8. Beam Driven Systems

Patric Muggli

Max Planck Institute for Physics

Munich

muggli@mpp.mpg.de

https://www.mpp.mpg.de/~muggli

Menu:

- Emittance
- Relativistic Particles/bunches
- Driving wakefields
- Focusing (-/+)
- Emittance preservation
- Bunch length and wakefields
- Self-modulation instability

Reminder: we are speech-lazy …

Plasma with electron density n_e (and equal ion density n_i)

Plasma electron frequency: $\omega_{pe} = \left(\frac{n_e e^2}{\varepsilon_0 m_e}\right)^{1/2} = 2\pi f_{pe}$

Plasma ion frequency: $\omega_{pi} = \left(\frac{n_i Z^2 e^2}{\varepsilon_0 M_i}\right)^{1/2} << \omega_{pe}$ => ions don't move at the $1/f_{pe}$ time scale (warning: SMI, very strong fields, …)

Cold plasma collisionless skin depth: c/ω_{pe} results from electrons inertia …

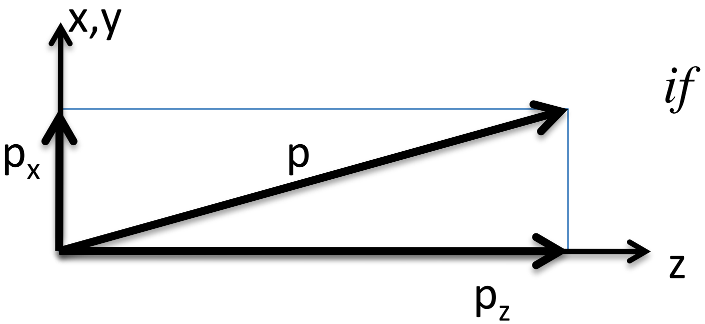
Use m for m_0 or m_e or m_{e0} or sometimes just for … m

Warning: references are … "selected" … not always THE reference …
Check references … inside my references …
Many more of my references at: https://www.mpp.mpg.de/~muggli/index.html

BEAM EMITTANCE

Assume there is no interaction between particles, emittance defined for each degree of freedom

Particle motion defined by (x, p_x) but (x, x') is more useful (remember optics: position and angle)



$$if\ B_z = 0 \quad \Rightarrow \quad \vec{A} = A_z \vec{z} \quad \Rightarrow \quad \left(\vec{p} - e\vec{A}\right)_x = p_x \underset{\uparrow}{\cong} \beta\gamma m x'$$

$$x' = \frac{p_x}{p_z} \cong \frac{p_x}{p} \qquad p = \beta\gamma m$$

$$\Rightarrow \quad x' = \frac{p_x}{\beta\gamma m} \propto \frac{p_x}{\gamma}$$

Emittance definition:

$$\varepsilon_{geo,x} = [\pi]\left(\langle x^2\rangle\langle x'^2\rangle - \underbrace{\langle xx'\rangle^2}_{\text{Correlation term}}\right)^{1/2} \qquad \langle\ldots\rangle \text{ average} \Rightarrow \langle x^2\rangle^{1/2} \text{ RMS size}$$

$$\Rightarrow \langle x'^2\rangle^{1/2} \text{ ~RMS transverse velocity or temp.}$$

$$\Rightarrow \varepsilon_{geo,x} = \sigma_x \sigma_{v_x} / c$$

- For a bi-Gaussian beam
- When the correlation term is 0, i.e., at a beam waist
- Emittance = area of the ellipse …
- Emittance is <u>preserved</u>, Liouville theorem in Hamiltonian mechanics
- Emittance? ⇔ how well can it be focused?

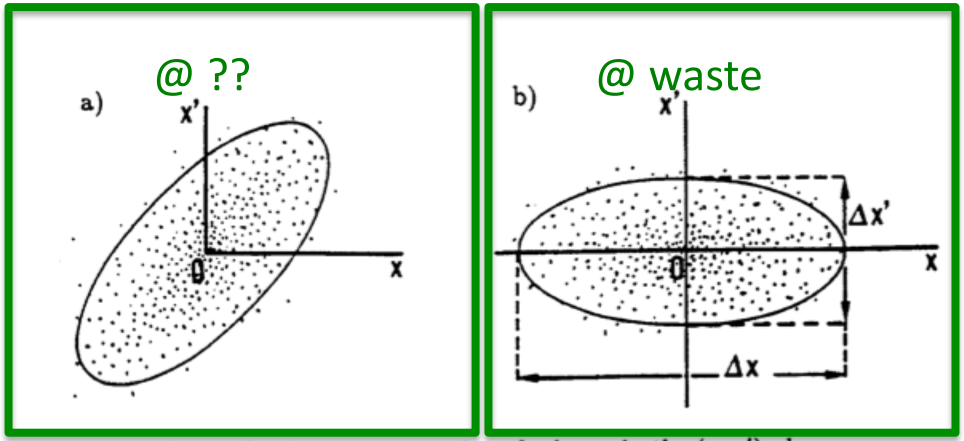


Fig. 1 : A set of points representative of a beam in the (x, x') phase space
a) Tilted emittance ellipse.
b) Upright emittance ellipse.

BEAM EMITTANCE

$$\varepsilon_{geo,x} = \left(\langle x^2\rangle\langle x'^2\rangle\right)^{1/2} \cong \sigma_x \frac{\sigma_{p_x}}{\beta\gamma mc} \propto \frac{1}{\gamma}$$ (at a waist)

=> The geometric emittance decreases upon acceleration …

Define a quantity that is preserved upon acceleration (no other effects):

Normalized emittance: $\varepsilon_{N,x} = \gamma\varepsilon_{geo,x}$

$\Rightarrow$A higher energy accelerator (preserving normalized emittance) produces lower geometric emittance beams that can be focused to smaller transverse size …

$\Rightarrow$Contributes to compensate for lower collision cross sections at higher energies …

$\Rightarrow$Emittance? ⇔ how well can it be focused?

CHARGES & BEAM

Characteristics of relativistic ($\gamma=E_{kin}/m_0c^2-1\sim E_{kin}/m_0c^2>>1$), charged particles (and bunches)

- Move essentially at the speed of light: $v=(1-1/\gamma^2)^{1/2}c\sim c$,
- Are not affected by propagation in plasma, there is scattering, but no index of refraction effect => v indep. of n_e! Important for dephasing …
- Have a large inertia ($\gamma m>>m$), transverse motion scales with: $k_\beta=k_{pe}/\gamma$ => $\lambda_\beta=\gamma\lambda_{pe}>>\lambda_{pe}$ in plasma
- Have essentially transverse fields: $E_r=\gamma E_0$, $E_z=E_0$, $E_z<<E_r$

Characteristics of relativistic, charged particle bunches

- Move essentially at the speed of light
- Have a large inertia ($\gamma m>>m$ for $\gamma>>1$)
- Are not affected by propagation in plasma, but by wakefields, collective-collective response
- Have essentially transverse fields: E_r, E_z, $E_z<<E_r$
- Can have long beta-function or equivalent of Rayleigh length

$$Z_R = \pi \frac{w_0^2}{\lambda_0} \quad \Leftrightarrow \quad \beta^* = \beta_0 = \frac{\sigma_0^{*2}}{\varepsilon_g} \qquad \varepsilon_g = \frac{\varepsilon_N}{\gamma}$$

$w_0=10\mu m$	$\varepsilon_N=5$mm-mrad
$\lambda_0=800nm$	$\gamma=40'000$ (20GeV e^-, SLAC)
$Z_R=393\mu m$	$\sigma_0=10\mu m$
	$\beta_0=0.8m$
Physics	Technology and physics

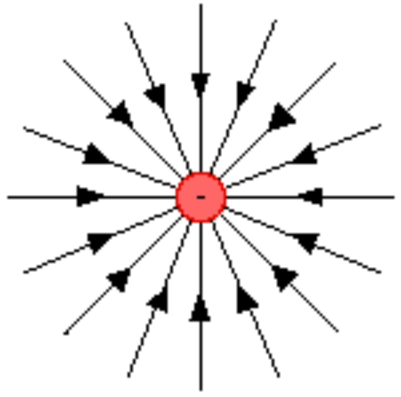
β^* or β_0 at a waist
But β in general $\beta=\beta$ (z)
Because $\sigma=\sigma$ (z), ε=cst

BEAM PROPAGATION

Influenced by the transverse fields $F_\perp = q\left(E_r + v_b \times B_\theta\right)$ from beam and plasma (wakefields)

Beam self-fields in vacuum:

In the e$^-$ rest frame: $\vec{E}_0 = \frac{1}{4\pi\varepsilon_0}\frac{-e}{r^2}\frac{\vec{r}}{|r|} = E_0\frac{\vec{r}}{|r|}$



The electric field from an isolated negative charge

http://physics.bu.edu/~duffy/PY106/Electricfield.html

In the lab frame (e$^-$ moves at v_b~c)

Joules-Bernoulli equation:

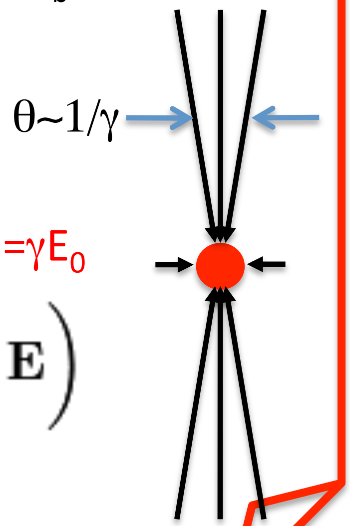
$\mathbf{E}_\parallel' = \mathbf{E}_\parallel$ $=E_0$

$\mathbf{B}_\parallel' = \mathbf{B}_\parallel$ $=0$

$\mathbf{E}_\perp' = \gamma\left(\mathbf{E}_\perp + \mathbf{v}\times\mathbf{B}\right)$ $=\gamma E_0$

$\mathbf{B}_\perp' = \gamma\left(\mathbf{B}_\perp - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}\right)$

$=(v/c^2)E_0$



=> $\theta=E_x/E_z=1/\gamma$

$F_\perp' = q\left(E_\perp' - v_{//}\times B_\perp'\right) = -e\gamma E_r'\left(1 - v_{//}\frac{v_{//}}{c^2}\right) = -eE_r'\frac{1}{\gamma^2}$ (jump from single particle to beam!)

The (total) self-force is $1/\gamma^2$ the "space charge force" (~E_r'), cancels to $1/\gamma^2$

1. In vacuum the relativistic beam diverges because of its divergence, not space charge forces
2. There is (~)no space charge effect when the particles are relativistic
3. Bunch (~)particles do not interact with each other

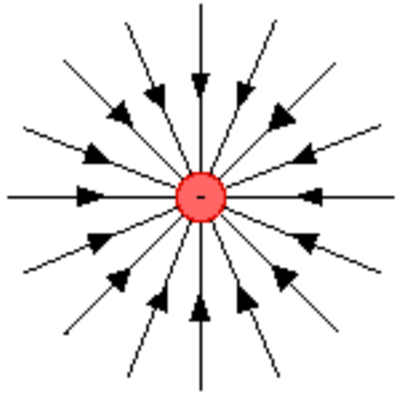
Q: compare the two for realistic beam density and emittance, e.g. SLAC bunch parameters

BEAM PROPAGATION

Influenced by the transverse fields $F_\perp = q\left(E_r + v_b \times B_\theta\right)$ from beam and plasma (wakefields)

Beam self-fields in vacuum:

In the e$^-$ rest frame: $\vec{E}_0 = \frac{1}{4\pi\varepsilon_0}\frac{-e}{r^2}\frac{\vec{r}}{|r|} = E_0\frac{\vec{r}}{|r|}$



The electric field from an isolated negative charge

http://physics.bu.edu/~duffy/PY106/Electricfield.html

In the lab frame (e$^-$ moves at v_b~c)

Joules-Bernoulli equation:

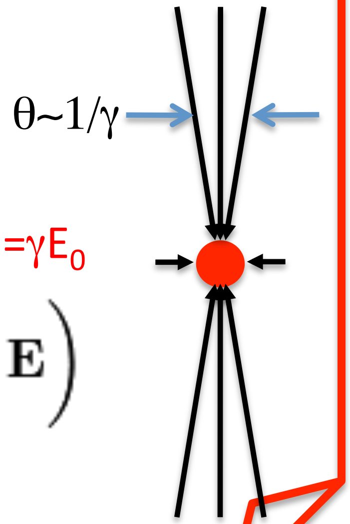
$\mathbf{E}_{\|}' = \mathbf{E}_{\|}$ =E_0

$\mathbf{B}_{\|}' = \mathbf{B}_{\|}$ =0

$\mathbf{E}_{\perp}' = \gamma\left(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}\right)$ =γE_0

$\mathbf{B}_{\perp}' = \gamma\left(\mathbf{B}_{\perp} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}\right)$

=$(v/c^2)E_0$



$$F_\perp' = q\left(E_\perp' - v_{//} \times B_\perp'\right) = -e\gamma E_r'\left(1 - v_{//}\frac{v_{//}}{c^2}\right) = -eE_r'\frac{1}{\gamma^2}$$

(jump from single particle to beam!)

The (total) self-force is $1/\gamma^2$ the “space charge force” (~E_r’), cancels to $1/\gamma^2$

1. In vacuum the relativistic beam diverges because of its divergence, not space charge forces
2. There is (~)no space charge effect when the particles are relativistic
3. Bunch (~)particles do not interact with each other

All this: for practical purposes only … (since only ~$1/\gamma^2$!!)

WAKEFILEDS EXCITATION

Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

Laser beams:

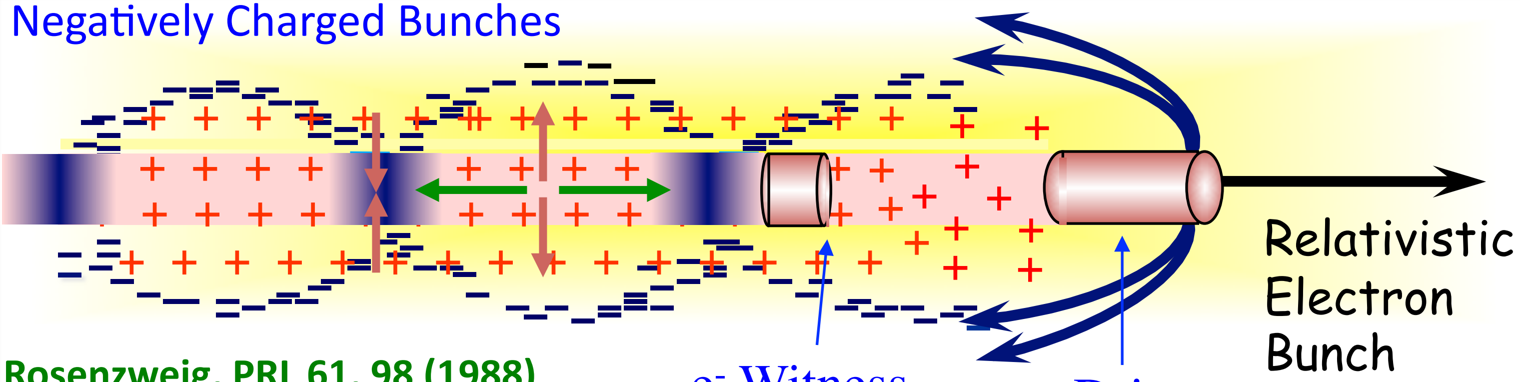
- Ponderomotive force $\propto \nabla E^2$ essentially 2D for real (bi-Gaussian) bunches
- Only one type (similar to e$^-$, i.e. blow-out)

Charged particle beams:

- Transverse space charge field
- Reverses sign for negatively (blow out) and positively (suck in) charged bunches

Charged particle beam-driven = plasma wakefield accelerator or PWFA **Chen, PRL 54, 693 (1985)**

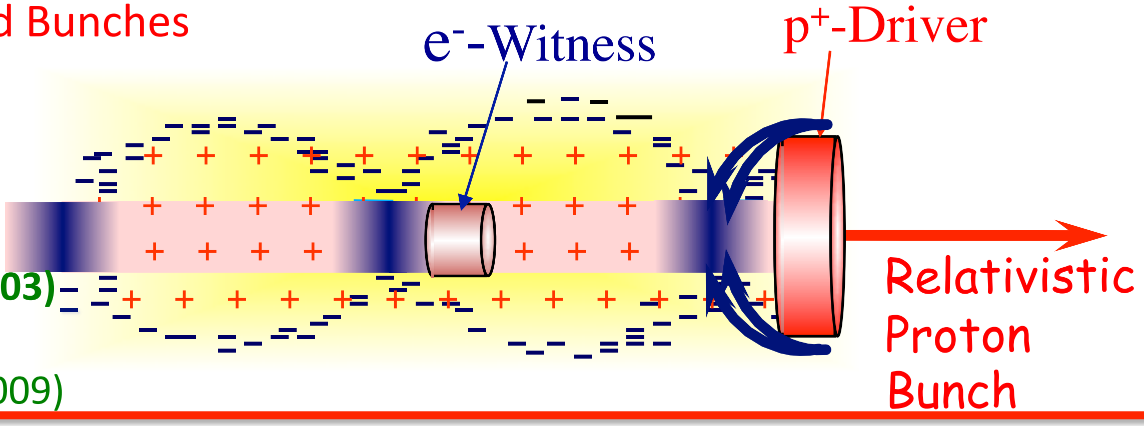
Negatively Charged Bunches



Pencil $(\sigma_r<<\sigma_z)$ or spherical $(\sigma_r\sim\sigma_z)$ beam

Rosenzweig, PRL 61, 98 (1988)
Lee, et al., PRE 61, 7014 (2000)

