

Resonances

- introduction: driven oscillators and resonance condition
- smooth approximation for motion in accelerators
- field imperfections and normalized field errors
- perturbation treatment
- Poincare section
- stabilization via amplitude dependent tune changes
- sextupole perturbation & slow extraction
- chaotic particle motion

Introduction: Damped Harmonic Oscillator

equation of motion for a damped harmonic oscillator:

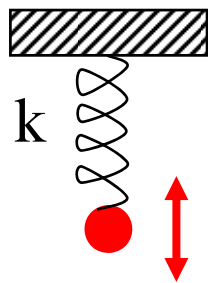
$$\frac{d^2}{dt^2} w(t) + \omega_0 \cdot Q^{-1} \cdot \frac{d}{dt} w(t) + \omega_0^2 \cdot w(t) = 0$$

Q is the damping coefficient

→ (amplitude decreases with time)

ω_0 is the Eigenfrequency of the HO

example: weight on a spring ($Q = \infty$)



$$\frac{d^2}{dt^2} w(t) + k \cdot w(t) = 0 \quad \rightarrow \quad w(t) = a \cdot \sin(\sqrt{k} \cdot t + \phi_0)$$

Introduction: Driven Oscillators

■ an external driving force can ‘pump’ energy into the system:

$$\frac{d^2}{dt^2} w(t) + \omega_0 \cdot Q^{-1} \cdot \frac{d}{dt} w(t) + \omega_0^2 \cdot w(t) = \frac{F}{m} \cdot \cos(\omega \cdot t)$$

■ general solution:

$$w(t) = w_{tr}(t) + w_{st}(t)$$

■ stationary solution:

$$w_{st}(t) = W(\omega) \cdot \cos[\omega \cdot t - \alpha(\omega)]$$



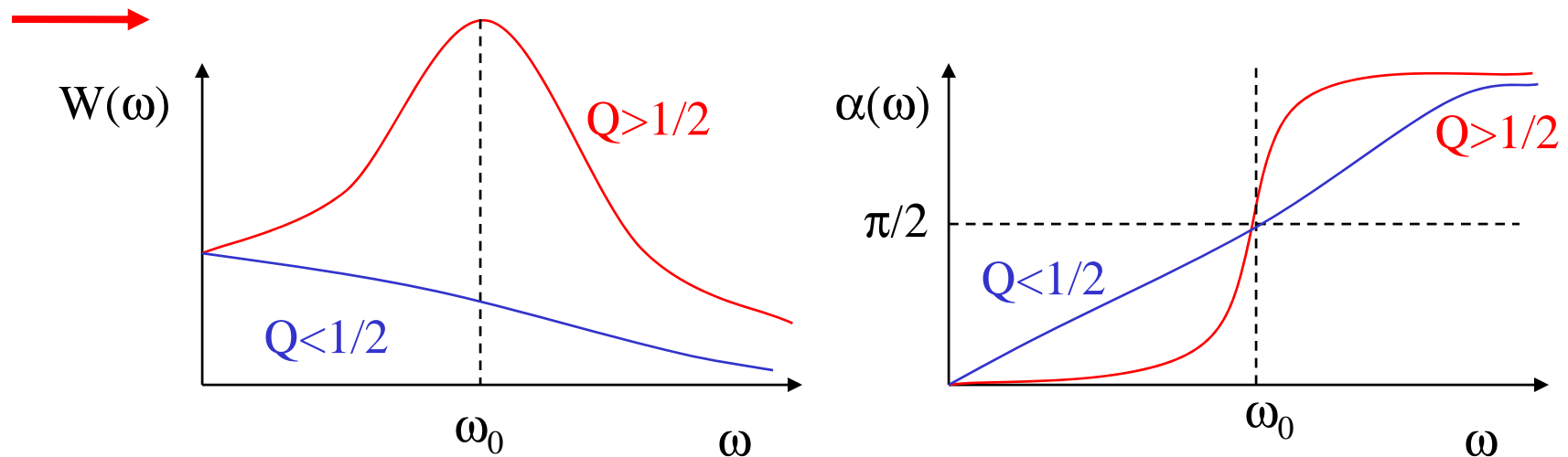
where ‘ ω ’ is the driving angular frequency!
and $W(\omega)$ can become large for certain frequencies!

Introduction: Driven Oscillators

stationary solution

stationary solution follows the frequency of the driving force:

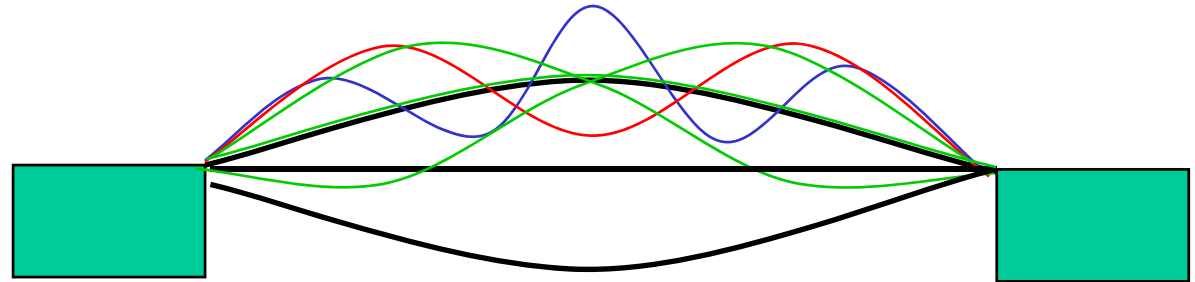
$$w_{st}(t) = W(\omega) \cdot \cos[\omega \cdot t - \alpha(\omega)]$$



oscillation amplitude can become large for weak damping

Introduction: Pulsed Driven Resonances Example

higher harmonics:



example of a bridge:

2nd harmonic:

3nd harmonic:

4th harmonic:



peak amplitude depends on the excitation frequency and damping

Introduction: Instabilities

■ resonance catastrophe without damping:

$$W(\omega) = W(0) \cdot \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left(\frac{\omega}{Q\omega_0}\right)^2}}$$

■ weak damping: resonance condition: $\omega = \omega_0$

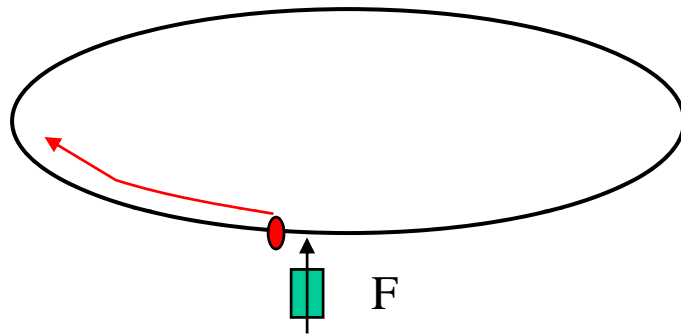
Tacoma Narrow bridge
1940



excitation by strong wind on the Eigenfrequencies

Smooth Approximation: Resonances in Accelerators

■ revolution frequency:

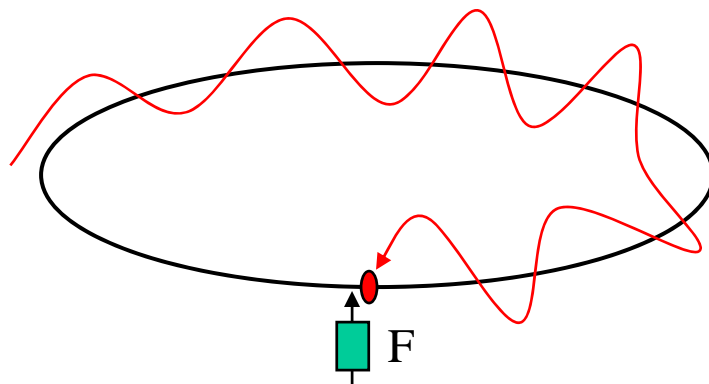


→ periodic kick

→ excitation with f_{rev}

$$(\omega_{\text{rev}} = 2\pi f_{\text{rev}})$$

■ betatron oscillations:



Eigenfrequency: $\omega_0 = 2\pi f_\beta$

$$Q = \omega_0 / \omega_{\text{rev}}$$

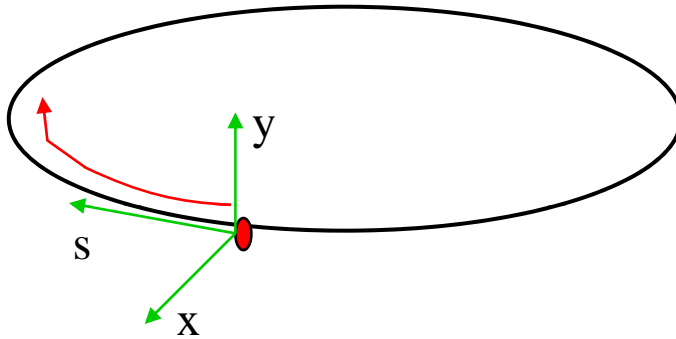
→ driven oscillator

→ weak or no damping!

(synchrotron radiation damping (single particle) or Landau damping distributions)

Smooth Approximation: Free Parameter

co-moving coordinate system:



→ choose the longitudinal coordinate as the free parameter for the equations of motion

equations of motion:

$$\frac{d}{dt} = \frac{ds}{dt} \cdot \frac{d}{ds}$$

with:

$$\frac{ds}{dt} = v$$

→
$$\frac{d^2}{dt^2} = v^2 \cdot \frac{d^2}{ds^2}$$

Smooth Approximation: Equation of Motion I

Smooth approximation for Hills equation:

$$\frac{d^2}{ds^2} w(s) + K(s) \cdot w(s) = 0 \xrightarrow{K(s) = \text{const}} \frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = 0$$

(constant β -function and phase advance along the storage ring)

$$\longrightarrow w(s) = A \cdot \cos(\omega_0 \cdot s + \phi_0) \qquad \omega_0 = 2\pi \cdot Q_0 / L$$

(Q is the number of oscillations during one revolution)

perturbation of Hills equation:

$$\frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = F(w(s), s) / (v \cdot p)$$

in the following the force term will be the Lorenz force of a charged particle in a magnetic field:

$$F = q \cdot \vec{v} \times \vec{B}$$

Field Imperfections: Origins for Perturbations

- linear magnet imperfections: derivation from the design dipole and quadrupole fields due to powering and alignment errors
- time varying fields: feedback systems (damper) and *wake fields due to collective effects (wall currents)*
- non-linear magnets: sextupole magnets for chromaticity correction and octupole magnets for Landau damping
- beam-beam interactions: strongly non-linear field!
- non-linear magnetic field imperfections: particularly difficult to control for super conducting magnets where the field quality is entirely determined by the coil winding accuracy

Field Imperfections: Localized Perturbation

periodic delta function:

$$\delta_L(s - s_0) = \begin{cases} 1 & \text{for 's' = } s_0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \oint \delta_L(s - s_0) ds = 1$$

equation of motion for a single perturbation in the storage ring:

$$\frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = \delta_L(s - s_0) \cdot l \cdot F(w, s) / (v \cdot p)$$

Fourier expansion of the periodic delta function:

$$\frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = \frac{l}{L} \sum_{r=-\infty}^{\infty} \cos(r \cdot 2\pi \cdot s / L) \cdot F(w, s) / (v \cdot p)$$

→ infinite number of driving frequencies

Field Imperfections: Constant Dipole

normalized field error: $\frac{F}{v \cdot p} = q \cdot \frac{\vec{v} \times \vec{B}}{v \cdot p} \xrightarrow{v \perp B} q \cdot B / p = k_0$

equation of motion for single kick:

$\longrightarrow \frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = \frac{lk_0}{L} \sum_{r=-\infty}^{\infty} \cos(r \cdot 2\pi \cdot s / L)$

\longrightarrow resonance condition: $\omega_0 = r \cdot 2\pi / L \xrightarrow{\omega_0 = 2\pi \cdot Q_0 / L} Q_0 = r$

\longrightarrow avoid integer tunes!

\longrightarrow remember the example of a single dipole imperfection from the 'Linear Imperfection' lecture yesterday!

Field Imperfections: Constant Quadrupole

 equations of motion:

$$\frac{d^2}{ds^2} x(s) + \omega_x^2 \cdot x(s) = k_1 \cdot x(s)$$

$$y(s) \equiv 0$$

with: $k_1 = \frac{q}{p} \cdot \frac{\partial B_y}{\partial x}$

→ $\frac{d^2}{ds^2} x(s) + (\omega_x^2 - k_1) \cdot x(s) = 0$

→ change of tune but no amplitude growth due to resonance excitations!

