Beam Instabilities in Circular Accelerators

Alex Chao

Robinson instability

Robinson instability (1964) is one of the most basic instability mechanisms. It is a longitudinal instability that occurs in circular accelerators. The main contributor to this instability is the longitudinal impedance due to the rf accelerating cavities. These cavities are tuned to have a resonant frequency ω_R for its fundamental accelerating mode. This mode is where the klystrons feed into, but at the same time, it is also a big source of impedance. Since we must have this rf mode to accelerate the beam, we must accept its big impedance and live with it.

The real part of this impedance narrowly peaks at ω_R, the width $\Delta\omega/\omega_R \approx \pm 1/Q$. Typically, $Q \sim 10^4$ (or 10^9 for superconducting cavities).

By design, ω_R is very close to an integer multiple of the revolution frequency ω_0. This necessarily means that the wakefield excited by the beam in the cavities contains a major frequency component near $\omega_R \approx h\omega_0$, and the impedance $Z_0^{\parallel}(\omega)$ has sharp peaks at $\pm h\omega_0$, where h is an integer called the *harmonic number*.

As we will soon show, the exact value of ω_R relative to $h\omega_0$ is of critical importance for the stability of the beam. Above transition, the beam will be unstable if ω_R is slightly above $h\omega_0$ and stable if slightly below. Below transition, it is the other way around.

Kenneth Robinson (1925-1979)

Consider a one-particle model. The beam is just a big point charge Ne, without internal structures, and consider its longitudinal motion under the influence of its own longitudinal wakefield. Let z_n be the longitudinal displacement of the beam at the accelerating rf cavity in the n-th revolution. The rate of change of z_n is related to the relative energy error δ_n of the beam in the same n-th revolution by

$$\frac{d}{dn} z_n = -\eta C \delta_n$$

The storage ring is above transition if $\eta > 0$ and below transition if $\eta < 0$.

The energy error also changes with time. In the absence of wakefields, its equation of motion is

$$\frac{d}{dn} \delta_n = \frac{(2\pi\nu_s)^2}{\eta C} z_n$$

where ν_s is the unperturbed synchrotron tune. If we combine these two equations, we get a simple harmonic oscillation for both z_n and δ_n, i.e the normal synchrotron oscillation.

But for an intense beam, we have to add the wakefield term,

$$\begin{aligned}\frac{d}{dn}\delta_n &= \frac{(2\pi\nu_s)^2}{\eta C} z_n + \frac{eV(z_n)}{E} \\ &= \frac{(2\pi\nu_s)^2}{\eta C} z_n - \frac{N r_0}{\gamma} \sum_{k=-\infty}^{n} W_0'(kC - nC + z_n - z_k)\end{aligned}$$

where W_0' is the longitudinal wake function accumulated over one turn of the accelerator. The summation over k is over the wakefields left behind by the beam from all revolutions prior to the n-th. The equation of motion now becomes

$$\frac{d^2 z_n}{dn^2} + (2\pi\nu_s)^2 z_n = \frac{N r_0 \eta C}{\gamma} \sum_{k=-\infty}^{n} W_0'(kC - nC + z_n - z_k)$$

In case the beam bunch has an oscillation amplitude much shorter than the wavelength of the fundamental cavity mode, one can expand the wake function,

$$W_0'(kC - nC + z_n - z_k) \approx W_0'(kC - nC) + (z_n - z_k) W_0''(kC - nC)$$

The first term is a static term independent of the motion of the beam. It describes the parasitic loss effect discussed earlier and can be taken care of by a constant shift in z_n. We will drop this term altogether because we are interested here only the dynamical effects. The second term does involve the dynamics of the beam. The quantity $z_n - z_k$ is the difference of z's and – although we will not make such an approximation – resembles a time derivative dz/dn, which in turn suggests an instability since a dz/dn term in a d^2z/dn^2 equation indicates a possible exponential growth (or damping) of z.

We need now to solve this equation for z_n as a function of n. To do so, let

$$z_n \propto e^{-in\Omega T_0}$$

where Ω is the mode frequency of the beam oscillation and is a key quantity yet to be determined.

Substituting into the equation of motion, we find an algebraic equation for Ω,

$$\Omega^2 - \omega_s^2 = -\frac{N r_0 \eta c}{\gamma T_0} \sum_{k=-\infty}^{\infty} (1 - e^{-ik\Omega T_0}) W_0''(kC)$$

where $\omega_s = \nu_s \omega_0$ is the unperturbed synchrotron ocillation frequency. Now the wake function can be expressed in terms of the impedance by a Fourier transform,

$$\Omega^2 - \omega_s^2 = -i\, \frac{N r_0 \eta}{\gamma T_0^2} \sum_{p=-\infty}^{\infty} \left[p\omega_0 Z_0^{\parallel}(p\omega_0) - (p\omega_0 + \Omega) Z_0^{\parallel}(p\omega_0 + \Omega) \right]$$

Given the impedance, this equation can in principle be solved for Ω. Note that Ω appears on both sides of the equation. Here, however, we take a perturbative approach and assume Ω does not deviate much from ω_s for modest beam intensities. We thus replace Ω by ω_s on the right hand side of the equation. Quantity Ω is then easily solved.

In general, Ω is complex. The real part of Ω is the perturbed synchrotron oscillation frequency of the collective beam motion, and the imaginary part gives the growth rate (or damping rate if negative) of the motion. We then obtain a *mode frequency shift*,

$$\begin{aligned} \Delta\Omega &= \mathrm{Re}(\Omega - \omega_s) \\ &= \frac{N r_0 \eta}{2\gamma T_0^2 \omega_s} \sum_{p=-\infty}^{\infty} \left[p\omega_0 \mathrm{Im} Z_0^{\parallel}(p\omega_0) - (p\omega_0 + \omega_s) \mathrm{Im} Z_0^{\parallel}(p\omega_0 + \omega_s) \right) \end{aligned}$$

and an *instability growth rate*,

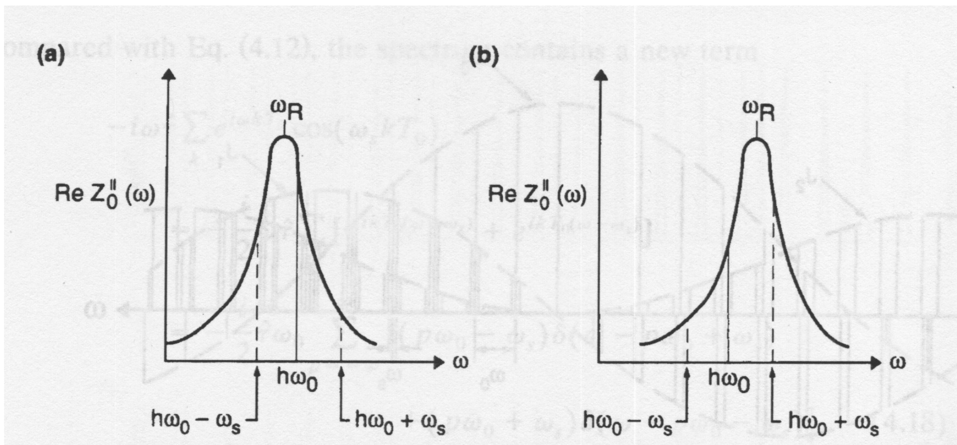
$$\tau^{-1} = \mathrm{Im}(\Omega - \omega_s) = \frac{N r_0 \eta}{2\gamma T_0^2 \omega_s} \sum_{p=-\infty}^{\infty} (p\omega_0 + \omega_s) \mathrm{Re} Z_0^{\parallel}(p\omega_0 + \omega_s)$$

Imaginary part of the impedance contributes to the collective frequency shift. Real part contributes to the instability growth rate. Note that when we measure the synchrotron frequency in an actual operation, what shows up in the beam spectrum is not ω_s, but Ω.

So far our results holds for arbitrary impedance. We now consider the resonator impedance for the fundamental cavity mode. The only significant contributions to the growth rate come from two terms in the summation, namely $p = \pm h$, assuming $\omega_R/Q \ll \omega_0$,

$$\tau^{-1} \approx \frac{N r_0 \eta h \omega_0}{2\gamma T_0^2 \omega_s} \left[\mathrm{Re} Z_0^{\parallel}(h\omega_0 + \omega_s) - \mathrm{Re} Z_0^{\parallel}(h\omega_0 - \omega_s) \right]$$

Beam stability requires $\tau^{-1} \leq 0$. That is, the real part of the impedance must be lower at frequency $h\omega_0 + \omega_s$ than at frequency $h\omega_0 - \omega_s$ if $\eta > 0$, and the other way around if $\eta < 0$. This condition gives the Robinson stability criterion that, above transition, the resonant frequency ω_R of the fundamental cavity mode should be slightly detuned downwards from an exact integral multiple of ω_0. Below transition, the other way around, as sketched below.