Positively Charged Bunches



Pancake $(\sigma_r>>\sigma_z)$ beam

Blue
PRL, 90, 214801 (2003)
Caldwell
Nat. Phys. 5, 363 (2009)

MAX-PLANCK-GESELLSCHAFT

WAKEFILEDS EXCITATION

Laser beams:

- Ponderomotive force $\propto \nabla E^2$ essentially 2D for real (bi-Gaussian) bunches
- Only one type (similar to e$^-$, i.e. blow-out)

Charged particle beams:

- Transverse space charge field
- Reverses sign for negatively (blow out) and positively (suck in) charged bunches

Charged particle beam-driven = plasma wakefield accelerator or PWFA

Negatively Charged Bunches

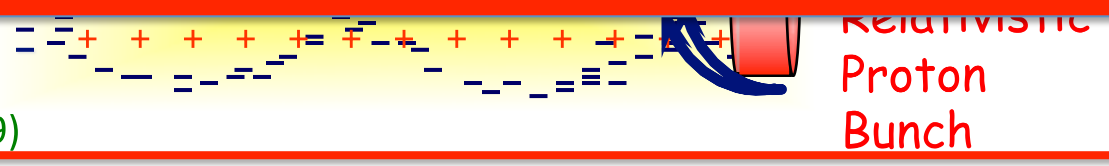
Rosenzweig, PR

Lee, et al., PRE 6

Pencil (σ_r<<σ_z) or spherical (σ_r~σ_z) beam

Plasma wakefields can be driven by
negatively (e$^-$, p$^-$, …)
and
positively (e$^+$, p$^+$, …)
charged bunches

Positively Cha

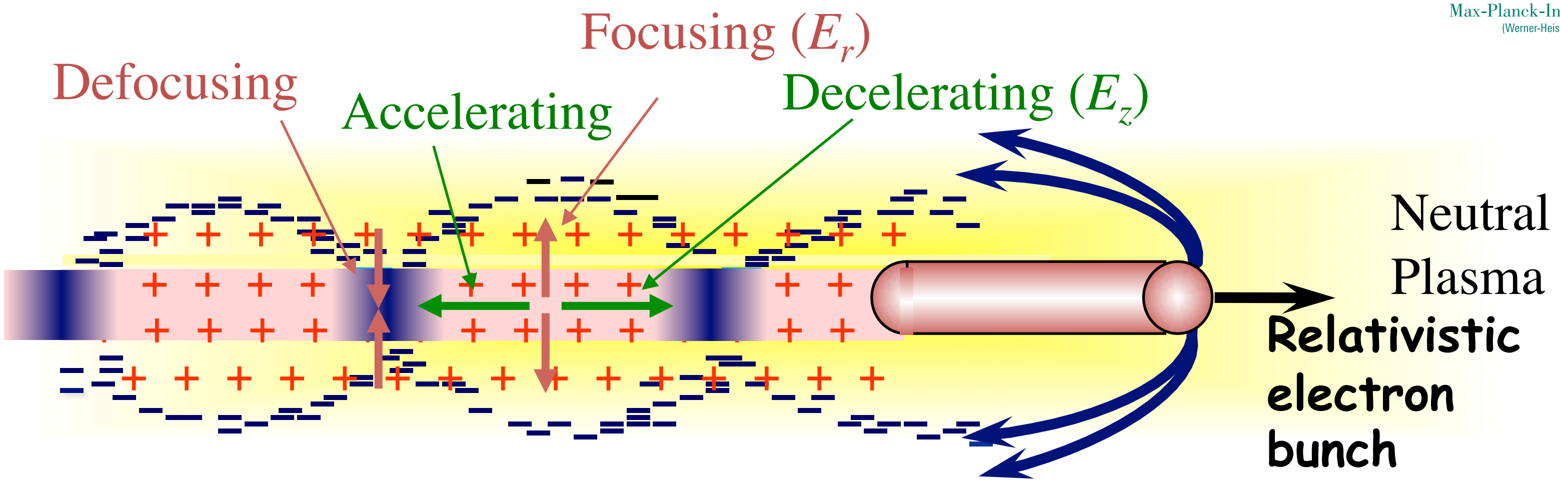


Blue
PRL, 90, 214801 (200
Caldwell
Nat. Phys. 5, 363 (2009)

Pancake (σ_r>>σ_z) beam

PLASMA WAKEFIELD ACCELERATOR (e^-)

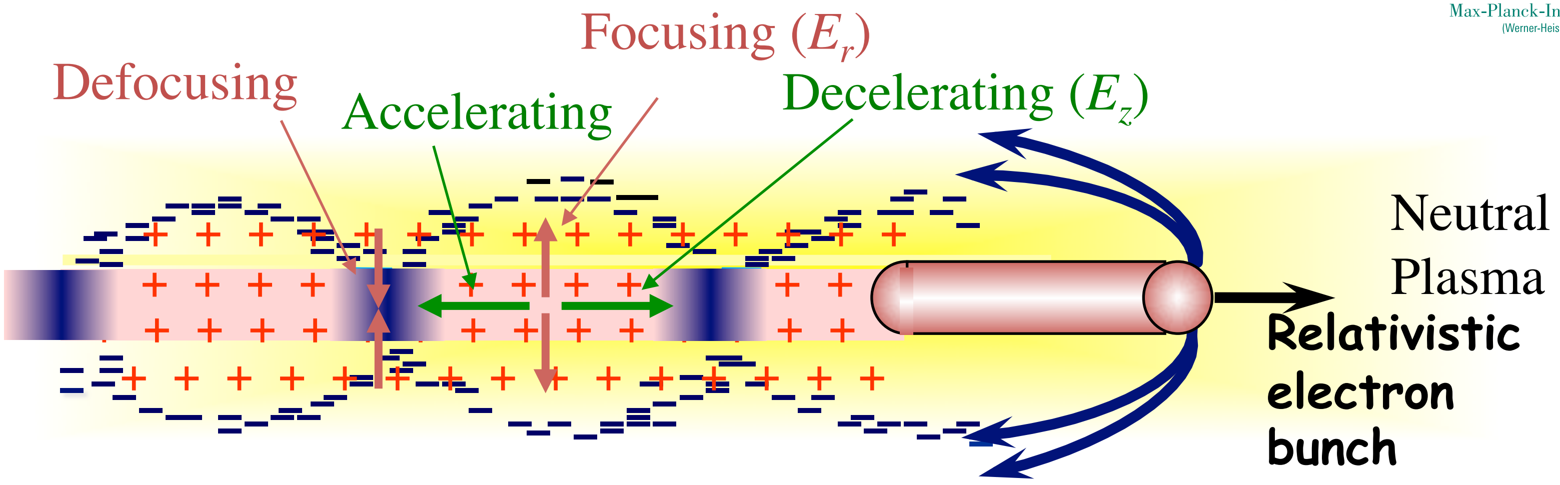
Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)



- Plasma wave/wake excited by a relativistic particle bunch
- Plasma e^- expelled by space charge force => deceleration + focusing (MT/m)
- Plasma e^- rush back on axis => acceleration, GV/m
- Ultra-relativistic driver => ultra-relativistic wake
  => no dephasing
- Particle bunches have long "Rayleigh length"
  (beta function $\beta^*=\sigma^{*2}/\varepsilon$~cm, m)
- Acceleration physics identical PWFA, LWFA

MAX-PLANCK-GESELLSCHAFT

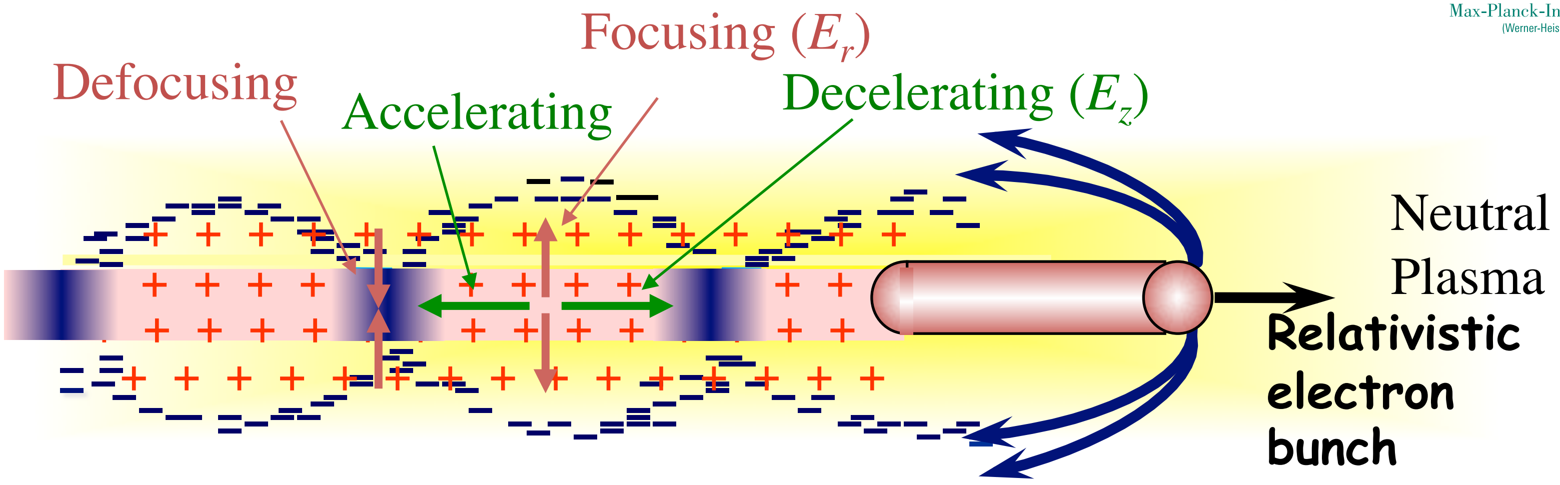
PLASMA WAKEFIELD ACCELERATOR (e^-)



- Plasma wave/wake excited by a relativistic particle bunch
- lled by space charge force => deceleration + focusing (MT/m)
- back on axis => acceleration, GV/m
- driver => ultra-relativistic wake
  no dephasing
- es have long Rayleigh lengths”
  $^*=\sigma^2/\varepsilon$~cm, m)
- Acceleration physics identical PWFA, LWFA

Very large energy gain possible with short, high-energy relativistic bunches!

PLASMA WAKEFIELD ACCELERATOR (e$^-$)



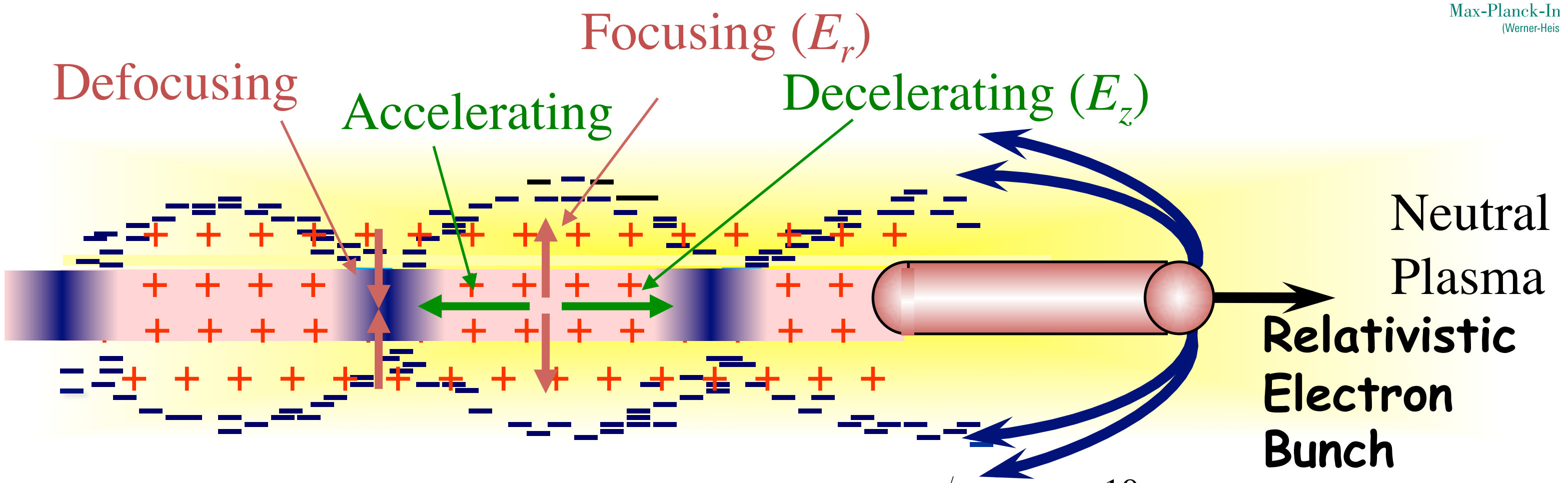
- Plasma wave/wake excited by a relativistic particle bunch
- …lled by space charge force => deceleration + focusing (MT/m)
- … back on axis => acceleration, GV/m
- …driver => ultra-relativistic wake

Very large energy gain possible with short, hig… rel… bu…

Calculate the dephasing length ΔL between two particles with relativistic factors γ and $\gamma+\Delta\gamma$ ($\Delta\gamma<<\gamma$)
Show it is: $\Delta L \sim (1/\gamma^2)(\Delta\gamma/\gamma)L$ (for $\Delta L<<L$)

- Acceleration physics identical PWFA, LWFA

PWFA NUMBERS (e⁻)



- ✧Linear theory (n_b<<n_e) scaling:

$$E_{acc} \cong 110(MV/m)\frac{N/2\times10^{10}}{(\sigma_z/0.6mm)^2} \approx N/\sigma_z^2$$

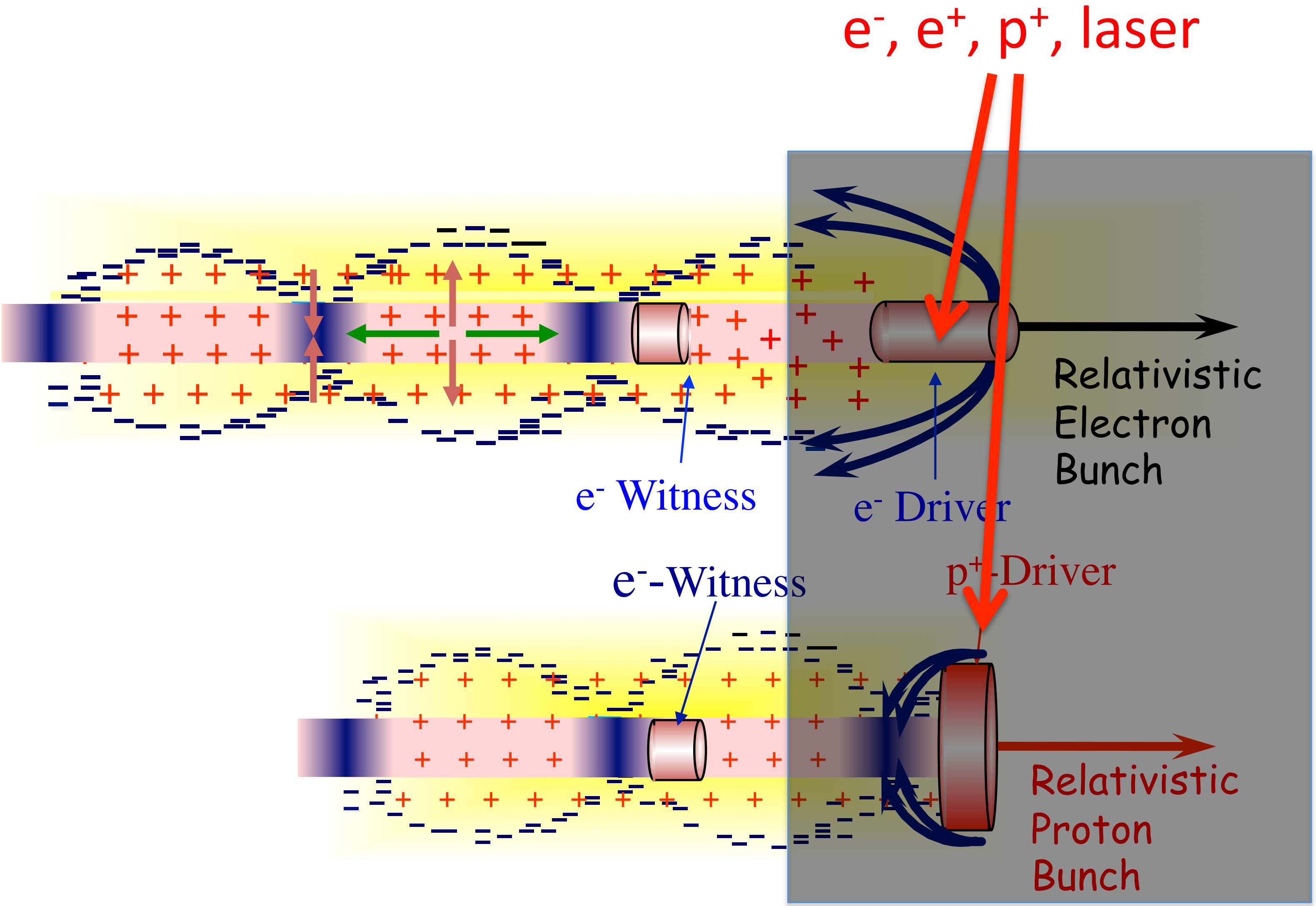
$$@\ k_{pe}\sigma_z \approx \sqrt{2}\ (with\ k_{pe}\sigma_r << 1)$$

Lee, et al., PRE 61, 7014 (2000)

- ✧Focusing strength: $\frac{B_\theta}{r} = \frac{1}{2}\frac{n_e e}{\varepsilon_0 c}$ (n_b>n_e)
- ✧N=2x10^{10}: σ_z=600 μm, n$_e$=2x10^{14} cm^{-3}, E$_{acc}$~100 MV/m, B_θ/r=6 kT/m
  σ_z= 20 μm, n$_e$=2x10^{17} cm^{-3}, E$_{acc}$~ 10 GV/m, B_θ/r=6 MT/m
- ✧Frequency: 100GHz to >1THz, “structure” size 1mm to 100μm
- ✧Conventional accelerators: MHz-GHz, E$_{acc}$<150 MV/m, B_θ/r<2 kT/m

ACCELERATION PROCESS

Independent of driver nature for same bubble parameters



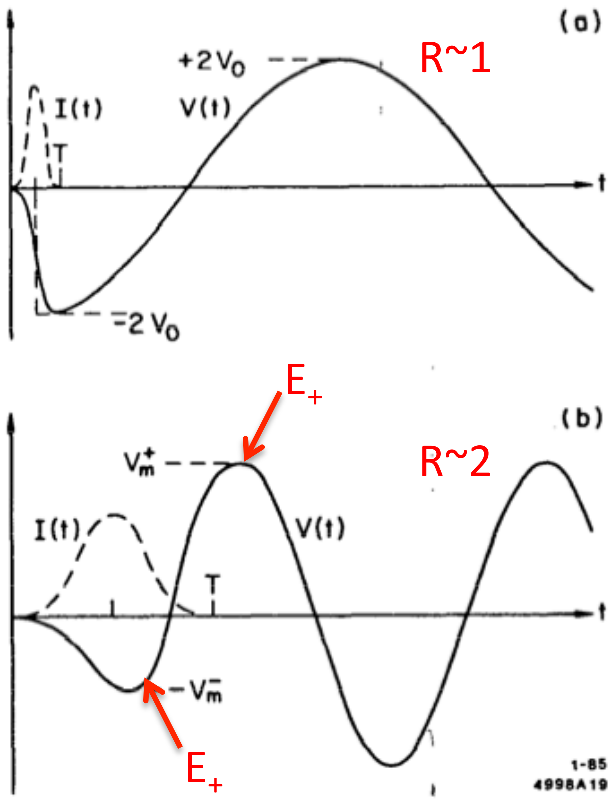
TRANSFORMER RATIO

Definition: Transformer ratio (R)

:=Peak accelerating field behind drive bunch(es) / peak decelerating field within drive bunch(es)

:=E_+/E_-

minimum



(a) Bunch too short!

