Numerical and Computational Tools in Accelerator Physics

An introduction

Werner Herr
CERN, AB Department

(http://cern.ch/Werner.Herr/CAS/CAS2008_Frascati/lectures/comp.pdf)

(http://cern.ch/Werner.Herr/CAS/CAS2008_Frascati/lectures/handout.pdf)

Why do we need computations and simulations ?

- To explore new fields
- To answer scientific or technical questions
- To make design choices
- To go from a theoretical to "real" machine
- → To deal with the "far from ideal" world

(B. Holzer)

Why do we need computations and simulations ?

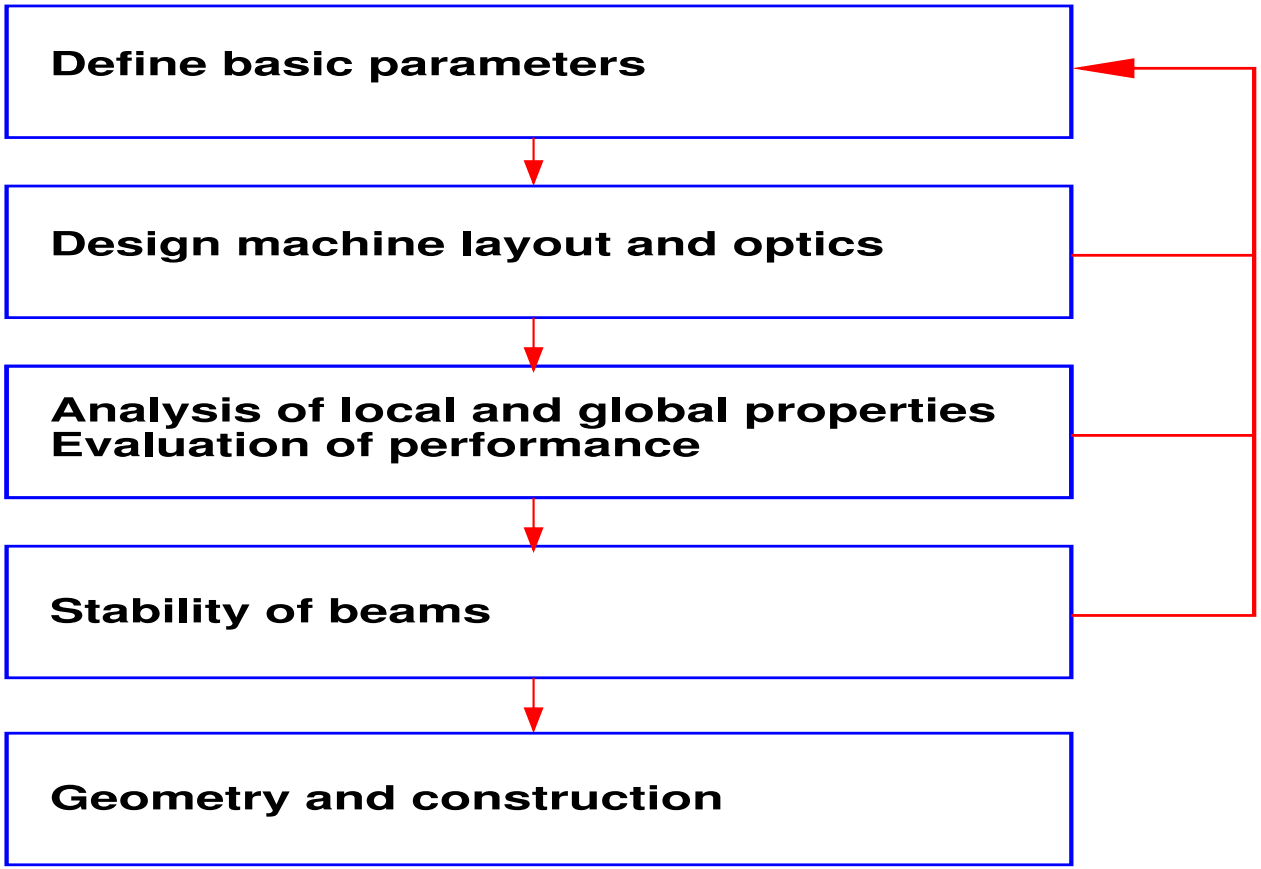
- **Why especially important nowadays ?**
  - Larger equipment, more people, more money
  - Many more applications
  - Safety issues
  - Complex control and operation
- **Eventually have to get the details right**

Where are computational tools needed ?

- Studies of Beam Dynamics
  - Design and simulation of an accelerator
  - Control and operation

- Design of accelerator equipment
  - Magnets, RF cavities ...
  - Vacuum components, cryogenics

Steps of accelerator design



Accelerator physics programs needed

- Initial parameter calculation
- Optics and lattice design program
- Geometry
- Single- and multi-particle dynamics modelling
- Analyse and understand the behaviour

→ Probably several programs needed

→ Probably written by other people (i.e. not you)

Working with accelerator physics programs

- Computer programs are only as good as the underlying concepts (and the way they are used):
  - May only work on certain class of idealized problems
  - May be optimized for special purpose
  - If something cannot be computed, does not mean that it does not exist
- Plenty of possible traps 

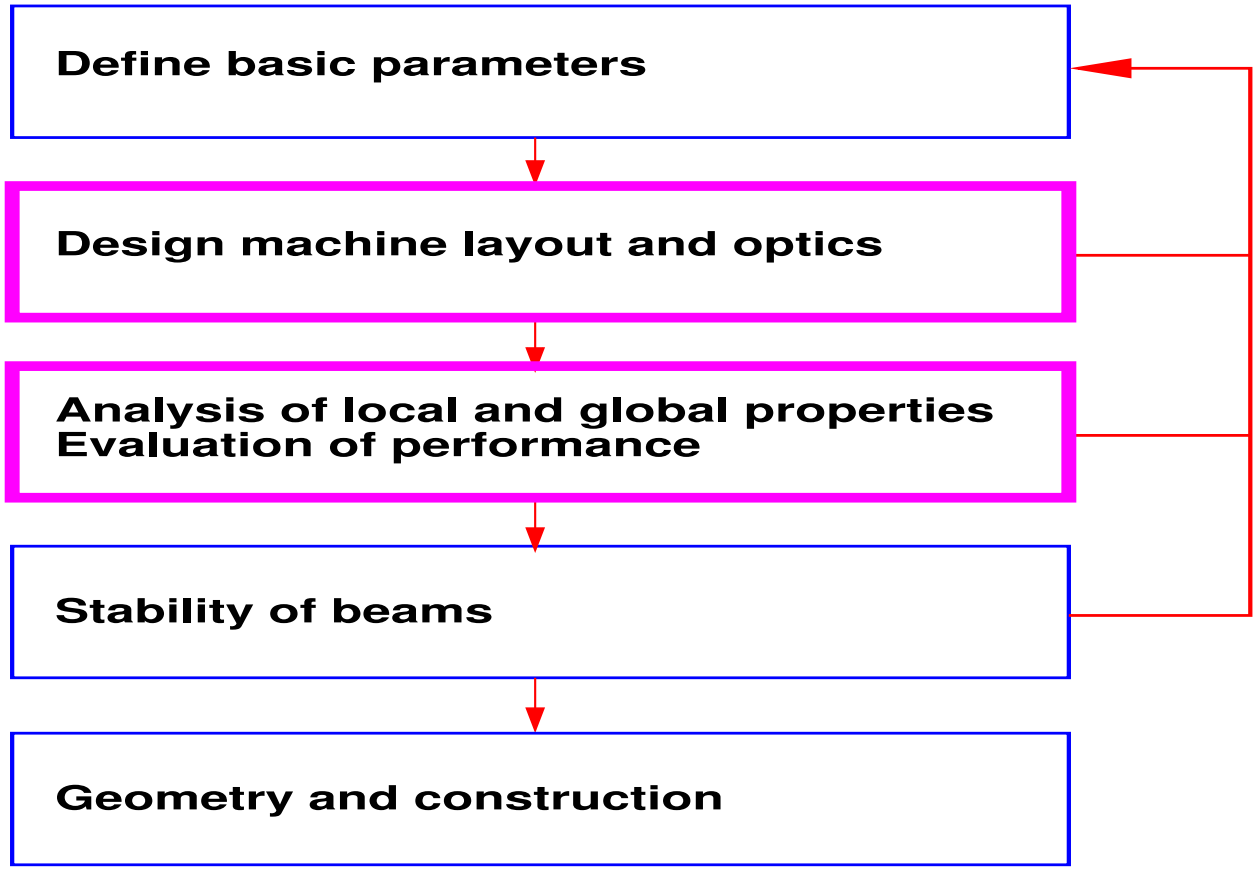
Working with accelerator physics programs

- Do we need to understand what they are doing (algorithms, technicalities, basic concepts) ?
- Using any program you must ask yourself:
  - Is the problem well defined ?
  - Is the description (model) adequate ?
  - What are the effects of approximations, etc. ?
  - Do I use the appropriate program ?
- To get the right conclusions → Yes !

Objectives of these lectures:

- Introduction to selected methods and programs
- Make you aware of possible problems and limitations to help understand the results correctly
- Get you acquainted with the terminology and to speak the same language
- Introduce some modern concepts and techniques
- Restrict myself to circular machines

Steps of accelerator design



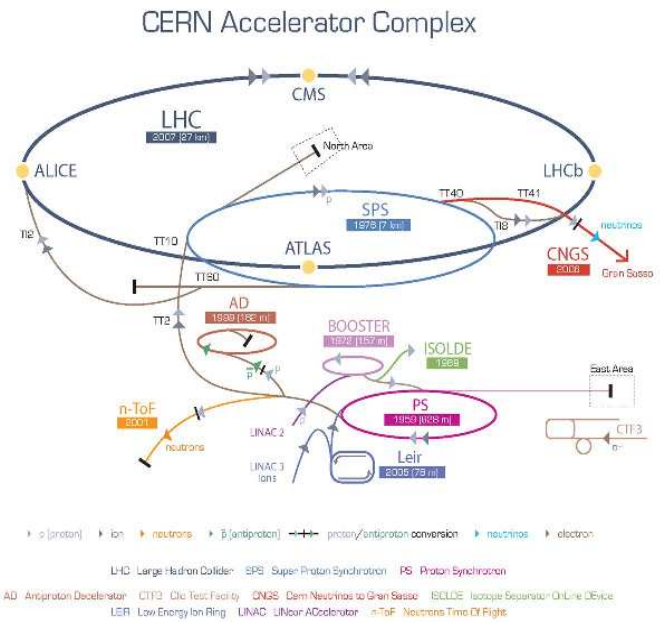
Accelerator design with an optics program

- It needs: Description of machine in standard format
- It does: Optics calculations
  - → Linear and non-linear optics computations
  - → Parameter matching
  - → Linear corrections (orbit, coupling, β, ..)
  - → Non-linear and chromatic corrections

How is an accelerator described to a computer ?

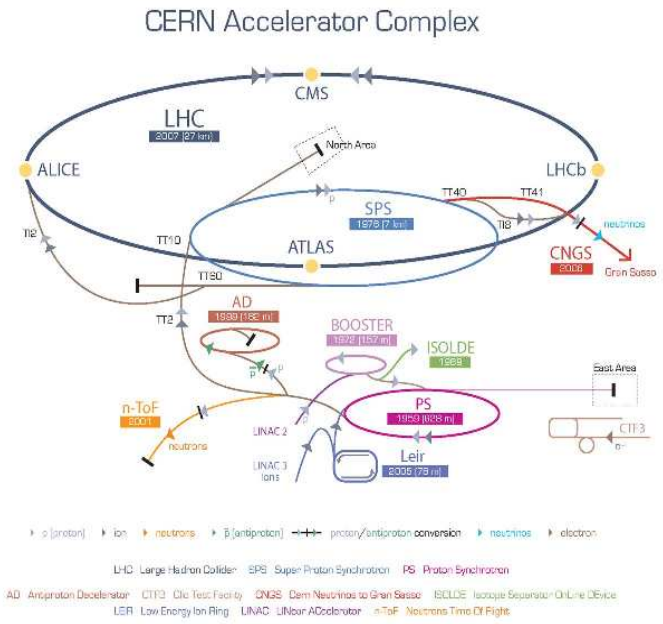
Not like:

$$\frac{d^2x}{ds^2} + K(s)\,x = 0$$



How is an accelerator described to a computer ?

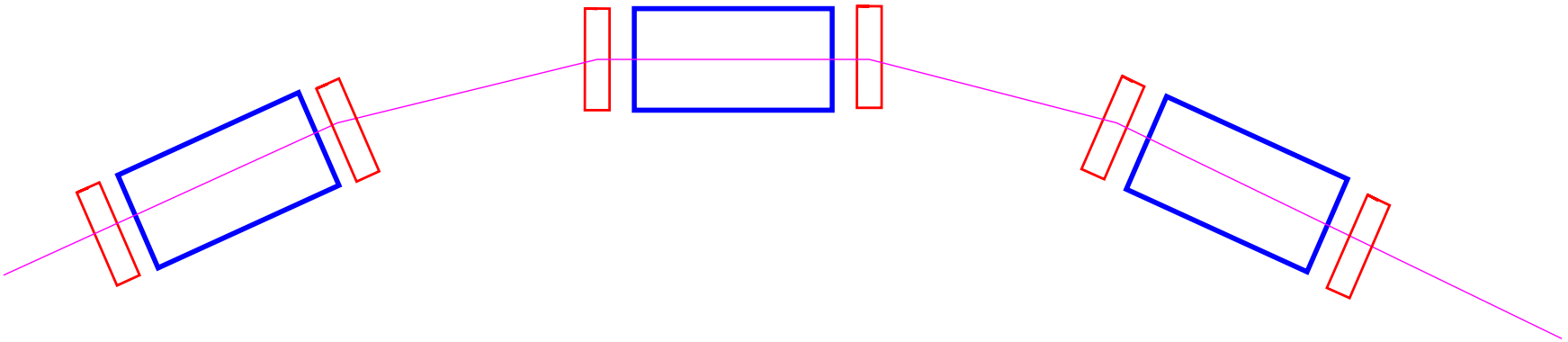
- **The challenge:**
  - **Describe a machine with several thousand elements**
  - **Describe a complicated structure**



How is an accelerator described to a computer ?

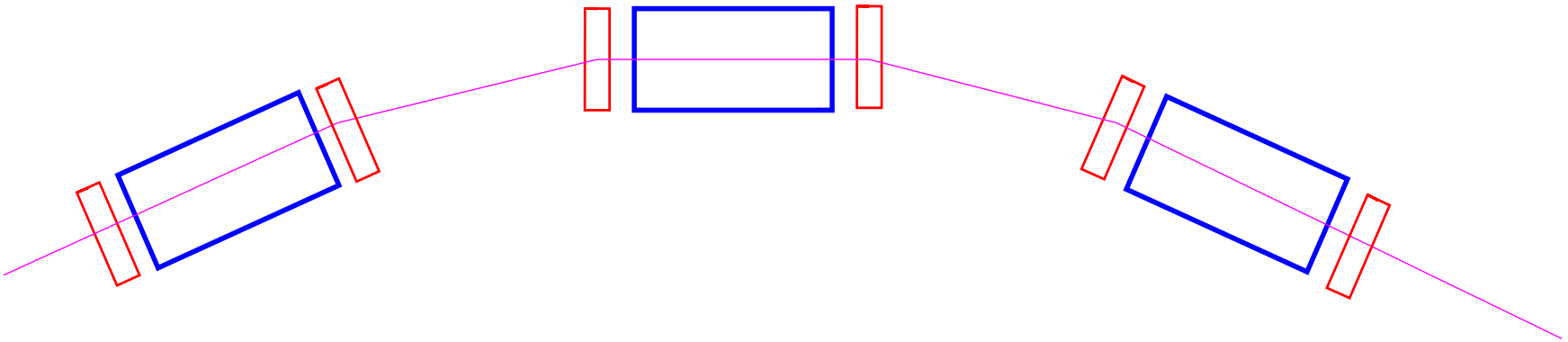
- **The main questions:**
  - **What kind of elements (magnets, etc.) are in the accelerator ?**
  - **Where are these elements in the accelerator ?**
  - **How do the elements act on a particle ?**
- **Other issues:**
  - **Simplicity, speed, accuracy ...**
  - **Description usually serves multiple purpose: optics calculation (β, ..), simulations (tracking), civil engineering**

Where to put the elements in an accelerator ?



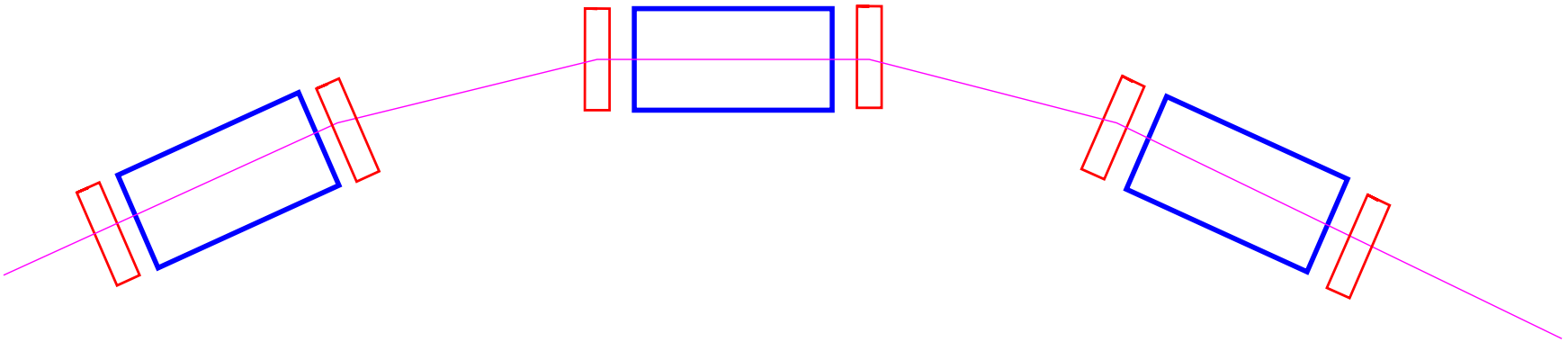
- **Take a simple structure:**
- **How to describe the position of the elements ?**

Where to put the elements in an accelerator ?