Physically, Robinson instability comes from the fact that the revolution frequency of an off-momentum beam is not given by ω_0 but by $\omega_0(1-\eta\delta)$. To illustrate the physical origin, consider a beam executing synchrotron oscillation above transition. Due to the energy error of the beam, the impedance samples the beam signal at a frequency slightly below $h\omega_0$ if $\delta > 0$, and slightly above $h\omega_0$ if $\delta < 0$. In order to damp this synchrotron oscillation, we need to let the beam lose energy when $\delta > 0$ and gain energy when $\delta < 0$. This can be achieved by having an impedance that decreases with increasing frequency in the neighborhood of $h\omega_0$. The Robinson stability criterion then follows.

When $\tau^{-1} > 0$, the beam is unstable because any accidental small synchrotron oscillation would grow exponentially. When $\tau^{-1} < 0$, the Robinson mechanism leads to exponential damping of any synchrotron oscillations of the beam. Robinson damping (or antidamping) can be rather strong. When the Robinson criterion is met, the synchrotron oscillation of the beam is "Robinson damped" and this damping will help stabilizing the beam against similar instabilities due to other impedance sources.

Strong head-tail instability

We next introduce another instability mechanism, the strong head-tail instability, to be discussed using a two-macroparticle model. It was first observed and analyzed at PEP. When intensity exceeds a threshold, the beam becomes unstable. Below it, the beam motion is perturbed but remains stable. A comparison between Robinson and strong head-tail instabilities:

	Robinson instability	Strong head-tail instability
Dimension	longitudinal	transverse
Mode	$m = 0$	$m = 1$
Wakefield	long-range	short-range
Impedance	sharply peaked	broad-band
Model	one-particle	two-particle
Threshold	no	yes

The physical mechanism of the strong head-tail instability is closely related to the beam break up in linacs. Consider an idealized beam with two macroparticles, each with charge $Ne/2$ and each executing synchrotron oscillation. We assume their synchrotron oscillations have equal amplitude but opposite phases. During time $0 < s/c < T_s/2$, where $T_s = 2\pi/\omega_s$ is the synchrotron oscillation period, particle 1 leads particle 2; the equations of motion are

$$\begin{aligned} y_1'' + \left(\frac{\omega_\beta}{c}\right)^2 y_1 &= 0 \\ y_2'' + \left(\frac{\omega_\beta}{c}\right)^2 y_2 &= \frac{Nr_0W_0}{2\gamma C} y_1 \end{aligned} \tag{7}$$

where ω_β is the unperturbed betatron oscillation frequency, whether horizontal or vertical. During $T_s/2 < s/c < T_s$, we have the same equations with indices 1 and 2 switched. Then during $T_s < s/c < 3T_s/2$, Eq.(7) applies again, etc.

In writing down Eq.(7), we have assumed for simplicity that the wake function (integrated over the accelerator circumference), $W_1(z)$, is a constant, and yet it vanishes before the beam completes one revolution,

$$W_1(z) = \begin{cases} -W_0 & \text{if } 0 > z > -(\text{bunch length}) \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

The property of wake functions requires that $W_0 > 0$. This short range wake function corresponds to a broad-band impedance.

Solution for y_1 in Eq.(7) is simply a free betatron oscillation,

$$\tilde{y}_1(s) = \tilde{y}_1(0)e^{-i\omega_\beta s/c}$$

where

$$\tilde{y}_1 = y_1 + i\frac{c}{\omega_\beta}\, y_1'$$

Both the real and imaginary parts are meaningful in the representation.

Substituting this $\tilde{y}_1(s)$ into the equation for y_2 yields the solution

$$\tilde{y}_2(s) = \tilde{y}_2(0)e^{-i\omega_\beta s/c} + i\frac{Nr_0W_0c}{4\gamma C\omega_\beta}\left[\frac{c}{\omega_\beta}\tilde{y}_1^*(0)\sin\frac{\omega_\beta s}{c} + \tilde{y}_1(0)se^{-i\omega_\beta s/c}\right] \tag{9}$$

The first two terms describe free betatron oscillation. The third term, proportional to s, is a resonantly driven response. This analysis is similar to the beam break up instability.

Equation (9) can be simplified if $\omega_\beta T_s/2 \gg 1$, or equivalently, $\omega_\beta \gg \omega_s$. The second term can then be dropped. The solution during the period $0 < s/c < T_s/2$ can be written in a matrix form,

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}_{s=cT_s/2} = e^{-i\omega_\beta T_s/2} \begin{bmatrix} 1 & 0 \\ i\Upsilon & 1 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}_{s=0}$$

with a positive, dimensionless parameter

$$\Upsilon = \frac{\pi N r_0 W_0 c^2}{4\gamma C \omega_\beta \omega_s}$$

The time evolution during $T_s/2 < s/c < T_s$ can be obtained by exchanging indices 1 and 2. The total transformation for one full synchrotron period is

$$\begin{aligned} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}_{cT_s} &= e^{-i\omega_\beta T_s} \begin{bmatrix} 1 & i\Upsilon \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ i\Upsilon & 1 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}_0 \\ &= e^{-i\omega_\beta T_s} \begin{bmatrix} 1-\Upsilon^2 & i\Upsilon \\ i\Upsilon & 1 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}_0 \end{aligned}$$

As time evolves, the phasors $\tilde{y}_1$ and $\tilde{y}_2$ are repeatedly transformed by the 2×2 matrix of this map. Stability of the system is determined by the eigenvalues of this matrix. The two eigenvalues (a $+$ mode and a $-$ mode) are

$$\lambda_\pm = e^{\pm i\phi} \quad , \quad \sin\frac{\phi}{2} = \frac{\Upsilon}{2} \tag{10}$$

with eigenvectors

$$V_\pm = \begin{bmatrix} \pm e^{\pm i\phi/2} \\ 1 \end{bmatrix}$$

Stability requires $\phi =$ real, or

$$\Upsilon \le 2$$

For weak beams, $\Upsilon \ll 1$, we have $\phi \approx \Upsilon$. Near the instability, ϕ approaches π as Υ approaches 2.

A moment of reflection indicates that the instability when $\Upsilon > 2$ causes a rather severe disruption of the beam, as seen by the fact that, during half a synchrotron period, the motion of the trailing particle has grown by an amount more than twice the amplitude of the free-oscillating leading particle. For $\Upsilon \le 2$, the growths made during the half synchrotron periods when the particle is trailing do not accumulate and the beam is stable. As the beam intensity increases so that $\Upsilon > 2$, the growths of the particles then do accumulate and bootstrap into an instability. This *threshold* behavior is very different from the linac case in which the beam head is always stable and the beam tail is always unstable. One can imagine that, by periodically exchanging the roles of leading and trailing particles, the two-particle beam is made more stable. The more frequently they are exchanged, the more stable is the beam, as evidenced by $\Upsilon \propto 1/\omega_s$. Synchrotron oscillation is thus an effective stabilizing mechanism in circular accelerators. Strong betatron focusing and a high beam energy also help stabilize the beam, as indicated by $\Upsilon \propto 1/(\gamma\omega_\beta)$.