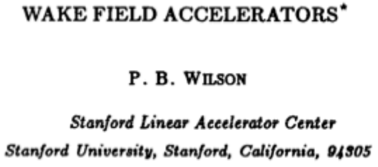
Maximum field after the bunch

For a single symmetric (in time) bunch: R≤2

(b) Bunch just right!

Maximum field inside the bunch

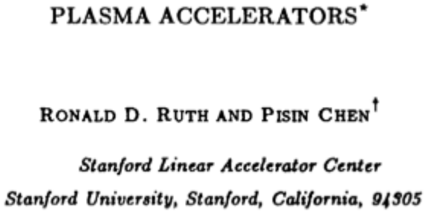
Figure 7. Potential in and behind a charge distribution interacting with a si
mode for (a) a short bunch, and (b) a long bunch.



WAKE FIELD ACCELERATORS*

P. B. WILSON

Stanford Linear Accelerator Center
Stanford University, Stanford, California, 94305

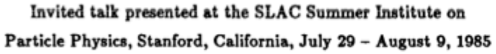


PLASMA ACCELERATORS*

RONALD D. RUTH AND PISIN CHEN†

Stanford Linear Accelerator Center
Stanford University, Stanford, California, 94305

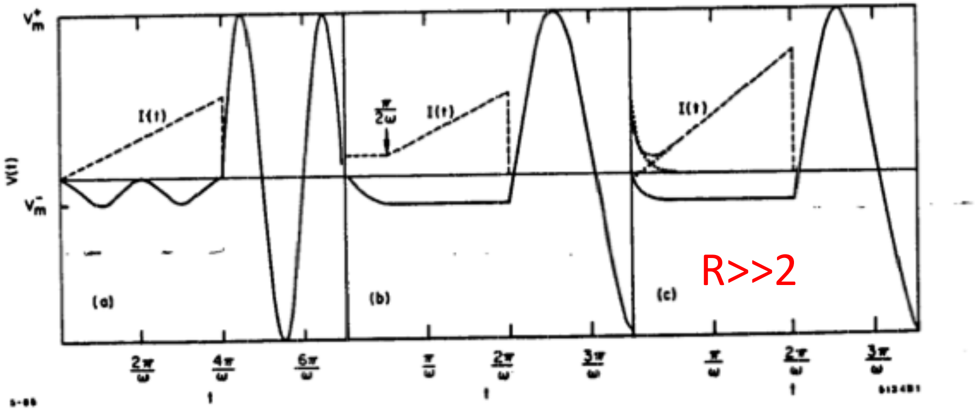
SLAC-R-296



Invited talk presented at the SLAC Summer Institute on
Particle Physics, Stanford, California, July 29 – August 9, 1985

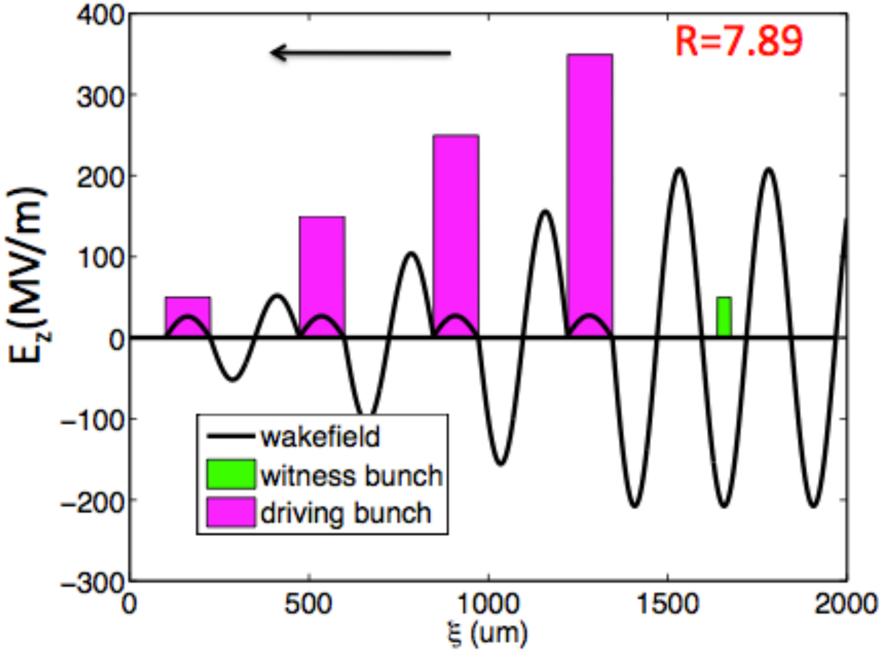
TRANSFORMER RATIO

Adjusting the current profile or with multiple bunches R>2 is possible:



R>>2

Figure 9. The voltage induced by three different asymmetric current distributions interacting with a single mode.



Conservation of energy => W=Q*U W~Q*E or

=> $\Delta W_W=Q_wE_+L_p \leq \Delta W_D=Q_DE_-L_p$

=> $Q_w \leq Q_D/R$

R applies also to nonlinear wakes and has important implications on energy transfer efficiency

Tzoufras, PRL 101, 145002 (2008).

When Q_w~Q_D beam loading (wakefield of W ~ wakefield of D), i.e., to the addition of the bunches wakefields

PLASMA FOCUSING

For large transverse size beams ($k_{pe}\sigma_r >> 1$) with low density ($n_b << n_{e0}$)

The plasma (tries to) neutralizes the bunch charge and current … (lecture 1)

The plasma return current flows through the bunch and is not relativistic

$$j_b = qn_b v_b = en_{e0} v_p = j_p \qquad n_b / n_{e0} << 1 \quad \Rightarrow \quad v_p = \left(n_b / n_{e0}\right) v_b << c$$

Linear regime: beam space-charge field exactly compensated by plasma e^- displacement (charge neutralization at scales $>c/\omega_{pe}$)

$$F_{\perp tot} = q\left(\underbrace{E_{rb} + v_b \times B_{\theta b}}_{\text{Cancel to } 1/\gamma^2} + E_{rp} + v_b \times B_{\theta p}\right)$$

Focusing term when return current flows through the bunch

Charge-neutralized when $n_b << n_{e0}$

Danger:
plasma responds at c/ω_{pe}-scale also in the transverse dimension
Beam and plasma return current are opposite currents => repel each other

Current
Filamentation
Instability
CFI
(transverse)

CFI => $k_{pe}\sigma_r < 1$ otherwise, transverse filamentation at the c/ω_{pe} scale

Allen, PRL 109, 185007 (2012)

PLASMA FOCUSING

$$F_{\perp tot} = q\left(\underbrace{E_{rb} + v_b \times B_{\theta b}}_{\text{Cancel to } 1/\gamma^2} + E_{rp} + v_b \times B_{\theta p}\right)$$

Focusing term when return current flows through the bunch

Charge-neutralized when $n_b << n_{e0}$

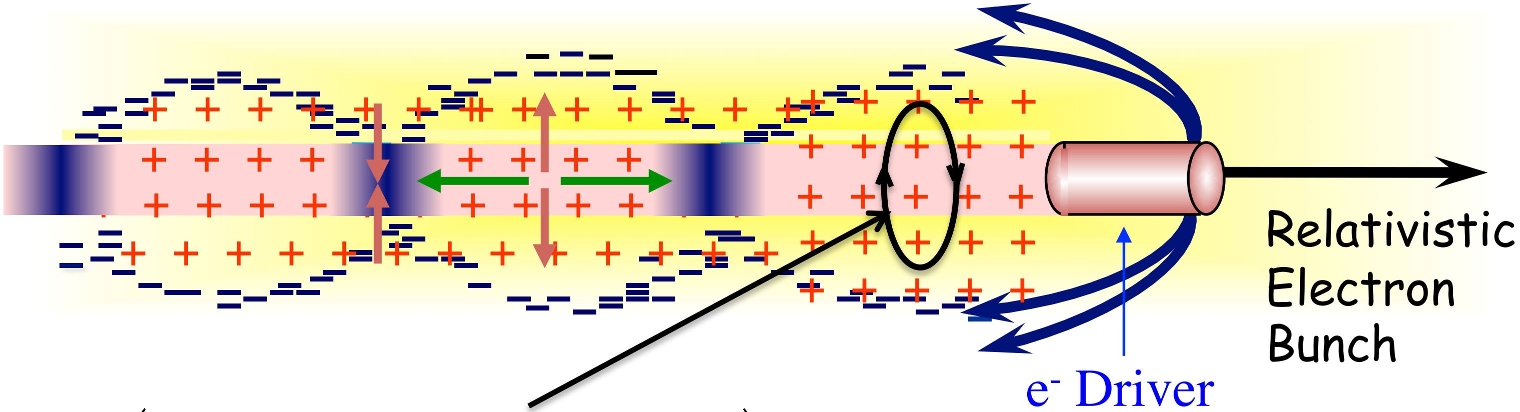
Linear regime ⇔ $n_b << n_{e0}$

- Focusing/defocusing in the linear regime
- Fields vary in r and x => no matching condition, no emittance preservation
- Symmetric for – and + charges

Non linear regime, "blow-out", bubble regime (LWFA) ⇔ $n_b >> n_{e0}$

- Pure ion column
- $E_r \sim r$, E_r indep of x => can preserve incoming emittance of e^- witness/accelerated bunch
- There is a matching condition

PURE (uniform) ION COLUMN FOCUSING



$$F_{\perp tot} = q\big(\underbrace{E_{rb} + v_b \times B_{\theta b}}_{\text{Cancel to }1/\gamma^2} + E_{rp} + v_b \times B_{\theta p}\big)$$

=0, no current inside the bunch

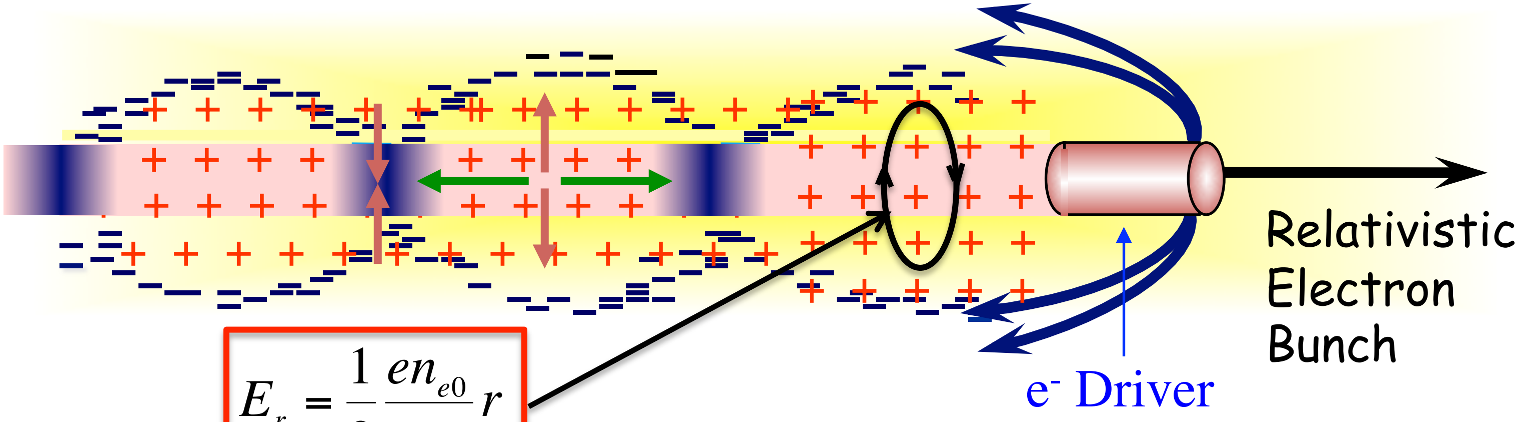
Gauss law for infinite cylinder (approximation)

n_i is uniform and $n_i=n_{e0}$

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0} \quad \Rightarrow \quad 2\pi r dz E_r = \frac{\pi r^2 e n_i}{\varepsilon_0} \quad \Rightarrow \quad E_r = \frac{1}{2}\frac{e n_{e0}}{\varepsilon_0} r$$

The focusing field varies linearly with radius => focusing free of geometric aberrations
=> can preserve incoming emittance

PURE (uniform) ION COLUMN FOCUSING



$$\boxed{E_r = \frac{1}{2}\frac{en_{e0}}{\varepsilon_0}r}$$

n_i is uniform and $n_{i0}=n_{e0}$

Motion of a particle in the ion column:

$$\gamma m\frac{dv_\perp}{dt} = F_\perp \quad \Rightarrow \quad \gamma mc^2\frac{d^2r}{dz^2} = e\frac{1}{2}\frac{en_{e0}}{\varepsilon_0}r \quad \Rightarrow \quad \frac{d^2r}{dz^2} = \frac{1}{2\gamma c^2}\frac{e^2n_{e0}}{m\varepsilon_0}r = \frac{\omega_{pe}^2}{2\gamma c^2}r = \frac{k_{pe}^2}{2\gamma}r = k_\beta^2 r$$

Harmonic motion (no energy gain/loss)

$$\frac{d^2r}{dz^2} = k_\beta^2 r \quad \Rightarrow \quad r(z) = r_0 e^{ik_\beta z}$$ => emission of betatron radiation (synchrotron)

$$\omega_\beta = \omega_{pe}/\sqrt{2\gamma}$$ => Relativistic (bunch) e$^-$ radiates at high frequencies

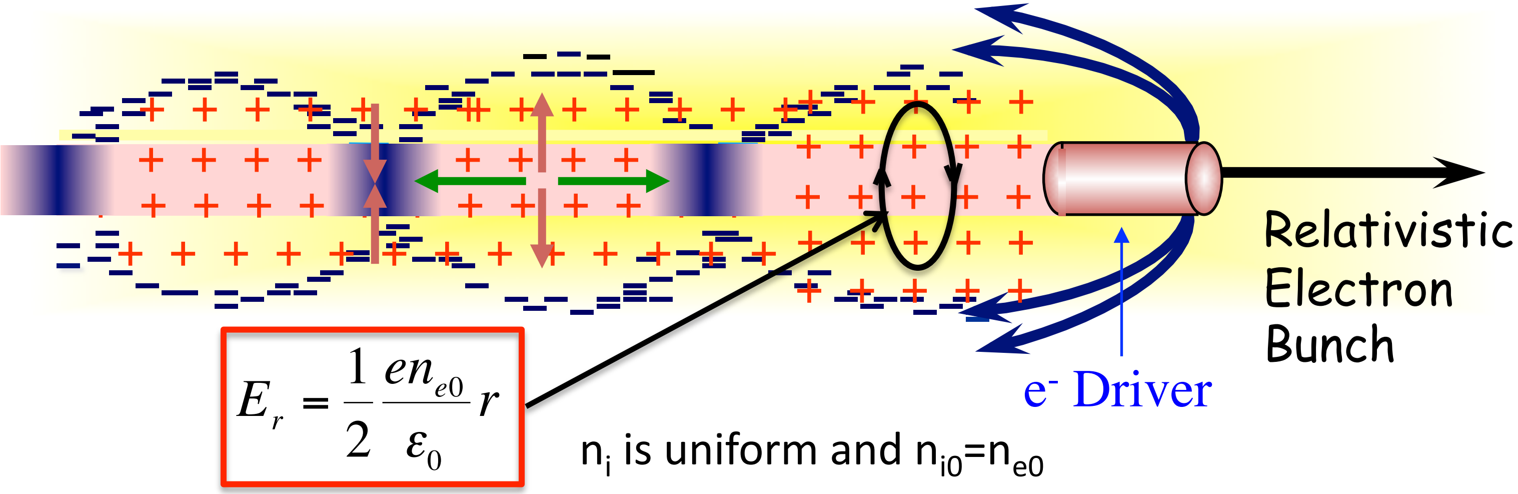
See K. Ta Phuoc on Friday

Examples: SLAC E_{kin}=28.56GeV => γ~56'000

n_e~$2x10^{14}cm^{-3}$ => KeV photons Wang, PRL 88, 135004 (2002)

n_e~$2x10^{17}cm^{-3}$ => MeV photons Johnson, PRL 97, 175003 (2006)