Field Imperfections: Single Quadrupole Perturbation

assume $y = 0$ and $B_x = 0$: $F(s)/(v \cdot p) = \delta_L(s - s_0) \cdot l \cdot k_1 \cdot x$

$$\longrightarrow \frac{d^2}{ds^2} x(s) + \omega_{x,0}^2 \cdot x(s) = \frac{lk_1}{L} \sum_{r=-\infty}^{\infty} \cos(2\pi \cdot r \cdot s / L) \cdot x(s)$$

$$[x(s) = A \cdot \cos(\omega_0 \cdot s)] \longrightarrow = \frac{lk_1}{2L} \sum_{r=-\infty}^{\infty} \cos(2\pi \cdot r \cdot s / L \pm \omega_0 \cdot s) \cdot x(s)$$

resonance condition: $\omega_{x,0} = r \cdot 2\pi / L \pm \omega_{x,0} \xrightarrow{\omega_0 = 2\pi \cdot Q_0 / L} Q_0 = r / 2$

avoid half integer tunes plus resonance width from tune modulation!

exact solution: variation of constants \rightarrow see the lecture yesterday

Field Imperfections: Time Varying Dipole Perturbation

 time varying perturbation:

$$F(t) = F_0 \cdot \cos(\omega_{kick} \cdot t) \xrightarrow{t \rightarrow s} F_0 \cdot \cos(2\pi \cdot \frac{\omega_{kick}}{\omega_{rev}} \cdot s / L) / (v \cdot p)$$

$$\longrightarrow \frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = \frac{lF_0}{2L} \sum_{r=-\infty}^{\infty} \cos(2\pi \cdot [r \pm \omega_{kick} / \omega_{rev}] \cdot s / L) / (v \cdot p)$$

resonance condition:

$$\omega_0 = 2\pi \cdot (r \pm \omega_{kick} / \omega_{rev}) / L \xrightarrow{\omega_0 = 2\pi \cdot Q_0 / L} f_{kick} = f_{rev} \cdot (Q_0 \pm r)$$

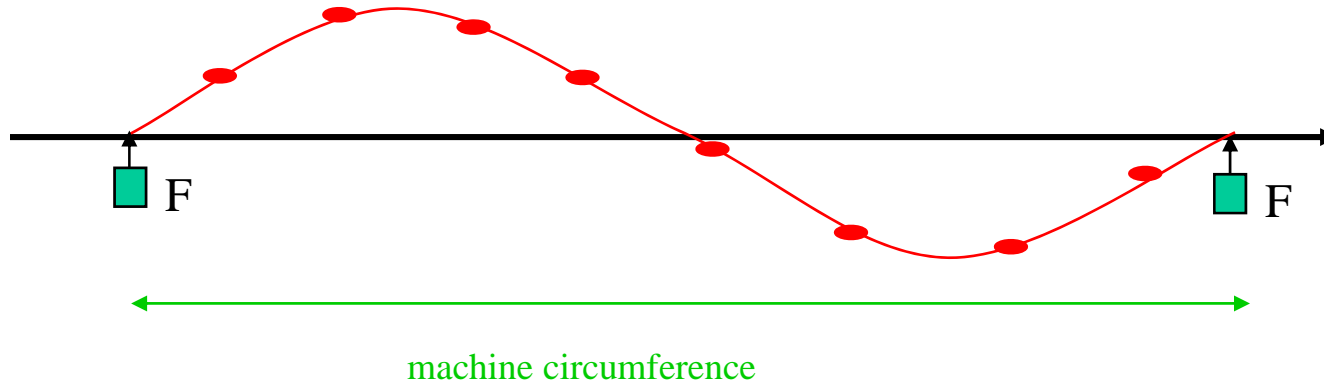
 avoid excitation on the betatron frequency!

(the integer multiple of the revolution frequency corresponds to the modes of the bridge in the introduction example)

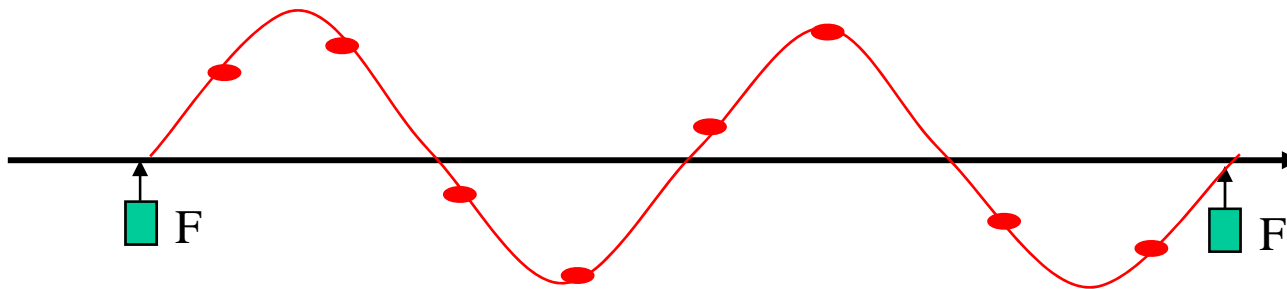
Field Imperfections: Several Bunches



$$F(t) = B \cdot \cos(\omega_{kick} \cdot t); \omega_{kick} \approx \omega_{rev} :$$



$$F(t) = B \cdot \cos(\omega_{kick} \cdot t); \omega_{kick} \approx 2 \cdot \omega_{rev} :$$



higher modes analogous to bridge illustration

Field Imperfections: Multipole Expansion

Taylor expansion of the magnetic field:

$$B_y + iB_x = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot f_n \cdot (x + iy)^n \quad \text{with:} \quad f_n = \frac{\partial^{n+1} B_y}{\partial x^{n+1}}$$

multipole	order	B_x	B_y
dipole	0	0	B_0
quadrupole	1	$f_1 \cdot y$	$f_1 \cdot x$
sextupole	2	$f_2 \cdot x \cdot y$	$\frac{1}{2} \cdot f_2 \cdot (x^2 - y^2)$
octupole	3	$\frac{1}{6} \cdot f_3 \cdot (3yx^2 - y^3)$	$\frac{1}{6} \cdot f_3 \cdot (x^3 - 3xy^2)$

normalized multipole gradients:

$$F(s)/(v \cdot p) = \frac{q \cdot (\vec{v} \times \vec{B})}{(v \cdot p)}$$

$$k_n = \frac{q}{p} \cdot f_n$$

$$k_n = 0.3 \cdot \frac{f_n [T / m^n]}{p [GeV / c]}$$

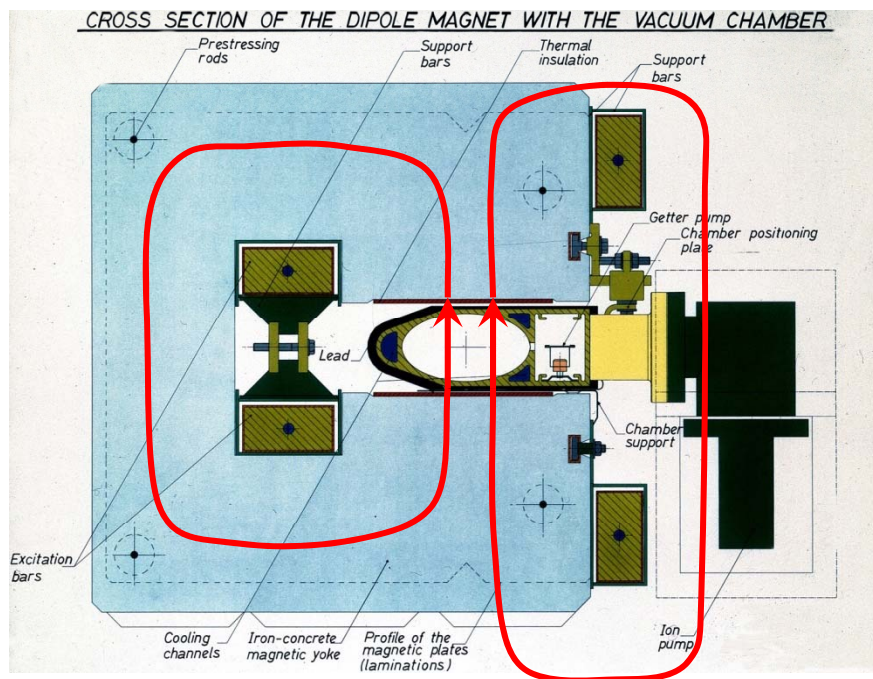
$$[k_n] = \frac{1}{m^{n+1}}$$

Field Imperfections: Dipole Magnets

■ dipole magnet designs:

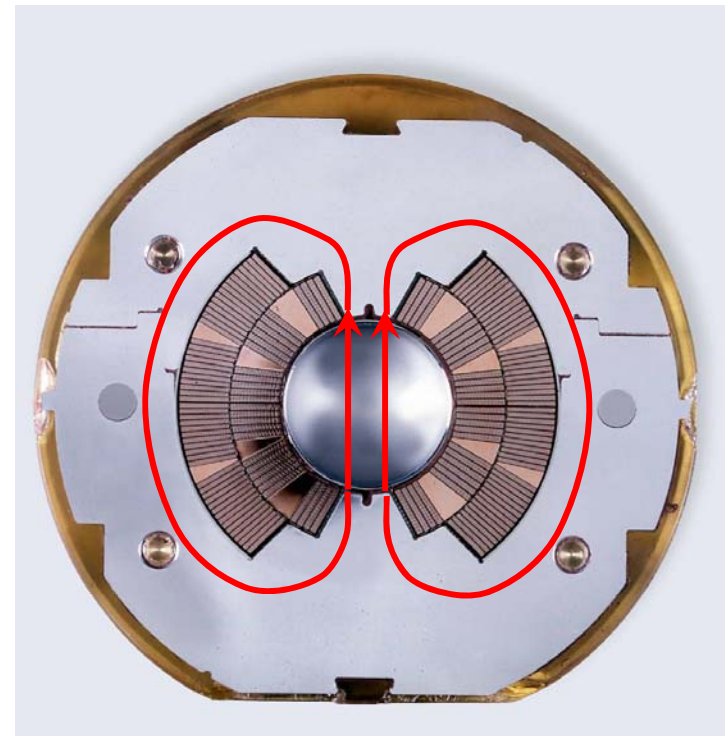
LEP dipole magnet:

conventional magnet design
relying on pole face accuracy
of a Ferromagnetic Yoke



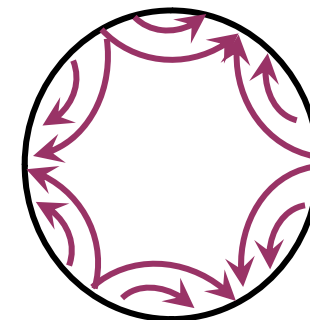
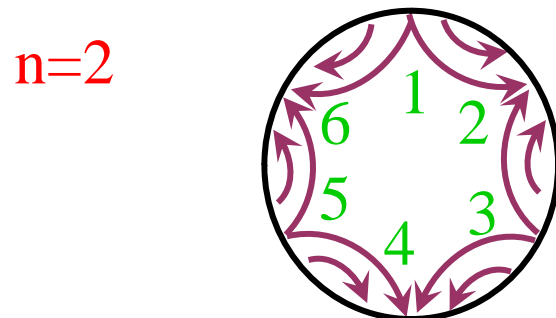
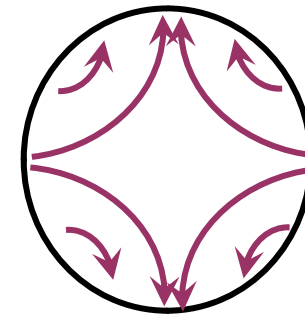
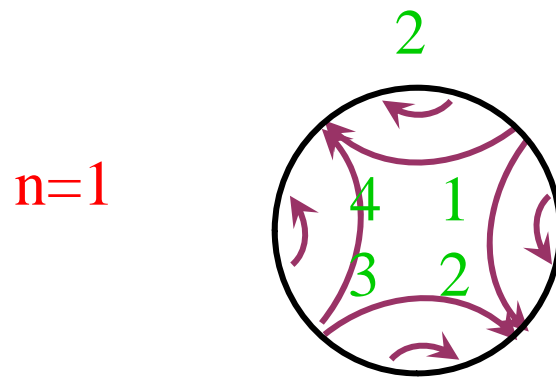
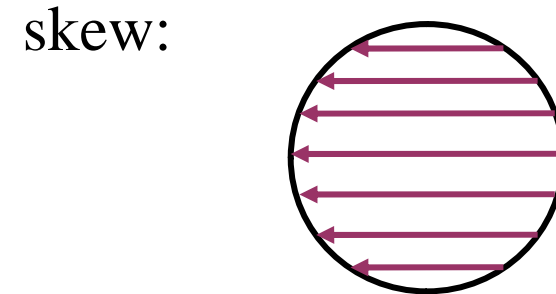
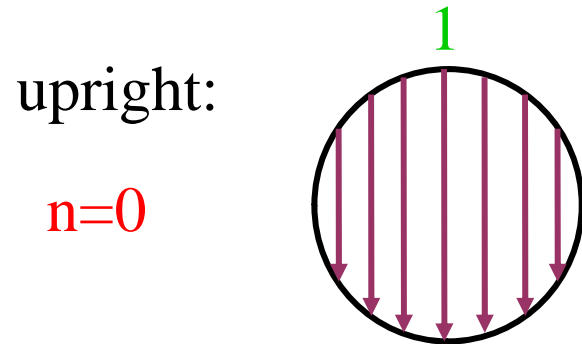
LHC dipole magnet:

air coil magnet design relying
on precise current distribution



Field Imperfections: Multipole Illustration

 upright and skew field errors



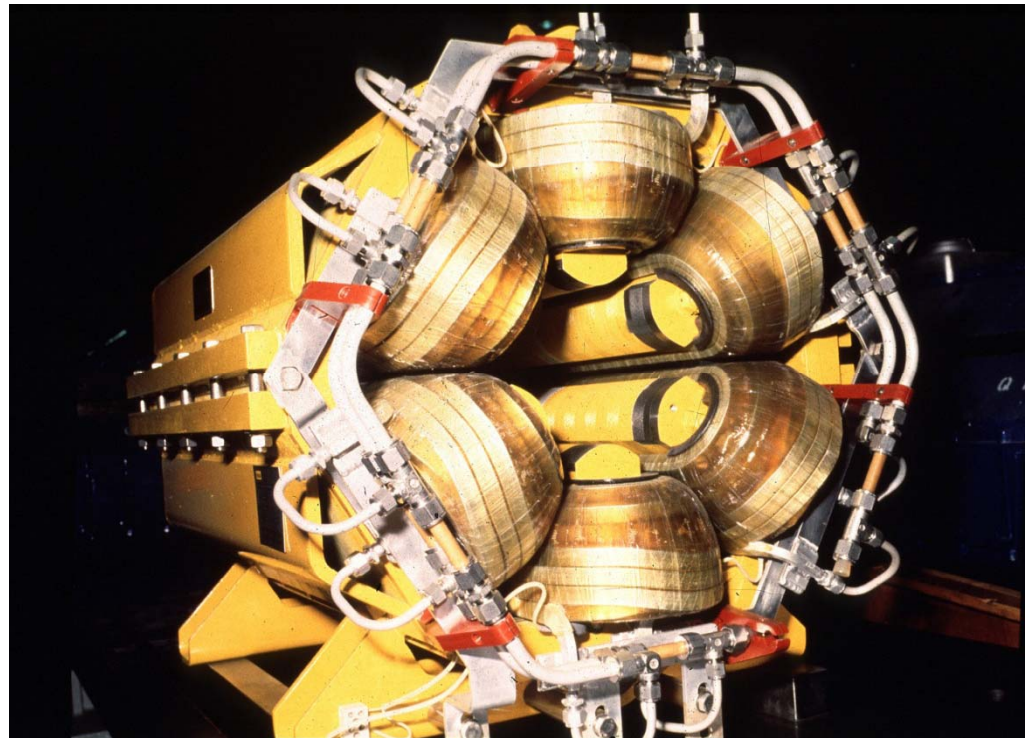
Field Imperfections: Multipole Illustrations

 quadrupole and sextupole magnets



ISR quadrupole

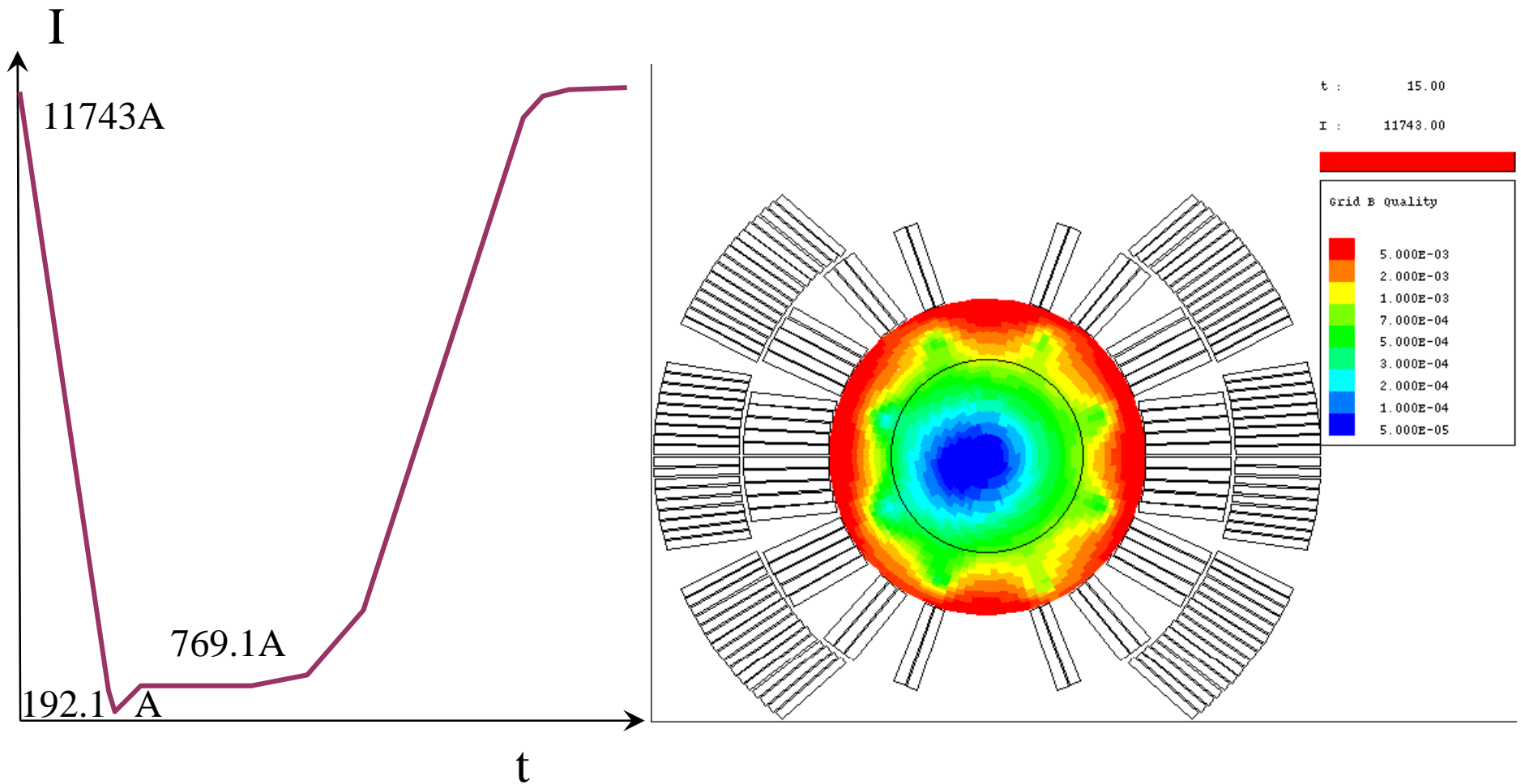
LEP Sextupole



Field Imperfections: Super Conducting Magnets

 time varying field errors in super conducting magnets

Luca Bottura CERN, AT-MAS



Perturbation Treatment: Resonance Condition

equations of motion:

(nth order Polynomial in x and y for nth order multipole)

$$\frac{d^2}{ds^2} w(s) + \omega_0^2 \cdot w(s) = \varepsilon \cdot \sum_{\substack{l+m < n, \\ r}} a_{n,m,r} \cdot x^l \cdot y^m \cdot \cos(2\pi \cdot r \cdot s / L)$$

with: $w = x, y$

perturbation treatment:

$$w(s) = w_0 + \varepsilon \cdot w_1 + \varepsilon^2 w_2 + \dots + O(\varepsilon^n)$$

$$\omega_0 = \frac{2\pi}{L} Q_0$$

with: $w_0(s) = w_0 \cdot \cos(2\pi \cdot Q_0 \cdot s / L + \phi_0)$

$w = x$:

$$\longrightarrow \frac{d^2}{ds^2} x_1 + \omega_0^2 \cdot x_1 = \sum_{\tilde{l} < l, \tilde{m} < m} a_{\tilde{n}, \tilde{m}, r} \cos\left(\frac{2\pi}{L} \cdot [\tilde{l} Q_{x,0} + \tilde{m} Q_{y,0} + r] \cdot s\right)$$

Perturbation Treatment: Tune Diagram I

resonance condition:

$$\frac{2\pi}{L} \cdot (\tilde{l} \cdot Q_x + \tilde{m} \cdot Q_y + r) = \frac{2\pi}{L} \cdot Q_{x,y}$$



$$l \cdot Q_x + m \cdot Q_y = r$$

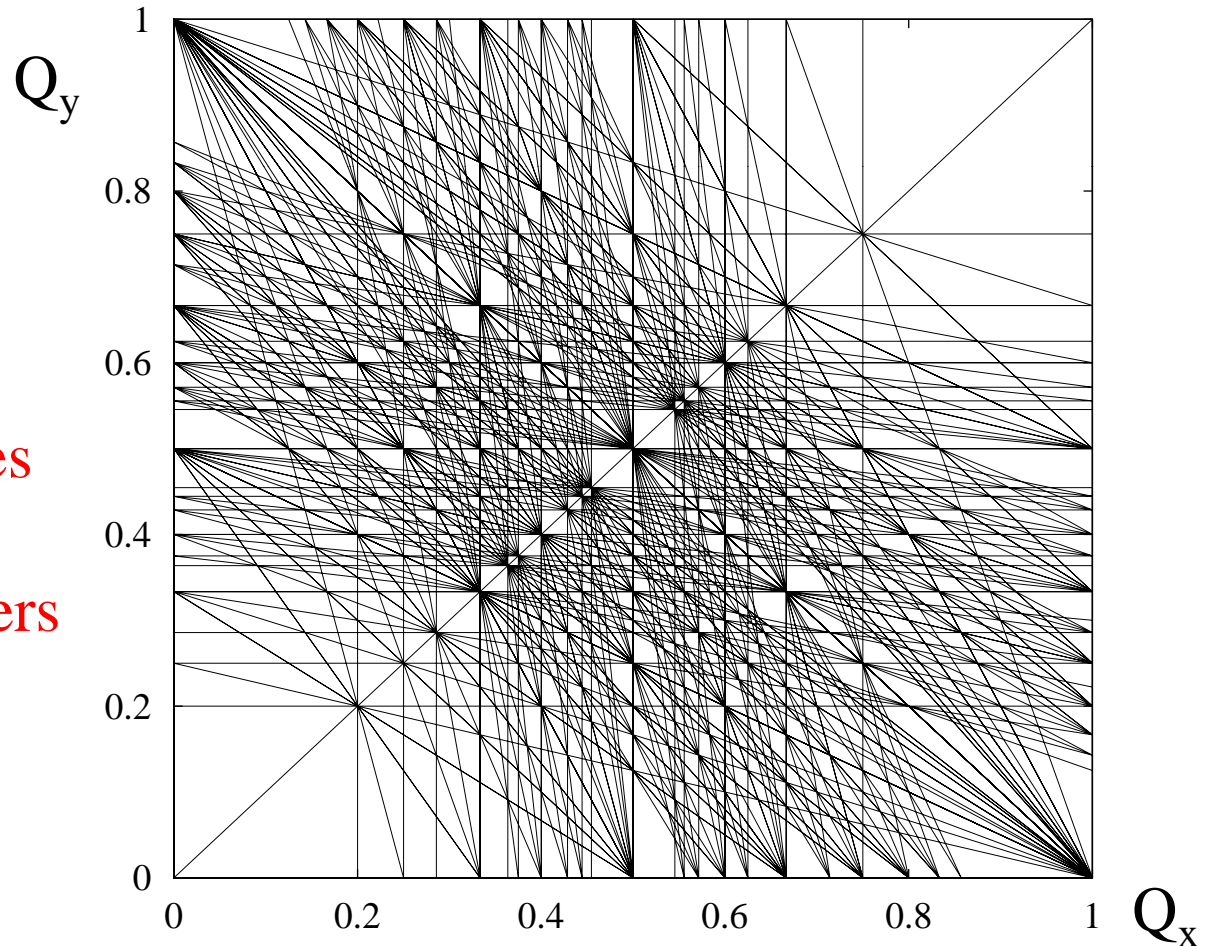
avoid rational tune values!

tune diagram:

up to 11 order ($p+1 < 12$)



there are resonances everywhere!
(the rational numbers lie dens within the real number)



Perturbation Treatment: Tune Diagram II

 regions with few resonances:

$$l \cdot Q_x + m \cdot Q_y = r$$

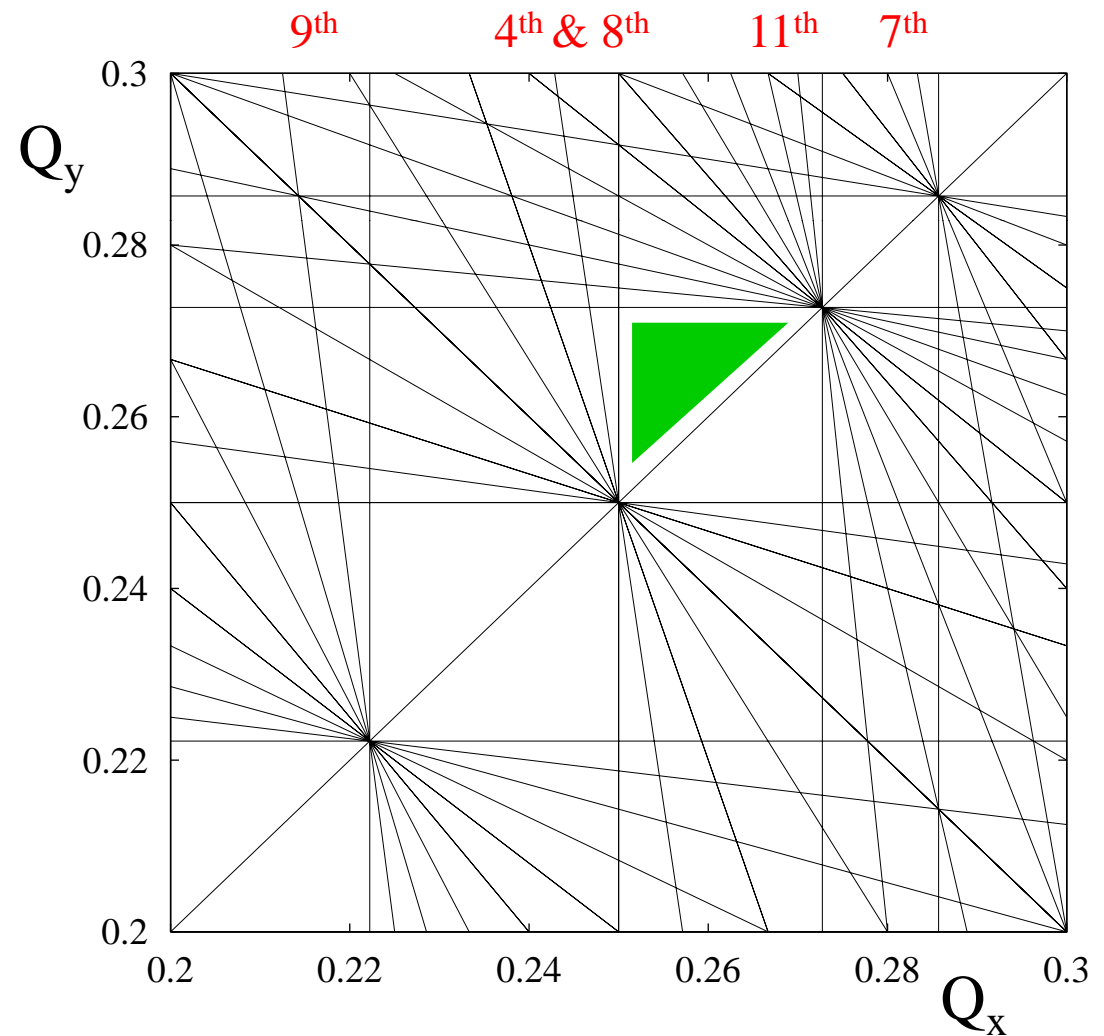
avoid low order resonances!

→ $< 12^{\text{th}}$ order for a proton beam without damping

→ $< 3^{\text{rd}} \Leftrightarrow 5^{\text{th}}$ order for electron beams with damping

 coupling resonance:

regions without low order resonances are relatively small!



Perturbation Treatment: Single Sextupole Perturbation

■ perturbed equations of motion: $F(s)/(v \cdot p) = \frac{1}{2} \cdot \delta_L(s - s_0) \cdot lk_2 \cdot x^2$

$$\rightarrow \frac{d^2}{ds^2} x_1(s) + \omega_0^2 \cdot x_1(s) = \frac{1}{2} \cdot lk_2 \cdot x_0^2 \cdot \frac{1}{L} \sum_{r=-\infty}^{\infty} \cos(2\pi \cdot r \cdot s / L)$$

$$\text{with: } x_0(s) = A \cdot \cos(\omega_{0,x} \cdot s + \phi_0) \quad \text{and} \quad \omega_{0,x} = 2\pi \cdot Q_{x,0} / L$$

$$\begin{aligned} \rightarrow \frac{d^2}{ds^2} x_1(s) + (2\pi Q_{x,0} / L)^2 \cdot x_1(s) &= \frac{lk_1}{2L} \cdot A^2 \cdot \sum_{r=-\infty}^{\infty} \cos(2\pi \cdot r \cdot s / L) \\ &+ \frac{lk_1}{8L} \cdot A^2 \cdot \sum_{r=-\infty}^{\infty} \cos(2\pi \cdot [r \pm 2Q_{x,0}] \cdot s / L) \end{aligned}$$

Perturbation Treatment: Sextupole Perturbation

resonance conditions:

$$\longrightarrow 2\pi Q_{x,0} = 2\pi \cdot (r) \longrightarrow Q_{x,0} = r$$

$$2\pi Q_{x,0} = 2\pi \cdot (r \pm 2Q_{x,0}) \xrightarrow{r-2Q_{x,0}} Q_{x,0} = r/3$$

$$\xrightarrow{r+2Q_{x,0}} Q_{x,0} = r$$

→

avoid integer and r/3 tunes!

perturbation treatment:

contrary to the previous examples no exact solution exist!

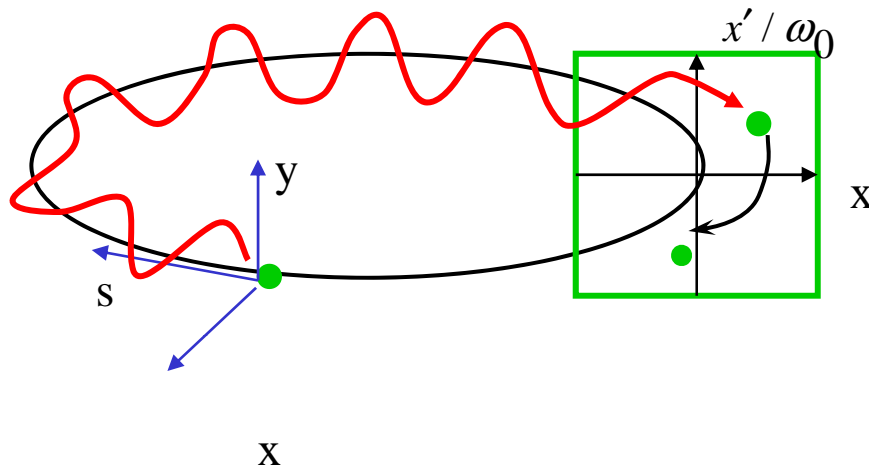
this is a consequence of the non-linear perturbation

(remember the 3 body problem?)