In an accelerator, the beam signal comes from the beam position monitors that detect the center of charge $y_1 + y_2$ of the beam, and it would be useful to examine its frequency spectrum. To do that, consider a two-particle beam in a pure eigenstate $V_\pm$ at time $s/c = 0$. In the stable region, the subsequent motion of the beam center of charge is

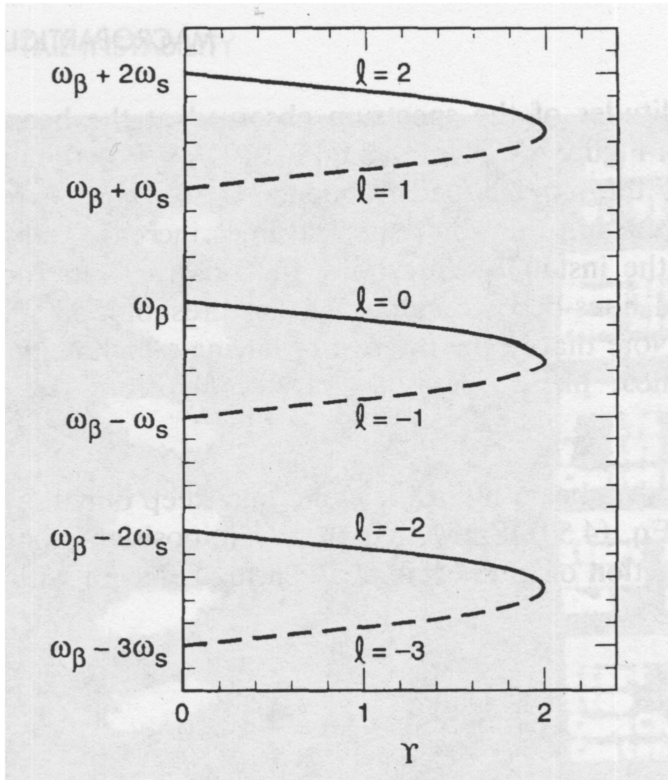
$$\begin{aligned} (\tilde{y}_1 + \tilde{y}_2)(s) &= \exp\left[-i(\omega_\beta \mp \frac{\phi\omega_s}{2\pi})\frac{s}{c}\right] \sum_{\ell=-\infty}^{\infty} C_\ell e^{-i\ell\omega_s s/c} \\ C_\ell &= 2i\Upsilon \, \frac{1 \pm (-1)^\ell}{(2\pi\ell \mp \phi)^2}(1 \mp e^{\pm i\phi/2}) \end{aligned}$$

The $\pm$ modes as observed by a beam position monitor therefore contain the following frequencies:

$$\begin{aligned} +\text{ mode} &\;:\; \omega_\beta + \ell\omega_s - \frac{\phi}{2\pi}\,\omega_s \;,\; \ell = \text{even} \\ -\text{ mode} &\;:\; \omega_\beta + \ell\omega_s + \frac{\phi}{2\pi}\,\omega_s \;,\; \ell = \text{odd} \end{aligned}$$

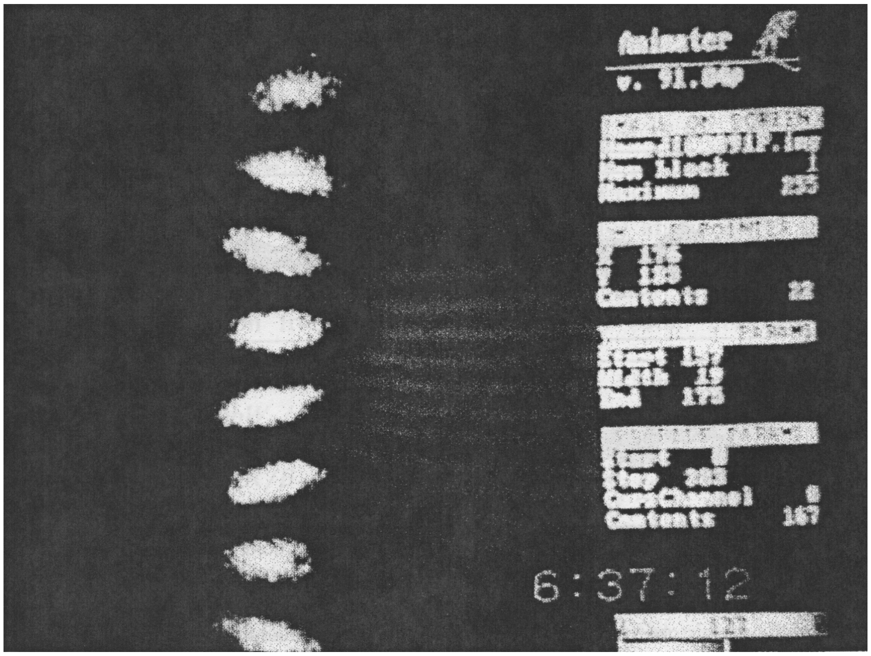
Note that each mode contains a multiplicity of frequencies when observed continuously in time.

For weak beams, the two macroparticles oscillate in phase in the $+$ mode and out of phase in the $-$ mode. As Υ increases, the mode frequencies shift and the particle motions become more complicated; each mode contains a combination of in-phase and out-of-phase motions. At the stability limit $\Upsilon = 2$, the frequencies of the two modes merge into each other and become imaginary, which means instability.



Solid curves are the spectrum of the + mode; dashed curves are that of the - mode. Instability occurs at the point where the mode frequencies merge.

To detect internal beam motion in addition to the center of charge motion, one applies a streak camera. One such observation, made on the electron storage ring LEP at CERN, is shown below. It shows the turn-by-turn pictures of a beam executing a transverse head-tail oscillation. The bunch is seen from the side and one observes a vertical head-tail oscillation ($\ell = 1$). The horizontal scale is 500 ps for the total image. The vertical scale is uncalibrated. The figure shows the same bunch each turn from top to bottom.



(Courtesy Albert Hofmann and Edouard Rossa, 1992)

The strong head-tail instability is one of the cleanest instabilities to observe in electron storage rings. One may measure the threshold beam intensity when the beam becomes unstable transversely and associate the observation with $\Upsilon = 2$. Another approach is to measure the "betatron frequency" (what is measured is the frequency of the $\ell = 0$ spectral line) as the beam intensity is varied. From our two-particle analysis, the initial slope of this frequency shift is

$$\left(\frac{d\omega_\beta}{dN}\right)_{N=0} = -\frac{\omega_s}{2\pi}\left(\frac{d\phi}{dN}\right)_{N=0} = -\frac{r_0 W_0 c^2}{8\gamma C \omega_\beta}$$

By measuring the instability threshold or the initial slope of the betatron frequency, information on the short-range wakefield or broad-band impedance can be obtained.

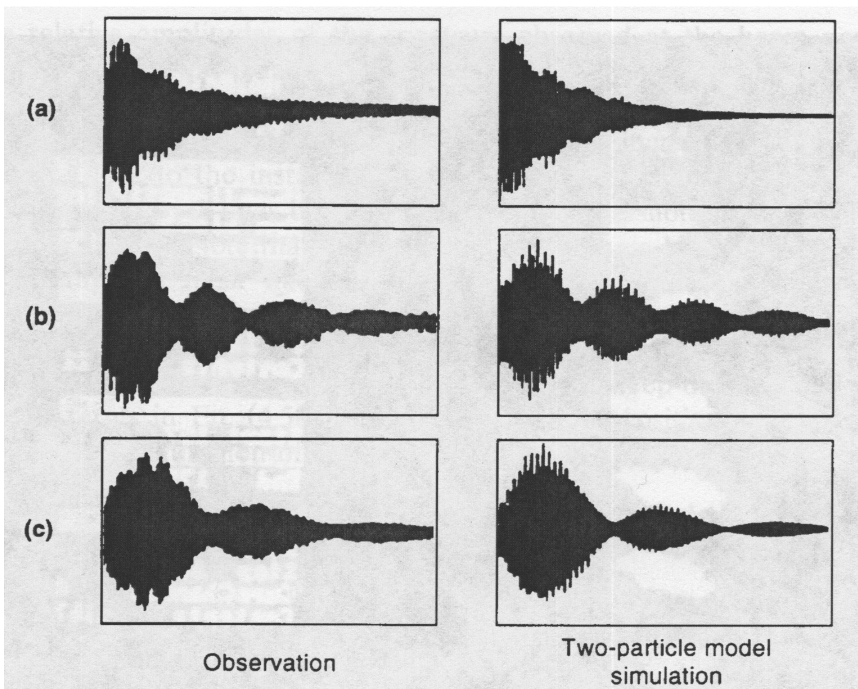
At the instability threshold, the measured betatron frequency has shifted by $\omega_s/2$, according to the two-particle model. The measured value of $(d\omega_\beta/dN)_{N=0}$ can be used to predict the instability threshold N_{th} by

$$N_{\text{th}} = -\frac{\omega_s}{\pi}\frac{1}{(d\omega_\beta/dN)_{N=0}}$$

By measuring ω_β at low beam intensities, the eventual instability threshold can be estimated.