PURE (uniform) ION COLUMN FOCUSING



$$E_r = \frac{1}{2}\frac{en_{e0}}{\varepsilon_0}r$$

n_i is uniform and $n_{i0}=n_{e0}$

$$\frac{d^2r}{dz^2} = k_\beta^2 r \quad \Rightarrow \quad r(z) = r_0 e^{ik_\beta z}$$

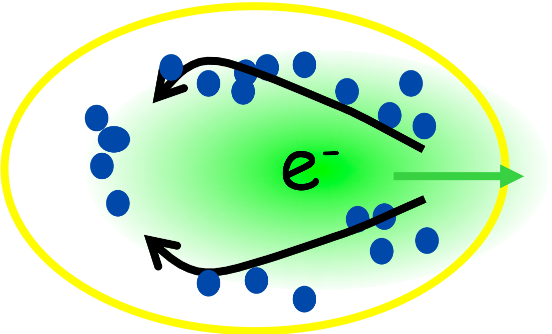
=> Particles oscillate at $\omega_\beta = \omega_{pe}/\sqrt{2\gamma} << \omega_{pe}$

=> $\lambda_\beta = \sqrt{2\gamma}\lambda_{pe} >> \lambda_{pe}$ $\Rightarrow$ the beam transverse size (envelope) over a length/time scale much longer than the plasma

$\Rightarrow$ Quasi-static approximation used in computer codes …
(with loss of some physics, must decide … and remember …)

e^- & e^+ BEAM NEUTRALIZATION

3-D QuickPIC simulations, plasma e⁻ density:

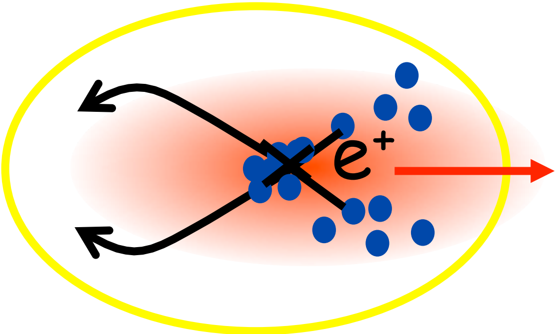


σ_r=35 μm

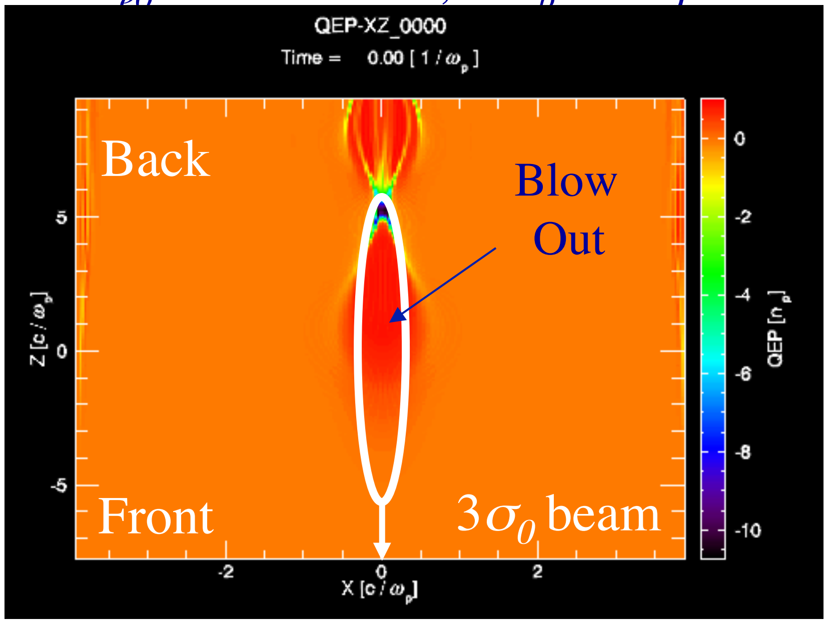
σ_r=700 μm

N=1.8×10^{10}

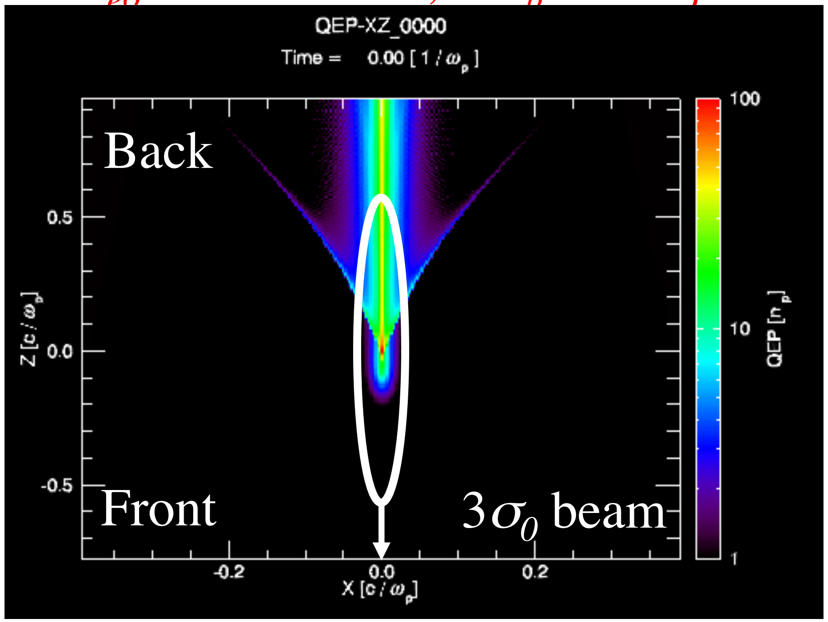
d=2 mm



e⁻: n_{e0}=2×10^{14} cm^{-3}, c/ω_p=375 μm



e⁺: n_{e0}=2×10^{12} cm^{-3}, c/ω_p=3750 μm

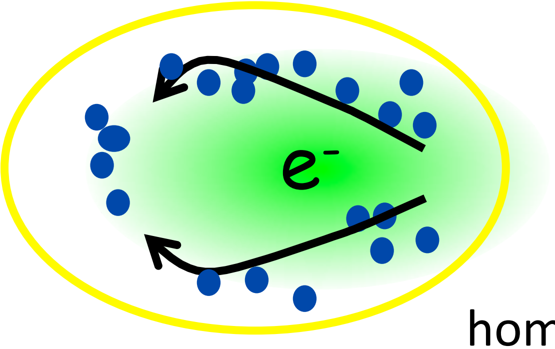


- Uniform focusing force (r,z)

- Non-uniform focusing force (r,z)

Patric Muggli, HEEAUP05, 06/08/05

WAKEFIELD FIELDS for e^- & e^+



σ_r=35 μm

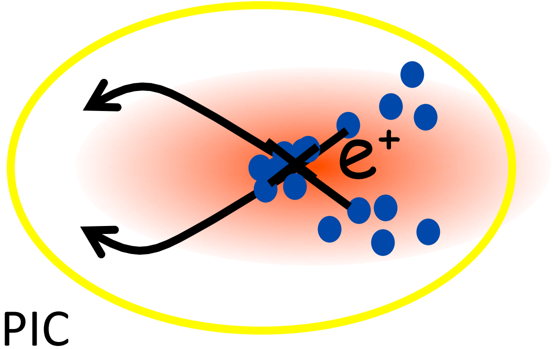
σ_r=700 μm

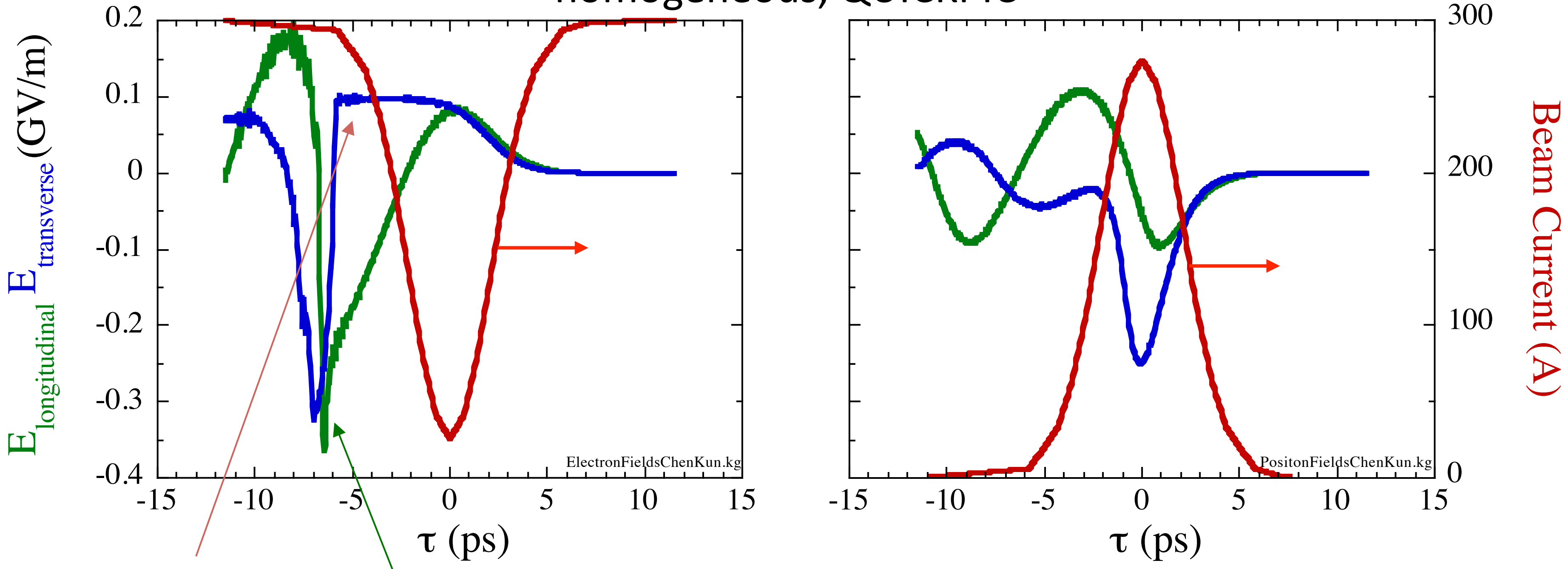
N=1.8×10^{10}

d=2 mm

n_e=1.5×10^{14} cm^{-3}

homogeneous, QUICKPIC

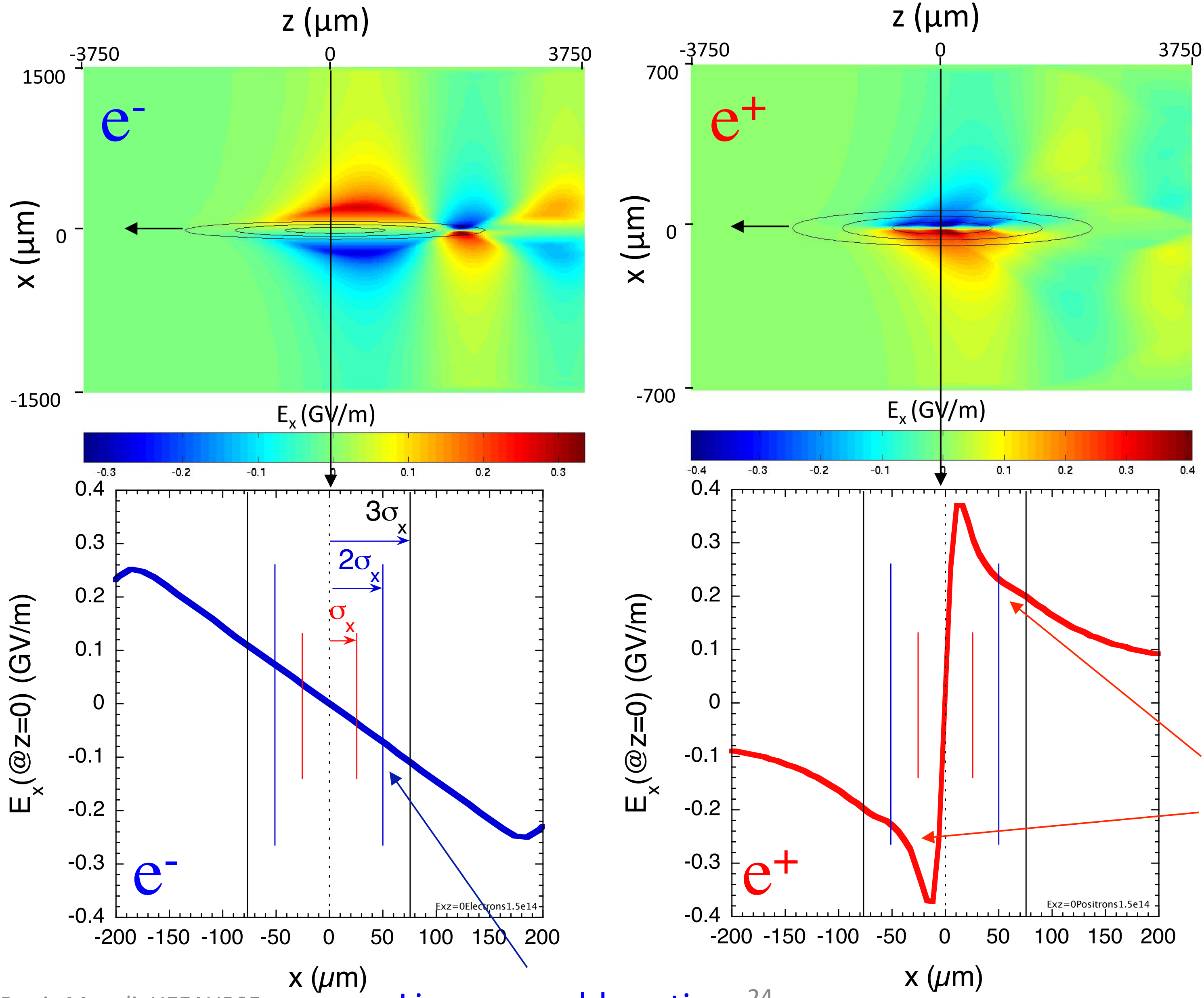




- Blow-Out
- Accelerating "Spike"

- Fields vary along r, stronger
- Less Acceleration, "linear-like"

e^- & e^+ FOCUSING FIELDS*



Non-linear, abberations

Linear, no abberations

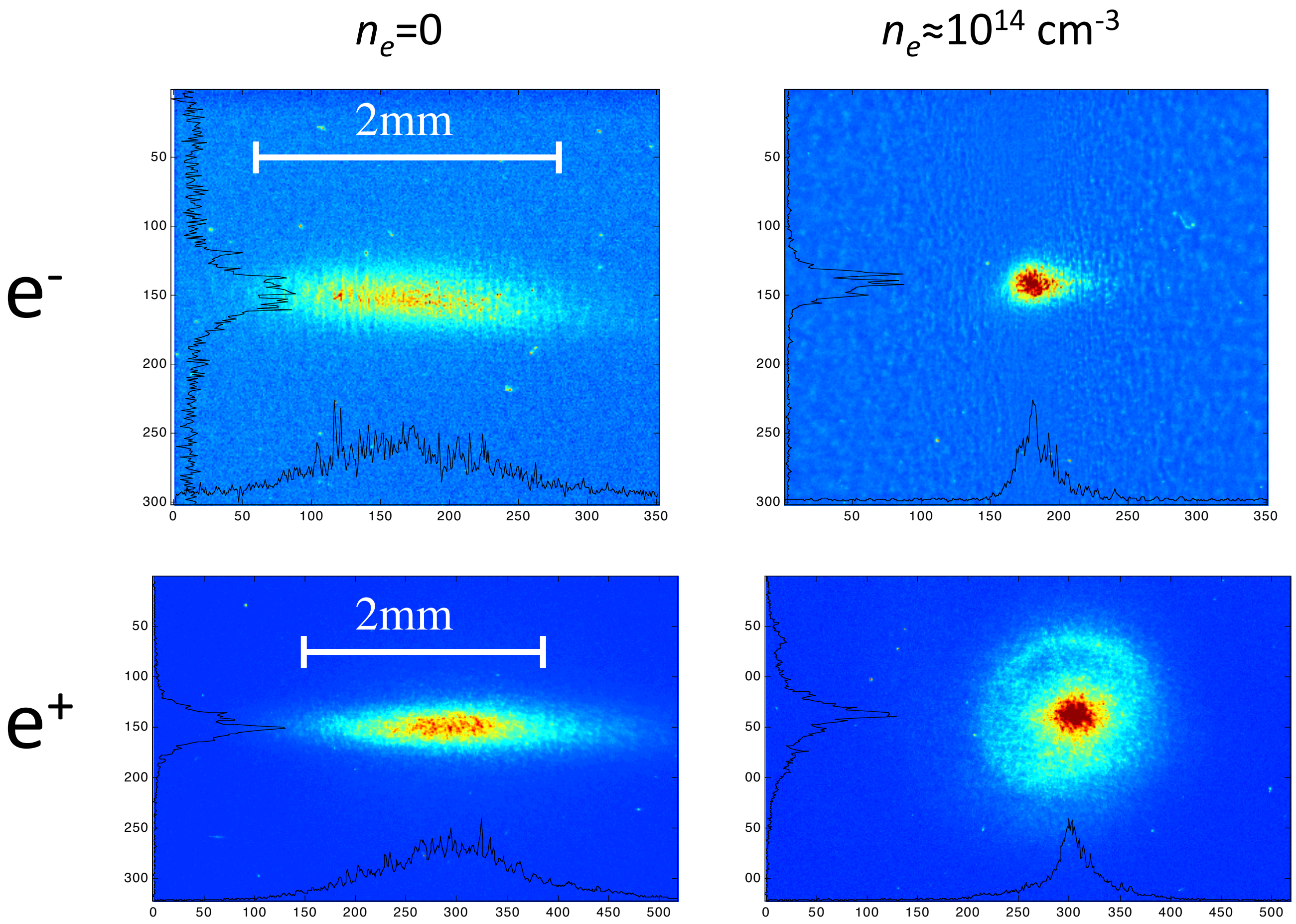
$\sigma_{x0}=\sigma_{y0}=25\ \mu m$
$\sigma_z=730\ \mu m$
$N=1.9\times10^{10}\ e^+/e^-$
$n_e=1.5\times10^{14}\ cm^{-3}$