- **Cartesian coordinates in space ??**

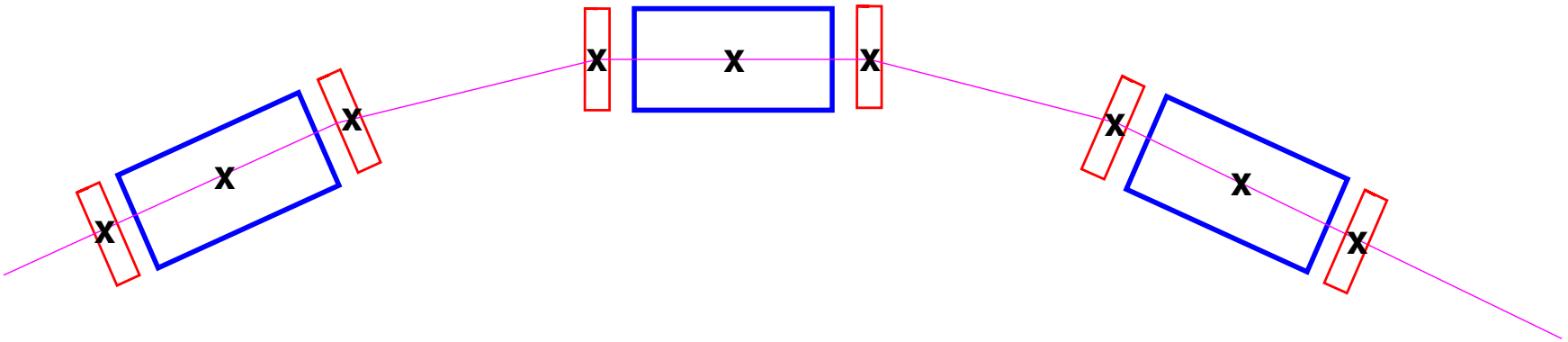
Where to put the elements in an accelerator ?



$$\frac{d^2x}{ds^2} + K(s)\,x = 0$$

- **Usually use s (pathlength) along "reference path"**
- **"Reference path" defined geometrically by straight sections and bending magnets**

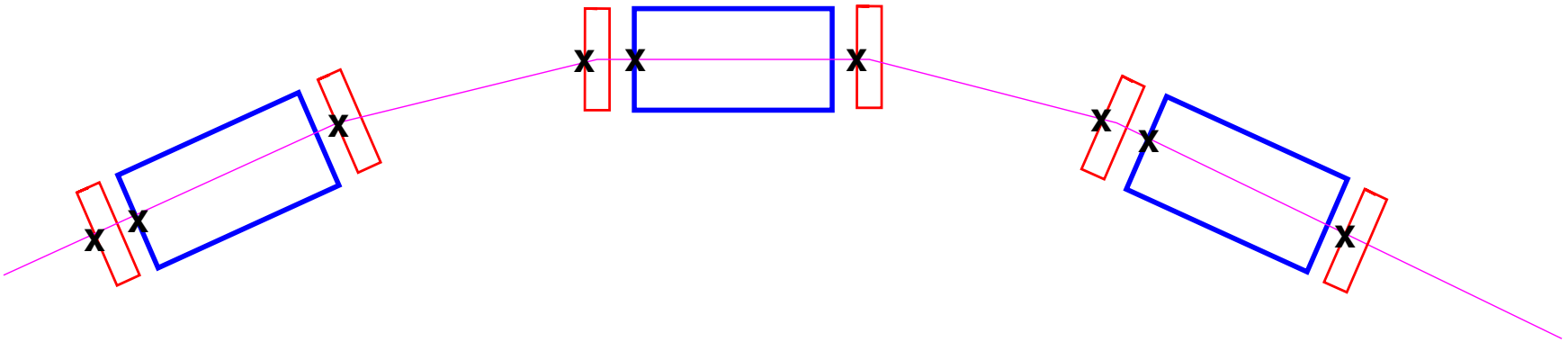
Where to put the elements in an accelerator ?



$$\frac{d^2x}{ds^2} + K(s)\, x = 0$$

- Usually use s (pathlength) along "reference path"
- Specify coordinates at centre of the element ...

Where to put the elements in an accelerator ?

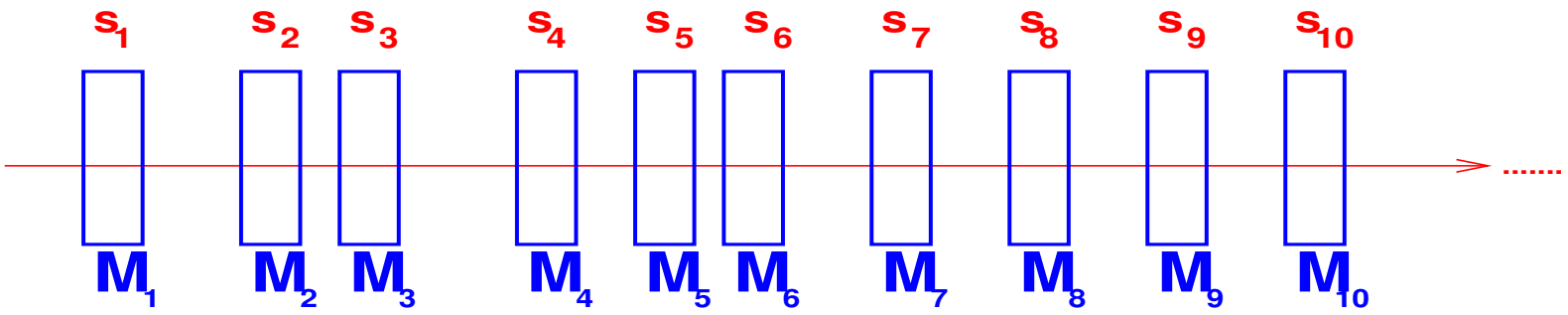


$$\frac{d^2x}{ds^2} + K(s)\, x = 0$$

- Usually use **s** (pathlength) along "reference path"
- Specify coordinates at **entry** of the element ...

How does an accelerator look like to a computer ?

- Bending and focusing is (in general) not a continuous function of s
- A (finite) sequence of machine elements M at longitudinal positions s_1, s_2, s_3,...:



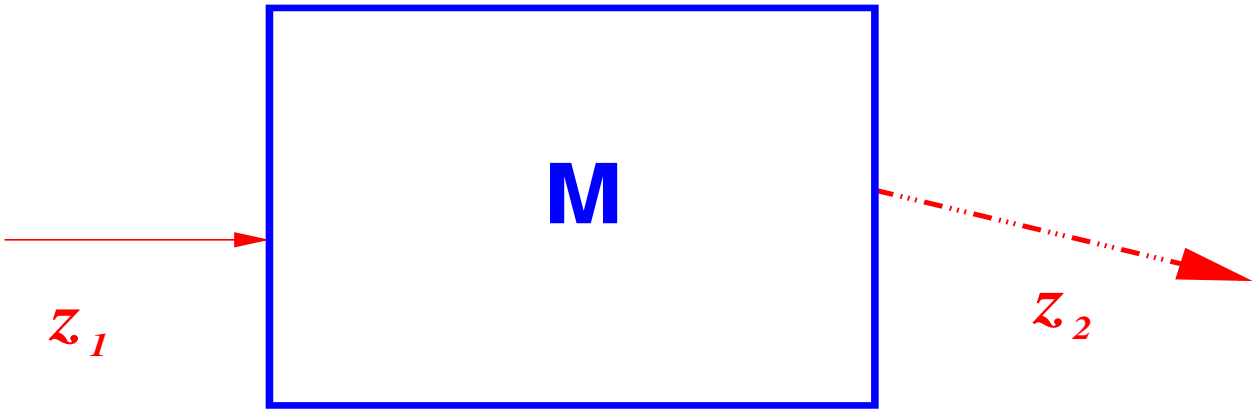
Are there problems with this description ?

Path length s for positioning may have problems:

- **Elements shared by several beams (colliders, switching magnets in beam lines ...)**
- **Same element at different "path length" (microtron etc.)**

How does an element look like to a computer ?

- You need to describe what happens to the particle in M
- Assume each element M (e.g. magnet) acts on the beam locally in a deterministic way, functionally independent of other elements



- In general: $\vec{z_2} \neq \vec{z_1}$

What is M ? It can represent:

- Single machine elements:
  - magnet: dipole, quadrupole
  - RF cavity
- Single machine elements (not only magnets):
  - collimators, targets, obstacles
  - vacuum chamber
  - drift
  - .....

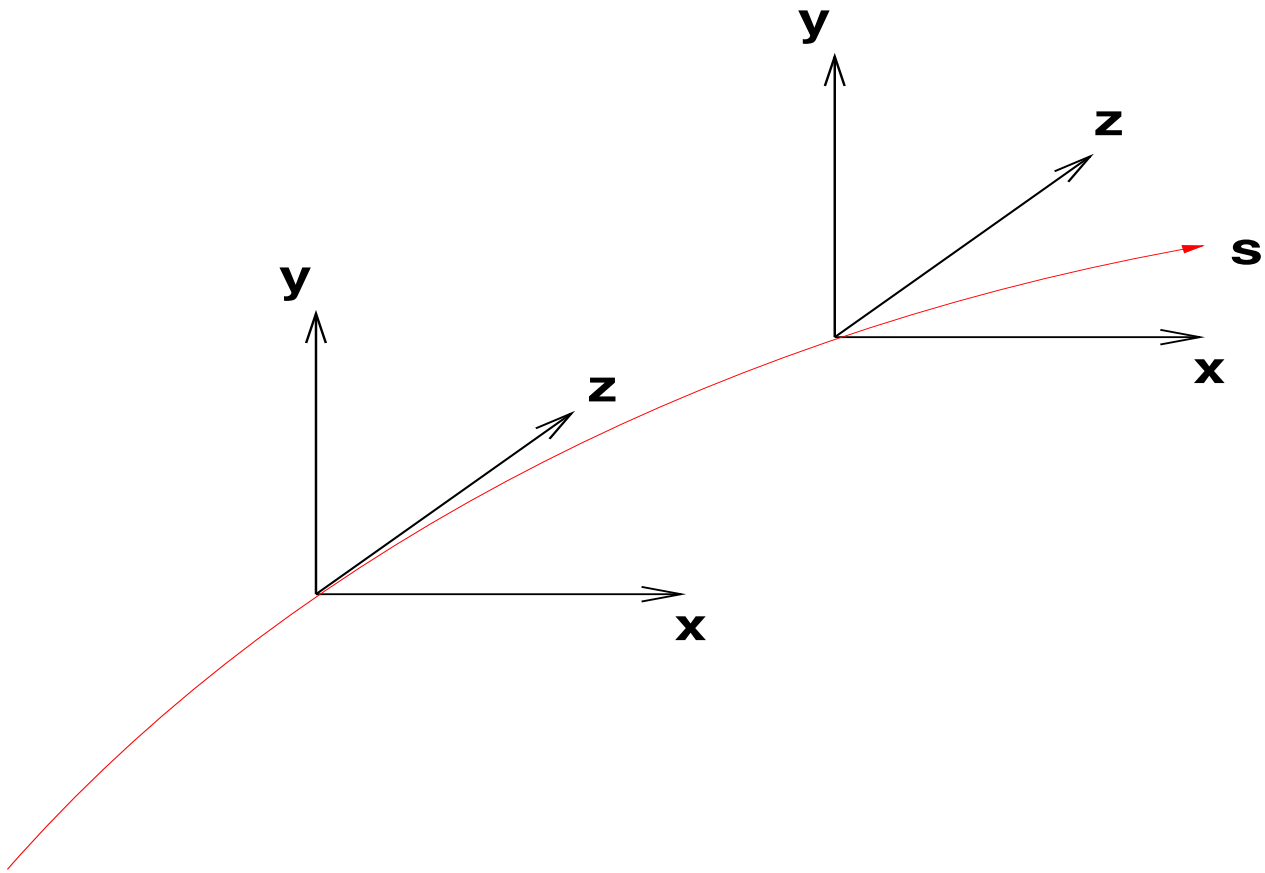
How is an element described ?

- Let $\vec{z_1}, \vec{z_2}$ describe a quantity (coordinates, beam sizes ...) before and after the element
- Take an machine element (e.g. magnet) and build a mathematical model $\mathcal{M}$
  - → $\mathcal{M}$ describes the element in terms of this quantity
  - → In general: $\vec{z_2} = \mathcal{M} \circ \vec{z_1}$
  - → $\mathcal{M}$ is a so-called map
  - → Very important: no need to know what happens in the rest of the machine !!
- The complete sequence of MAPS connects the pieces together to make a ring (or beam line)

Coordinates used

- **4 coordinates needed for 2 transverse dimensions**
- **Describe the deviations from the reference path**
- **Coordinate vector:**
  - $\vec{z} = (x, \quad px, \quad y, \quad py)$ **or:**
  - $\vec{z} = (x, \quad x' = \frac{\delta x}{\delta s}, \quad y, \quad y' = \frac{\delta y}{\delta s})$ **(see B. Holzer)**
- **Coordinate vector follows the reference path along s**

Coordinates used



Coordinates used

- **4 coordinates needed for 2 transverse dimensions**
- **Coordinate vector:**
  - $\vec{z} = (x, \quad px, \quad y, \quad py)$ **or:**
  - $\vec{z} = (x, \quad x' = \frac{\delta x}{\delta s}, \quad y, \quad y' = \frac{\delta y}{\delta s})$
- **another trap: how are the variables defined, which variables are used ?**
  - p_x **and** p_y **are canonical variables, are** x' **and** y' **?**
  - $x' = p_x/p_z$ **or** $x' = p_x/p_0$ **?**
  - **What if the beam has a momentum spread** $\Delta(p_z)/p_z \neq 0$ **?**

MAPS transform coordinates through an element

- **We use coordinate vector:** $\vec{z} = (x, \quad x' = \frac{\delta x}{\delta s}, \quad y, \quad y' = \frac{\delta y}{\delta s})$
- $\mathcal{M}$ **transforms the coordinates** $\vec{z_1}(s_1)$ **at position** s_1 **to new coordinates** $\vec{z_2}(s_2)$ **at position** s_2**:**

$$\vec{z_2}(s_2) = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \mathcal{M} \circ \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1} = \mathcal{M} \circ \vec{z_1}(s_1)$$

... or OPTICAL functions

- **6 optical functions used for 2 dimensions:**

$$\vec{\nu_2}(s_2) = \begin{pmatrix} \beta_x \\ \alpha_x \\ \gamma_x \\ \beta_y \\ \alpha_y \\ \gamma_y \end{pmatrix}_{s_2} = \mathcal{M} \circ \begin{pmatrix} \beta_x \\ \alpha_x \\ \gamma_x \\ \beta_y \\ \alpha_y \\ \gamma_y \end{pmatrix}_{s_1} = \mathcal{M} \circ \vec{\nu_1}(s_1)$$

How does $\mathcal{M}$ look like ?

The map $\mathcal{M}$ describes local properties of a machine element and can be:

- **Any "description" to go from $\vec{z_1}$ to $\vec{z_2}$**
- **Must be derived from physics, i.e. equation of motion**
- **Sounds maybe complicated, but we should try to develop tools that have obvious extensions to more complicated situations**
- **Is this description unique for a given element ?**

Is $\mathcal{M}$ unique for a given element ?

$$\vec{z_2} = \begin{pmatrix} cos(kL) & \frac{sin(kL)}{k} & 0 & 0 \\ -ksin(kL) & cos(kL) & 0 & 0 \\ 0 & 0 & cosh(kL) & \frac{sinh(kL)}{k} \\ 0 & 0 & ksinh(kL) & cosh(kL) \end{pmatrix} \vec{z_1}$$

$$\vec{z_2} = e^{(-L:\frac{k}{2}(x^2-y^2)+\frac{1}{2}(x'^2+y'^2):)}\,\vec{z_1}$$

- **Absolutely not: the representation depends on the purpose, can be very different in different programs**

What can $\mathcal{M}$ be ?

- This "description" can be:
  - A simple linear matrix or transformation
  - A non-linear transformation (Taylor series, Lie Transform ...)
  - High order integration algorithm
  - A computer program, subroutine etc.
- Let us look at linear theory first !