The two-particle model also predicts that the $\ell = 0$ frequency always shifts *down* as the beam intensity is increased. Physically, this is because, for short bunches, the sign of the wake force is such that the bunch tail is always deflected further away from the vacuum chamber axis if the beam is transversely displaced. With the head and the tail moving together in the $\ell = 0$ mode, the wake force acts as a defocusing effect and the mode frequency shifts down.

The center of charge signal of the beam as a function of time after the beam receives an initial transverse kick can be analyzed for a two-particle model. The figure below shows a result compared with experimental observation at PEP. The agreement indicates that the highly idealized two-particle model describes this instability mechanism remarkably well. The fact that the signal exhibits damping is due to radiation damping.



The beam-position-monitor signal as a function of time after the beam is kicked. Left: PEP data with (a) $N/N_{\rm th} = 0.86$, (b) $N/N_{\rm th} = 0.93$, and (c) $N/N_{\rm th} = 0.988$. Right: two-particle model with (a) $\Upsilon/2 = 0.77$, (b) $\Upsilon/2 = 0.96$, and (c) $\Upsilon/2 = 0.99$.

The instability threshold observed at PEP was $N_{\rm th} = 6.4 \times 10^{11}$ with $\omega_\beta/\omega_0 = 18.19$, $\omega_s/\omega_0 = 0.044$, $E = 14.5$ GeV, and $\omega_0 = 0.86 \times 10^6$ s^{-1}. By relating these parameters to $\Upsilon = 2$, one obtains $W_0 = 58$ cm^{-2}. This translates into an effective angular kick at the bunch tail of 18 μrad per millimeter of bunch head displacement per revolution. As mentioned, W_0 can also be obtained by measuring the betatron frequency as a function of beam intensity.

These data can be used to estimate the impedances. Using the property of impedance, we have $Z_1^\perp \approx (R/\beta_Z\nu_\beta)bW_0/c$, where $R/\beta_Z\nu_\beta$ is the weighting factor due to β_Z, the β-function at the location of the impedance. Taking a beam pipe radius $b = 5$ cm and $R/\beta_Z\nu_\beta = 0.5$ for PEP, we find $Z_1^\perp = 0.44$ MΩ/m. Using the approximate connection between $Z_1^\perp$ and $Z_0^\parallel$ then gives $Z_0^\parallel/n \approx 1.6\ \Omega$. This value of $Z_0^\parallel/n$ indicates that about 0.8% of the accelerator circumference is effectively occupied by cavities or their equivalents.

Head-tail instability

In our analysis of the strong head-tail instability, we assumed that the betatron and the synchrotron motions are decoupled. In doing so, we have ignored an important source of instability known as the *head-tail instability*, to which we now turn.

The head-tail instability is one of the cleanest to be observed experimentally. Although it involves a mechanism more subtle than that of the strong head-tail instability, this instability can occur at a much lower beam intensities. This may explain the fact that it was actually observed and explained earlier than the strong head-tail instability.[5]

The betatron oscillation frequency of a particle in a circular accelerator depends on the energy error $\delta = \Delta E/E$ of the particle. If we denote that betatron frequency of an on-momentum particle as ω_β, the betatron frequency for an off-momentum particle can be written as

$$\omega_\beta(\delta) = \omega_\beta(1 + \xi\delta)$$

where ξ is the chromaticity parameter determined by the accelerator design. To assure that the beam has a small betatron frequency spread due to a spread in δ, the absolute value of ξ must not be too large. A consequence of the head-tail consideration, as we will soon see, is that in addition to this requirement, ξ must also have a definite sign. The main reason for introducing sextupoles in circular accelerators is, in fact, to control ξ.

So far we have used s, the longitudinal coordinate along the accelerator, as the independent variable and time t is related to s simply by $s = ct$. It is no longer so simple here because now we have to consider synchrotron motions and the varying time-of-arrival confounds the connection between s and t, and it turns out to be simpler to use s as the independent variable, as will be done below.

Let us first examine the free betatron oscillation in the absence of wake fields. The accumulated betatron phase is given by an integration of the betatron frequency, i.e.,

$$\begin{aligned} \phi_\beta(s) &= \int \omega_\beta(\delta)\frac{ds}{c} \ = \ \omega_\beta\left(\frac{s}{c} + \xi\int\delta\ \frac{ds}{c}\right) \\ &= \omega_\beta\left[\frac{s}{c} - \frac{\xi}{c\eta}z(s)\right] \end{aligned}$$

[5] It may also explain the fact that this instability has preempted the name of "head-tail instability" although almost any other collective instability could justify the same name.

where η is the slippage factor, and we have used $z' = -\eta\delta$.

This is already a remarkable result. It says that the deviation of the betatron phase of a particle from the nominal value $\omega_\beta s/c$ is determined by its longitudinal position. In other words, the chromatic modulation of the betatron phase depends only on z and not on other dynamic variables, such as δ. In fact, one observation to be made is that modulation of the betatron phase by z or the betatron frequency by δ leads to an instability. Modulation of the betatron phase by δ or the betatron frequency by z does not lead to an instability. The modulation, of course, is slow and weak.

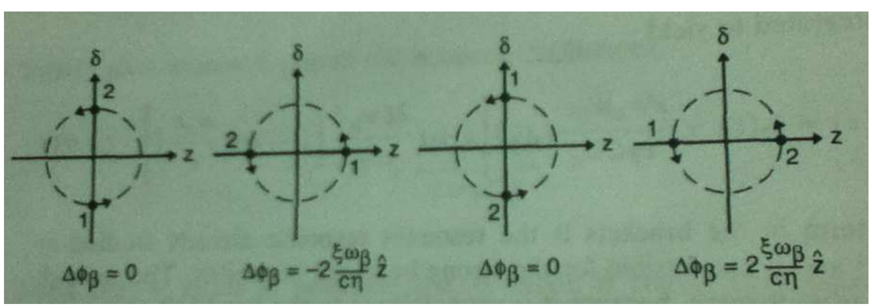
We now consider two macroparticles whose synchrotron oscillations are given by

$$z_1 = \hat{z}\sin\frac{\omega_s s}{c} \quad \text{and} \quad z_2 = -z_1$$

where ω_s is the synchrotron oscillation frequency. Particle 1 leads particle 2 during $0 < s/c < \pi/\omega_s$ and trails it during $\pi/\omega_s < s/c < 2\pi/\omega_s$. The free betatron oscillations of the two particles are described by

$$\begin{aligned} y_1(s) &= \tilde{y}_1 e^{-i\phi_{\beta 1}(s)} = \tilde{y}_1 \exp\left(-i\omega_\beta \frac{s}{c} + i\,\frac{\xi\omega_\beta}{c\eta}\,\hat{z}\sin\frac{\omega_s s}{c}\right), \\ y_2(s) &= \tilde{y}_2 e^{-i\phi_{\beta 2}(s)} = \tilde{y}_2 \exp\left(-i\omega_\beta \frac{s}{c} - i\,\frac{\xi\omega_\beta}{c\eta}\,\hat{z}\sin\frac{\omega_s s}{c}\right) \end{aligned} \tag{11}$$

As the particles exchange the roles of leading particle and trailing particle, the betatron phases are such that the leading particle always lags in phase relative to the trailing particle if $\xi/\eta > 0$ and the situation reverses if $\xi/\eta < 0$, as illustrated in the following figure.



The synchrotron oscillations of a two-particle beam observed in the longitudinal phase space. The quantity $\Delta\phi_\beta = \phi_{\beta 1} - \phi_{\beta 2}$ is the difference of the betatron phases of the two particles; it is modulated by the synchrotron motion as shown. The sense of rotation of particle motion in the phase space is for the case above transition, i.e. $\eta > 0$.

The factor $\xi\omega_\beta\hat{z}/c\eta$ is called the *head-tail phase*. It is the physical origin of the head-tail instability. As a numerical example, one may have an electron accelerator with $\eta = 0.003$, $\xi = 0.2$, $\hat{z} = 3$ cm, and $\omega_\beta = 1.4 \times 10^7$ s^{-1}, which gives a head-tail phase of $2\pi \times 0.016$.

Recalling the strong head-tail instability studied previously, the trailing particle is always unstable due to the resonant driving by the wake field of the leading particle; the growths of the trailing particle during the half synchrotron periods are strong, but below a certain threshold the synchrotron oscillation washes away the growths and the net result is that the beam becomes stable. The additional chromatic term that we are considering now does not have this fortunate property. As we will see, the weak growths associated with chromaticity do accumulate persistently from one half synchrotron period to the next, and thus slowly build up an instability without a threshold.