*QuickPIC

Patric Muggli, HEEAUP05, 06/08/05

FOCUSING OF e^-/e^+

- OTR images ≈1m from plasma exit ($\varepsilon_x \neq \varepsilon_y$)



- Ideal Plasma Lens in Blow-Out Regime

- Plasma Lens with Aberrations
- Halo formation

➡ Qualitative differences

Muggli, PRL 101, 055001 (2008)

Beam Equilibrium Distribution Function in a Focusing Channel

Consider the case of a linear focusing force acting on a beam of charged particles. The 2-D (1-D in space) distribution function *f(x,v,t)* for the beam particles satisfies Vlasov equation (non-relativistic case):

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}+\frac{F}{m}\frac{\partial f}{\partial v}=\left.\frac{\partial f}{\partial t}\right|_{coll.} \qquad (1)$$

Assuming that there are no collisions and looking for a stationary solution, i.e., looking for the equilibrium distribution function *f(x,v)*:

$$v\frac{\partial f}{\partial x}+\frac{F}{m}\frac{\partial f}{\partial v}=0 \qquad (2)$$

A solution can be found for the case of a separable distribution function *f(x,v)=X(x)V(v)*. In this case, the previous equation can be re-written:

$$vV\frac{\partial X}{\partial x}+\frac{F}{m}X\frac{\partial V}{\partial v}=0 \qquad (3)$$

Writing the term *F/m* as *g(x)*, i.e., assuming that the focusing force is a function of *x* only, and dividing by *vVX*:

$$\frac{1}{gX}\frac{\partial X}{\partial x}+\frac{1}{vV}\frac{\partial V}{\partial v}=0 \qquad (4)$$

which can be separated and set equal to a constant $-\alpha^2$, since the two sides are functions of independent variables:

$$\frac{1}{gX}\frac{\partial X}{\partial x}=-\frac{1}{vV}\frac{\partial V}{\partial v}=-\alpha^2 \qquad (5)$$

The spatial part of the equation leads to:

$$\frac{\partial X}{\partial x} = -\alpha^2 g X \quad \Rightarrow \quad X(x) = X_0 e^{-\alpha^2 \int_0^x g(x')dx} \tag{6}$$

The velocity part of the equation leads to:

$$\frac{\partial V}{\partial v} = \alpha^2 v V \quad \Rightarrow \quad V(v) = V_0 e^{\alpha^2 v^2/2} \tag{7}$$

In the case of an ion column with an ion density equal to the electron plasma density n_i=n_e, the electric field is given by:

$$E_x = \frac{1}{2}\frac{n_e e}{\varepsilon_0} x \tag{8}$$

The force acting on electrons is:

$$F_x = -\frac{1}{2}\frac{n_e e^2}{\varepsilon_0} x \tag{9}$$

The term *g(x)* can therefore be written as:

$$g(x) = -\frac{1}{2}\frac{n_e e^2}{m\varepsilon_0} x = \beta x, \quad \beta = -\frac{1}{2}\frac{n_e e^2}{m\varepsilon_0} \tag{10}$$

The term *X(x)* is then given by:

$$\int_0^x g(x')dx = \frac{\beta x^2}{2} \quad \Rightarrow \quad X(x) = X_0 e^{-\frac{\alpha^2 \beta x^2}{2}} \tag{11}$$

The beam transverse emittance ε is:

$$\varepsilon^2 = \sigma_x^2\theta^2 \cong \sigma_x^2 \frac{\sigma_v^2}{c^2} \tag{12}$$

If *X(x)* has a Gaussian dependency, then its rms width is:

$$\sigma_x^2 = \frac{1}{\alpha^2\beta} \tag{13}$$

(Note that since $\beta<0$, one must also have $a^2<0$). Therefore,

$$\sigma_v^2 = \frac{\varepsilon^2 c^2}{\sigma_x^2} = \varepsilon^2 c^2 \alpha^2 \beta \tag{14}$$

Replacing β, and putting the beam parameters on the lhs and the constants on the rhs one obtains:

$$\frac{\sigma_x^4 n_e}{\varepsilon^2} = \frac{2\varepsilon_0 mc^2}{e^2} \tag{15}$$

Adding the fact that the beam is relativistic, one can replace *m* by *γm*, and obtain the matching condition for a Gaussian beam in a linear (ideal ion column) focusing system:

$$\frac{\sigma_x^4 n_e}{\gamma\varepsilon^2} = \frac{2\varepsilon_0 mc^2}{e^2} \tag{16}$$

which can be obtained, for example, from the beam envelope equation. Introducing the factor γ is an approximation that is valid only if one assumes that the perpendicular energy of the particles oscillating in the focusing channel is small compared to their total (or longitudinal) energy. Otherwise, even Vlasov equation must be modified (keep the F/m term inside the $\partial/\partial v$).

EMITTANCE PRESERVATION

An optical system with focusing forces varying linearly with radius (magnet, plasma ion column, …) preserves the emittance of the incoming beam, with bi-Gaussian distributions, when matched …

There is a matching condition, for example for the beam to the pure ion column of a PWFA (or LWFA)

This was shown in the simple case …

But these statements are general …

Let's find an equation for the beam propagation, i.e., for $\sigma_{x,y}(r)$

Associate the velocity distribution of the bunch particles to a transverse temperature
(a bit strange since we determined that relativistic particles do not interact …)

$$\varepsilon^2 = \sigma^2 \frac{\sigma_v^2}{c^2} = \sigma^2 \frac{k_B T_e}{\gamma m c^2}$$

Same distribution and rms, but usually, temperature associated with collisions
The initial emission (cathode) may be dominated by thermal motion, cathode temperature

ENVELOPE EQUATION

Motion of a single particle in a radial focusing force (e. g., ion column)

$$\frac{d^2r}{dz^2} = k_\beta^2 r \quad \Rightarrow \quad r(z) = r_0 e^{ik_\beta z}$$ => emission of betatron radiation (synchrotron)

Beam/bunch described by a distribution function f(r,v,t)

Choose a particular distribution function:

$$f(r,v,t=0) = f_0 e^{-r^2/2\sigma_r^2} e^{-v^2/2\sigma_v^2}$$

Characterized by rms widths σ_x, σ_x and corresponding emittance $\varepsilon_x=\sigma_x\sigma_v$ /c (at waist)

What is the evolution of the distribution?

Find an equation for the evolution of σ_x, assuming ε_x is conserved

Note: this assumes the distribution (e.g., Gaussian) is also preserved …

=> Envelope equation

$$\frac{d^2r}{dz^2} - k_\beta^2 r = 0 \quad or \quad \frac{d^2r}{dz^2} - \left(\frac{F_r}{r}\right) r = 0$$ and write equation for a fluid (i.e., distribution)

Envelope Equation Derivation

The equation for the evolution in space (along z) of the envelope size σ of a particle beam is derived from the equation of motion for a fluid of density n with a transverse temperature T_e. The fluid equation reads, neglecting the convective term $v\nabla v$:

$$n\gamma m\frac{\partial v}{\partial t} = nF_{foc} - \nabla P \qquad (1)$$

where $v=v(r)$ is the transverse oscillatory velocity, F_{foc} is a focusing force and P is the (transverse) pressure resulting from the (transverse) fluid temperature. The fluid relativistic in its longitudinal motion, but not in its transverse motion. This corresponds to assuming $v_x << v_z \approx c$. This also means that to first order the energy of the particles is constant along the motion, hence γ is outside of the derivative, and the mass can be replaced with γm. Assuming that the focusing force is independent of time, the time derivative can be replaced by a derivative along z ($\partial/\partial t = c\partial/\partial z$):

$$n\gamma mc\frac{\partial v}{\partial z} = nF_{foc} - \nabla P \qquad (2)$$

The fluid is assumed to follow an ideal gas law with a single, uniform temperature:

$$P = nk_BT_e \qquad (3)$$

The fore the pressure gradient term becomes:

$$-\nabla P = -k_BT_e\nabla n \qquad (4)$$

Considering an electron beam with a Gaussian transverse profile or density, the density gradient is:

$$n(r) = n_0e^{-r^2/2\sigma^2} \quad \rightarrow \quad -\nabla n = \frac{r}{\sigma^2}n \qquad (5)$$

Therefore Eq. 2 becomes:

$$\gamma mc\frac{\partial v}{\partial z} = F_{foc} + k_BT_e\frac{r}{\sigma^2} \qquad (6)$$

Assuming that the focusing force is linear with radius, it can be written as:

$$F_{foc} = -kr \qquad (7)$$

where k is a constant. The transverse velocity is also the derivative of the transverse position: $v=\partial r/\partial t=c\partial r/\partial z$, and Eq. 6 becomes:

$$\frac{\partial^2 r}{\partial z^2} = -\frac{k}{\gamma mc^2} r + \frac{k_B T_e}{\gamma mc^2}\frac{r}{\sigma^2} \qquad (8)$$

The transverse temperature and the transverse velocity are related through the velocity distribution. For a Maxwelian distribution with a temperature T_e this relation is:

$$\frac{1}{2}\gamma m\sigma_v^2 = \frac{1}{2}k_B T_e \qquad (9)$$

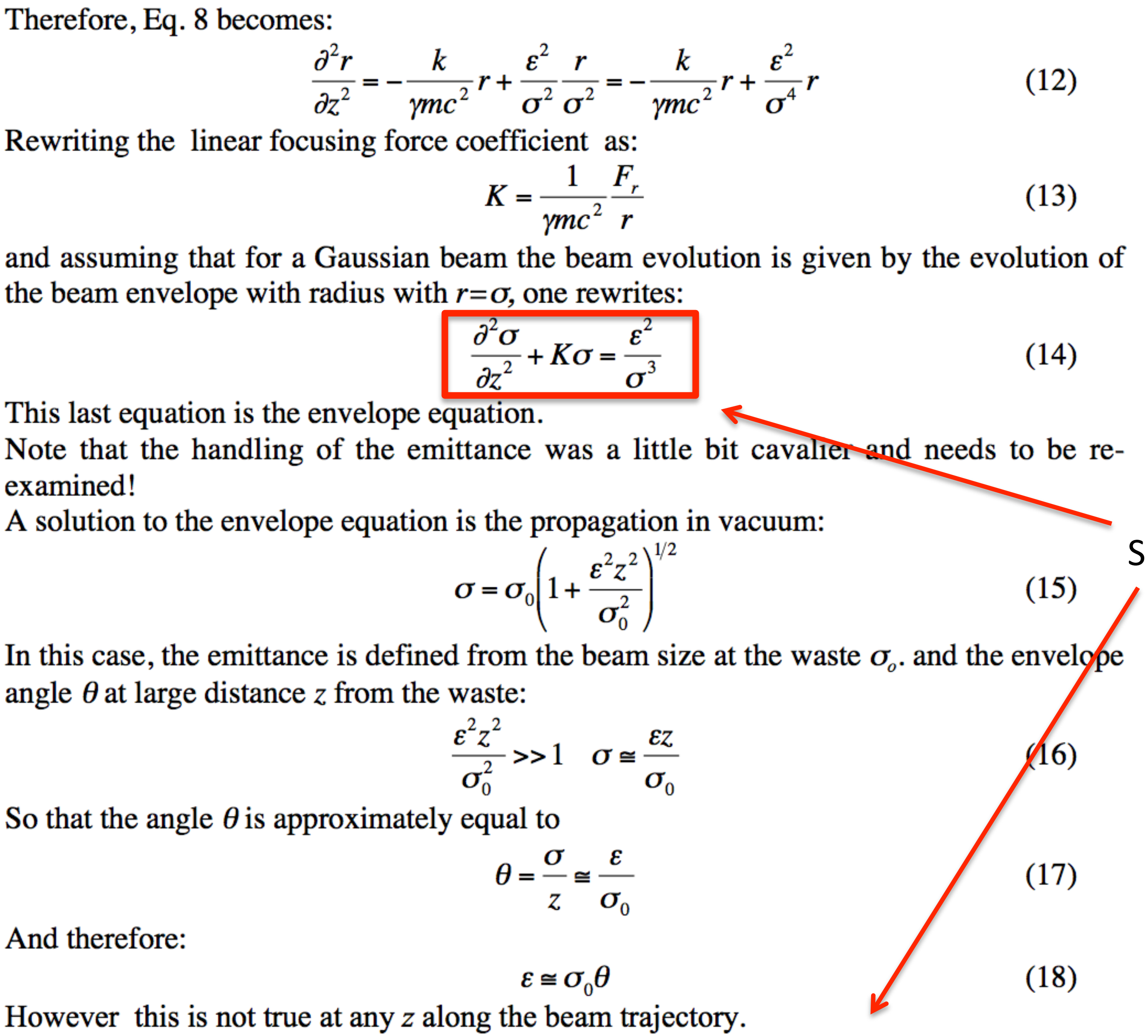
where σ_v^2 is the mean of the square of the thermal velocity.

The emittance of the beam can be defined as:

$$\varepsilon = \sigma\theta = \sigma\frac{v_r}{v_z} \cong \sigma\frac{\sigma_v}{c} \qquad (10)$$

or:

$$\varepsilon^2 = \sigma^2\frac{\sigma_v^2}{c^2} = \sigma^2\frac{k_B T_e}{\gamma mc^2} \qquad (11)$$



Therefore, Eq. 8 becomes:

$$\frac{\partial^2 r}{\partial z^2} = -\frac{k}{\gamma mc^2} r + \frac{\varepsilon^2}{\sigma^2}\frac{r}{\sigma^2} = -\frac{k}{\gamma mc^2} r + \frac{\varepsilon^2}{\sigma^4} r \qquad (12)$$

Rewriting the linear focusing force coefficient as:

$$K = \frac{1}{\gamma mc^2}\frac{F_r}{r} \qquad (13)$$

and assuming that for a Gaussian beam the beam evolution is given by the evolution of the beam envelope with radius with $r=\sigma$, one rewrites:

$$\boxed{\frac{\partial^2 \sigma}{\partial z^2} + K\sigma = \frac{\varepsilon^2}{\sigma^3}} \qquad (14)$$

This last equation is the envelope equation.

Note that the handling of the emittance was a little bit cavalier and needs to be re-examined!

A solution to the envelope equation is the propagation in vacuum:

$$\sigma = \sigma_0 \left(1 + \frac{\varepsilon^2 z^2}{\sigma_0^2}\right)^{1/2} \qquad (15)$$

In this case, the emittance is defined from the beam size at the waste σ_o. and the envelope angle θ at large distance z from the waste:

$$\frac{\varepsilon^2 z^2}{\sigma_0^2} >> 1 \quad \sigma \cong \frac{\varepsilon z}{\sigma_0} \qquad (16)$$

So that the angle θ is approximately equal to

$$\theta = \frac{\sigma}{z} \cong \frac{\varepsilon}{\sigma_0} \qquad (17)$$

And therefore:

$$\varepsilon \cong \sigma_0 \theta \qquad (18)$$

However this is not true at any z along the beam trajectory.