  (i.e. Transverse Dynamics, B. Holzer)

Simple examples (linear, one dimensional)
(Matrix formulation for linear* elements)

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22}+m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \circ \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_1}$$

➞ **The maps become so-called "transport matrices"**

* **The changes depend on x or x' only**

(Interlude: Σ-matrix)

The transformation of the optical functions can also be written using the Σ-matrix formalism:

$$\Sigma_{s_2} = \mathcal{M} \circ \Sigma_{s_1} \circ \mathcal{M}^T$$

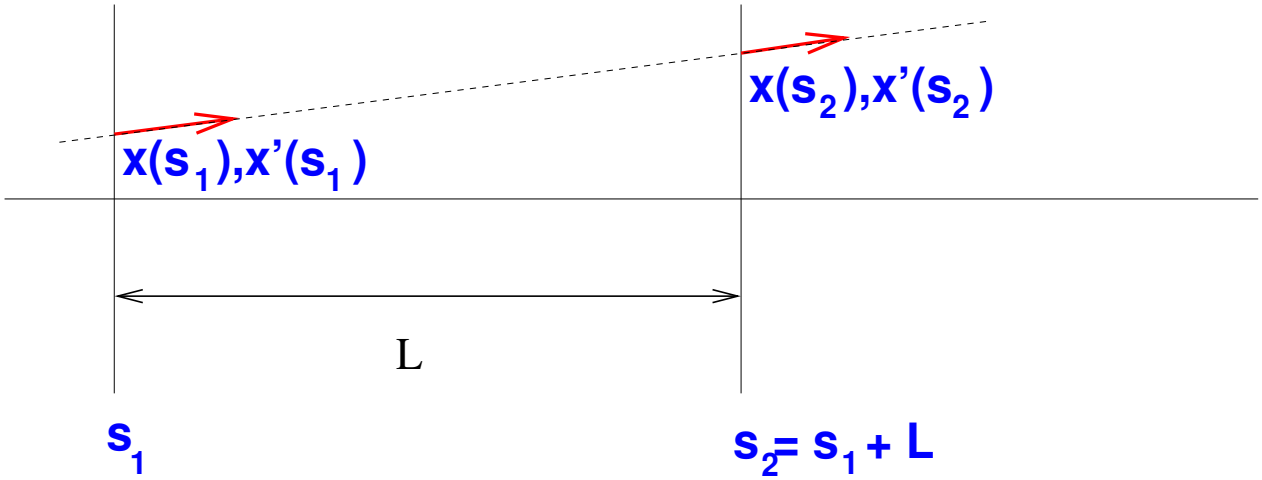
i.e. for example in the linear case:

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \circ \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_{s_1} \circ \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}$$

- **Allows formal extension to higher order effects (e.g. synchrotron radiation)**
- **Prove that it is equivalent to previous formula**

Transformation of coordinates (one dimension)

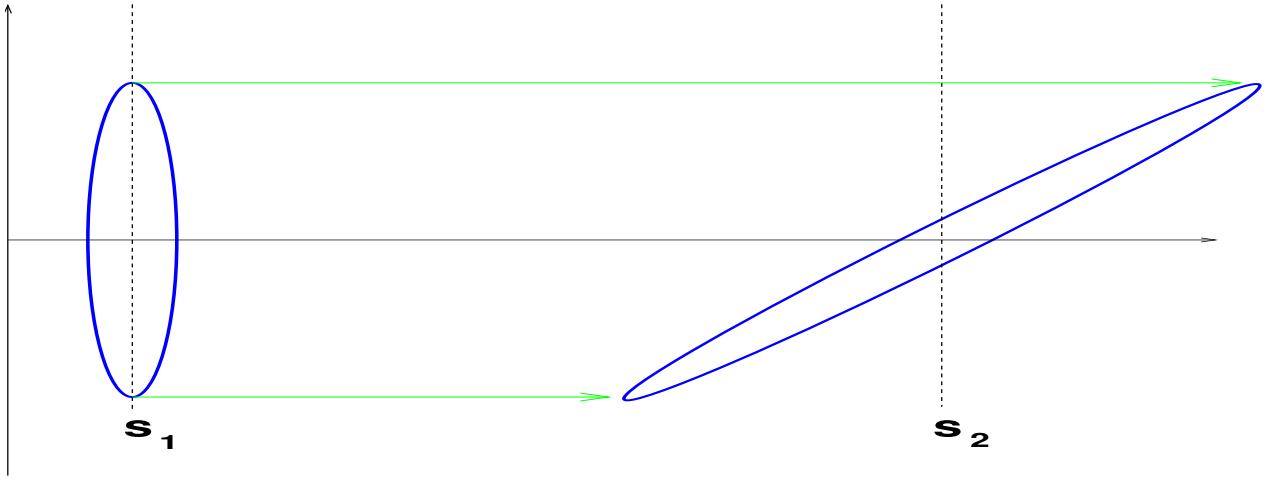
Drift space of length $L = s_2 - s_1$:



$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} + \begin{pmatrix} x' \cdot L \\ 0 \end{pmatrix}_{s_1} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

Transformation of beam ellipse

Drift space of length $L = s_2 - s_1$:



$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_2} = \begin{pmatrix} 1 & -2L & L^2 \\ 0 & 1 & -L \\ 0 & 0 & 1 \end{pmatrix} \circ \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_1} = \begin{pmatrix} \beta_0 - 2L\alpha_0 + L^2\gamma_0 \\ \alpha_0 - L\gamma_0 \\ \gamma_0 \end{pmatrix}_{s_2}$$

Simple examples (one dimensional)

Focusing quadrupole of length L and strength K:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = \begin{pmatrix} \cos(L \cdot K) & \frac{1}{K} \cdot \sin(L \cdot K) \\ K \cdot \sin(L \cdot K) & \cos(L \cdot K) \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

Quadrupole with short length L (i.e.: $1 \gg L^2 \cdot K^2$)

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = \begin{pmatrix} 1 & 0 \\ K^2 \cdot L(= -\frac{1}{f}) & 1 \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

Initial steps for optics calculation

- The optics program reads the sequence of elements of a machine (their order, their positions ..)
- It reads properties of the elements, i.e. type (dipole, quadrupole, drift ...)
- It reads strength of the elements
- It sets up the maps (matrices)

→ A "standard" for the input language exists, plus converters (do not forget this issue !)

Simplest machine description (MADX format)

```
// description of elements and their strengths
// dipoles and quadrupoles only ...
   mb: dipole,     l=6.0, angle=0.03570;
   qf: quadrupole, l=3.0, k1= 0.013426;
   qd: quadrupole, l=3.0, k1=-0.013426;

// centre position of elements in the ring
   start:    at=0;
   qf.1: qf, at=1.5000e+00;
   mb: mb,   at=9.0000e+00;
   mb: mb,   at=1.9000e+01;
   qd.1: qd, at=2.6500e+01;
   mb: mb,   at=3.4000e+01;
   mb: mb,   at=4.4000e+01;
   qf.2: qf, at=5.1500e+01;
             ...
   end:      at=2.2000e+03;
```

Putting the "pieces" together

Starting from a position s_0 and applying all maps (for N elements) in sequence around a ring with circumference C to get the One-Turn-Map (OTM) for the position s_0 (for one dimension only):

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0 \; + \; C} = \mathcal{M}_1 \circ \mathcal{M}_2 \circ \ldots \circ \mathcal{M}_N \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

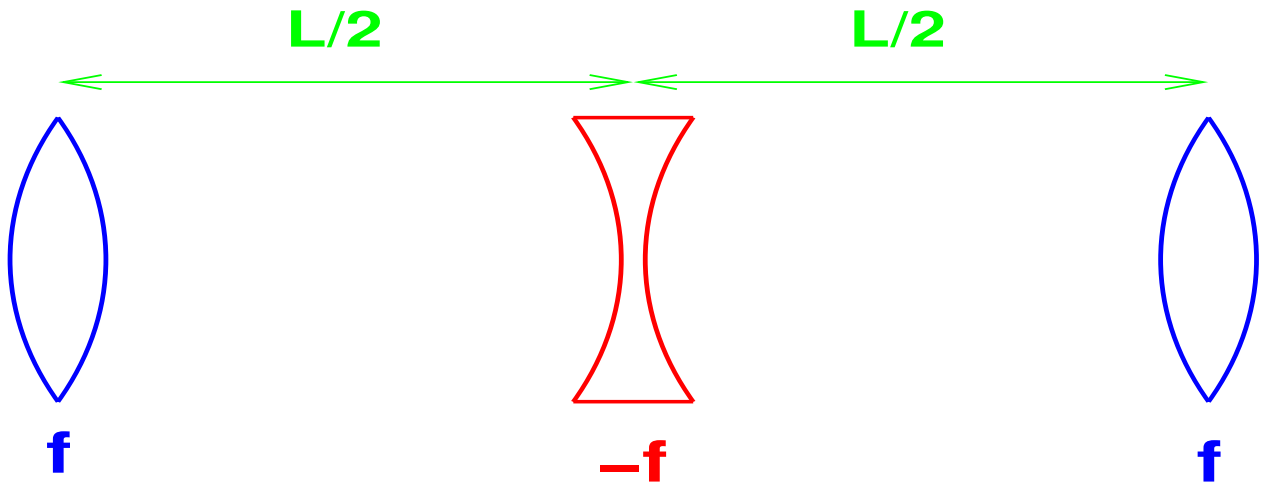
$$\Longrightarrow \quad \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0 \; + \; C} = \mathcal{M}_{ring}(s_0) \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

Combining maps together

$$\mathcal{M}_{ring}(s_0) = \quad \mathcal{M}_1 \quad \circ \quad \mathcal{M}_2 \quad \circ \quad ... \quad \circ \quad \mathcal{M}_N$$

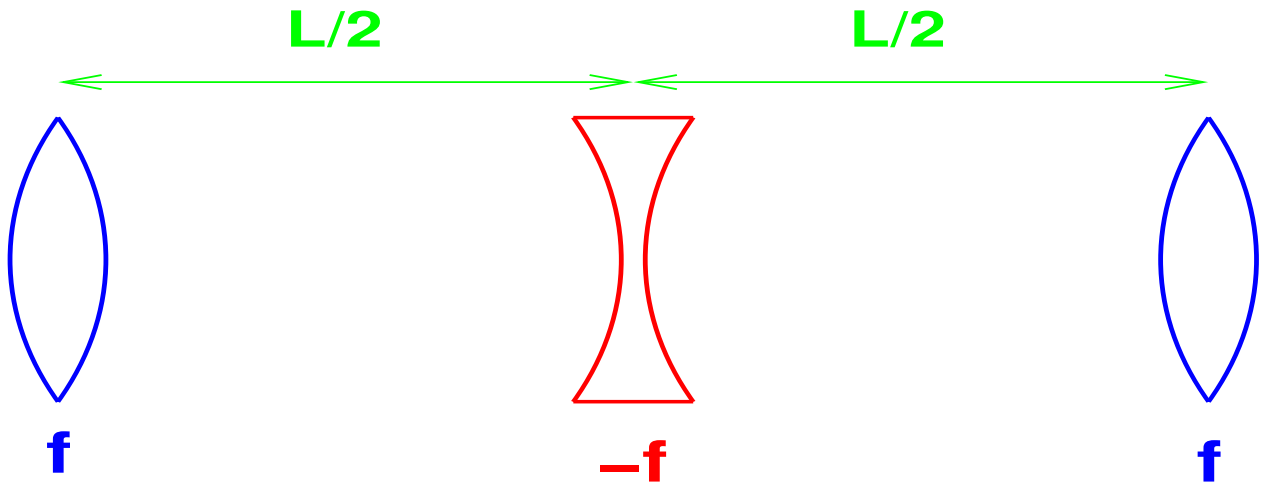
- **Need to combine the N maps together**
- **The results is a map for one turn**
- **How to combine maps ?**
- **Start with the simplest example:**

Composition of elements (FODO cell)
(here: simple matrix multiplications)



$$\mathcal{M}_{cell} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix}$$

Composition of elements (FODO cell)
(here: simple matrix multiplications)



$$\mathcal{M}_{cell} = \begin{pmatrix} 1 + \frac{L}{2f} & L + \frac{L^2}{4f} \\ -\frac{L}{2f^2} & 1 - \frac{L}{2f} - \frac{L^2}{4f^2} \end{pmatrix}$$

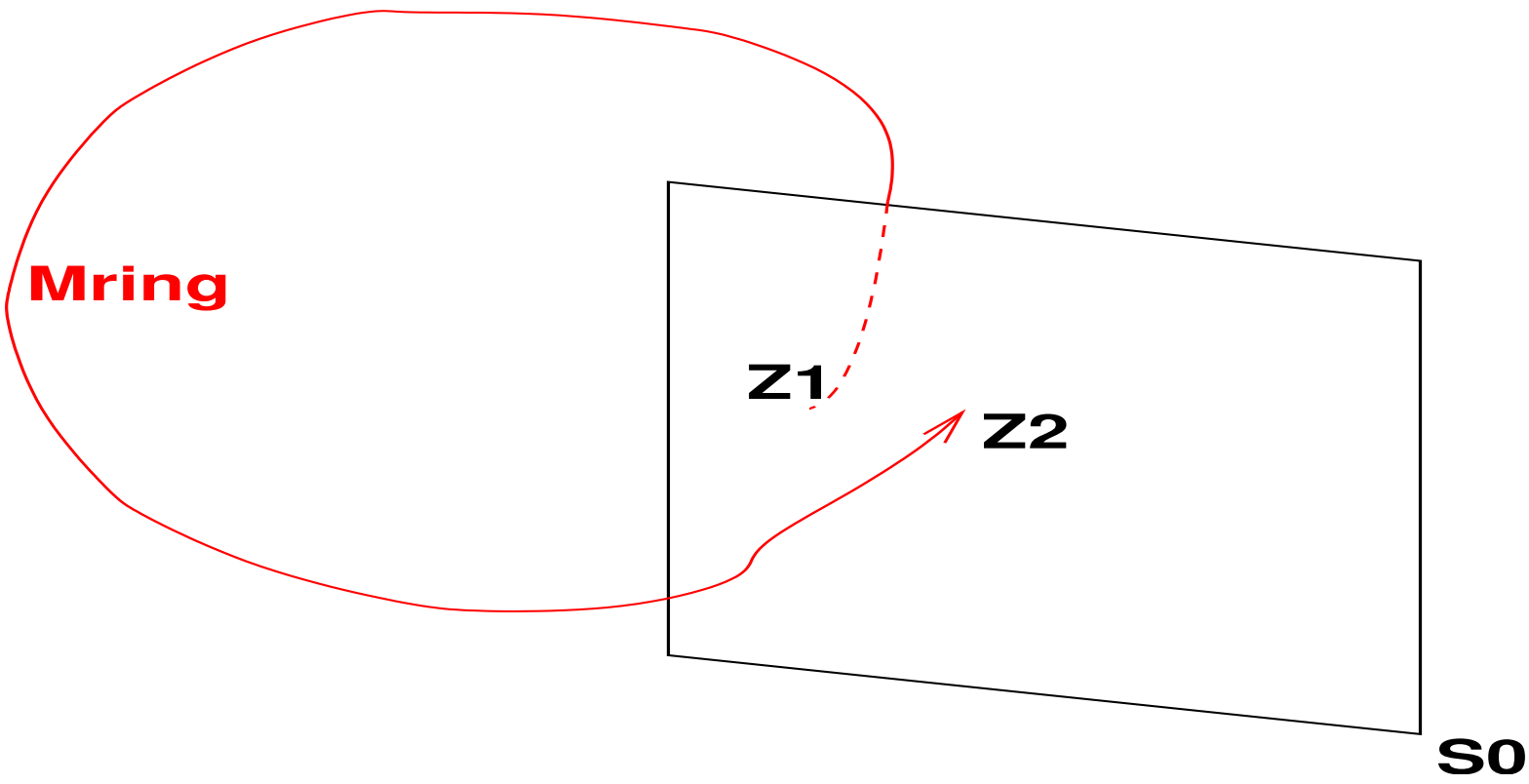
Combining maps together

For our FODO ring with N cells:

$$\mathcal{M}_{ring}(s_0) = \mathcal{M}_1 \circ \mathcal{M}_2 \circ \ldots \circ \mathcal{M}_N = \mathcal{M}_{cell}^N$$

- **Simple matrix multiplications for regular ring**
- **If maps are more complicated ? (see later)**
- **What is the physical picture of $\mathcal{M}_{ring}(s_0)$?**

$\mathcal{M}_{ring}$?



Why are we interested in $\mathcal{M}_{ring}$?

- The map $\mathcal{M}_{ring}$ is extremely important:
  - A computer does not know Hill's equation
  - Courant-Snyder ansatz (formalism) assumes motion is linearly stable, periodic, confined, and has a closed orbit.
  - A priori we do not know that ...
  - The OTM $\mathcal{M}_{ring}$ contains all information about global behaviour in the ring, i.e. stability, tune, β, closed orbit etc.
- No need for assumptions

What else can we do with $\mathcal{M}_{ring}$?

- $\mathcal{M}_{ring}$ (or $\mathcal{M}_{part}$) allow to derive global quantities
  - In the Courant-Snyder analysis of linear systems $\mathcal{M}_{ring}$ corresponds to a matrix
  - "Straightforward" to formally extend it to complicated (e.g. non-linear) problems
  - Allow the analysis of imperfections (and their correction !)
  - Additional tools and concepts needed (invariants, fixpoints, normal forms etc.)
- Demonstrate the Courant-Snyder analysis first

(Interlude: Fixed Points)

- Some points in phase space $\vec{z_1}$ are repeated after **n** completed turns (remember resonances !)

$$\mathcal{M}_{ring}^{n} \circ \vec{z_1} = \vec{z_2} \equiv \vec{z_1}$$

- Defines a **Fixed Point** of order **n**
- Fixed Point of order **1** is the **closed orbit**
- Stability requires existence of such a fixed point
- Closed orbit is found (or not !) in optics programs by searching for the first order fixed point (i.e. $\vec{z_1}$ at s_0)

Analysis of the One-Turn-Map (Matrix)

- Start simple: all maps are matrices (i.e. only linear elements)
- Usually the case for initial design

→ The One-Turn-Map is a One-Turn-Matrix:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0 \; + \; C} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

→ After all multiplications we get the One-Turn-Matrix which depends on the starting point s_0.

Find the tune Q

We can find the tune Q from the One-Turn-Matrix $\mathcal{M}_{ring}(s_0)$ by computing the eigenvalues of $\mathcal{M}_{ring}(s_0)$:

$$\det(\mathcal{M}_{ring}(s_0) - \lambda) = 0$$

gives

$$\lambda = \cos(2\pi Q) \pm i \cdot \sin(2\pi Q)$$

(verify with the One-Turn-Matrix you know from previous lecture !)

Analysis of the One-Turn-Matrix

What else can we do with the One-Turn-Matrix ?