Let us look at the motion of particle 2 during $0 < s/c < \pi/\omega_s$ in the presence of the wake field. The wake function, we assume, is that given by Eq.(8). The equation of motion is

$$y_2'' + \left[\frac{\omega_\beta(\delta_2)}{c}\right]^2 y_2 = \frac{Nr_0W_0}{2\gamma C} y_1$$
$$\omega_\beta(\delta_2) = \omega_\beta\Big(1 + \frac{\xi\hat{z}\omega_s}{c\eta}\cos\frac{\omega_s s}{c}\Big)$$

The y_1 on the right hand side is given by the free oscillation result. If we let y_2 also be given by Eq.(11), but allow $\tilde{y}_2$ to be slowly varying in time, we obtain an equation for $\tilde{y}_2$,

$$\tilde{y}_2'(s) \approx \frac{iNr_0W_0c}{4\gamma C\omega_\beta}\,\tilde{y}_1(0)\exp\left(2i\frac{\xi\omega_\beta\hat{z}}{c\eta}\sin\frac{\omega_s s}{c}\right)$$

For most practical cases, the head-tail phase $\xi\omega_\beta\hat{z}/c\eta$ is much less than unity, the exponential factor in the expression can be Taylor expanded and y_2 can be integrated to yield

$$\tilde{y}_2(s) = \tilde{y}_2(0) + \frac{iNr_0W_0c}{4\gamma C\omega_\beta}\tilde{y}_1(0)\left[s + i\frac{2\xi\omega_\beta\hat{z}}{\eta\omega_s}\left(1 - \cos\frac{\omega_s s}{c}\right)\right]$$

The first term in the brackets is the resonant response already studied before and is responsible for the strong head-tail instability. The second, chromatic term is small, because it is proportional to the head-tail phase and also because it is not a resonant response. On the other hand, the important fact here is that the chromatic term is 90° out of phase from the resonant term.

The transformation from $s = 0$ to $s = \pi c/\omega_s$ is thus described by

$$\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}_{\pi c/\omega_s} = \begin{bmatrix}1 & 0\\ i\Upsilon & 1\end{bmatrix}\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}_0$$

where

$$\Upsilon = \frac{\pi Nr_0W_0c^2}{4\gamma C\omega_\beta\omega_s}\left(1 + i\,\frac{4\xi\omega_\beta\hat{z}}{\pi c\eta}\right)$$

This Υ, of course, reduces to the strong head-tail *Upsilon* if $\xi = 0$, but now it has acquired an imaginary part if $\xi \neq 0$. A similar procedure applied to the period $\pi c/\omega_s < s < 2\pi c/\omega_s$ leads to the transformation

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}_{2\pi c/\omega_s} = \begin{bmatrix} 1 & i\Upsilon \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}_{\pi c/\omega_s}$$

Stability of the system is determined by the total transformation matrix

$$\begin{bmatrix} 1 & i\Upsilon \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ i\Upsilon & 1 \end{bmatrix} = \begin{bmatrix} 1-\Upsilon^2 & i\Upsilon \\ i\Upsilon & 1 \end{bmatrix}$$

The eigenvalues of this matrix have been obtained before in Eq.(10). For a weak beam intensity, $|\Upsilon| \ll 1$, the two eigenvalues are

$$\lambda_\pm \approx e^{\pm i\Upsilon}$$

The $+$ mode ($-$ mode) is the mode when the two macroparticles oscillate in phase (out of phase) in the limit of weak beam intensity. The imaginary part of Υ thus gives a growth rate of the betatron oscillations,

$$\tau_\pm^{-1} = \mp \frac{N r_0 W_0 c \xi \hat{z}}{2\pi\gamma C\eta}$$

When the $+$ mode is unstable, the $-$ mode is stable; the transverse displacement of the beam center of charge grows with time but the transverse size of the beam essentially remains constant. When the $-$ mode is unstable, the $+$ mode becomes stable; the beam center of charge does not oscillate, but the beam size grows exponentially.

The $+$ mode is damped if $\xi/\eta > 0$ and antidamped if $\xi/\eta < 0$. The $-$ mode is damped if $\xi/\eta < 0$ and antidamped if $\xi/\eta > 0$. We conclude from this that the only value of ξ that assures a stable beam is $\xi = 0$. However, it can be shown by using a Vlasov equation technique, the two-particle model has overestimated the growth rate of the $-$ mode. This consideration, together with the presence of some stabilizing mechanisms (such as Landau damping, or radiation damping in the case of circular electron accelerators) leads us to choose slightly positive values for ξ for operation above transition, and slightly negative ξ below transition.

The growth rate is proportional to N and ξ, and inversely proportional to γ as one would expect. The linear dependence on the bunch length $\hat{z}$, however, is a consequence of the constant wake model. Had we assumed a different wake model, the dependence of τ^{-1} on $\hat{z}$ would change.

Note that the same transverse wakefield is responsible for both the strong head-tail instability and the head-tail instability. Continuing the PEP example mentioned earlier, and further taking $\hat{z} = 3$ cm and $\xi = 0.2$, we find the head-tail growth rate ∓ 0.6 ms at the threshold for strong head-tail instability, $N = 6.4 \times 10^{11}$.[6] The head-tail damping or antidamping can be rather fast.

[6]Strictly speaking, Eq.(4.99) applies only when $|\Upsilon| \ll 1$. We apply it here, even though ReΥ=2, to obtain an order of magnitude estimate.

In addition to the methods mentioned earlier, the head-tail growth rate provides another way to measure the transverse wake function and the impedance of an accelerator. To do so, ξ is made slightly positive (above transition), a beam center-of-charge motion (in a + mode) is excited by a kicker, and its subsequent damped motion is observed. The linear dependence of the damping rate on ξ allows the extraction of the wake function information. The various methods of measuring the wake function are not expected to give identical values for the transverse impedance $Z_1^\perp$, but the results should at least be comparable.

Landau damping

There are many collective instability mechanisms acting on a high intensity beam in an accelerator, demanding a wide range of sometimes conflicting stability conditions. Yet the beam as a whole seems basically stable, as evidenced by the existence of a wide variety of working accelerators. One of the reasons for this fortunate outcome is *Landau damping*, which provides a natural stabilizing mechanism against collective instabilities if particles in the beam have a small spread in their natural (synchrotron or betatron) frequencies.

The spread in ω_β has several sources. A dependence of ω_β on the energy of the particle, together with an energy spread in the beam, leads to a spread in ω_β. Nolinearities in the focusing system cause a dependence of ω_β on the particle's betatron amplitude. A spread in betatron amplitudes then leads also to a spread in ω_β.

The source of spread in ω_s depends on whether the beam is bunched or unbunched. For bunched beams, a spread can result from nonlinearity in the rf focusing voltage. For unbunched beams, dependence of the revolution frequency on the particle energy plays a similar role.

Consider a simple harmonic oscillator with natural frequency ω driven by a sinusoidal force of frequency Ω,

$$\ddot{x} + \omega^2 x = A\cos\Omega t$$

with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$. The solution is

$$x(t>0) = -\frac{A}{\Omega^2 - \omega^2}(\cos\Omega t - \cos\omega t) \tag{12}$$

The $\cos\Omega t$ term gives the main term responding to the driving force; the $\cos\omega t$ term comes from matching the initial conditions.

The explicit inclusion of the initial conditions plays an important role. Otherwise, one could have carelessly written the solution

$$x(t) = -\frac{A}{\Omega^2 - \omega^2}\cos\Omega t, \quad \text{or} \quad x(t) = -\frac{A}{\Omega^2 - \omega^2}e^{-i\Omega t} \tag{13}$$

Eq.(13) contains a singularity at $\Omega = \omega$ while (12) is well behaved there. This singularity is the source of subtleties and at this point is to be avoided. As we will see later, by applying some mathematical tricks, it is possible to bypass the explicit inclusion of the initial conditions and go straight to (13), but at this point, we stay with (12).