Similar equations!

$$\boxed{\frac{d^2 r}{dz^2} - k_\beta^2 r = 0 \quad or \quad \frac{d^2 r}{dz^2} - \left(\frac{F_r}{r}\right) r = 0}$$

ENVELOPE EQUATION

Meaning of the envelope equation: $\sigma_r'' + K^2\sigma_r = \frac{\varepsilon_g^2}{\sigma_r^3}$

When σ$_r$ is large … $K^2\sigma_r >> \frac{\varepsilon_g^2}{\sigma_r^3} \Rightarrow \sigma_r'' \cong -K^2\sigma_r$ the beam focuses

When σ$_r$ is small … $K^2\sigma_r << \frac{\varepsilon_g^2}{\sigma_r^3} \Rightarrow \sigma_r'' \cong \frac{\varepsilon_g^2}{\sigma_r^3}$ the beam expands

In between … $\sigma_r'' = \frac{\varepsilon_g^2}{\sigma_r^3} - K^2\sigma_r = 0$ matched …

=> the beam radius does not change if $K = cst \quad and \quad \sigma'(z=0) = 0$

with $K^2 = k_\beta^2 = \omega_\beta^2 c^2 = \frac{\omega_{pe}^2 c^2}{2\gamma} \Rightarrow ??$ matching condition

Note: there is a more general envelope equation that includes acceleration, …

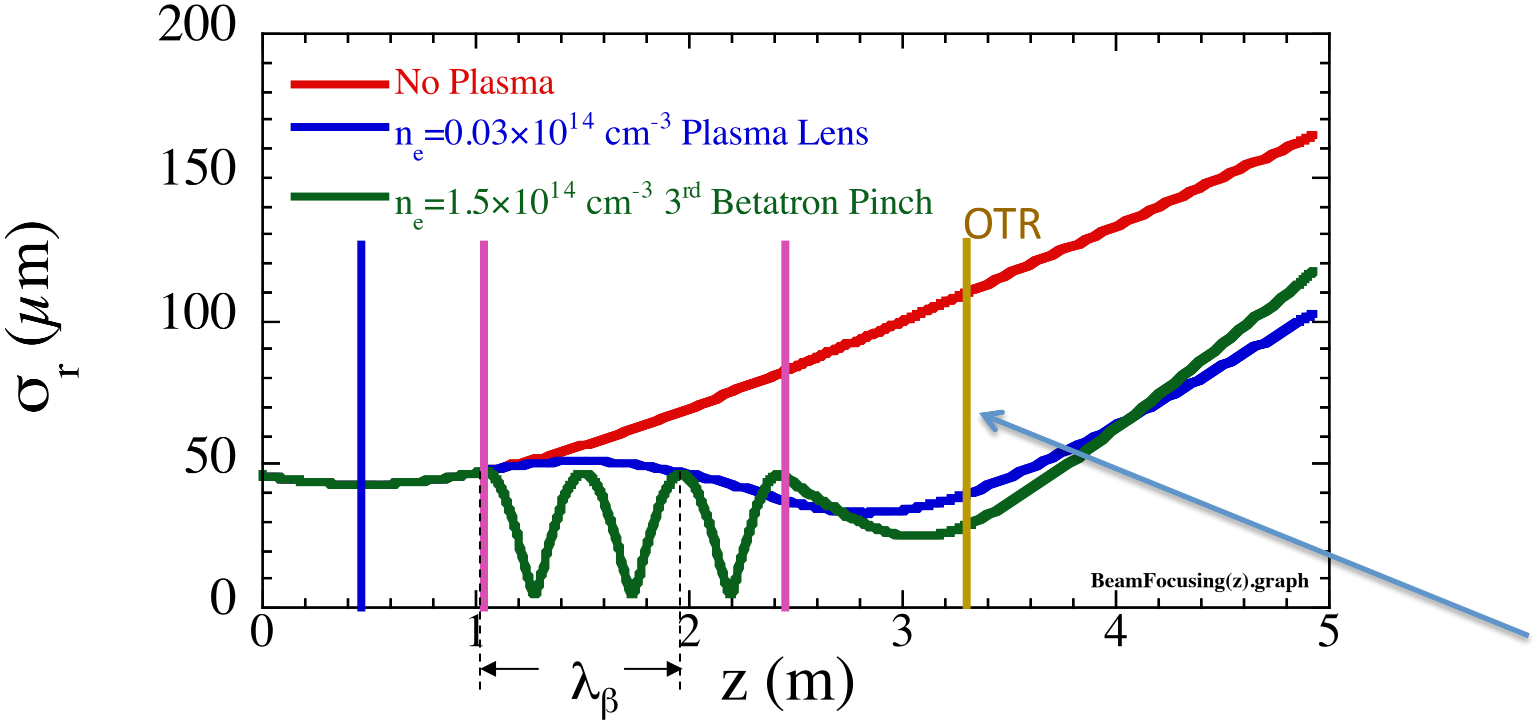
ENVELOPE EQUATION

Solution to the envelope equation (for the Gaussian beam transverse size σ_r):

$$\frac{d^2\sigma_r}{dz} + K^2\sigma_r = \frac{\varepsilon_g^2}{\sigma_r^3} \quad \Leftrightarrow \quad \sigma_r'' + K^2\sigma_r = \frac{\varepsilon_g^2}{\sigma_r^3} \quad \Rightarrow \quad \sigma_r(z) = \sigma_{r0}\left(1 + \frac{z^2}{\beta_0^2}\right)^{1/2} = \sigma_{r0}\left(1 + \frac{\varepsilon_g^2 z^2}{\sigma_0^4}\right)^{1/2}$$

$$\sigma_{r0} = \sigma_r(z=0), \quad \beta_0 = \frac{\sigma_{r0}^2}{\varepsilon_g}$$

Plasma Focusing Force > Beam "Emittance Force" ($\beta_{beam} > \beta_{plasma}$)



Screen

- σ at plasma exit ≤ σ at entrance
- Particles exit angle ∝ $\sigma_{x,y}/\lambda_\beta \propto \sigma_{x,y} n_e^{1/2}$ Remove!

Simple Beam Envelope Model for Plasma Focusing

ENVELOPE EQUATION

Solution to the envelope equation (for the Gaussian beam transverse size σ_r):

$$\frac{d^2\sigma_r}{dz} + K^2\sigma_r = \frac{\varepsilon_g^2}{\sigma_r^3} \quad \Leftrightarrow \quad \sigma_r'' + K^2\sigma_r = \frac{\varepsilon_g^2}{\sigma_r^3} \quad \Rightarrow \quad \sigma_r(z) = \sigma_{r0}\left(1+\frac{z^2}{\beta_0^2}\right)^{1/2} = \sigma_{r0}\left(1+\frac{\varepsilon_g^2 z^2}{\sigma_0^4}\right)^{1/2}$$

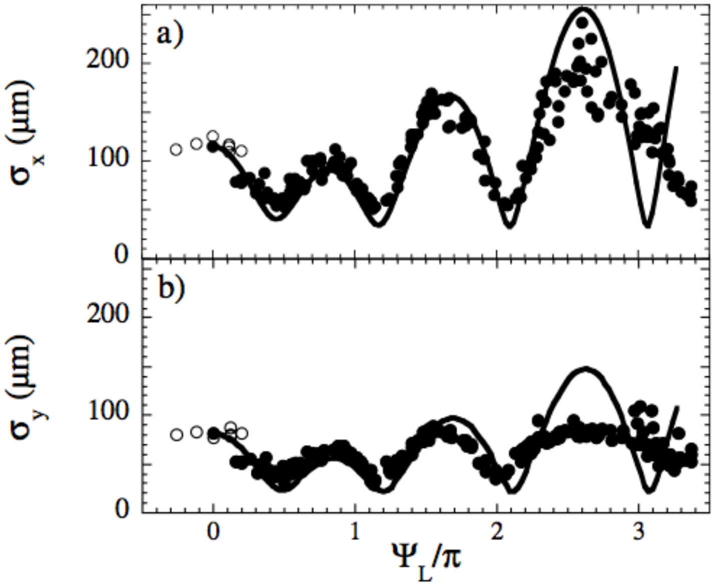


FIG. 3. Beam spot size (a) in the horizontal (σ_x), and (b) in the vertical plane (σ_y) measured at the downstream OTR location as a function of the plasma density expressed as the phase advance $\Psi_L \sim n_p^{1/2}L$ of the beam in the plasma normalized to π. Each symbol corresponds to a measurement of a single event. The empty circles correspond to beam propagation in vacuum ($n_p = 0$, ionization laser off). The solid lines are the best fits of the solutions to the envelope equation [Eq. (1)] obtained for a beam with a waist at the SiO$_2$ pellicle, and with: (a) $\sigma_{x0} = 39\ \mu\text{m}$, $\epsilon_{N,x} = 8 \times 10^{-5}$ m · rad ($\beta_{x0} = 1.06$ m), and (b) $\sigma_{y0} = 20\ \mu\text{m}$, $\epsilon_{N,y} = 3 \times 10^{-5}$ m · rad ($\beta_{y0} = 0.73$ m).

Clayton, PRL 88, 154801 (2002)

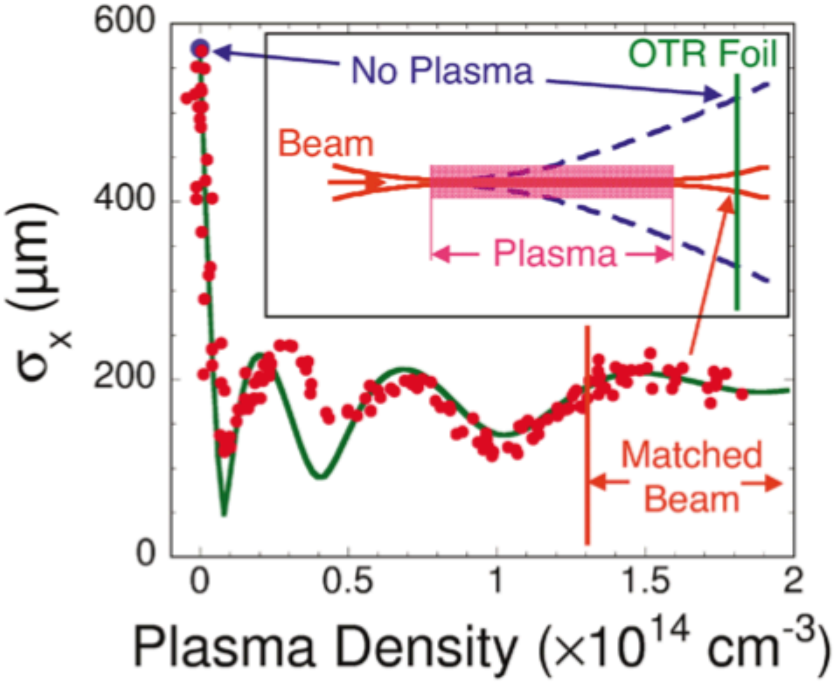


FIG. 2 (color). Transverse size σ_x of the beam (red points) in the x plane measured on the downstream OTR foil (see illustration inset) as a function of plasma density. The green line is the best fit to the data using a beam envelope model in which $\sigma_{x0} = 30\ \mu\text{m}$, $\epsilon_x = 9 \times 10^{-9}$ m rad, and $\beta_0 = 0.11$ m.

Muggli, PRL 93, 014802 (2004)

To efficiently drive wakefields: $k_{pe}\sigma_z \cong \sqrt{2}$ and $k_{pe}\sigma_r \leq 1$

Can reach: $E_{WB} = \frac{m_e c\omega_{pe}}{2}$ or a fraction n_b/n_{e0} of that

Rewrite:

$$E_{WB} = \frac{m_e c\omega_{pe}}{2} = \frac{m_e c^2 k_{pe}}{2} \cong m_e c^2 \frac{\sqrt{2}}{\sigma_z} \cong 0.7MeV \frac{1}{\sigma_z}$$

$$k_{pe}\sigma_z = \frac{\omega_{pe}}{c}\sigma_z \cong \sqrt{2} \quad \Rightarrow \quad \frac{n_e e^2}{\varepsilon_0 m_e c^2}\sigma_z^2 \cong 2 \quad \Rightarrow \quad n_e \cong 2\frac{\varepsilon_0 m_e c^2}{e^2}\frac{1}{\sigma_z^2} \cong 5.65\times 10^{13}\frac{1}{\sigma_z^2}$$

$$\sigma_r \leq \frac{1}{k_{pe}} \cong \frac{\sigma_z}{\sqrt{2}}$$

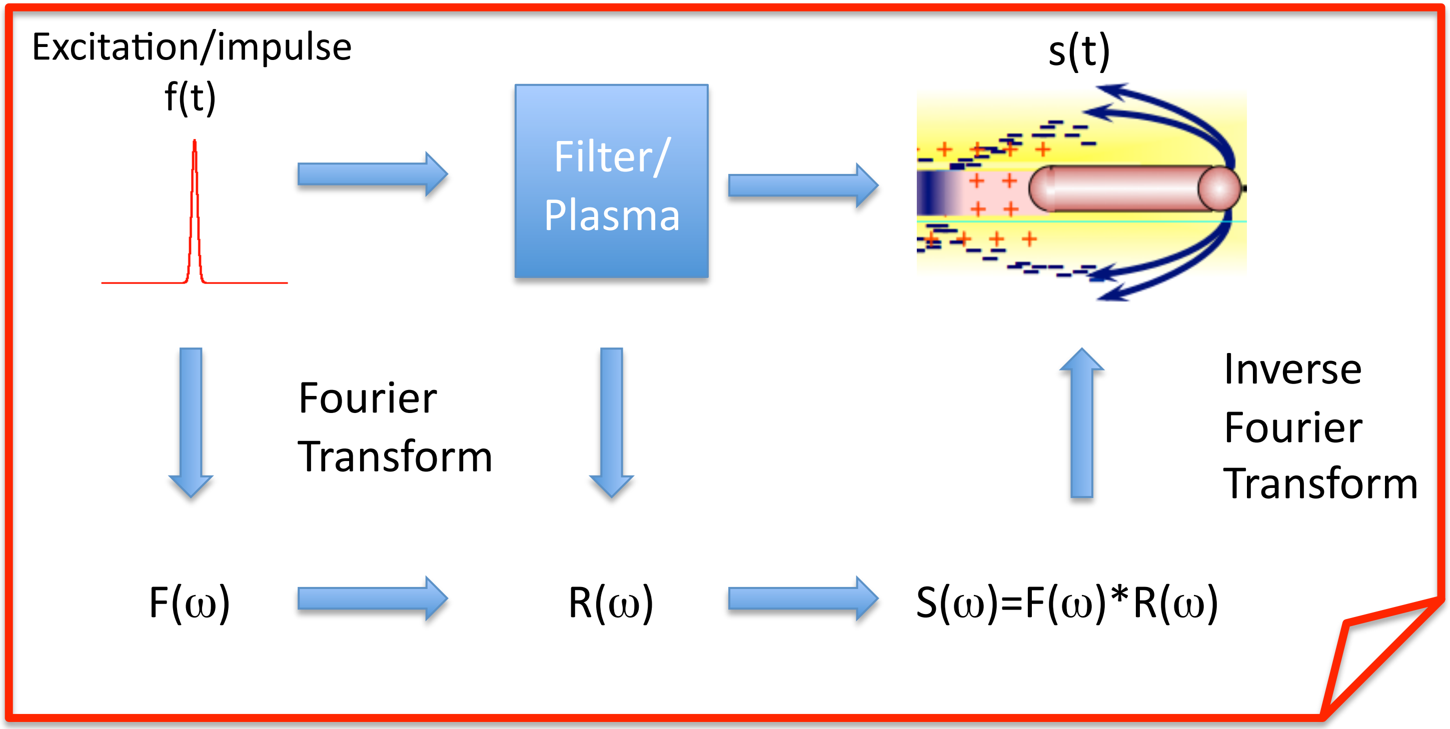
Example: SPS proton (p^+) bunch, σ_z=12cm
=> σ_r~8.5cm
=> $n_{e0}=3.9\text{x}10^9$ cm^{-3}
=> E_{WB}~6MV/m

With SMI
=> σ_r~200µm
=> σ_z~283µm
=> n_{e0}~$7\text{x}10^{14}$ cm^{-3}
=> E_{WB}~2.54GV/m

SMI takes the wakefields from the MV/m to the GV/m range!!

Note: must also consider n_b/n_{eo} and the evolution of the instability ….