We can express the One-Turn-Matrix $\mathcal{M}_{ring}(s_0)$ in terms of Courant-Snyder parameters:

We know that $\mathcal{M}_{ring}(s_0)$ for one dimension must be:

$$\mathcal{M}_{ring}(s_0) \equiv \begin{pmatrix} \cos\mu + \alpha(s_0)\sin\mu & \beta(s_0)\sin\mu \\ -\gamma(s_0)\sin\mu & \cos\mu - \alpha(s_0)\sin\mu \end{pmatrix}$$

and we also know that (for a ring):

$$\alpha(s_0+C) \equiv \alpha(s_0), \quad \beta(s_0+C) \equiv \beta(s_0), \quad \gamma(s_0+C) \equiv \gamma(s_0)$$

Get optical functions

Comparison of:

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \mathcal{M}_1 \circ \mathcal{M}_2 \circ \ldots \circ \mathcal{M}_N$$

and :

$$\mathcal{M}_{ring}(s_0) = \begin{pmatrix} \cos\mu + \alpha(s_0)\sin\mu & \beta(s_0)\sin\mu \\ -\gamma(s_0)\sin\mu & \cos\mu - \alpha(s_0)\sin\mu \end{pmatrix}$$

gives optical functions at position s_0:

- $\beta(s_0)$, $\alpha(s_0)$, $\gamma(s_0)$ **(depend on position s_0)**
- μ **is independent of s_0:** **(2πQ)**

We have now:

- Values for β_x, β_y, α_x, ... etc. at the position s_0
- Tunes for both planes, closed orbit

The next step:

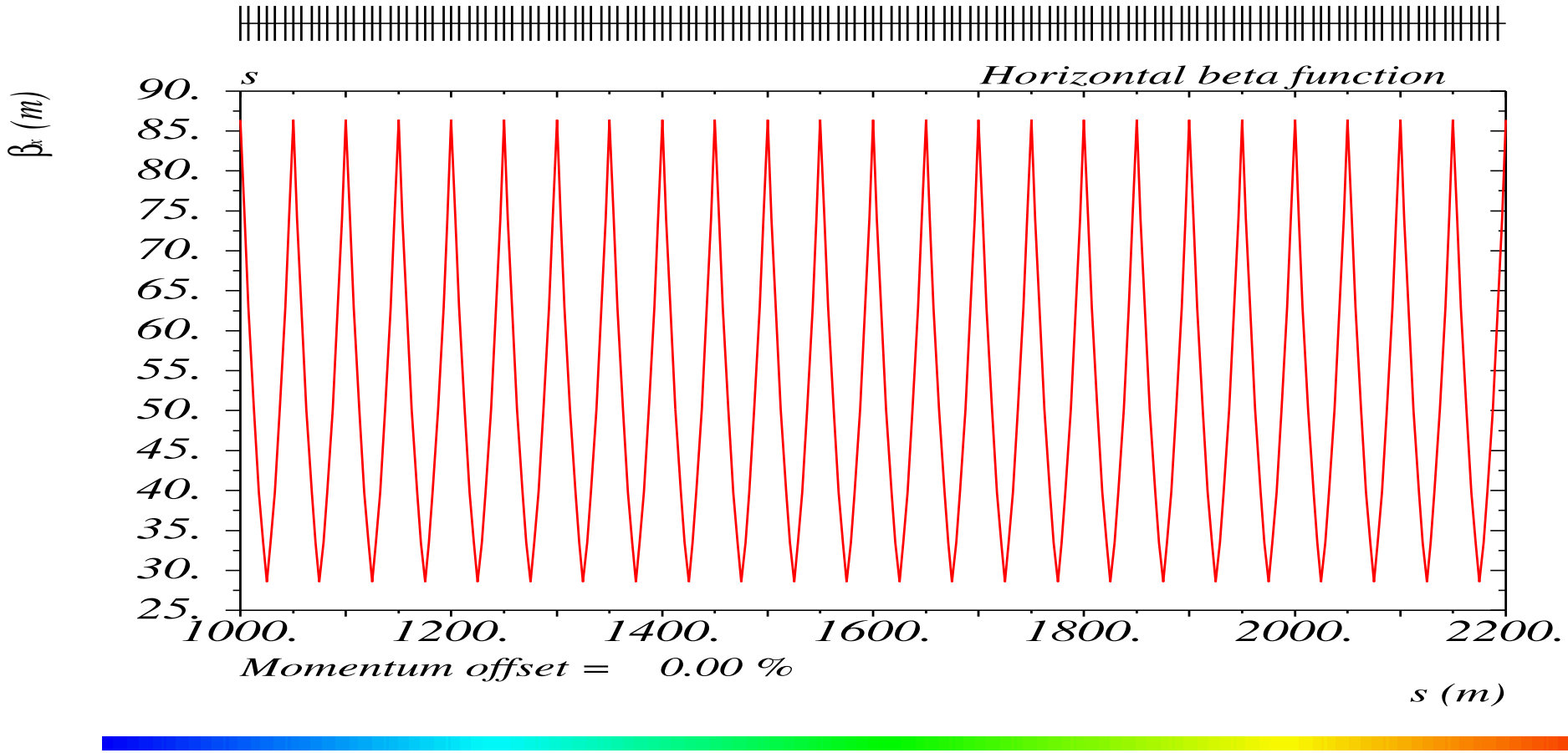
- Starting from initial optical (Twiss) functions at s_0, transforming β_x, β_y, α_x, ... through the lattice gives functions at all positions s.
- Question: what are the β-functions etc. of a linear accelerator or a beam line ???

Get the optical functions around the ring

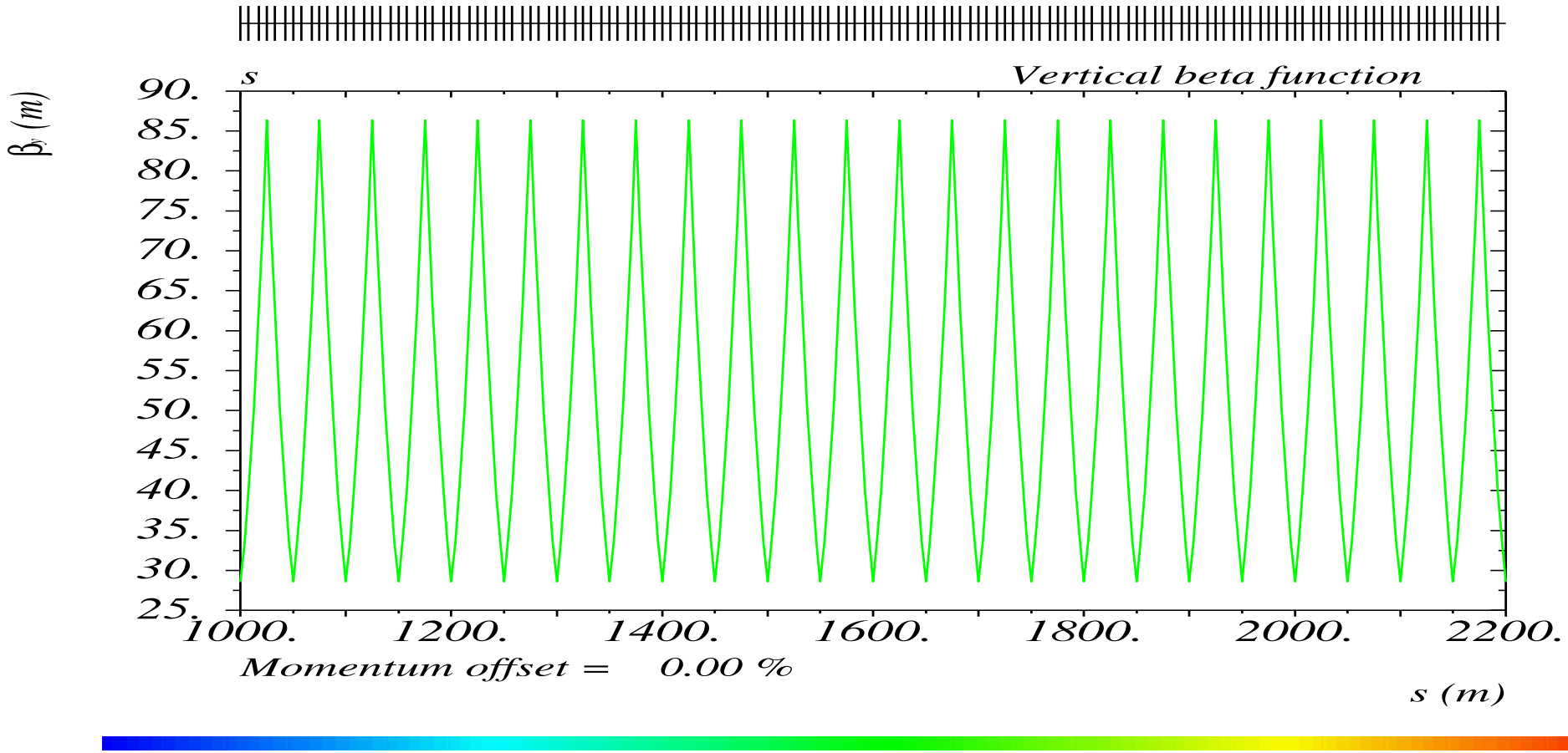
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{11}m_{22}+m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \circ \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_0}$$

Successive application of matrices give Twiss functions at each element around the ring and at each position s ➞

Optical functions (horizontal β):



Optical functions (vertical β):



Extension to two dimensions

- **Can be written as separate equations:**

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

$$\begin{pmatrix} y \\ y' \end{pmatrix}_{s_2} = \begin{pmatrix} m_{33} & m_{34} \\ m_{43} & m_{44} \end{pmatrix} \circ \begin{pmatrix} y \\ y' \end{pmatrix}_{s_1}$$

Extension to two dimensions

- **Extend vectors for coordinates or optical parameters**
- **Extend transfer maps/matrices**

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11} & m_{12} & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{43} & m_{44} \end{pmatrix} \circ \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1}$$

Extension to two dimensions (coupling)

- **The horizontal and vertical motion can be coupled:**

→ **Additional elements in matrix**

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \circ \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1}$$

- **Q: what about the β-functions now ??**

Going to three dimensions

- **Formally extended by adding two new variables:**
  - **(x, x', y, y', Δs, $\frac{\Delta p}{p}$)**
  - **$\Delta s = c\Delta t$: longitudinal displacement with respect to reference particle**
  - **$\frac{\Delta p}{p}$: relative momentum difference with respect to reference particle**

Going to three dimensions

→ **With (x, x', y, y', Δs, $\frac{\Delta p}{p}$)**

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ c\Delta t \\ \frac{\Delta p}{p} \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} \\ m_{32} & m_{32} & m_{33} & m_{34} & m_{35} & m_{36} \\ m_{42} & m_{42} & m_{43} & m_{44} & m_{45} & m_{46} \\ m_{32} & m_{32} & m_{33} & m_{34} & m_{55} & m_{56} \\ m_{62} & m_{62} & m_{63} & m_{64} & m_{65} & m_{66} \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \\ c\Delta t \\ \frac{\Delta p}{p} \end{pmatrix}_{s_1}$$

Going to three dimensions

- **Formally extended by adding two new variables:**
  - **(x, x', y, y',** Δs**,** $\frac{\Delta p}{p}$**)**
  - $\Delta s = c\Delta t$**: longitudinal displacement with respect to reference particle**
  - $\frac{\Delta p}{p}$**: relative momentum difference with respect to reference particle**

- **Not all programs use this, but rather canonical variables**
  - **(x,** p_x/p_s**, y,** p_y/p_s**,** $-c\Delta t$**,** $p_t = \frac{\Delta E}{p_s c}$**)**

- p_s **may be:** $p_s = p_0$ **or** $p_s = p_0(1+\delta_s) = m\beta_s\gamma_s$

  δ_s**: difference between reference momentum and design momentum**

Off momentum effects

- **Correct Hamiltonian for a magnetic element:**

$$\mathcal{H} = -(1 + \frac{x}{\rho})\sqrt{(1+\delta)^2 - x'^2 - y'^2} + V(x, y)$$

- **Non-linear terms due to kinematics:** $(1+\delta)^2$, **(even with quadrupoles only !)**
- **After expansion and keep second order (e.g. large machines):**

$$\mathcal{H} = \frac{(x'^2 + y'^2)}{2(1+\delta)} + V(x, y)$$

Off momentum effects

Strengths k of elements modified by non-zero $\delta = \frac{\Delta p}{p}$:

$$k \Longrightarrow k/(1 + \frac{\Delta p}{p}) = k/(1 + \delta)$$

- **Closed orbit and tune are usually different for non-zero $\frac{\Delta p}{p}$**

→ **Dispersion**

→ **Chromatic effects**

⚠ **Beware of small rings or large δ ....**

Matching optical functions

- Modify machine optics to get desired properties around the machine or in specific places
- For example you may want special conditions
  - for equipment: RF, collimators, diagnostics
  - for experiments: in colliding beam machines
- Algorithms to adjust parameters and layout

→ This process is called MATCHING !

→ Available in most optics programs (for lines and circular machines)

General purpose optics programs

- Always allow to:
  - Compute optical parameters (Twiss functions)
  - Match the required properties
- Often allow to:
  - Simulate machine imperfections
  - Correct imperfections

Popular Optics Programs

- BeamOptics (based on Mathematica)
- TURTLE (Beam lines)
- WINAGILE (WINDOWS, interactive, originally for teaching)
- TRANSPORT (General purpose, third-order matrix)
- DIMAD (Second-order matrix, tracking)
- TEAPOT (General purpose, thin element approximation)

Popular Optics Programs (cont.)

- COSY (Multi purpose, high order maps, differential algebra)
- SYNCH (General purpose)
- MAD (versions: 8,9,X) (Multi purpose optics and tracking)
- SAD (Multi purpose optics and tracking)
- MARYLIE (Lie algebra, tracking)
- PTC (MAP based, object oriented)
- MADX-PTC (combined MADX-PTC)

Which Optics Program should I use ?

Very application dependent, you have:

- **Beam line**
- **Large ring**
- **Small ring**
- **Large momentum offset or changing momentum (e.g. FFAG, acceleration)**
- **Linear accelerator**
- **Unconventional geometry**
- **Collider (one or more rings/lines)**

Which Optics Program should I use ?

Very application dependent, you want to do:

- **Design linear optics**
- **Linear optical matching**
- **Introduce and correct imperfections**
- **Non-linear optical matching**
- **Particle tracking**
- **Evaluate the dynamic aperture**
- **Study collective effects**

(Interlude: Course on optics design)

- Intermediate level CAS 2003, 2005, 2007 (and maybe 2009) offers a course on optics design
  - Purpose is to develop a realistic accelerator optics
  - Includes correction elements, optical matching, dispersion suppressors ...
  - MAD is used for practical implementation
- The course is available on CD-ROM (on request) or from the web

(Reminder: Symplecticity)

- **Not all possible maps are allowed !**
- **Requires for a matrix** $\mathcal{M} \rightarrow \mathcal{M}^T \cdot S \cdot \mathcal{M} = \mathcal{S}$

  with:

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

- **It basically means:** $\mathcal{M}$ **is area preserving and**

$$\lim_{n \to \infty} \mathcal{M}^n = \text{finite} \qquad \Longrightarrow \qquad \det \mathcal{M} = 1$$

Introducing non-linear elements

Effect of a (short) quadrupole depends linearly on amplitude (re-written from the matrix form):

$$\vec{z}(s_2) = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1} + \begin{pmatrix} 0 \\ k_1 \cdot x_{s_1} \\ 0 \\ k_1 \cdot y_{s_1} \end{pmatrix}$$

→ $\vec{z}(s_2) = \mathbf{M} \cdot \vec{z}(s_1)$

→ **M is a matrix**

Non-linear elements (e.g. sextupole)

Effect of a (thin) sextupole with strength k_2 is:

$$\vec{z}(s_2) = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1} + \begin{pmatrix} 0 \\ k_2 \cdot (x_{s_1} \cdot y_{s_1}) \\ 0 \\ \frac{1}{2} k_2 \cdot (x_{s_1}^2 - y_{s_1}^2) \end{pmatrix}$$

- $\vec{z}(s_2) = \mathcal{M} \circ \vec{z}(s_1)$
- $\mathcal{M}$ **is not a matrix, i.e. cannot be expressed by matrix multiplication**

Non-linear elements

Cannot be written in linear matrix form !

We need something like:

$$
\begin{aligned}
z_1(s_2) = x(s_2) \quad = \quad & R_{11} \cdot x \quad + R_{12} \cdot x' \quad + R_{13} \cdot y + ... \\
& + T_{111} \cdot x^2 + T_{112} \cdot xx' + T_{122} \cdot x'^2 + \\
& + T_{113} \cdot xy + T_{114} \cdot xy' + ... \\
& + U_{1111} \cdot x^3 + U_{1112} \cdot x^2 x' + ...
\end{aligned}
$$

and the equivalent for all other variables ...

Higher order (Taylor -) MAPS:

We have (for: j = 1...4):

$$z_j(s_2) \;=\; \sum_{k=1}^{4} R_{jk} z_k(s_1) \;+ \sum_{k=1}^{4}\sum_{l=1}^{4} T_{jkl} z_k(s_1) z_l(s_1)$$

Let's call it : $\quad \mathcal{A}_2 \;=\; [R, T] \quad$ (second order map $\mathcal{A}_2$)

Higher orders can be defined as needed ...

$$\mathcal{A}_3 \;=\; [R, T, U] \quad \Longrightarrow \quad + \sum_{k=1}^{4}\sum_{l=1}^{4}\sum_{m=1}^{4} U_{jklm} z_k(s_1) z_l(s_1) z_m(s_1)$$

Second order MAPS concatenation

Assume now 2 maps of second order:

$\mathcal{A}_2 = [R^A, T^A]$ **and** $\mathcal{B}_2 = [R^B, T^B]$

the combined second order map

$\mathcal{C}_2 = \mathcal{A}_2 \circ \mathcal{B}_2$ **is** $\mathcal{C}_2 = [R^C, T^C]$ **with:**

$$R^C = R^A \cdot R^B$$

and (after truncation of higher order terms !!):

$$T^C_{ijk} = \sum_{l=1}^{4} R^B_{il} T^A_{ljk} + \sum_{l=1}^{4} \sum_{m=1}^{4} T^B_{ilm} R^A_{lj} R^A_{mk}$$

Symplecticity for higher order MAPS

- **Truncated Taylor expansions are not matrices !!**
- **It is the associated Jacobian matrix $\mathcal{J}$ which must fulfil the symplecticity condition:**

$$\mathcal{J}_{ik} = \frac{\delta z_2^i}{\delta z_1^k}$$

$$\mathcal{J} \quad \textbf{must fulfil:} \quad \mathcal{J}^t \cdot S \cdot \mathcal{J} = \mathcal{S}$$

- **In general: $\mathcal{J}_{ik} \neq$ const → for truncated Taylor map can be difficult to fulfil for all z**

Symplecticity for higher order MAPS

try truncated Taylor map in 2D, second order:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} R_{11}x_0 + R_{12}x'_0 + T_{111}x_0^2 + T_{112}x_0x'_0 + T_{122}x'^2_0 \\ R_{21}x_0 + R_{22}x'_0 + T_{211}x_0^2 + T_{212}x_0x'_0 + T_{222}x'^2_0 \end{pmatrix}$$

The Jacobian becomes:

$$\mathcal{J} = \begin{bmatrix} R_{11} + 2T_{111}x_0 + T_{112}x'_0 & R_{12} + T_{112}x_0 + 2T_{122}x'_0 \\ R_{21} + 2T_{211}x_0 + T_{212}x'_0 & R_{22} + T_{212}x_0 + 2T_{222}x'_0 \end{bmatrix}$$

symplecticity condition requires that:

det $\mathcal{J}$ = 1 for all x_0 and all x'_0

Symplecticity for higher order MAPS

This is only possible for the conditions:

$$\left(\begin{array}{c} R_{11}R_{22} - R_{12}R_{21} = 1 \\ R_{11}T_{212} + 2R_{22}T_{111} - 2R_{12}T_{211} - R_{21}T_{112} = 0 \\ 2R_{11}T_{222} + R_{22}T_{112} - R_{12}T_{212} - 2R_{21}T_{122} = 0 \end{array}\right)$$

- **10 coefficients, but 3 conditions**
- **number of independent coefficients only 7 !**
- **Taylor map requires more coefficients than necessary**
- **e.g. 4D, order 4: coefficients 276 instead of 121**

Using higher order MAPS

- Needed to study long term stability and derive dynamics quantities
- Basic questions:
  - Is it sufficiently high order (i.e. accurate) ?
  - Is it sufficiently symplectic ?
  - How do we generate a high order map efficiently (accurate and symplectic) ?
- ⚠ Many programs and algorithms available, not all are always appropriate

Various types of higher order MAPS

- Choice depends on the application
  - Taylor maps (large number of coefficients)
  - Lie transformations (minimum number of coefficients)
  - Truncated power series algebra (TPSA), can also generate Taylor map from tracking
  - Symplectic integration techniques (see later)
  - ...
- The catch: which is the truncation order ??