Consider now an ensemble of oscillators (each oscillator represents a single particle in the beam) which have a spectrum $\rho(\omega)$ satisfying $\int_{-\infty}^{\infty} d\omega \rho(\omega) = 1$. Now subject this ensemble of particles to the driving force $A \cos \Omega t$ with all particles starting with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$. The ensemble average response is

$$\langle x \rangle(t > 0) = -\int_{-\infty}^{\infty} d\omega \rho(\omega) \frac{A}{\Omega^2 - \omega^2} (\cos \Omega t - \cos \omega t)$$

For simplicity, let us consider a narrow beam spectrum around a frequency ω_x and a driving frequency near the spectrum, i.e. $\Omega \approx \omega_x$. The beam response is then

$$\langle x \rangle(t) = -\frac{A}{2\omega_x} \int_{-\infty}^{\infty} d\omega \rho(\omega) \frac{1}{\Omega - \omega} (\cos \Omega t - \cos \omega t)$$

Changing variable from ω to $u = \omega - \Omega$ leads to

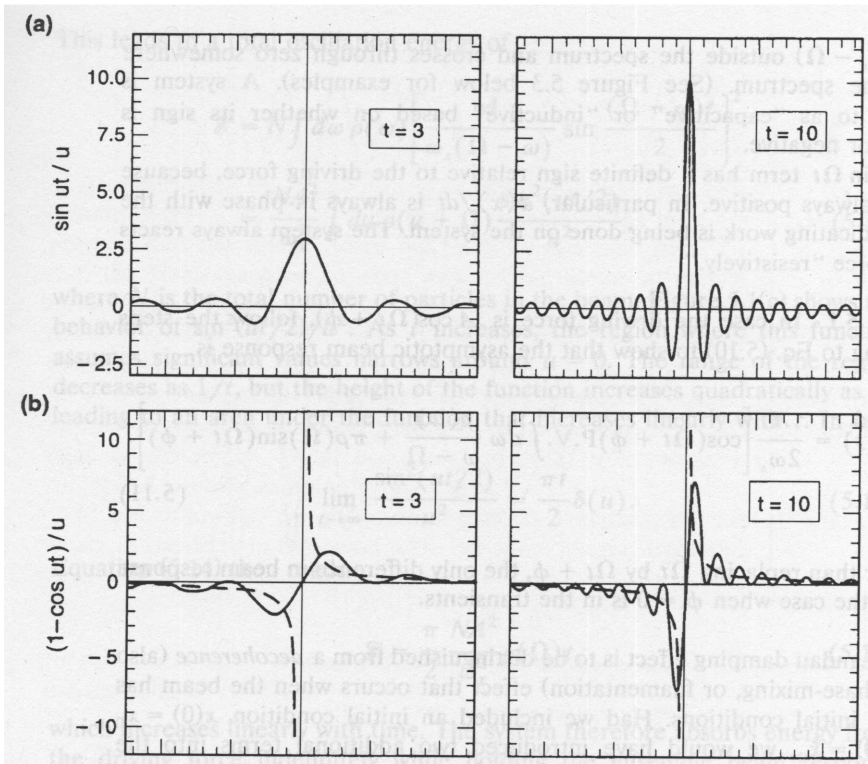
$$\begin{aligned}
\langle x \rangle(t) &= \frac{A}{2\omega_x} \int_{-\infty}^{\infty} du \frac{\rho(u + \Omega)}{u} [\cos \Omega t - \cos(\Omega t + ut)] \\
&= \frac{A}{2\omega_x} \left[\cos \Omega t \int_{-\infty}^{\infty} du \rho(u + \Omega) \frac{1 - \cos ut}{u} + \sin \Omega t \int_{-\infty}^{\infty} du \rho(u + \Omega) \frac{\sin ut}{u} \right]
\end{aligned}$$

All integrals are well behaved at $u = 0$.

The beam response contains a $\cos \Omega t$ term and a $\sin \Omega t$ term, but their coefficients are time dependent. The next step is to show that those coefficients approach well behaved limits. To do so, one first observes

$$\begin{aligned}
\lim_{t \to \infty} \frac{\sin ut}{u} &= \pi \delta(u) \\
\lim_{t \to \infty} \frac{1 - \cos ut}{u} &= \text{P.V.} \left(\frac{1}{u} \right)
\end{aligned}$$

The proof is illustrated in figure below.



The functions $\sin(ut)/u$, $(1-\cos ut)/u$ are shown in (a), (b) for two values $t=3$ and 10. The dashed curves in (b) are for the function $1/u$. The sole function of $(1-\cos u)$ in (b) is to suppress the singularity at $u=0$.

If we are not interested in the transient effects immediately following the onset of the driving force, we obtain

$$\langle x\rangle(t) = \frac{A}{2\omega_x}\left[\cos\Omega t \text{ P.V.}\int d\omega \frac{\rho(\omega)}{\omega-\Omega} + \pi\rho(\Omega)\sin\Omega t\right]$$

This expression contains explicitly a $\cos\Omega t$ term and a mysterious $\sin\Omega t$ term.

The sign of the $\cos\Omega t$ term relative to the driving force depends on the sign of P.V. $\int d\omega\rho(\omega)/(\omega-\Omega)$. A system is referred to as "capacitive" or "inductive" based on whether its sign is positive or negative.

The $\sin\Omega t$ term has a definite sign relative to the driving force because $\rho(\Omega)$ is always positive. In particular, $d\langle x\rangle/dt$ is always in phase with the force, indicating work is being done on the system. The system always reacts to the force "resistively."

To proceed, write the beam response in a complex notation,

$$\text{driving force} = Ae^{-i\Omega t}$$

$$\langle x \rangle = \frac{A}{2\omega_x \Delta\omega} e^{-i\Omega t} \left[f(u) + i g(u) \right]$$

where $u = (\omega_x - \Omega)/\Delta\omega$ with $\Delta\omega$ the width of the spectral spread, and

$$\begin{aligned} f(u) &= \Delta\omega \text{ P.V.} \int d\omega \frac{\rho(\omega)}{\omega - \Omega} \\ g(u) &= \pi \Delta\omega \; \rho(\omega_x - u\Delta\omega) \end{aligned}$$

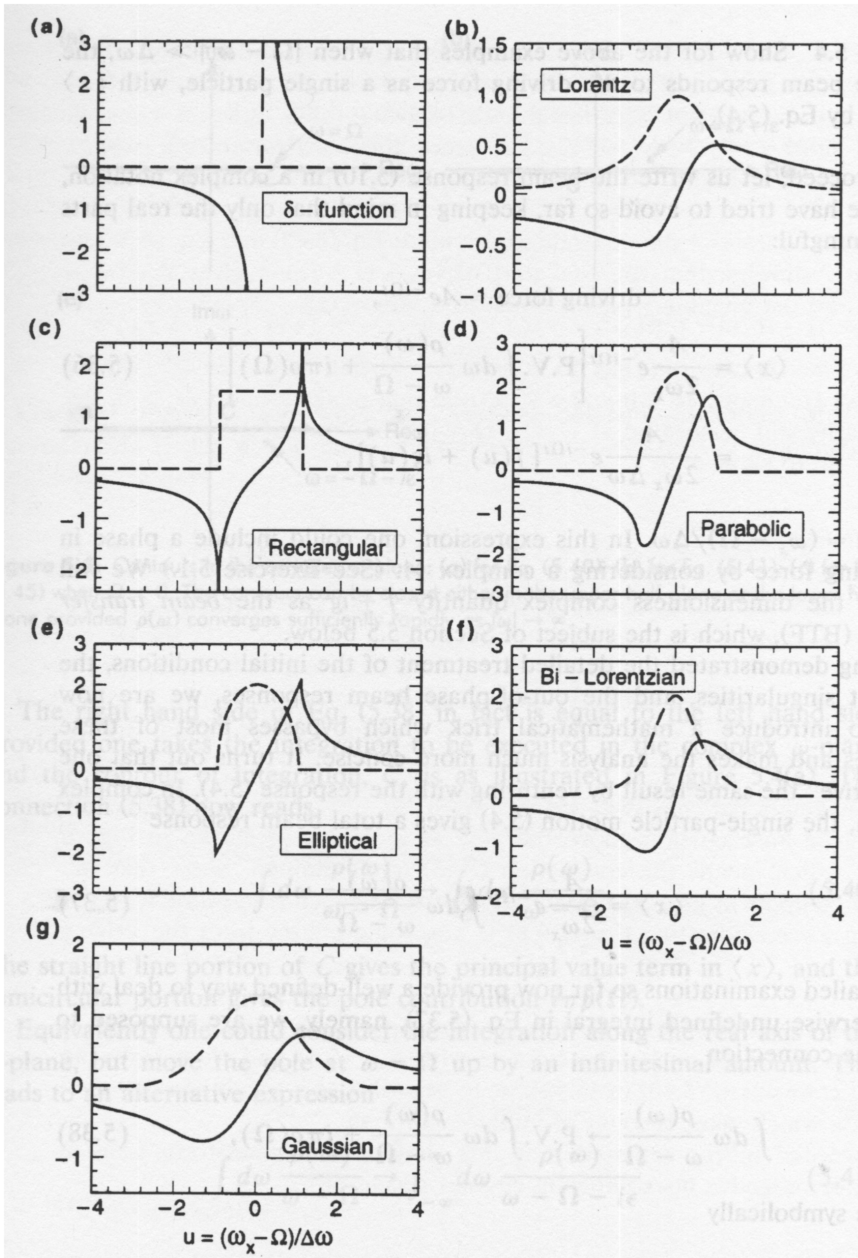
The dimensionless complex quantity $f + ig$ is the *beam transfer function*.