Consider the plasma as a harmonic oscillator with an eigen-frequency $f_{pe}=\omega_{pe}/2\pi$ (p^+: f_{pe}~237GHz, ω_{pe}~1.23mm)



Excitation/impulse

f(t)

Filter/ Plasma

s(t)

Fourier Transform

Inverse Fourier Transform

F(ω)

R(ω)

S(ω)=F(ω)*R(ω)

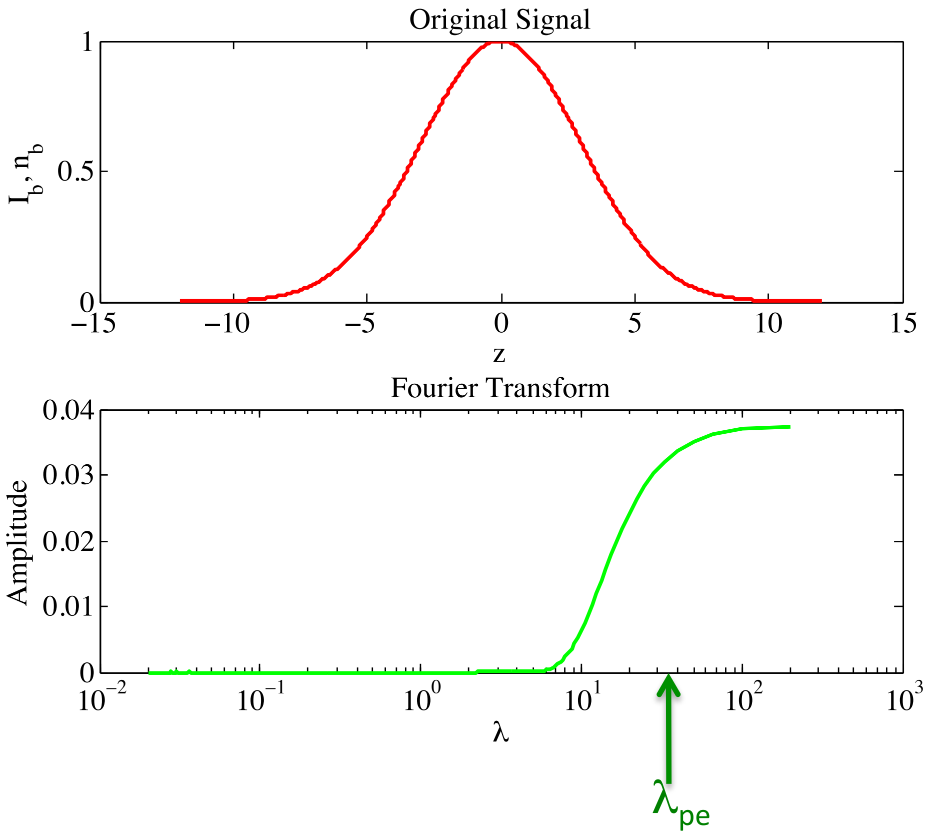
The oscillator is driven by a time signal given by the bunch current profile

Therefore, the effectiveness of the driving is given by the amount of energy at frequency f_{pe} (in a narrow bandwidth around) …

Remember linear equation for the plasma density:

BUNCH LENGTH

Single, short bunch … $k_{pe}\sigma_z$~√2~1

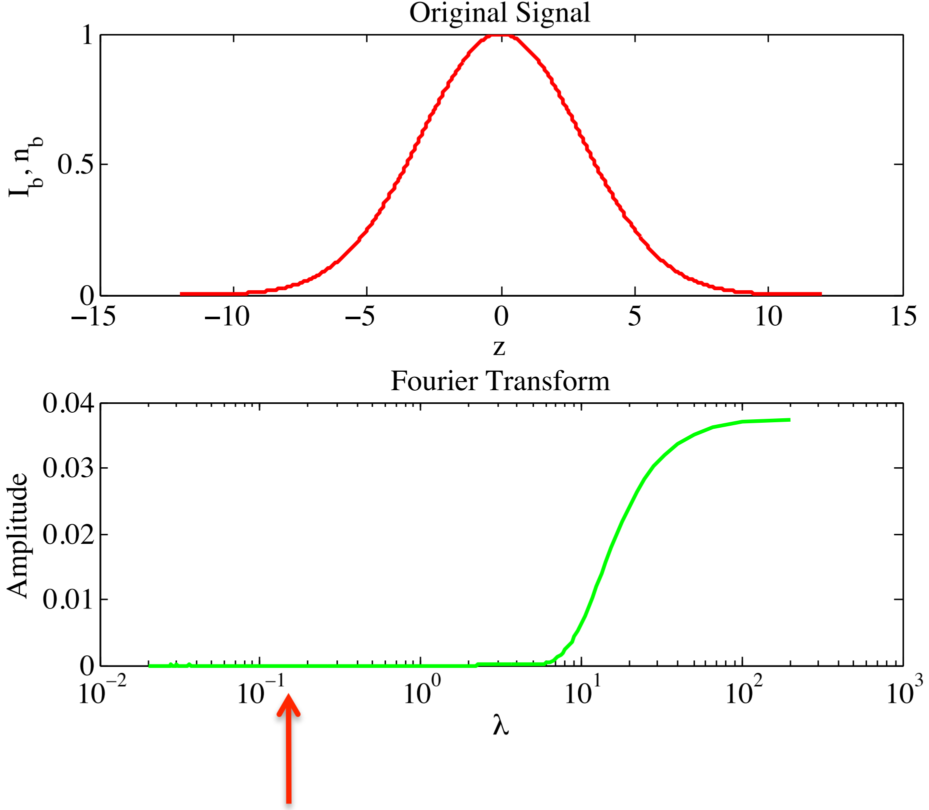


Effective
When $k_{pe}\sigma_z$~√2

BUNCH LENGTH

Long bunch in high plasma density … $k_{pe}\sigma_z$>>1

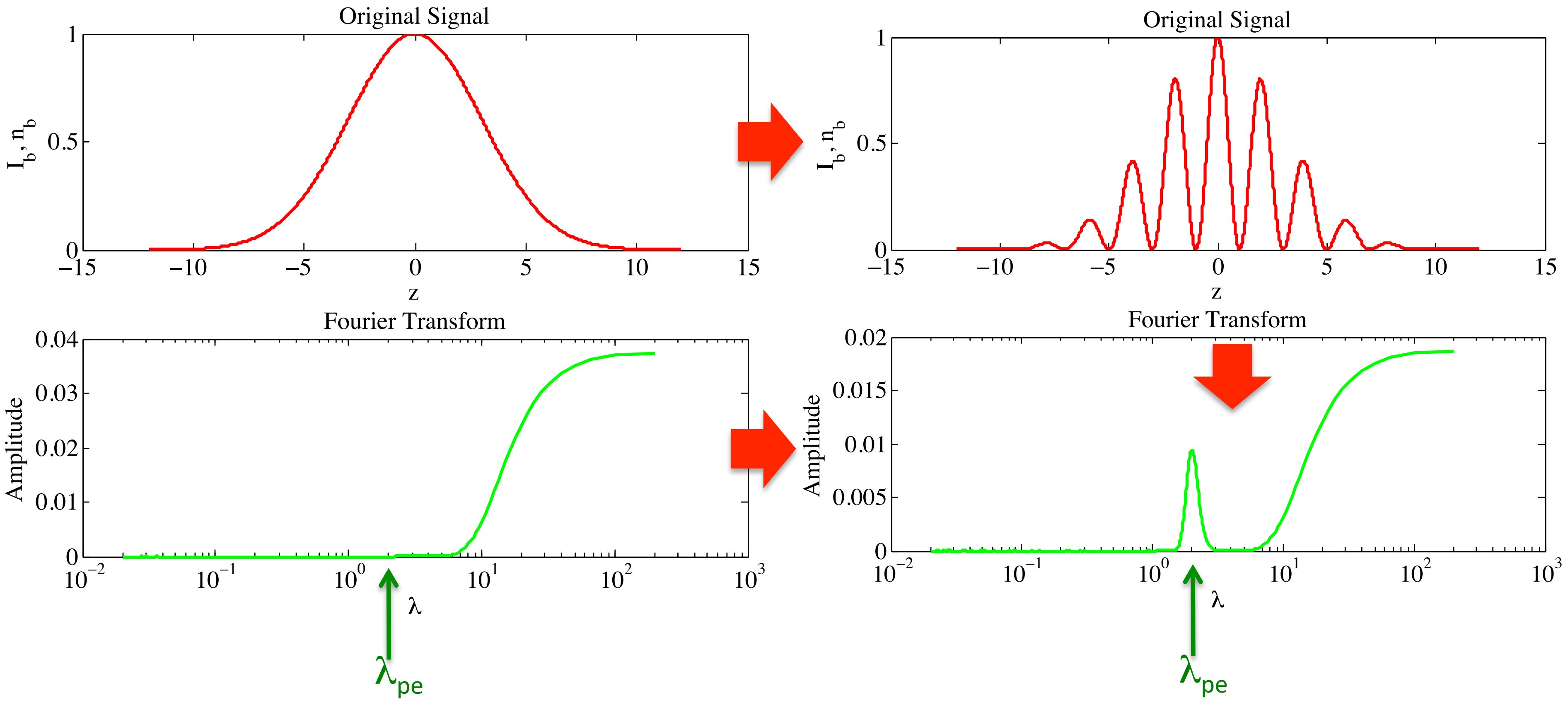
Example: bunch with σ=3 in a plasma with plasma wavelength λ_{pe}=2



Not effective
for $k_{pe}\sigma_z$>>1
or λ_{pe}<<σ_z

BUNCH LENGTH

Example: bunch with σ=3, modulated with λ=2 in a plasma with plasma wavelength λ_{pe}=2



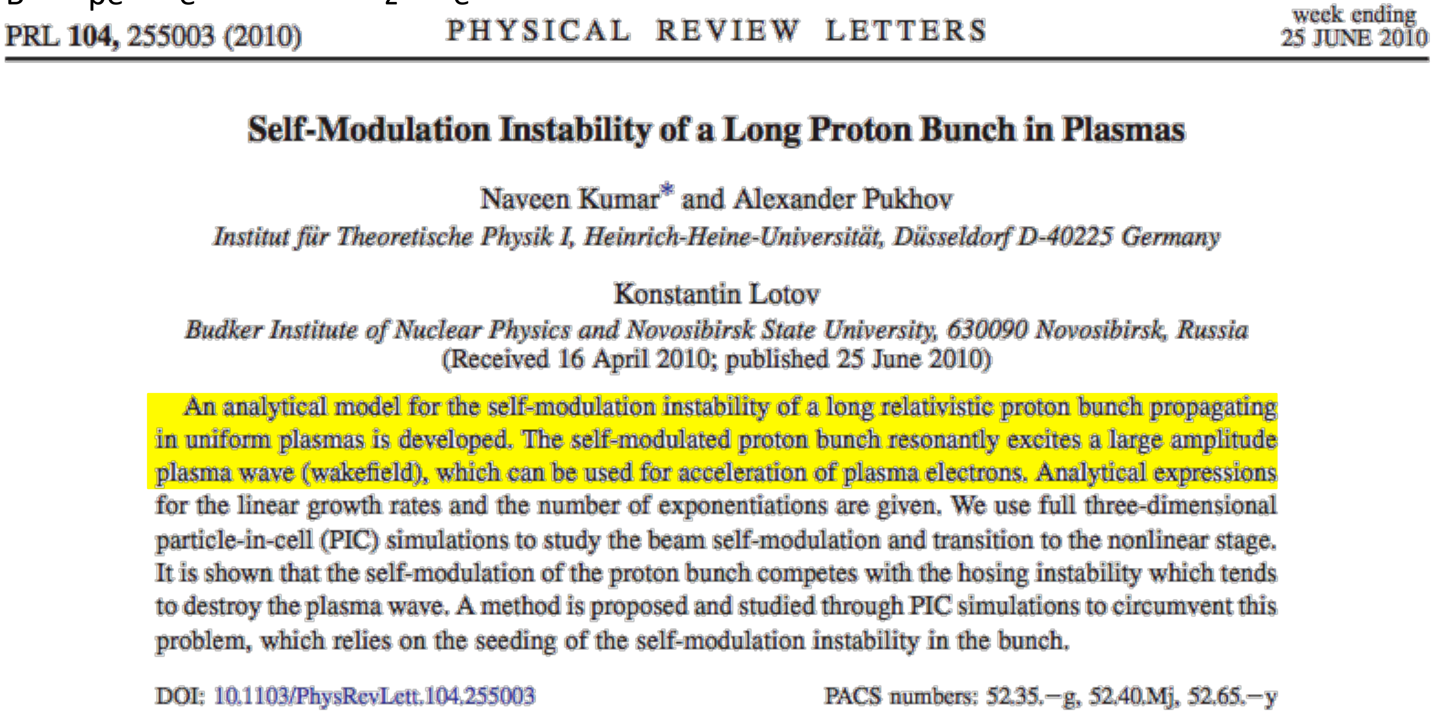
Not effective

Effective

=> (self-)Modulating the bunch density ($n_b \sim 1/\sigma_r^2\sigma_z$) at $\sim\lambda_{pe}$ is a very effective way to drive larger (than in the $k_{pe}\sigma_z \sim \sqrt{2}$ case) wakefields

SELF-MODULATION INSTABILITY (SMI)

- ✧CERN p^+ bunches (PS, SPS, LHC) ~12cm long
- ✧$E_{WB}\sim\omega_{pe}\sim n_e^{1/2}$ and $\sigma_z\sim n_e^{-1/2}$



PRL 104, 255003 (2010) PHYSICAL REVIEW LETTERS week ending 25 JUNE 2010

Self-Modulation Instability of a Long Proton Bunch in Plasmas

Naveen Kumar* and Alexander Pukhov
Institut für Theoretische Physik I, Heinrich-Heine-Universität, Düsseldorf D-40225 Germany

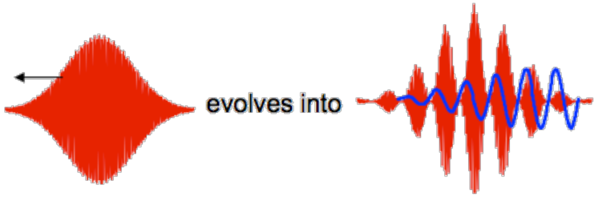
Konstantin Lotov
Budker Institute of Nuclear Physics and Novosibirsk State University, 630090 Novosibirsk, Russia

(Received 16 April 2010; published 25 June 2010)

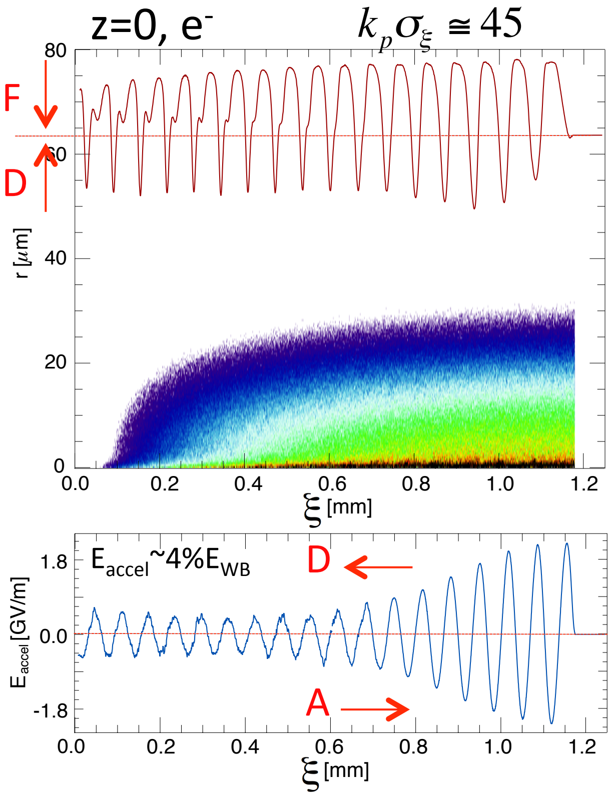
An analytical model for the self-modulation instability of a long relativistic proton bunch propagating in uniform plasmas is developed. The self-modulated proton bunch resonantly excites a large amplitude plasma wave (wakefield), which can be used for acceleration of plasma electrons. Analytical expressions for the linear growth rates and the number of exponentiations are given. We use full three-dimensional particle-in-cell (PIC) simulations to study the beam self-modulation and transition to the nonlinear stage. It is shown that the self-modulation of the proton bunch competes with the hosing instability which tends to destroy the plasma wave. A method is proposed and studied through PIC simulations to circumvent this problem, which relies on the seeding of the self-modulation instability in the bunch.

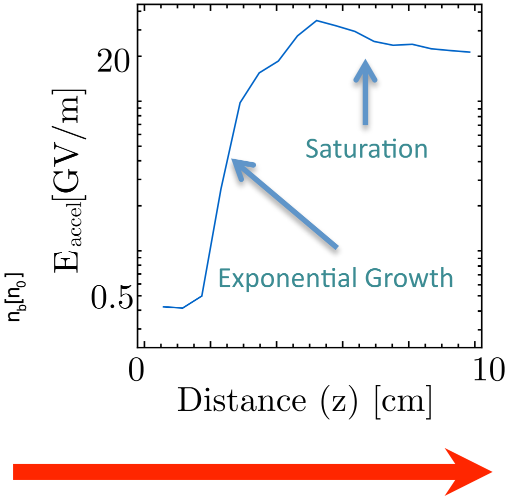
DOI: 10.1103/PhysRevLett.104.255003 PACS numbers: 52.35.−g, 52.40.Mj, 52.65.−y

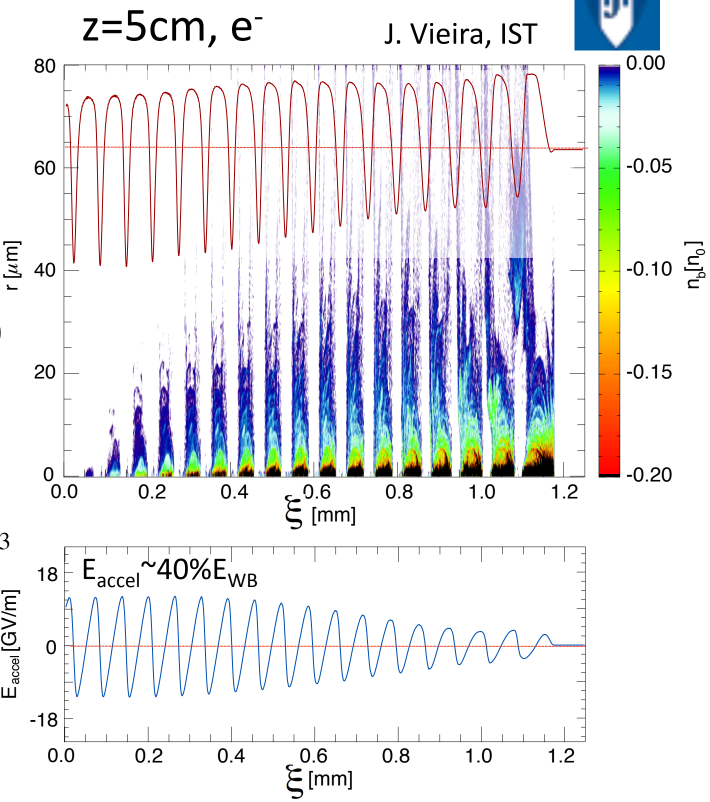
- ✧Idea developed “thanks” to the non-availability of short p^+ bunches
- ✧Very similar to Raman self-modulation of long laser pulses (LWFA of the 20th century)



SELF-MODULATION INSTABILITY (SMI)







J. Vieira, IST

$$N_{\exp} \cong \frac{3\sqrt{3}}{4}\left(\frac{n_b}{n_e}\frac{m_e}{\gamma M_b}\left(k_p|\xi|\right)\left(k_p z\right)^2\right)^{1/3}$$