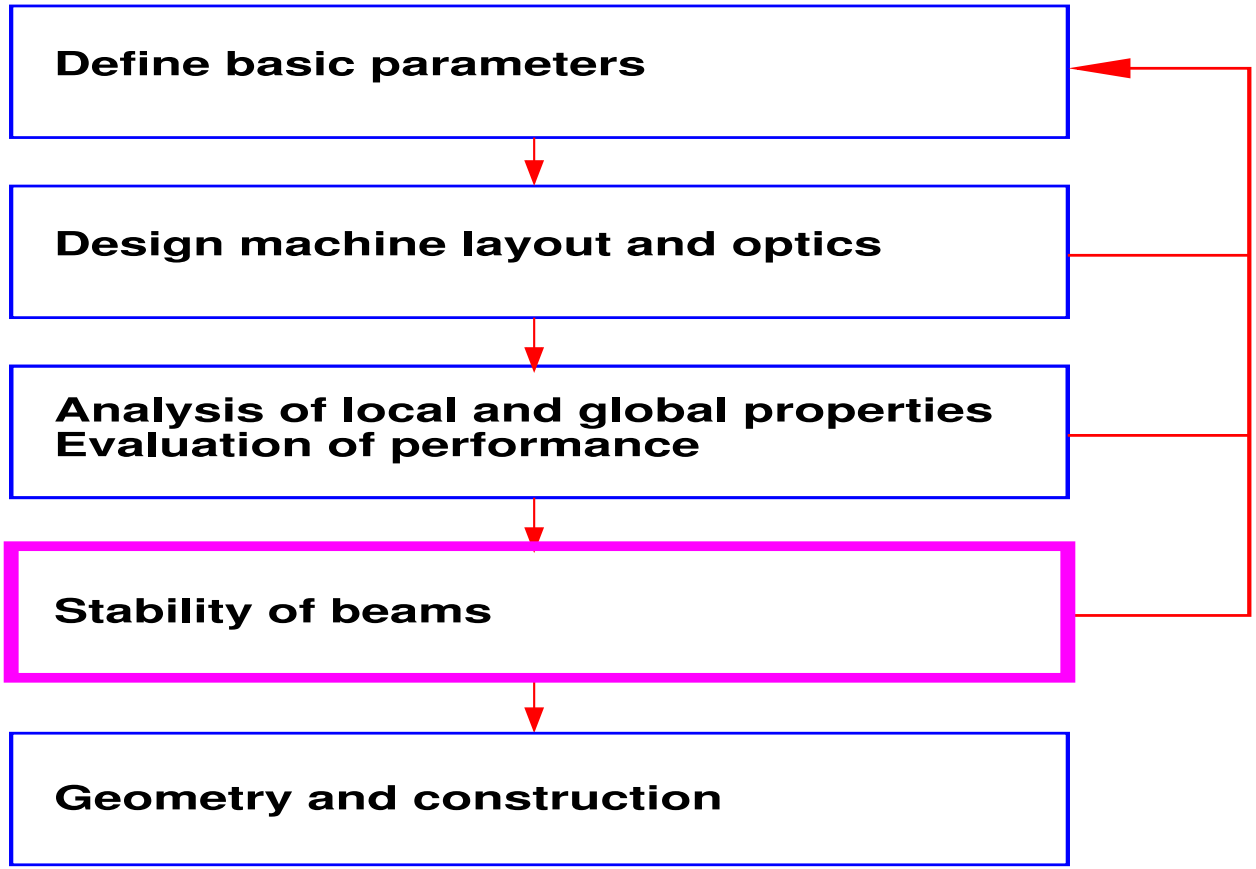
Single particle dynamics in a nutshell

- Try to compute a (one turn) MAP
- It contains everything
- Its analysis will tell you what you need to know
- It doesn't matter how you got it (analytically, tracking ..) !
- Benchmark against simulation, i.e. element by element tracking, to ensure that approximations do not affect physics

You can derive:

- Tunes (we already did)
- Betatron functions (we already did)
- Stability borders, dynamic aperture (within limits)
- Detuning with amplitudes
- Invariants of the motion
- Fixpoints, closed orbit, resonance strength
- More difficult: long term stability of the beam !

Steps of accelerator design



Simulation of an accelerator

- **Purpose is to imitate the behaviour of a particle or a beam in an accelerator**
- **Use local properties of the machine element to describe its interaction with a particle**
- **The aim is to derive the global behaviour**
  - **Long term stability**
  - **Lifetime**
  - **....**
- **An exact simulation code is the most reliable "map"**

Evaluation by simulation (1)

- Single particle behaviour
- Usually concerns long term behaviour such as beam loss, effect of the accelerator components on a single particle
  - → Non-linear elements
  - → Machine imperfections (e.g. field, alignment errors)
  - → External distortions (e.g. scattering)

Evaluation by simulation (2)

- Multi particle behaviour
- Usually concerns collective behaviour: coherent motion, emittance increase, damping etc.
  - → The effect of the accelerator components on an ensemble of particles (e.g. impedances)
  - → Interactions of particles between each other. These are usually dictated by the properties of the accelerator (e.g. space charge, beam-beam effects ...)

Most popular: single particle tracking

- **The motion of a test particle through the elements of a machine is simulated for a large number of turns → "tracking"**
  - **Use appropriate coordinates, start with $\vec{z_0}$.**
  - **In each element (or part of the machine), the coordinates are transformed by $\vec{z_{n+1}} = \mathcal{M} \circ \vec{z_n}$**
  - **$\mathcal{M}$ must be symplectic of course**
- **We follow the evolution of the coordinates $\vec{z_{n+1}}$, not of the maps $\mathcal{M}$**
- **We must distinguish thick and thin elements**

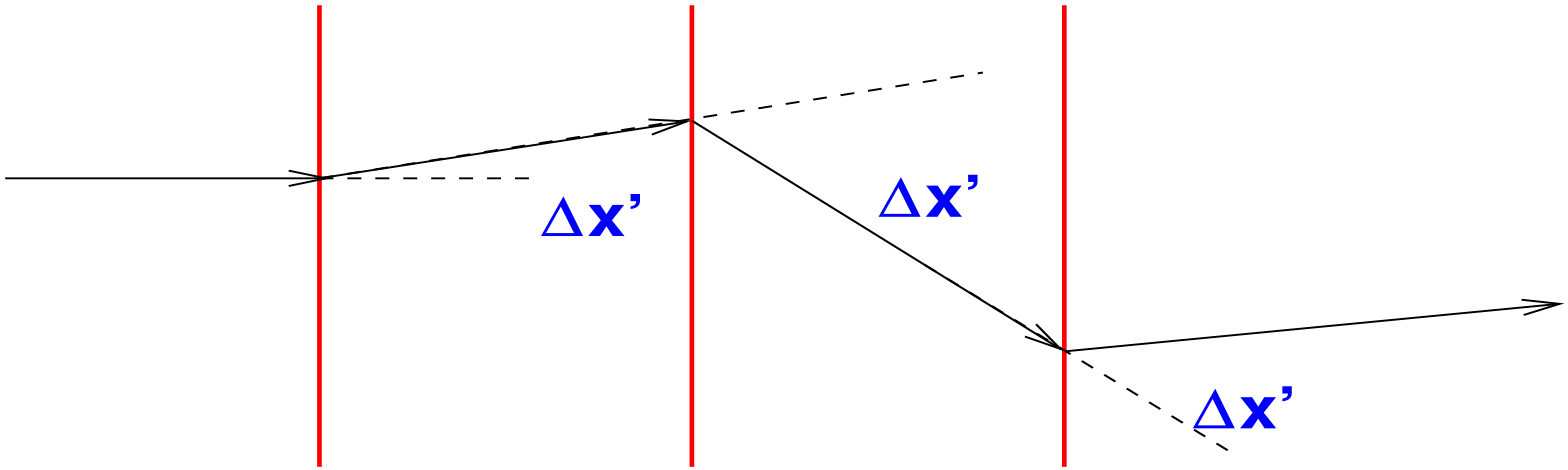
Thick versus thin magnets

- Real magnets have a finite length, i.e. thick magnets
- Thick magnet: field and length used to compute effect, i.e. the map
- Consequence: they are not always linear elements (also not dipoles, quadrupoles)
- For thick, non-linear magnets closed solution for maps often does not exist !

Thick versus thin magnets

- Thin "magnet": let the length go to zero, but keep field integral finite (constant)
- Thin dipoles and quadrupoles are linear elements
- Thin elements are much easier to use ...

Tracking through thin elements



- **No change of amplitudes x and y**
- **The momenta x' and y' receive an amplitude dependent deflection (kick)**

→ $x' \rightarrow x' + \Delta x'$ **and** $y' \rightarrow y' + \Delta y'$

Tracking through thin elements

- These programs are so-called kick-codes
- Kick can be non-linear
- Always symplectic ! (homework)
- Usually rather fast on computers
- Most tracking programs are of this type

Using thin elements

- Can we approximate a thick element by thin element(s) ?
  - Yes, when the length is small or does not matter
  - Yes, when we can model the thick magnet correctly
  - What about accuracy, symplecticity etc. ??
  - Demonstrate with some simple examples

Simple example (1D, see B. Holzer)

Focusing quadrupole of length L and strength K:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1+L} = \begin{pmatrix} \cos(L \cdot K) & \frac{1}{K} \cdot \sin(L \cdot K) \\ -K \cdot \sin(L \cdot K) & \cos(L \cdot K) \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

Quadrupole with short length L (i.e.: $1 \gg L^2 \cdot K^2$)

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1+L} = \begin{pmatrix} 1 & 0 \\ -K^2 \cdot L(= -\frac{1}{f}) & 1 \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

Thick → thin quadrupole

$$\mathcal{M}_{s\to s+L} = \begin{pmatrix} \cos(L\cdot K) & \frac{1}{K}\cdot\sin(L\cdot K) \\ -K\cdot\sin(L\cdot K) & \cos(L\cdot K) \end{pmatrix}$$

- **Exact map (matrix) for quadrupole**
- **For efficiency:**
  - **Can we avoid trigonometric functions ?**
  - **What do we lose ?**
- **(What follows is valid for all elements)**

Accuracy of thin lenses

$$\mathcal{M}_{s\to s+L} = \begin{pmatrix} \cos(L\cdot K) & \frac{1}{K}\cdot\sin(L\cdot K) \\ -K\cdot\sin(L\cdot K) & \cos(L\cdot K) \end{pmatrix}$$

- **Start with exact map**
- **Taylor expansion in "small" length L:**

$$L^0\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + L^1\cdot\begin{pmatrix} 0 & 1 \\ -K^2 & 0 \end{pmatrix} + L^2\cdot\begin{pmatrix} -\frac{K^2}{2} & 0 \\ 0 & -\frac{K^2}{2} \end{pmatrix} + \ldots$$

Accuracy of thin lenses (B)

- **Keep up to first order term in L**

$$\mathcal{M}_{s\to s+L} = L^0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + L^1 \cdot \begin{pmatrix} 0 & 1 \\ -K^2 & 0 \end{pmatrix}$$

$$\mathcal{M}_{s\to s+L} = \begin{pmatrix} 1 & L \\ -K^2 \cdot L & 1 \end{pmatrix} + \mathcal{O}(L^2)$$

- **Precise to first order $\mathcal{O}(1)$**
- **$\det \mathcal{M} \neq 1$, non-symplectic**

Accuracy of thin lenses (C)

$$\mathcal{M}_{s\to s+L} = \begin{pmatrix} 1 & L \\ -K^2 \cdot L & 1 \end{pmatrix} + \mathcal{O}(L^2)$$

$$\mathcal{M}_{s\to s+L} = \begin{pmatrix} 1 & L \\ -K^2 \cdot L & 1-K^2L^2 \end{pmatrix}$$

- **Precise to first order $\mathcal{O}(1)$**
- **"symplectified" by adding term $-K^2L^2$**

  (it is wrong to $\mathcal{O}(2)$ anyway ...)

Accuracy of thin lenses

- **Keep up to second order term in L**

$$\mathcal{M}_{s\to s+L} = \begin{pmatrix} 1-\frac{1}{2}K^2L^2 & L \\ -K^2\cdot L & 1-\frac{1}{2}K^2L^2 \end{pmatrix} + \mathcal{O}(L^3)$$

- **Precise to second order $\mathcal{O}(2)$**
- **More accurate than (C), but not symplectic**

Accuracy of thin lenses (D)

- **Symplectification like:**

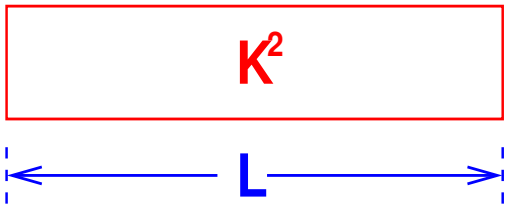
$$\mathcal{M}_{s\to s+L} = \begin{pmatrix} 1-\frac{1}{2}K^2L^2 & L-\frac{1}{4}K^2L^3 \\ -K^2\cdot L & 1-\frac{1}{2}K^2L^2 \end{pmatrix} + \mathcal{O}(L^3)$$

- **Precise to second order $\mathcal{O}(2)$**
- **Fully symplectic**

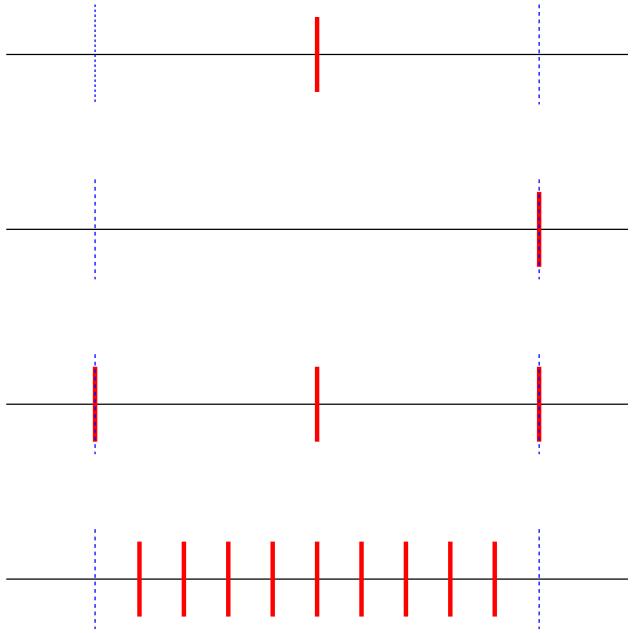
Accuracy of thin lenses

- Is there a physical picture behind the approximations ?
- Yes, geometry of thin lens kicks ...
- A thick element is split into thin elements with drifts between them

Thick → thin quadrupole

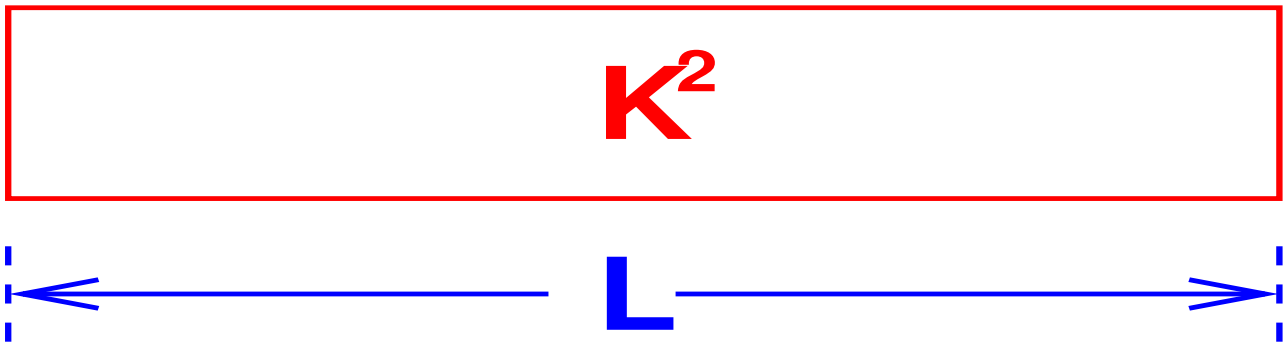


quadrupole of
finite length

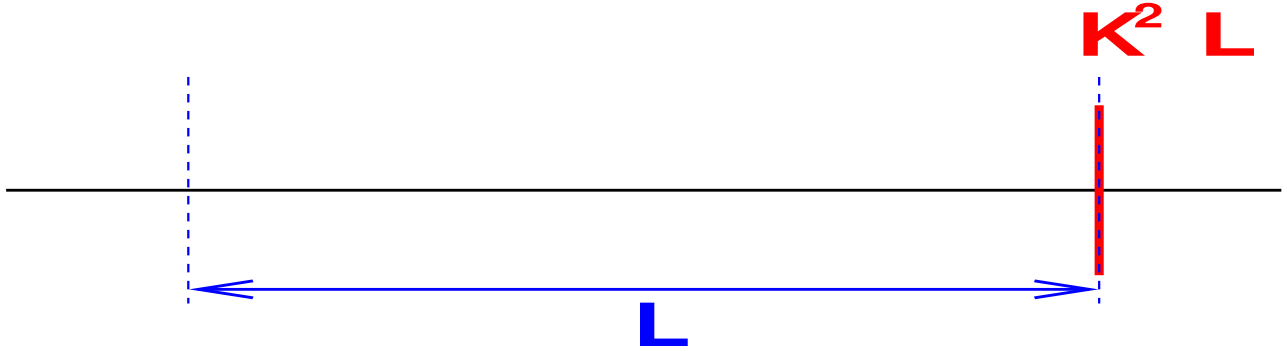


options:

Thick quadrupole ..