For the δ-function spectrum, there is no frequency spread, Landau damping is lost,

$$f(u) = \frac{1}{u}, \quad \text{and} \quad g(u) = \pi\delta(u)$$

For the Lorentz spectrum,

$$f(u) = \frac{u}{1+u^2}, \quad \text{and} \qquad g(u) = \frac{1}{1+u^2}$$



The functions $f(u)$ (solid curves) and $g(u)$ (dashed curves) for various spectral distributions.

We now introduce a mathematical trick. It turns out that one can "derive" the same result by venturing with (13). In complex notation, Eq.(13) gives

$$\langle x \rangle = \frac{A}{2\omega_x} e^{-i\Omega t} \int d\omega \frac{\rho(\omega)}{\omega - \Omega}$$

Our detailed examinations provides a well-defined way to deal with the otherwise undefined integral, i.e.

$$\int d\omega \frac{\rho(\omega)}{\omega - \Omega} \quad \to \quad \text{P.V.} \int d\omega \frac{\rho(\omega)}{\omega - \Omega} + i\pi\rho(\Omega) \tag{14}$$

or more symbolically

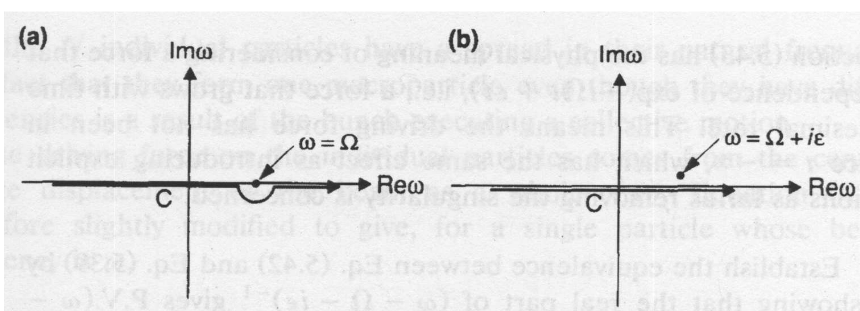
$$\frac{1}{\omega - \Omega} \quad \to \quad \text{P.V.} \left(\frac{1}{\omega - \Omega} \right) + i\pi\delta(\omega - \Omega)$$

Again, it is necessary to include an out-of-phase term—with a definite sign—as evidenced by the imaginary term $i\pi\rho(\Omega)$, even though the expression on the left hand side seems to be for a real quantity.

The right hand side of Eq.(14) in fact is equal to the left hand side provided one takes the integration to be executed in the complex ω-plane and the contour of integration, C, is as illustrated in the figure (a) below. The connection (14) now reads

$$\int d\omega \frac{\rho(\omega)}{\omega - \Omega} \quad \to \quad \int_C d\omega \frac{\rho(\omega)}{\omega - \Omega}$$

The straight line portion of C gives the principal value term in $\langle x \rangle$ and the semicircular portion gives the pole contribution $i\pi\rho(\Omega)$.



Equivalently one could consider the integration along the real axis of the ω-plane, but move the pole at $\omega = \Omega$ up by an infinitesimal amount,

$$\int d\omega \frac{\rho(\omega)}{\omega - \Omega} \quad \to \quad \int_{-\infty}^{\infty} d\omega \frac{\rho(\omega)}{\omega - \Omega - i\epsilon}$$

or

$$\frac{1}{\omega - \Omega} \quad \to \quad \frac{1}{\omega - \Omega - i\epsilon}$$

or simply

$$\Omega \quad \to \quad \Omega + i\epsilon \tag{15}$$

It is now a matter of taste whether to regard our main conclusion (14) as a result of a simple derivation starting with Eq.(13) and then make a profound connection (15), or to regard it as a result of a detailed calculation which takes into account of initial conditions.

<u>One-particle model for bunched beams – transverse</u>

Results obtained in the previous section applied to circular accelerators lead to Landau damping of collective instabilities. To demonstrate this for a bunched beam, consider a one-particle model, except now the N individual particles have a spread in their natural frequencies. The fact that they form one macroparticle even though they have different frequencies is a result of the bunch executing a collective motion.

The driving force on the individual particles comes from the center-of-charge displacement of the beam as a whole, $\langle y \rangle$, through the wakefield. For a single particle whose betatron frequency is ω,

$$y''(s) + \left(\frac{\omega}{c}\right)^2 y(s) = -\frac{Nr_0}{\gamma C}\sum_{k=1}^{\infty}\langle y\rangle(s-kC)\ W_1(-kC)$$

Consider the situation when y-motion of the macroparticle is just at the edge of exponential growth due to a collective instability. We have

$$\langle y\rangle(s) = Be^{-i\Omega s/c} \tag{16}$$

where Ω carries an imaginary part $i\epsilon$, where ϵ is infinitesimally positive.

It is not very interesting to search for damped, stable solutions. Finding stable solutions does not assure beam stability, but finding one unstable solution reveals the beam to be unstable.

We now have

$$y''(s) + \left(\frac{\omega}{c}\right)^2 y(s) = -\frac{BNr_0}{\gamma C}\mathcal{W}e^{-i\Omega s/c}$$

where

$$\mathcal{W} = \sum_{k=1}^{\infty} W_1(-kC)e^{i\omega_\beta kT_0}$$

or in terms of impedance,

$$\mathcal{W} = -\frac{i}{T_0}\sum_{p=-\infty}^{\infty} Z_1^{\perp}(p\omega_0 + \omega_\beta)$$

We have assumed the mode frequency shift is small so that $\Omega \approx \omega_\beta$, where ω_β is the center of the beam frequency spectrum.

The beam is driven by a sinusoidal driving force. Our analysis of Landau damping gives the beam response,

$$\langle y \rangle = -\frac{BNr_0\mathcal{W}c}{2\omega_\beta\gamma T_0} e^{-i\Omega s/c} \left[\text{P.V.} \int d\omega \frac{\rho(\omega)}{\omega - \Omega} + i\pi\rho(\Omega) \right]$$

But we had already assumed that the collective beam motion is given by Eq.(16). This means the mode frequency Ω is not arbitrary. In order for the beam motion to be nontrivial, Ω must satisfy a self-consistency condition, the *dispersion relation*,

$$1 = -\frac{Nr_0\mathcal{W}c}{2\omega_\beta\gamma T_0} \left[\text{P.V.} \int d\omega \frac{\rho(\omega)}{\omega - \Omega} + i\pi\rho(\Omega) \right]$$

or

$$-\frac{Nr_0\mathcal{W}c}{2\omega_\beta\gamma T_0\Delta\omega} = \frac{1}{f(u) + ig(u)}$$

In case the beam does not have a natural frequency spread, we have $f(u) = 1/u, g(u) = 0$. The complex mode frequency shift is found to be

$$(\Omega - \omega_\beta)_{\text{no Landau damping}} = \frac{Nr_0 c\mathcal{W}}{2\omega_\beta\gamma T_0}$$

We shall designate this quantity as ξ_1; it contains essentially the beam intensity, multiplied by the impedance, divided by the focusing strength and the magnetic rigidity.