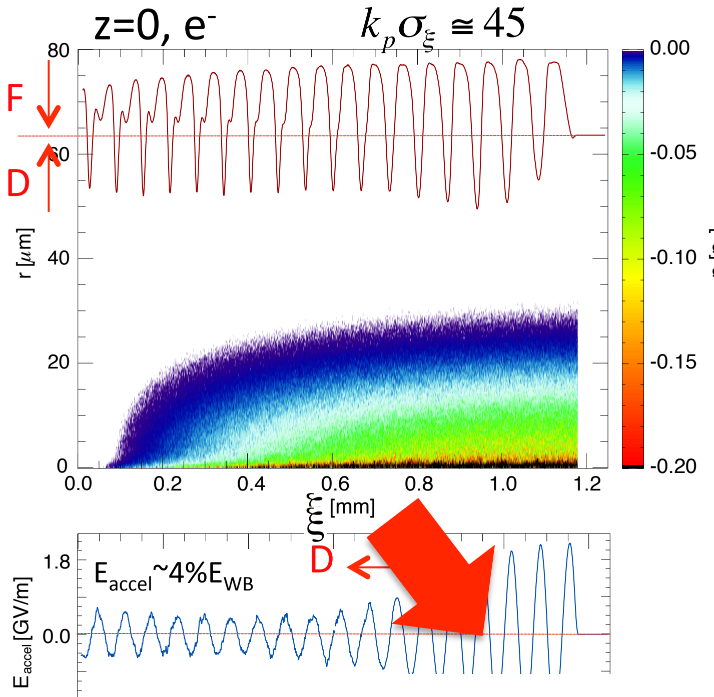
Grows along the bunch & along the plasma

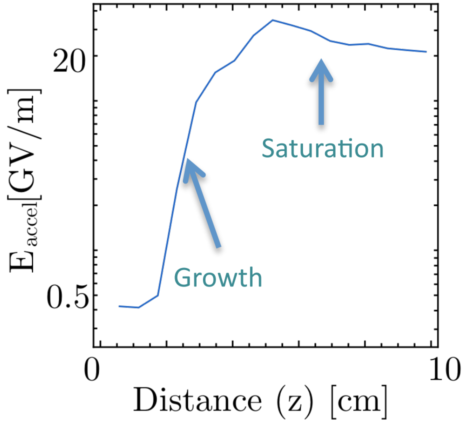
Pukhov et al., PRL 107, 145003 (2011)
Schroeder et al., PRL 107, 145002 (2011)

- ✧Initial small transverse wakefields modulate the bunch density
- ✧Longitudinal, transverse wakefields ~n_b
- ✧Associated longitudinal wakefields reach large amplitude through resonant excitation

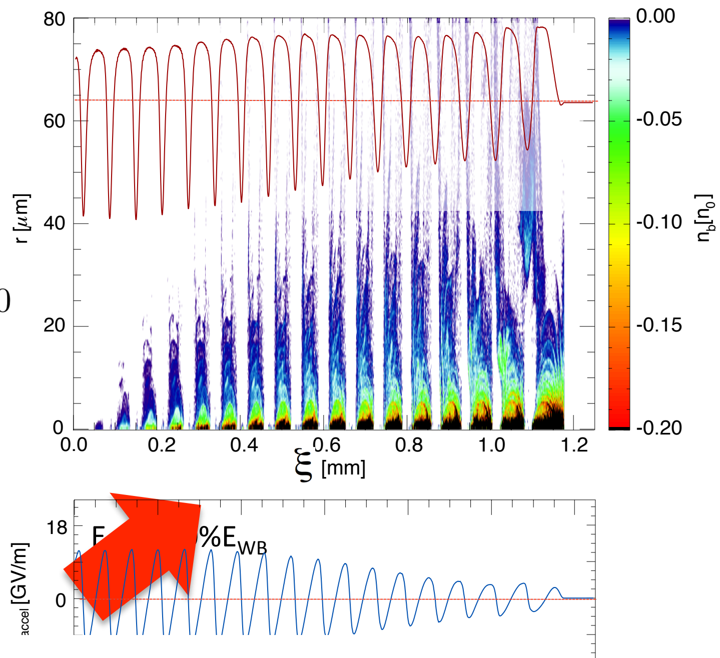
J. Vieira et al., Phys. Plasmas 19, 063105 (2012)

SELF-MODULATION INSTABILITY (SMI)



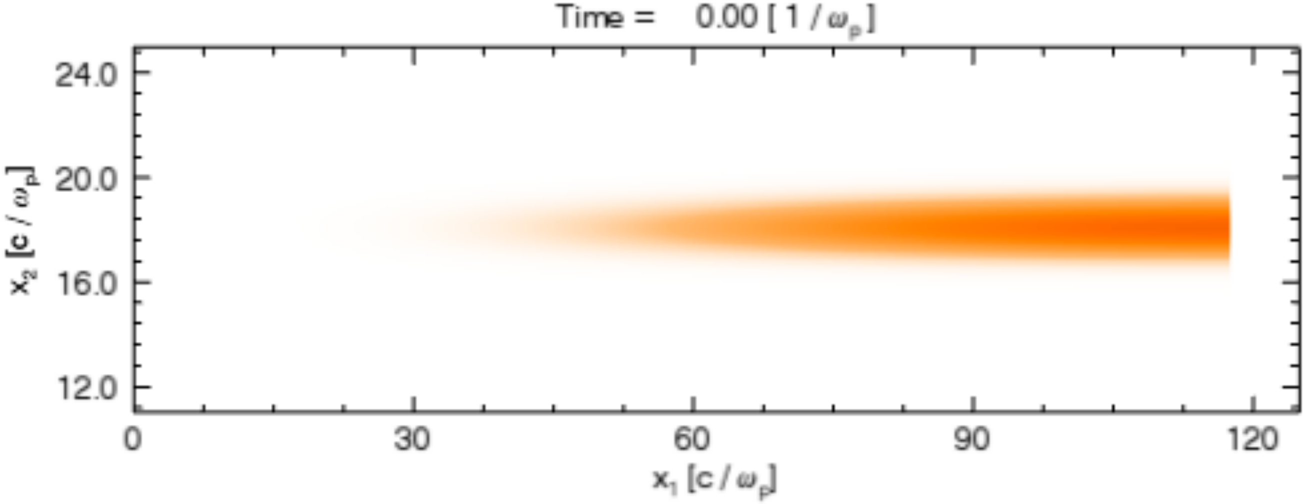


Vieira et al., Phys. Plasmas 19, 063105 (2012).



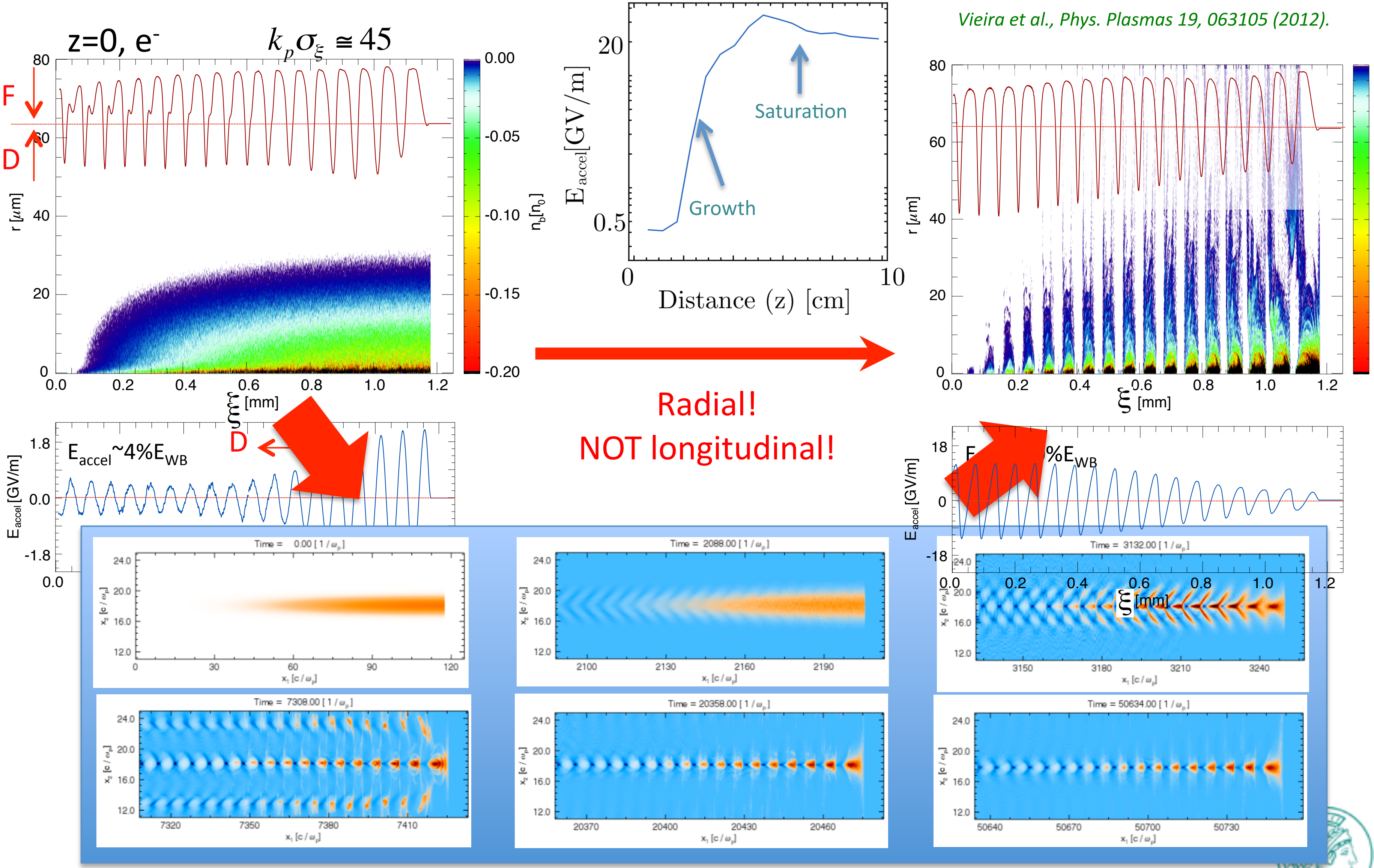
Radial!

NOT longitudinal!



SELF-MODULATION INSTABILITY (SMI)

Vieira et al., Phys. Plasmas 19, 063105 (2012).



Radial!
NOT longitudinal!

SELF-MODULATION INSTABILITY (SMI)

SMI/SMI-based experiments:

Accelerator Test Facility, Brookhaven National Laboratory, USA

E_0=60MeV e^-, Q=50-1000pC, n_e=10^{15}-$10^{16}$$cm^{-3}$

- Driving of wakefields with bunches 1, 2, …, 5λ_{pe} (E_z)
- Seeding of the SMI
- Comparison linear PWFA theory, simulations, experiments

Fang, PRL 112, 045001 (2014).

SLAC National Accelerator Laboratory, USA

E_0=20GeV e^-/e^+, Q=3nC, n_e=10^{16}-$10^{17}$$cm^{-3}$

- Radial modulation
- Energy loss gain (GeVs)
- Transverse evolution
- Seeding/ mode competition

Vieira, PoP 19, 063105 (2012)

DESY-PITZ Zeuthen, Germany

E_0=20MeV e^-, Q=pC, n_e=$10^{15}$$cm^{-3}$

- Radial modulation
- Seeding

Gros, NIMA 740, 74 (2014)

AWAKE @ CERN, CH

AWAKE Colab. PPCF 56 084013 (2014)

E_0=400GeV p^+, Q=25nC, n_e=10^{15}-$10^{15}$$cm^{-3}$

- Radial modulation
- Acceleration of externally injected e^-

=> Edda's talk on …

PLASMA WAKEFIELD ACCELERATOR

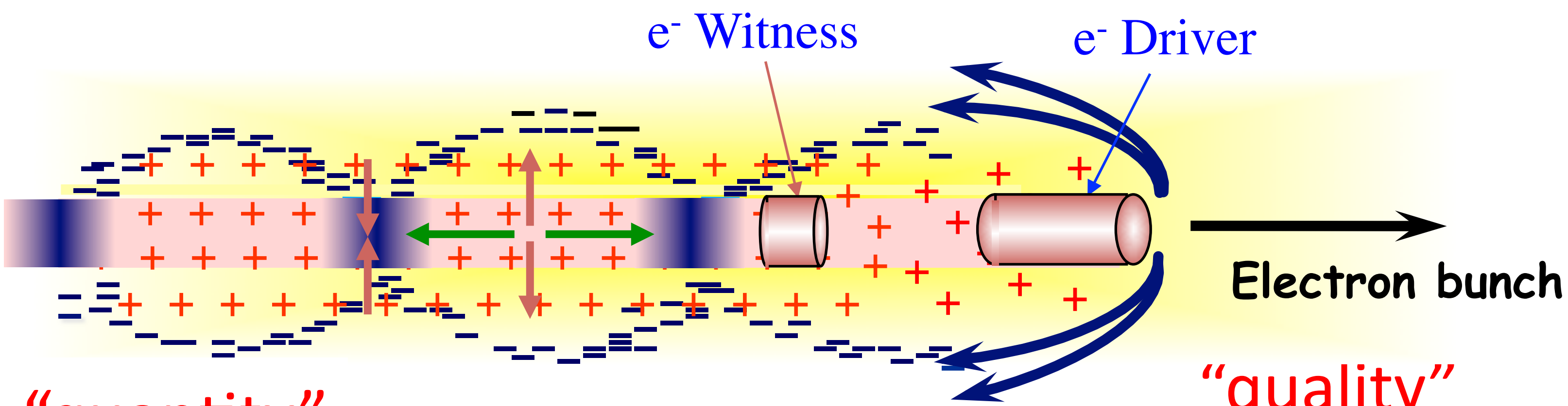
FACET @ SLAC: Flagship Experimental Program

- Acceleration of witness e^- bunch with GeV gain
- Narrow energy spread
- Large energy transfer efficiency
- Beam loading

- Acceleration of e^+ with GeV gain
- Emittance preservation

- External injection of e^- bunch
- Acceleration of e^+ witness bunch on e^-driven wake …

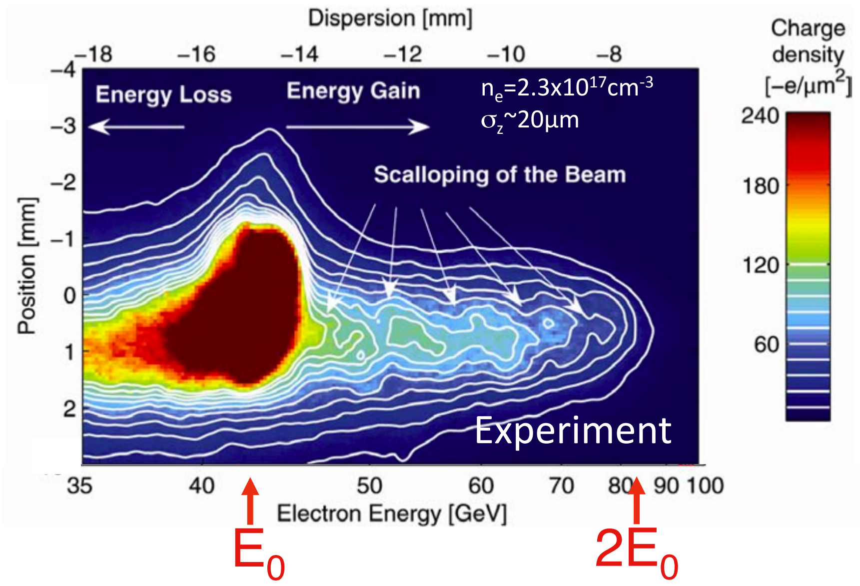
PLASMA WAKEFIELD ACCELERATOR



"quantity"

"quality"

Blumenfeld, Nature 445, 741 (2007)



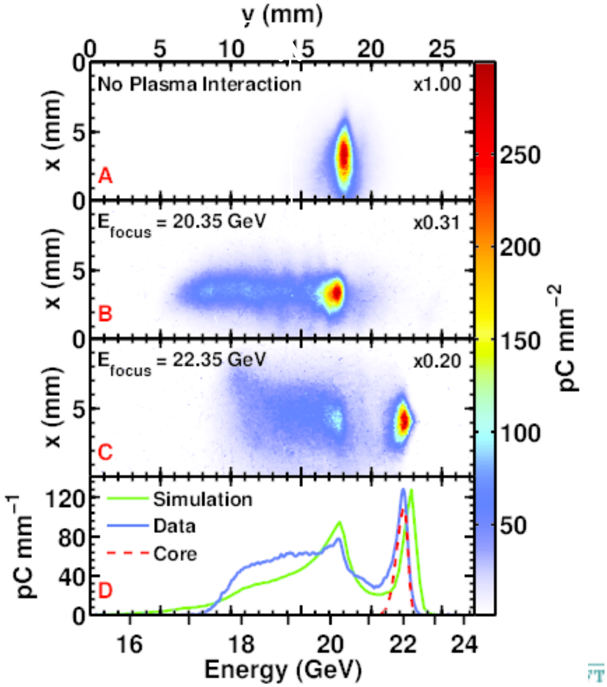
42 => 84GeV in 85cm! 50GeV/m

SLAC FACET

E200

Hogan, NJP 12, 055030 (2010)

Litos, Nature 515, 92 (2014)



SUMMARY

The PWFA is a very interesting and active research topic

Barely scratched the surface of the topic

There is a large body of results … (theory, simulations, experiments)

Many topics were not addressed (but may be in other lectures):
- Linear – non linear – quasi-linear theory/regimes
- Transformer ratio
- Beam loading
- Energy transfer efficiency
- Collider-related issues …
- Etc. …

Thank you!