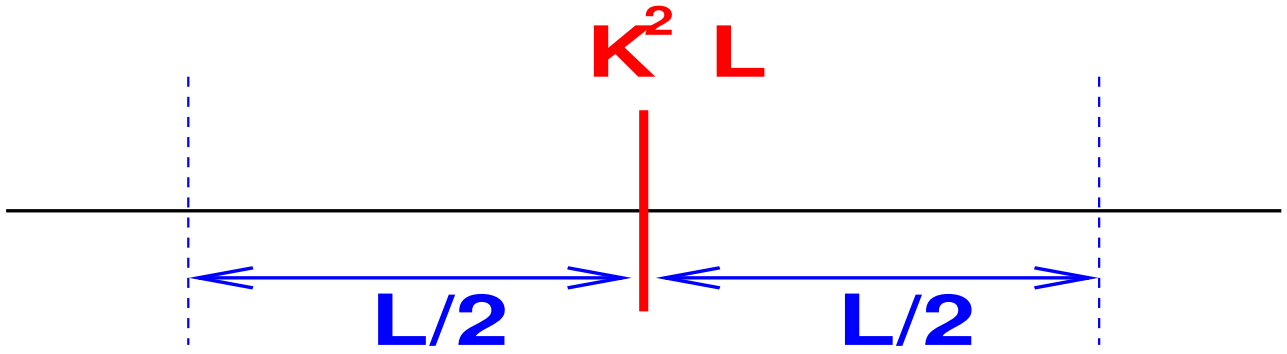
First order ..



- One thin quadrupole "kick" and one drift combined
- Resembles "symplectification" of type (C)

$$\begin{pmatrix} 1 & 0 \\ -K^2 \cdot L & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & L \\ -K^2 \cdot L & 1 - K^2 L^2 \end{pmatrix}$$

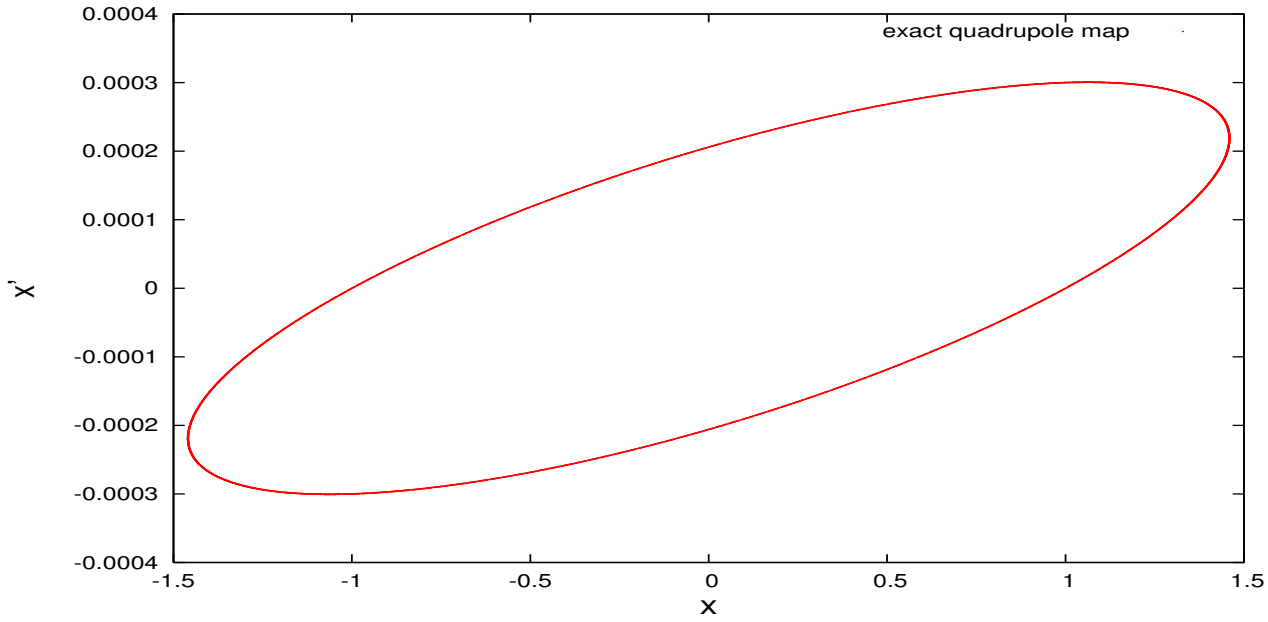
Second order ..



- One thin quadrupole "kick" between two drifts
- Resembles more accurate, symplectic model of type (D)

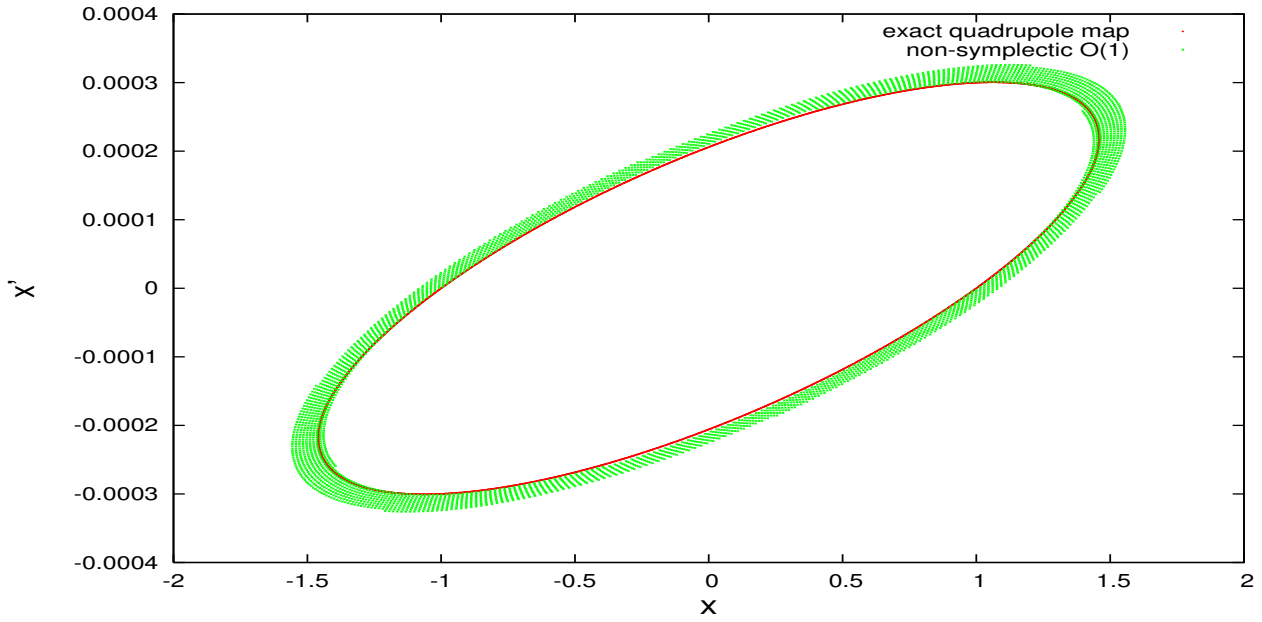
$$\begin{pmatrix} 1 & \frac{1}{2}L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -K^2 \cdot L & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2}L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}K^2L^2 & L - \frac{1}{4}K^2L^3 \\ -K^2 \cdot L & 1 - \frac{1}{2}K^2L^2 \end{pmatrix}$$

What is the point ???



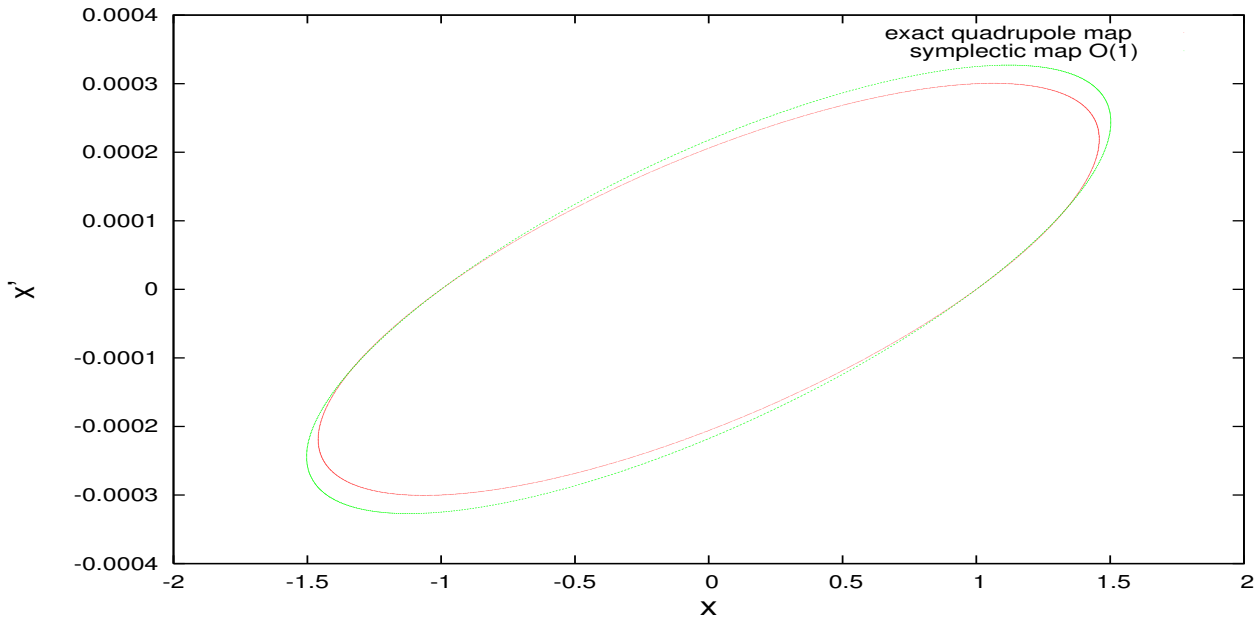
- **Phase ellipse - quadrupole exact solution**

Quadrupole non-symplectic solution



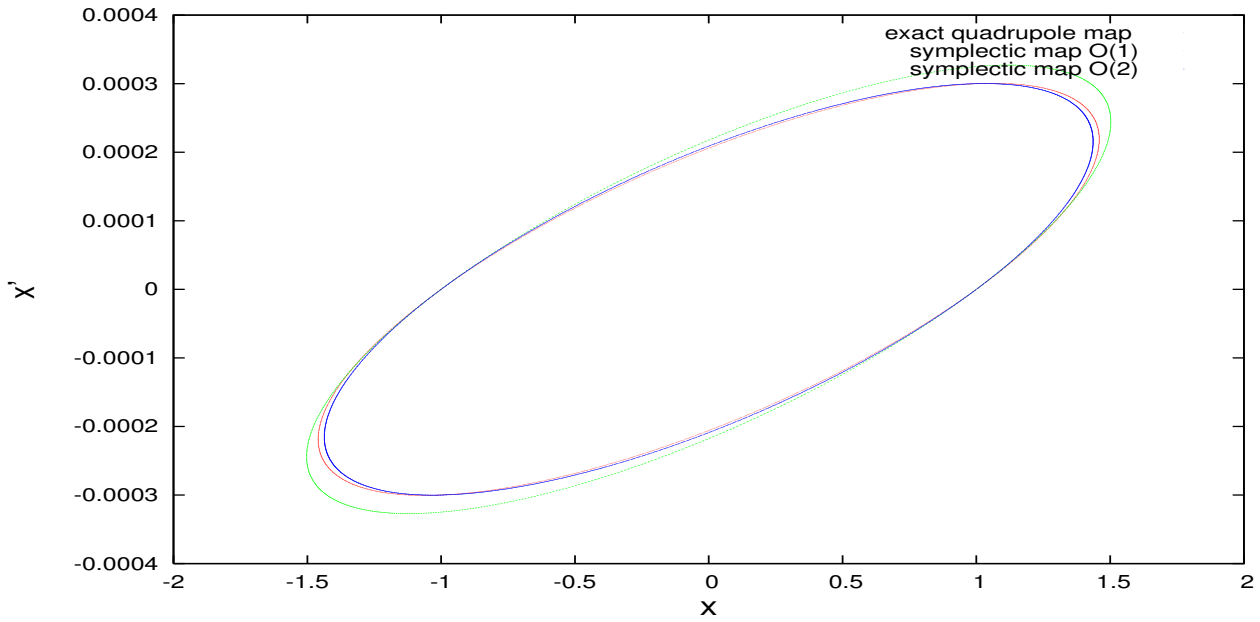
- **Non-symplecticity: particles spiral towards outside**

Quadrupole symplectic $\mathcal{O}(1)$ solution



- **symplecticity: but phase space ellipse not accurate**

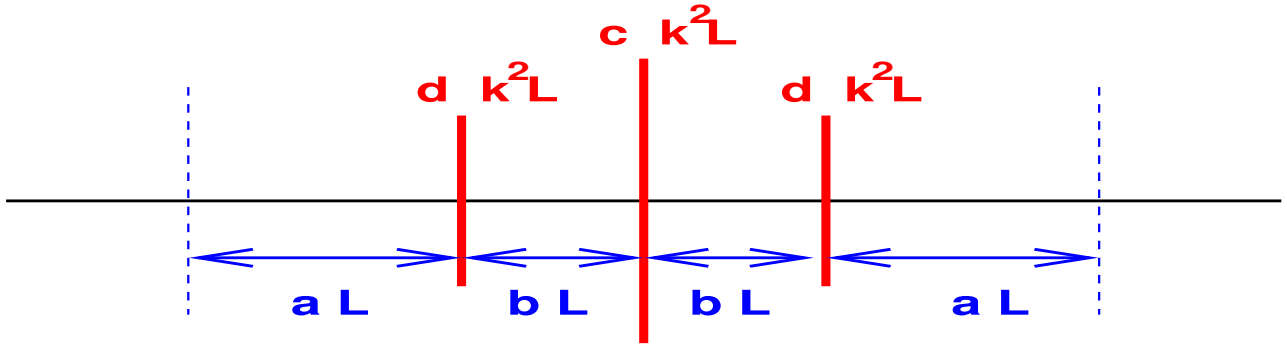
Quadrupole symplectic $\mathcal{O}(2)$ solution



- **symplecticity: phase space ellipse accurate enough**

Can we do better ?

- **Try a model with 3 kicks:**



- **with:**

  a ≈ 0.6756, b ≈ -0.1756, c ≈ -1.7027, d ≈ 1.3512

- **we have a $\mathcal{O}(4)$ integrator ...**

- **(a $\mathcal{O}(6)$ integrator would require 9 slices ...)**

Using thin elements in tracking

- Can we use this for studying long term behaviour ?
  - Symplecticity is important
  - If map is approximate:
    - → Phase space slightly distorted
    - → Long term stability conserved
  - Needs to be verified !

Accuracy of thin lenses

What about other (e.g. non-linear) elements ?

assume a general case:

$$x'' \quad = \quad f(x)$$

using the thin lens approximation gives:

$$x'(L) \quad \approx \quad x'_0 + L f(x_0 + \frac{L}{2} x'_0)$$

$$x(L) \quad \approx \quad x_0 + \frac{L}{2}(x'_0 + x'(L))$$

- **This is also called "leap frog" algorithm/integration**
- **It is symplectic !!**

Accuracy of thin lenses

Accuracy of "leap frog" algorithm/integration"

the Taylor expansion gives:

$$x(L) \quad = \quad x_0 + x_0' L + \frac{1}{2} f(x_0) L^2 + \frac{1}{6} x_0' f'(x_0) L^3 + ...$$

the "leap frog" algorithm gives:

$$x(L) \quad = \quad x_0 + x_0' L + \frac{1}{2} f(x_0) L^2 + \frac{1}{4} x_0' f'(x_0) L^3 + ...$$

- **Errors are $\mathcal{O}(L^3)$**

Accuracy of thin lenses

For bar/coffee discussions:

why did I write:

$$x'' \quad = \quad f(x)$$

and not:

$$x'' \quad = \quad f(x, x')$$

Accuracy of thin lenses

Assume a (one-dimensional) sextupole with:

assume:

$$x'' \quad = \quad k \cdot x^2$$

using the thin lens approximation (type D) gives:

$$\begin{aligned} x(L) &= x_0 + x_0' L + \frac{1}{2} k x_0^2 L^2 + \frac{1}{2}(\frac{1}{3}) k x_0 x_0' L^3 + \frac{1}{8}(\frac{1}{12}) k x_0'^2 L^4 \\ x'(L) &= x_0' + k x_0^2 L + k x_0 x_0' L^2 + \frac{1}{4}(\frac{1}{3}) k x_0'^2 L^3 \end{aligned}$$

Map for thick sextupole in thin lens approximation, accurate to $\mathcal{O}(L^2)$

Some comments:

- We have interleaved kicks with drifts
- Is that always necessary ?
  - No !
  - Can be any map with an exact expression
  - In most cases the exact map is a linear map (matrix)
- We have derived element maps for tracking from the equation of motion using this technique → can track now

Analysis tools

- Fourier analysis, diffusion coefficients, chaos detection ...
- Phase space plots from (simple) tracking:
  - Start with a "particle" with initial coordinates x and x' at a position s_0
  - Pass through the One-Turn-Map (for position s_0 !)
  - Plot the new x and x' coordinates at position s_0 after every turn

A simple example

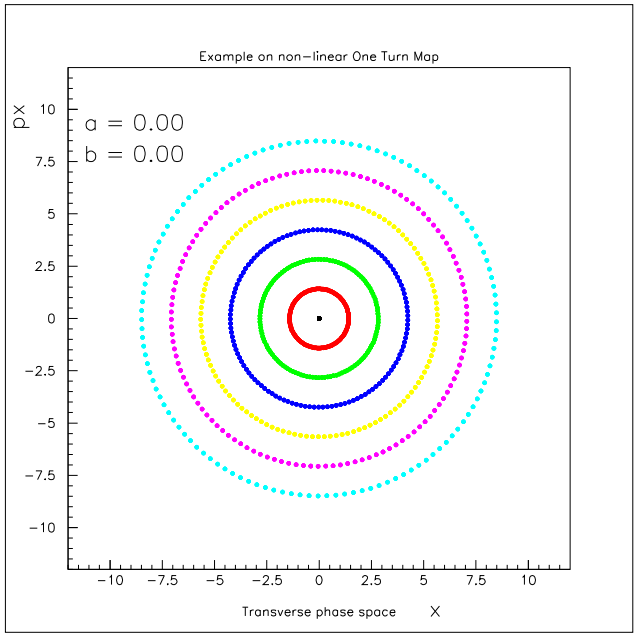
Linear transformation plus a constant deflection (i.e. orbit kick from displaced quadrupole)

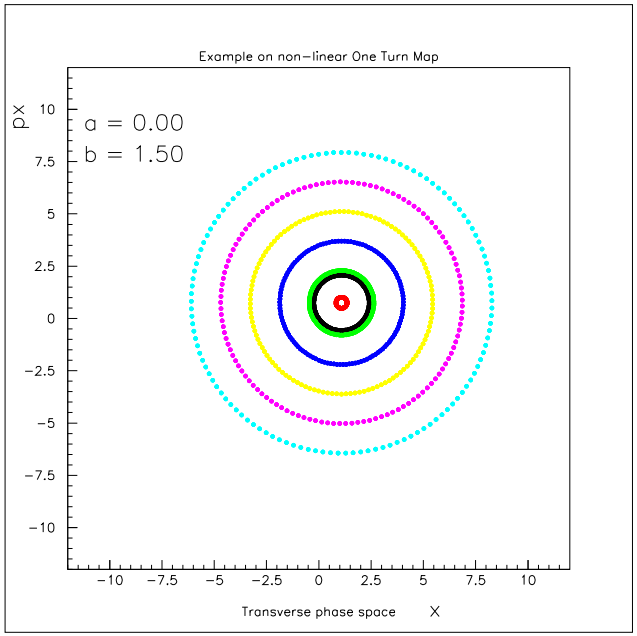
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{n+1} = \begin{pmatrix} \cos(\mu) & \sin(\mu) \\ -\sin(\mu) & \cos(\mu) \end{pmatrix}_{s_0} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{n} + \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$\mu = 2\pi \mathbf{Q}_x = \mathbf{2\pi \cdot 0.19}$,
constant b is a free parameter

➡ **Find the fixed point(s) (closed orbit)**

A simple example ..





