlinearities and resonances is very complex and may lead to diffusional effects

Impact of space charge on project (FAIR, LIU) is not negligible if target intensity approaches space charge limit, and if one wants to keep a beam for long time in a ring.

For each scenario, and specific beam dynamics gymnastics, dedicated studies have to be performed.

Space charge history timeline

Incoherent

Incoherent tuneshift
KV
Laslett

coherent/ core stability

Emittance preservation
multiparticle simulations
Sacherer's theory

few % beam loss

Particle core models
simulations codes

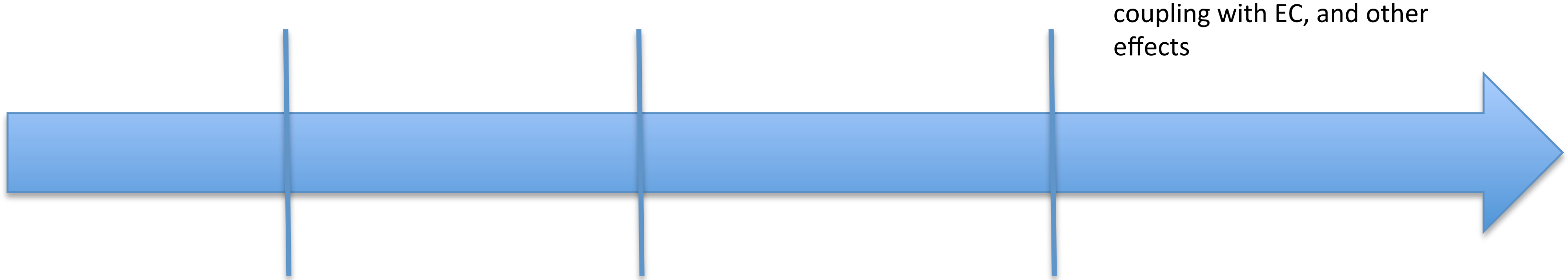
return of the incoherent

Machines studies
CERN,GSI, J-PARC, SNS

machine modeling in codes

long term tracking (frozen)

coupling with EC, and other effects



~ 1985 ~ 1990 ~ 2000

Thanks to Shinji

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