For a beam with natural frequency spread, the dispersion relation is

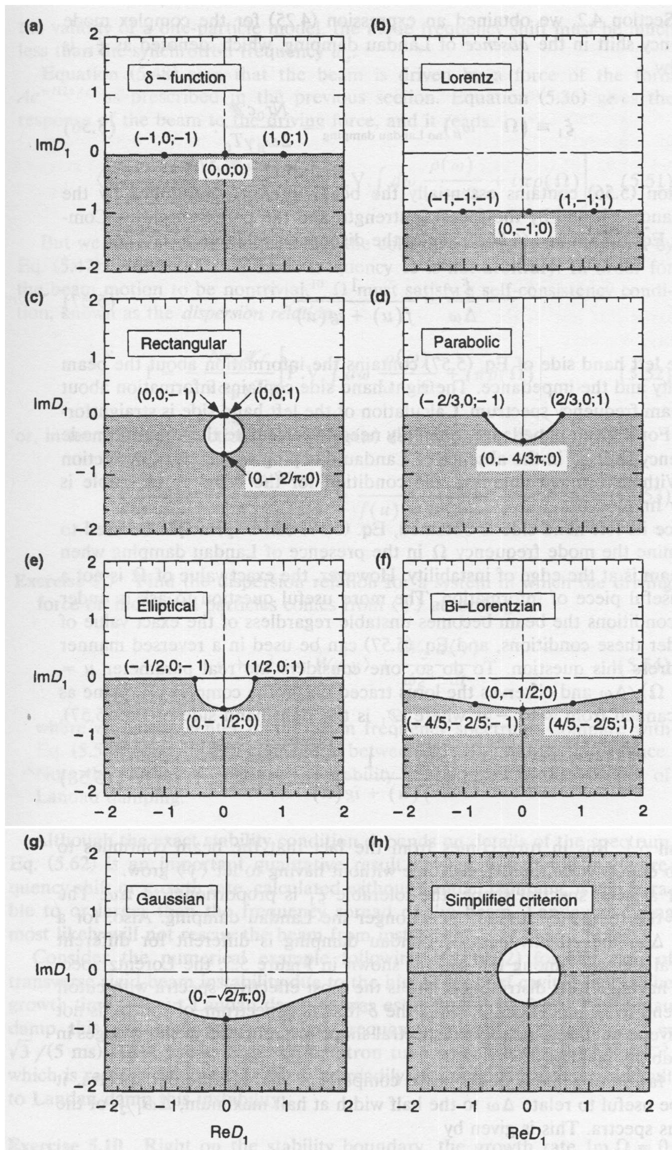
$$-\frac{\xi_1}{\Delta\omega} = \frac{1}{f(u) + ig(u)} \tag{17}$$

The left hand side of Eq.(17) contains information about the beam intensity and the impedance. The right hand side contains information about the beam frequency spectrum. For a given impedance, the LHS is obtained by calculating the complex mode frequency shift ξ_1 in the absence of Landau damping. Without Landau damping, the stability condition is simply Im $\xi_1 < 0$.

Once its LHS is obtained, Eq.(17) can in principle be used to determine Ω in the presence of Landau damping when the beam is at the edge of instability. However, the exact value of Ω is not useful. The useful question to ask is under what conditions the beam becomes unstable regardless of the exact value of Ω. Eq.(17) can be used in a reversed manner to address this question. To do so, consider the real parameter $u = (\omega_\beta - \Omega)/\Delta\omega$ and observes the lotus traced out in the complex $\mathcal{D}_1$-plane as u is scanned from ∞ to $-\infty$, where

$$\mathcal{D}_1 = \frac{1}{f(u) + ig(u)}$$

This lotus defines a *stability boundary diagram.* The LHS of Eq.(17), a complex quantity, is then plotted in the complex $\mathcal{D}_1$-plane as a single point. If this point lies on the lotus, it means the solution of Ω for Eq.(17) is real, and this ξ_1 value is such that the beam is just at the edge of instability. If it lies on the inside of the lotus (the side which contains the origin of the $\mathcal{D}_1$-plane), the beam is stable. If it lies on the outside of the lotus, the beam is unstable.



The stability boundary diagrams for various spectra. Shaded regions are unstable. The coordinates labeled refer to ($\mathrm{Re}\mathcal{D}_1$, $\mathrm{Im}\mathcal{D}_1$; u). The value of u can be used to obtain Ω. (a) δ-function spectrum, no Landau damping. (h) is the simplified criterion (18).

The dispersion relation is particularly simple for the Lorentz spectrum (case (b)),

$$\Omega = \omega_\beta + \xi_1 - i\Delta\omega$$

The stability condition $\mathrm{Im}\Omega < 0$ therefore becomes

$$\mathrm{Im}\xi_1 < \Delta\omega$$

The fact that the stable region is always enlarged by the frequency spread demonstrates the Landau damping mechanism. Its origin can be traced back to the fact that $g(u)$ is always positive, which in turn comes from the fact that the beam continues to absorb energy from the driving force without having to let $\langle y\rangle$ grow.

For a given spectral shape, the tolerable $\xi_1 \propto \Delta\omega$; the larger the frequency spread, the stronger the Landau damping. For a given $\Delta\omega$, the effectiveness of Landau damping is different for different spectral shapes. The Lorentz spectrum, having a long distribution tail, is most forgiving, while the δ-function spectrum is not effective.

For practical accelerator operations, there may be information on the value of the half-width-at-half-height $\Delta\omega_{\frac{1}{2}}$, but not enough detailed information on the shape of the frequency spectrum. For those applications, we introduce a simplified stability criterion

$$|\xi_1| = \frac{Nr_0c}{2\omega_\beta\gamma T_0^2}\left|\sum_{p=-\infty}^{\infty} Z_1^{\perp}(p\omega_0+\omega_\beta)\right| < \frac{1}{\sqrt{3}}\Delta\omega_{\frac{1}{2}} \tag{18}$$

where the factor $1/\sqrt{3}$ is chosen so that it coincides with the semicircular portion of the boundary for the elliptical spectrum. Stability diagram of this simplified model is shown in figure (h) above.

Eq.(18) says that if the mode frequency shift or growth rate, calculated without Landau damping, is larger than the frequency spread of the beam, Landau damping will not rescue the beam from instability.

<u>One-particle model for bunched beams – longitudinal</u>

A similar analysis can also be performed for the longitudinal Robinson instability using a one-particle model,

$$\begin{aligned} z''(s) + \left(\frac{\omega_s}{c}\right)^2 z(s) &= \frac{Nr_0\eta}{\gamma C}\sum_{k=1}^{\infty}\left[\langle z\rangle(s) - \langle z\rangle(s-kC)\right]W_0''(-kC) \\ &= \frac{Nr_0\eta}{\gamma C}Be^{-i\Omega s/c}\mathcal{W} \end{aligned}$$

where we have introduced

$$\langle z\rangle(s) = Be^{-i\Omega s/c}$$

and

$$\begin{aligned}
\mathcal{W} &= \sum_{k=1}^{\infty} \left(1 - e^{i\omega_s k T_0}\right) W_0''(-kC) \\
&= \frac{i}{C} \sum_{p=-\infty}^{\infty} \left[p\omega_0 Z_0^{\parallel}(p\omega_0) - (p\omega_0 + \omega_s) Z_0^{\parallel}(p\omega_0 + \omega_s) \right]
\end{aligned}$$

Self-consistency then gives rise to a dispersion relation

$$\frac{N r_0 \eta \mathcal{W} c^2}{2\omega_s \gamma C \Delta\omega} = \frac{1}{f(u) + i g(u)}$$

similar to the transverse case except that the frequency spectrum now refers to synchrotron frequency, and the complex mode frequency shift in the absence of Landau damping is

$$\xi_1 = -\frac{N r_0 \eta \mathcal{W} c^2}{2\omega_s \gamma C}$$

The simplified stability criterion reads

$$|\xi_1| = \frac{N r_0 \eta c^2}{2\omega_s \gamma C^2} \left| \sum_{p=-\infty}^{\infty} \left[p\omega_0 Z_0^{\parallel}(p\omega_0) - (p\omega_0 + \omega_s) Z_0^{\parallel}(p\omega_0 + \omega_s) \right] \right| < \frac{1}{\sqrt{3}} \Delta\omega_{\frac{1}{2}}$$

The conclusion that the longitudinal Landau damping behaves analogously to the transverse case, however, is valid only for *bunched* beams for which $\omega_s \neq 0$. The analyses depend on the assumption that the mode frequency shift $|\Omega|$ is small compared with the unperturbed natural frequency ω_β, ω_s. For unbunched beams, $\omega_s = 0$, the longitudinal analysis gives results very different from its transverse counterpart.