- **Start at different amplitudes and "observe" x and x' at position s_0**

A (still) simple example

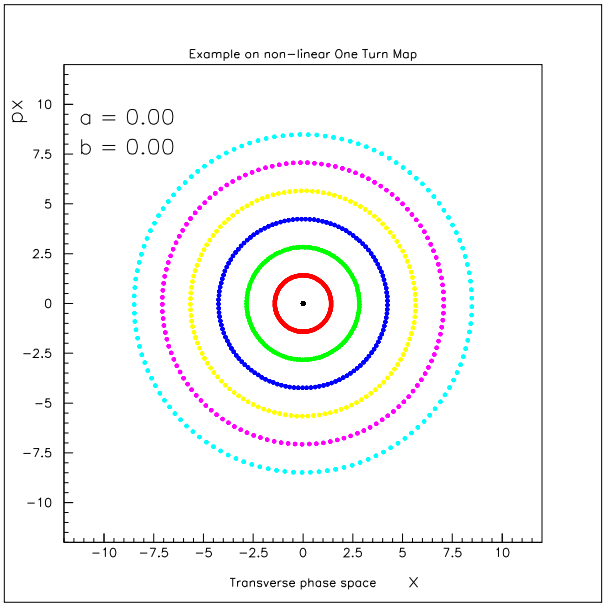
Linear transformation plus a quadratic non-linearity (e.g. (thin) sextupole) plus a constant deflection

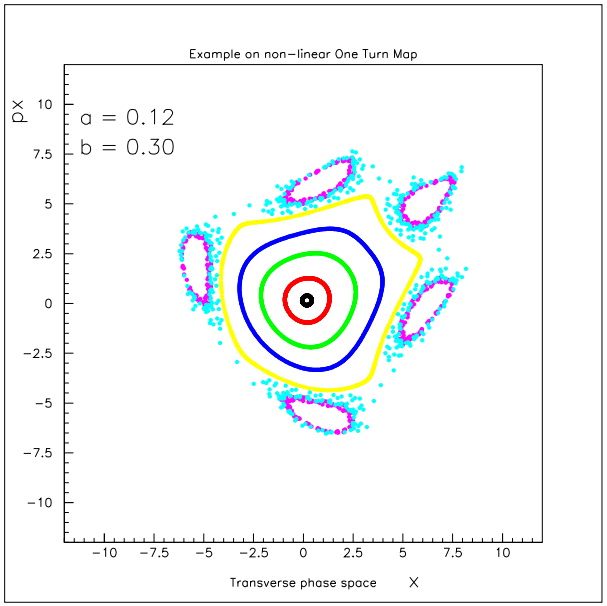
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{n+1} = \begin{pmatrix} \cos(\mu) & \sin(\mu) \\ -\sin(\mu) & \cos(\mu) \end{pmatrix}_{s_0} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{n} + \begin{pmatrix} 0 \\ ax^2 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$\mu = 2\pi \mathbf{Q}_x = \mathbf{2\pi \cdot 0.22}$,
constants a and b are free parameters

➡ **Find the fixed points (how many do you see ?)**

A (still) simple example ..





- **Motion at different amplitudes distorted: sextupole driving a 5th order resonance**
- **Stability region reduced by non-linear effect**

Analysis of tracking results

- Inspection of phase space structure
- Regular motion and stability boundary
- Identification of chaotic motion
- Tracking results (algorithms, output) can be used to "construct" analytical, approximate Taylor map for the ring: TPSA (Berz, Forest, 1989), Requires numerical differentiation → Differential Algebra (Berz, 1988)

Single particle tracking codes

- Several optics programs can also perform single particle tracking
- Many optics programs have thick to thin element converters
- Specialized codes exist (SIXTRACK, PTC, ...)
- Some codes have analysis tools (normal forms, chaos detection etc.)
- For predicting reliably long term behaviour essential !

Complications: light particles

- Light particles (e^-, e^+ etc.) emit synchrotron radiation and motion is damped
  - → Stochastic component
  - → No symplecticity, no invariants (but equilibrium parameters, e.g. emittance)
- Synchrotron motion must be simulated
- Computation of damping properties

Simulation of multi particle effects

- Often requires the simulation of a beam: simulate many (up to 10^8) particles simultaneously and study their behaviour:
  - Beam shape (density distribution)
  - Centre of mass motion of all particles
- Must be self-consistent: changes of the beams must be taken into account
- Fields generated by the beam need to be computed

Complications: many particles

- Particles have different amplitudes
- Particles have different tunes
- Particles have different momenta !
- Definition of emittance becomes more complicated

Strategy for multi-particle simulations (1)

- **Generate and simulate many particles (10^4 – 10^8 per beam) simultaneously**
- **Every particle interacts with the machine elements individually**
- **The whole ensemble interacts with the machine elements**
- **Every particle interacts with other particles !**

→ **Feed back into motion of individual particles**

Strategy for multi-particle simulations (2)

- All particles must be treated in parallel

→ For realistic LHC: 10^7 to 10^8 particles to simulate

→ Already for storage requires $\approx$ 10 Gb memory

→ Parallel processing needed for reasonable computing time

- Often requires (intelligent) simplifications

Simulation of interactions with environment

This means: interaction of the individual particles and the whole beam with:

- **Machine elements (e.g. magnets, RF, ...)**
- **Wake fields**
- **Impedances**
- **Electron cloud**
- **Intercepting elements (e.g. collimators, ...)**

→ **Strategies have changed with fast computers ...**

Simulation of interactions with itself

Particles inside a beam interact with other particles from the same beam:

- Space charge effects
- Intra-beam scattering
- Multi-bunch effects

Simulation of interactions with other beams

So-called beam-beam effects

- Other beam acts like a (very) non-linear lens
- Incoherent beam-beam effects (on each individual particle)
- Coherent beam-beam effects (on ensemble of particles)

→ Requires the knowledge of fields generated by the other beam

Matrix formulation (linear models)

- **One-Turn-Maps can be written for two bunches or two beams (e.g. 1 and 2)**

→ **Here only 1 dimension for illustration**

$$\begin{pmatrix} x_1 \\ x_1' \\ x_2 \\ x_2' \end{pmatrix}_{n+1} = \begin{pmatrix} m_{11} & m_{12} & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{43} & m_{44} \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_1' \\ x_2 \\ x_2' \end{pmatrix}_{n}$$

Matrix formulation (linear models)

- **One-Turn-Maps can be written for two bunches or two beams (e.g. 1 and 2)**

→ **Additional elements couple two beams**

$$\begin{pmatrix} x_1 \\ x_1' \\ x_2 \\ x_2' \end{pmatrix}_{n+1} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_1' \\ x_2 \\ x_2' \end{pmatrix}_{n}$$

→ **Allows computation of eigenmodes, eigenfrequencies of multi bunch systems**

Field computation

- **Some simulations require the computation of fields (or forces) produced by a beam from Poisson equation (here 2-dimensional):**

$$\Delta V = -4\pi \cdot \rho(x, y)$$

- **The density of the beam is $\rho(x, y)$**
- **For simple distributions (Gaussian, uniform ...) can be solved analytically**
- **In general (i.e. in the interesting cases !) it is done numerically**

Some basic methods

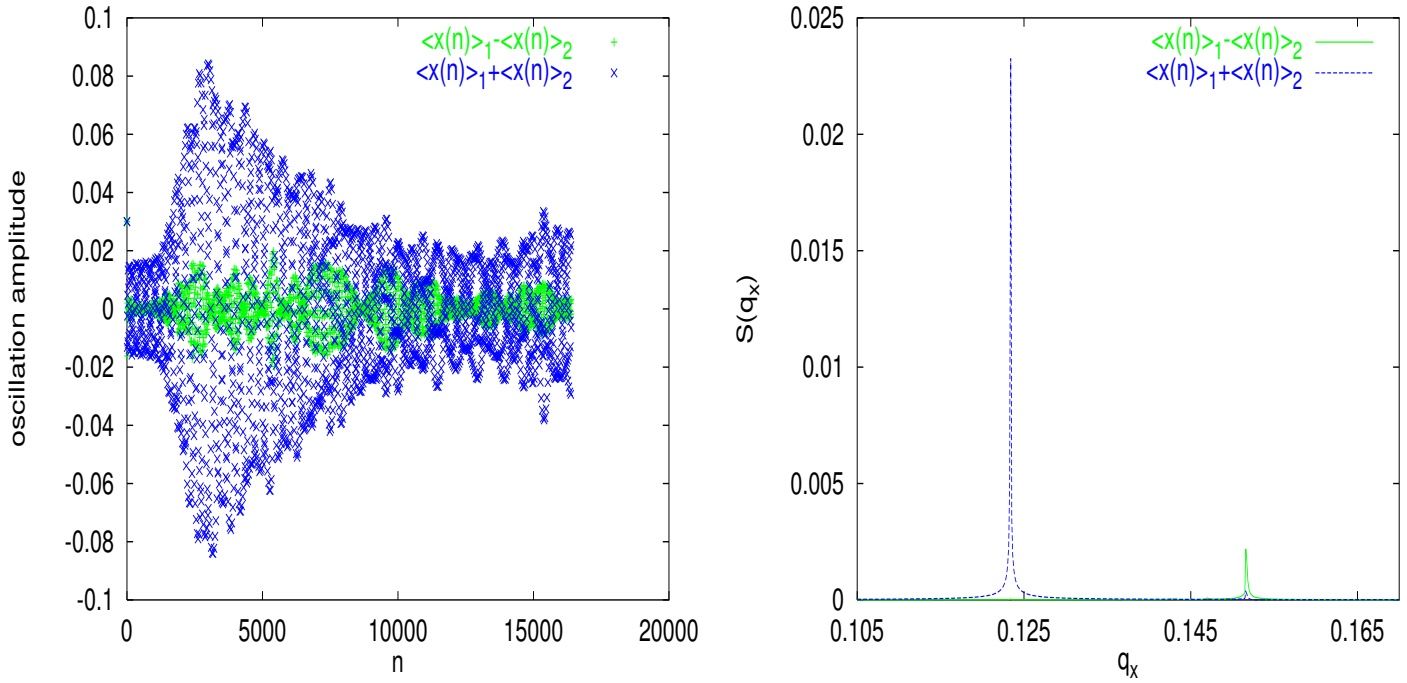
Particle - particle methods: compute field between each particle pair and add up (not practical for large number of particles, sometimes used in celestial mechanics)

Particle - mesh methods: distribute particles on a mesh (grid) and solve the Poisson equation for discrete points

Multipole methods: develope potentials/fields as multipole expansion

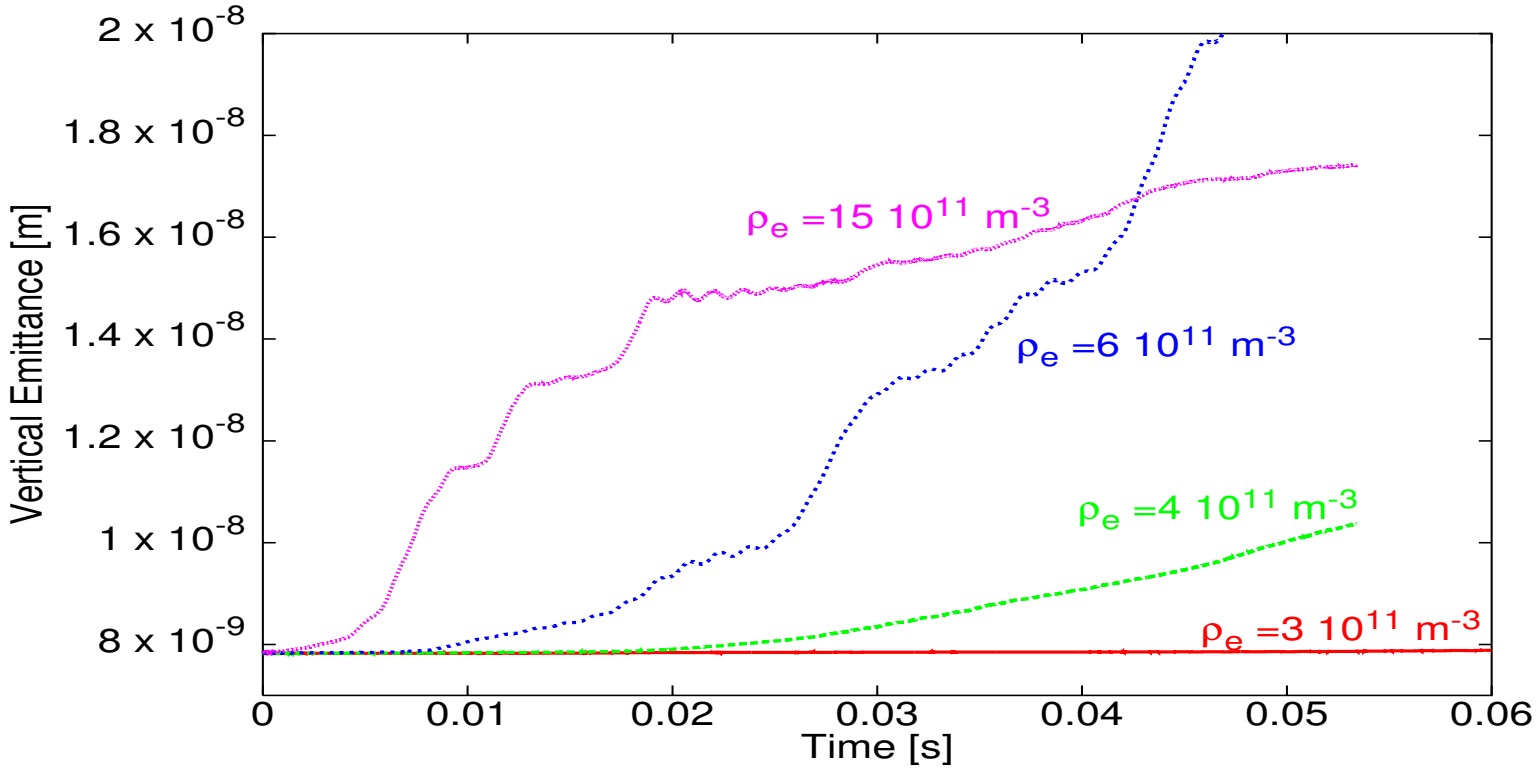
→ Choice depends on application and parameters

Example: Centre of mass motion as function of time



→ From beam-beam simulations: Bunch oscillations and frequency spectrum

Example: Beam size as function of time



→ **From electron-cloud simulation (courtesy: E. Benedetto)**

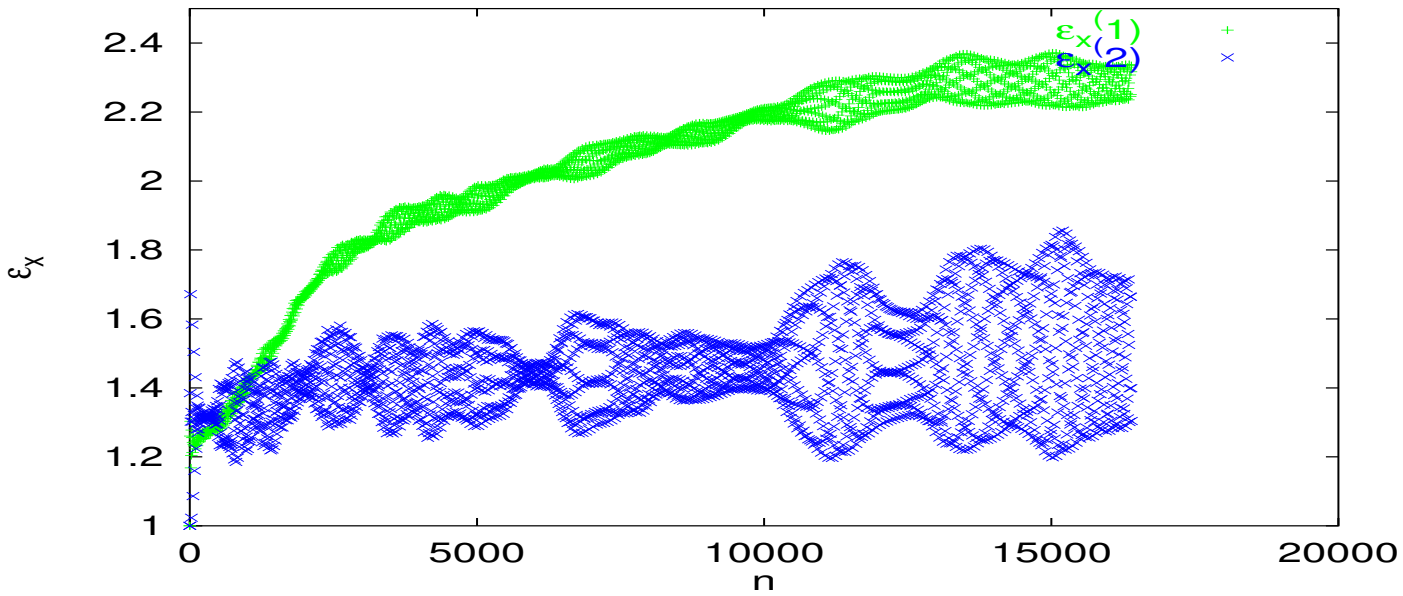
Alternatives

- Sometimes multi-particle simulations are too time consuming
- Numerical solution of the Vlasov-equation

→ E.g. finite difference methods

→ Intermediate level school (Sevilla, 2001)

Example: Beam size as function of time



→ **Beam-beam simulation: numerical solution of Vlasov equation gives evolution of beam sizes**

Multi particle simulation codes

- Many codes exist, always specialized:
  - → Collective instabilities
  - → Beam-beam effects
  - → Electron cloud effects
  - → etc. ...
- Often compact and linked to optics codes

(Personal) Comments on simulations:

- Here I gave only a selective overview of what can be done
- Techniques and tools in dedicated schools and courses (some in Intermediate CAS Course)
- What can be done has changed a lot in the last decade
- It is easy to write a program !
- Analysis and interpretation is usually the difficult part

What can go wrong ?

- Wrong or missing physics in the program
- Numerical problems
- Different results on different computers
- Programming bugs ...
- Biased analysis

→ Be aware of the limitations of the program

→ Make sure it is reproducible

Control and operation

- Basic aim: optimize performance

As operator or accelerator physicist:

- Provide and improve model of machine
- Measure and interpret beam parameters
- Correct and control beam parameters
- Conduct machine experiments

Control and operation of an accelerator

Basic problem: measure and control beam parameters

- Control (orbit, chromaticities ..) depend on machine model which may be incomplete
- Feedback from measurements improves the model and simulations
- Should use the same strategies and methods as during design (Remember: matching !)

→ May be an iterative process

Control and operation of an accelerator

- Very similar to simulation or design of a machine except:
  - → Interface to hardware and control (e.g. power converters)
  - → Beam instrumentation !
  - → Communication (networks, etc.)
  - → Issues such as: timing, alarms, interlocks,
- Treated in dedicated workshops and schools

Summary: what is needed ?

- Appropriate description of the accelerator
- Tools to get a One-Turn-Map to derive global quantities
- Simulation tools to study long term behaviour
  - Single particles
    (dynamic aperture, life time ..)
  - Multi particles
    (collective effects, emittance, coherent motion ...)

backup slides

backup slides

- Lie transforms (principles and examples)
- Normal forms (principles and examples)
- Truncated Power Series Algebra (TPSA) (principles and examples)
- Differential algebra (principles and examples)

Hamiltonian ..

$$\mathcal{H} = \overbrace{\frac{p_x^2 + p_y^2}{2(1+\delta)}}^{kinematic} - \overbrace{\underbrace{k_0 x\delta}_{dispersive} + \underbrace{\frac{k_0^2}{2}x^2}_{focusing}}^{dipole} + \overbrace{\frac{k_1}{2}(x^2 - y^2)}^{quadrupole} + \overbrace{\frac{k_2}{3}(x^3 - 3xy^2)}^{sextupole}$$

- **Useful formalism when invariants are investigated**
- **Perturbative formalism can lead to complete nonsense, if wrongly interpreted ...**
- **Very useful in the context of Lie transformations**

Lie transformations

Introduce Poisson bracket:

$$[f,g] = \sum_{i=1}^{n} \left(\frac{\delta f}{\delta q_i} \frac{\delta g}{\delta p_i} - \frac{\delta f}{\delta p_i} \frac{\delta g}{\delta q_i} \right)$$

Here the variables q_i, p_i are canonical variables, f and g are functions of q_i and p_i.

Is $\mathcal{H}$ is the Hamiltonian of a system, then:

$$[f,\mathcal{H}] = 0$$

implies that f is an invariant of the motion !

This is classical mechanics ...

Lie transformations

We can define:

$$: f : g = [f, g]$$

where $: f :$ **is an operator acting on the function** g**.**

We can define powers as:

$$(: f :)^2 g =: f : (: f : g) = [f, [f, g]] \qquad \text{etc.}$$

in particular:

$$e^{:f:} = \sum_{i=0}^{\infty} \frac{1}{i!} (: f :)^i$$

Lie transformations

Lie transforms acting on functions like x, p (canonical momentum, instead of x'):

$$: p^2 :x = -2p \qquad : p^2 :p = 0$$

or more complicated:

$$\begin{aligned} e^{:-Lp^2/2:}x &= x - \frac{1}{2}L : p^2 : x + \frac{1}{8}L^2(: p^2 :)^2 x + .. \\ &= x + Lp \\ e^{:-Lp^2/2:}p &= p - \frac{1}{2}L : p^2 : p + ... \\ &= p \end{aligned}$$

This is the transformation of a drift space of length L !!

Lie transformations

$$e^{:f:}(x,p)_0 = (x,p)_1$$

Lie transforms describe how to go from one point to another.

Homework try:

$$e^{:-\frac{1}{2f}x^2:}$$

or:

$$e^{:-\frac{1}{2}L(k^2x^2+p^2):}$$

Why all that ???

concatenation very easy:

$$e^{:h:} = e^{:f:} e^{:g:} = e^{:f+g:}$$

when f and g commute.

- **Otherwise formalism exist**
- **Exponential form $e^{:h:}$ is always symplectic**

the one turn map is the exponential of the effective Hamiltonian:

$$\mathcal{M}_{ring} = e^{:-C\mathcal{H}_{eff}:}$$

Why all that ???

when f_2, f_3, f_4, are 2nd, 3rd, 4th order polynomials (factorization theorem):

$$e^{:f:} = e^{:f_2:} e^{:f_3:} e^{:f_4:}$$

- **Each term is symplectic**
- **Truncation at any order does not violate symplecticity**

Lie transformations are the natural extension of linear to non-linear formalism !

Example: thick sextupole

The map of an element of length L and the Hamiltonian H is:

$$e^{-L:H:} = \sum_{i=0}^{\infty} \frac{1}{i!}(-L : H :)^i$$

The Hamiltonian of a thick sextupole is:

$$H = \frac{1}{3}k(x^3 - 3xy^2) + \frac{1}{2}(p_x^2 + p_y^2)$$

we search for:

$$e^{-L:H:}x \quad \text{and} \quad e^{-L:H:}p_x \quad \text{etc.}$$

$$e^{-L:H:}x = \sum_{i=0}^{\infty} \frac{1}{i!}(-L : H :)^i x$$

Example: thick sextupole

we can compute:

$$: H :^{i} x$$

to get:

$$: H :^{1} x = -p_x$$

$$: H :^{2} x = -k(x^2 - y^2)$$

$$: H :^{3} x = 2k(xp_x - yp_y)$$

$$....$$

and get:

$$e^{-L:H:} x = x + p_x L - \frac{1}{2} k L^2 (x^2 - y^2) - \frac{1}{3} k L^3 (xp_x - yp_y) + ...$$

(more accurate than "leap frog" thin lens approximation !

Why all that ???

Very useful to find invariants!!

Homework → what is:

a) the invariant

b) the transformation

with the effective Hamiltonian h (1D):

$$e^{:h:} = e^{:f_2:} = e^{:-\frac{\mu}{2}(\gamma x^2 + 2\alpha xp + \beta p^2):}$$

Hints

with a second order polynomial as h:

$$e^{:h:} = e^{:f_2:} = e^{:-\frac{1}{2}(ax^2+2bxp+cp^2):}$$

the solution for the matrix form is:

$$\mathcal{R} = cos(\sqrt{ac-b^2}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{sin(\sqrt{ac-b^2})}{\sqrt{ac-b^2}} \begin{pmatrix} b & c \\ -a & -b \end{pmatrix}$$

- **(can also be used to find f_2 from matrix form, a quadratic form f_2 always corresponds to a matrix)**

Normal forms

If $\mathcal{M}$ is our usual one-turn-map:

$$\mathcal{N} \;=\; \mathcal{A}\mathcal{M}\mathcal{A}^{-1}$$

We can choose $\mathcal{A}$ to get:

$$\mathcal{N} \;=\; e^{-:h:}$$

with:

$$\begin{aligned} h = \quad & (w_x + w_x' p_\tau + w_x'' p_\tau^2) h_x \quad + \quad (w_y + w_y' p_\tau + w_y'' p_\tau^2) h_y \\ + \quad & (a \cdot h_x^2 + b \cdot h_x h_y + c \cdot h_y^2 + d \cdot p_\tau^2) \quad + \ldots . \end{aligned}$$

and

$$h_x = (x^2 + p_x^2)/2, \quad h_y = (y^2 + p_y^2)/2, \quad p_\tau = (p_t - p_t^0)/(p^0 c)$$

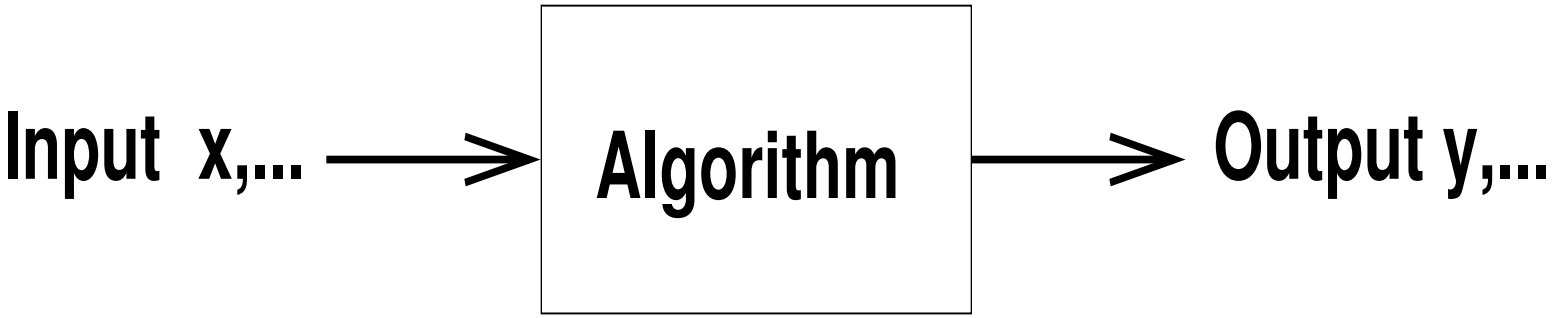
Why is THAT interesting ?

- **The map $\mathcal{N}$ sends circles into circles**
- **The function h contains:**
  - **w_x, w_y are the tunes**
  - **w'_x, w'_y are the chromaticities**
  - **w''_x, w''_y are the second order chromaticities**
  - **a, b, c, d anharmonicities, i.e. amplitude dependence of tunes**

Why is THAT interesting ?

- **The quantities $I_{x,y} = \mathcal{A}h_{x,y}$ are invariant i.e.:**
  - $\mathcal{M}I_{x,y} = I_{x,y}$
  - **They generalize Courant-Snyder invariants to the non-linear case**

Truncated Power Series Algebra



- What if we can somehow calculate the derivatives of the output with respect to input ??
- Then we have the Taylor series, i.e. the Taylor map!
- Algorithm can be formula or subroutine or program ..
- Formula → by hand
- Subroutine or program etc. → by numerical differentiation

Numerical differentiation

The problem getting the derivative of $f(x)$ at a:

$$f'(a) = \frac{f(a+\epsilon) - f(a)}{\epsilon}$$

- **Need to subtract almost equal numbers and divide by small number.**
- **For higher orders f'', f'''.., accuracy hopeless !**
- **We can use Differential Algebra (DA) (Berz, 1988)**

Differential Algebra

1. **Define a pair (q_0, q_1), with q_0, q_1 real numbers**
2. **Define operations like:**

$$(q_0, q_1) + (r_0, r_1) = (q_0 + r_0, q_1 + r_1)$$

$$c \cdot (q_0, q_1) = (c \cdot q_0, c \cdot q_1)$$

$$(q_0, q_1) \cdot (r_0, r_1) = (q_0 \cdot r_0, q_0 \cdot r_1 + q_1 \cdot r_0)$$

3. **And some ordering:**

$$(q_0, q_1) < (r_0, r_1) \quad \text{if} \quad q_0 < r_0 \quad \text{or} \quad (q_0 = r_0 \quad \text{and} \quad q_1 < r_1)$$

$$(q_0, q_1) > (r_0, r_1) \quad \text{if} \quad q_0 > r_0 \quad \text{or} \quad (q_0 = r_0 \quad \text{and} \quad q_1 > r_1)$$

4. **This implies something strange:**

$$(0, 0) < (0, 1) < (r, 0)$$

Differential Algebra

This means that (0,1) is between 0 and ANY real number

➡ infinitely small !!!

We call this therefore "differential unit" $d = (0, 1)$**.**

Of course $(q, 0)$ **is just the real number** q **and we define "real" and "differential part" (a bit like complex numbers..):**

$$q_0 = \mathcal{R}(q_0, q_1) \quad \text{and} \quad q_1 = \mathcal{D}(q_0, q_1)$$

With our rules we can further see that:

$$(1, 0) \cdot (q_0, q_1) = (q_0, q_1)$$

$$(q_0, q_1)^{-1} = \left(\frac{1}{q_0}, -\frac{q_1}{q_0^2} \right)$$

Differential Algebra

For a function $f(x)$ without proof:

$$\mathcal{D}[f(x+d)] = \mathcal{D}[f((x,0)+(0,1))] = f'(x)$$

An example instead:

$$f(x) = x^2 + \frac{1}{x}$$

then using school calculus:

$$f'(x) = 2x - \frac{1}{x^2}$$

For $x = 2$ we get then:

$$f(2) = \frac{9}{2}, f'(2) = \frac{15}{4}$$

Differential Algebra

For x in:

$$f(x) = x^2 + \frac{1}{x}$$

we substitute: x $\rightarrow$ (x, 1) = (2, 1) and use our rules:

$$\begin{aligned} f[(2,1)] &= (2,1)^2 + (2,1)^{-1} \\ &= (4,4) + (\frac{1}{2}, -\frac{1}{4}) \\ &= (\frac{9}{2}, \ \frac{15}{4}) = (f(2), \ f'(2)) \quad \text{!!!} \end{aligned}$$

The computation of derivatives becomes an algebraic problem, no need for small numbers, exact !

Differential Algebra

If we had started with:

$$x = (a, 1, 0, 0, 0...)$$

we would get:

$$f(x) = (\ f(a),\ f'(a),\ f''(a),\ f'''(a), ...\ f^{(n)}(a)\ )$$

Differential Algebra

If we had started with:

$$x = (a, 1, 0, 0, 0...)$$

we would get:

$$f(x) = (\ f(a),\ f'(a),\ f''(a),\ f'''(a), ...\ f^{(n)}(a)\ )$$

can be extended to more variables x, y:

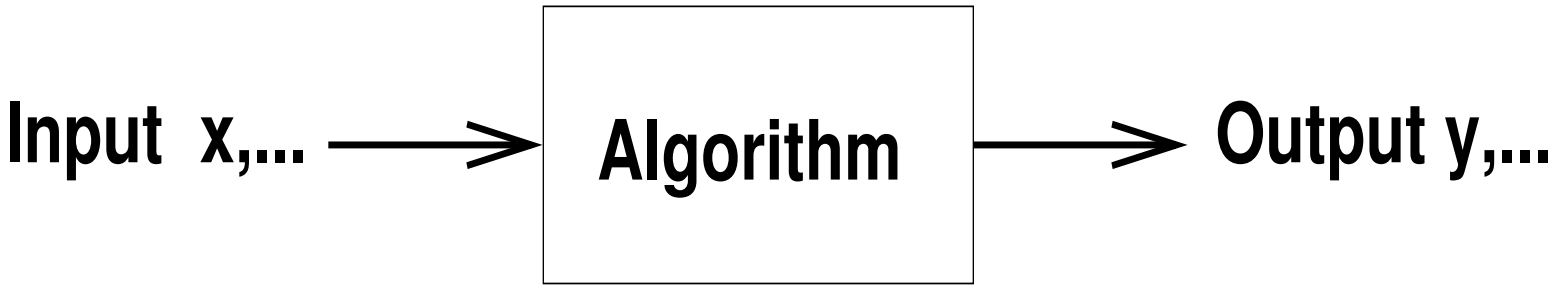
$$x = (a, 1, 0, 0, 0...)$$

$$y = (b, 0, 1, 0, 0...)$$

and get (with more complicated multiplication rules):

$$f((x + dx), y + dy)) = \left(f, \frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}, \frac{\delta^2 f}{\delta x^2}, \frac{\delta^2 f}{\delta x \delta y}, ...\right)(x, y)$$

What is the use of that:



Push $f(x) = (a, 1, 0, 0, 0...)$ through the algorithm (no matter what it is), using our rules and we get all derivatives around a, i.e. we get the Taylor coefficients and therefore the map !

The maps are provided with the desired accuracy and to any order.

Concatenation of Lie transformations

To concatenate two Lie-transforms, use BCH (Baker-Campbell-Hausdorff) formula (without proof):

$$e^{:f:}\ e^{:g:} \;=\; e^{:h:}$$

$$h = f + g + \frac{1}{2} : f : g + \frac{1}{12} : f :^2 g + \frac{1}{12} : g :^2 f + \frac{1}{24} : f :: g :^2 f + \frac{1}{720} ...$$

when g is not too large we can write the concatenation easier as:

$$e^{:f:}\ e^{:g:} \;=\; e^{:h:} \;=\; \exp\left[: f + \left(\frac{: f :}{1 - e^{-:f:}} \right) g + \mathcal{O}(g^2) : \right]$$

Concatenation of Lie transformations

Example: linear transfer (f_2) plus local, non-linear kick F(x) (in action and angle variables A, Φ):

$$f_2 \;=\; -\frac{\mu}{2}\left(\frac{x^2}{\beta}+\beta p_x^2\right) \;=\; -\mu A$$

$$F(x) \Rightarrow \sum_{n=-\infty}^{\infty} c_n(A) e^{in\Phi}$$

gives:

$$h = -\mu A + \sum_n c_n(A)\frac{in\mu}{1-e^{-in\mu}}e^{in\Phi} = -\mu A + \sum_n c_n(A)\frac{n\mu}{2\sin(\frac{n\mu}{2})}e^{(in\Phi + i\frac{n\mu}{2})}$$

now h is the invariant of motion.