→ graphic tools for analyzing the particle motion

Poincare Section: Definition

Poincare Section:



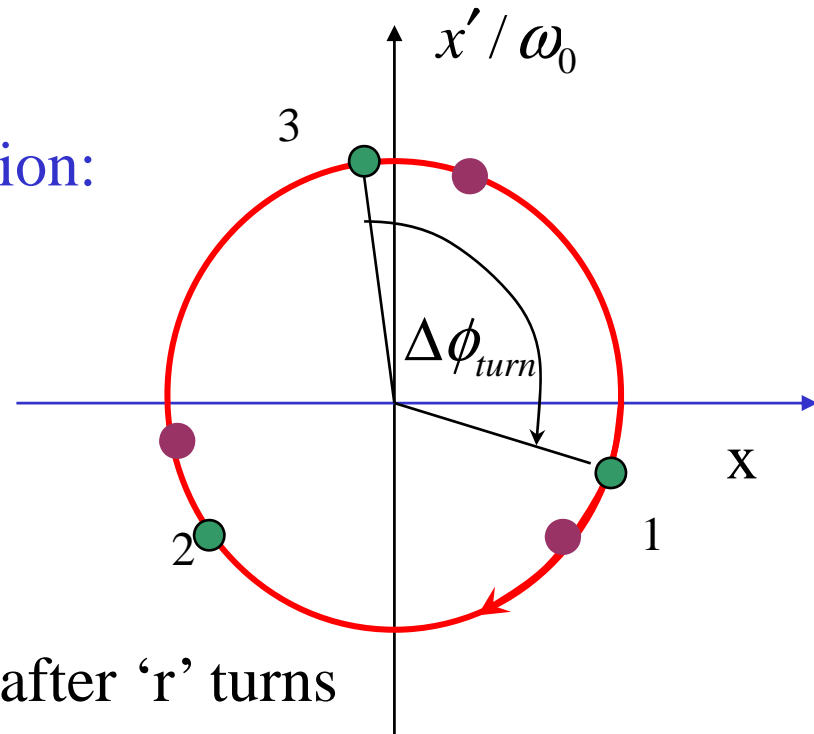
→ record the particle coordinates at one location in the storage ring

resonance in the Poincare section:

→ $\Delta\phi_{turn} = 2\pi \cdot Q$

fixed point condition: $Q = n/r$

points are mapped onto themselves after 'r' turns



Poincare Section: Linear Motion

unperturbed solution:

$$x(s) = \sqrt{R} \cdot \cos(\phi) \quad \text{with} \quad \frac{d}{ds} \phi = \omega_0$$

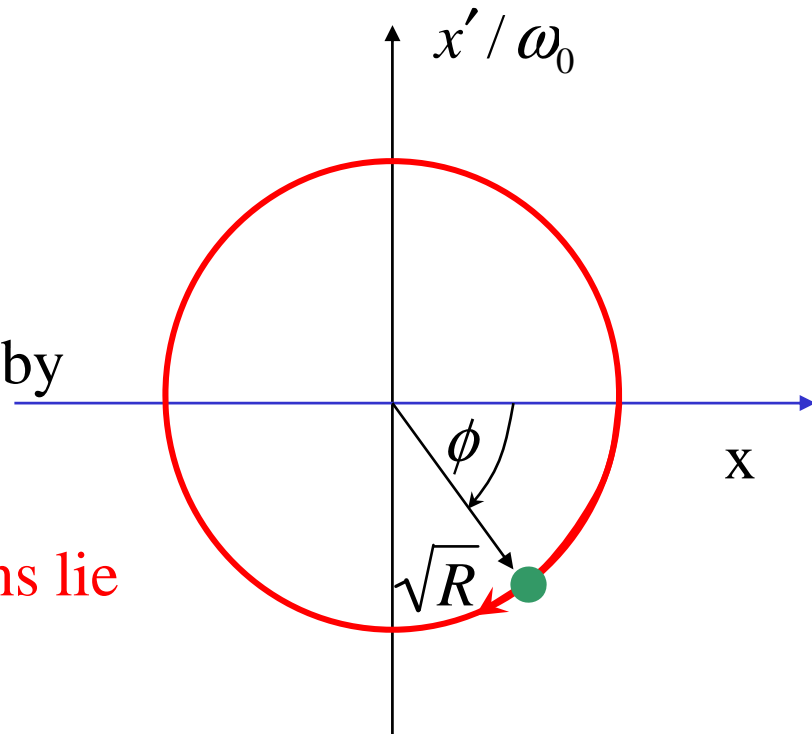
$$x' = \frac{d}{ds} x = -\sqrt{R} \cdot \omega_0 \cdot \sin(\phi)$$

phase space portrait:

→ the motion lies on an ellipse

→ linear motion is described by a simple rotation

→ consecutive intersections lie on closed curves



Poincare Section: Non-Linear Motion

momentum change due to perturbation: $\Delta x' = \oint \frac{F(s)}{v \cdot p} \cdot ds$

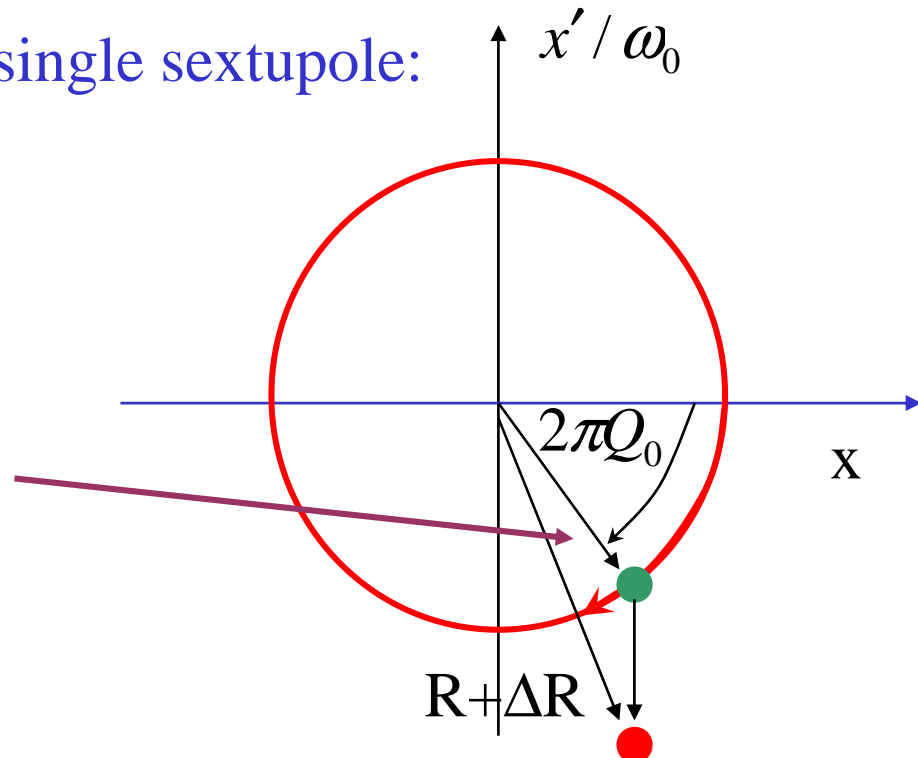
single n-pole kick: $\Delta x' = \frac{1}{n!} \cdot lk_n \cdot x^n$

phase space portrait with single sextupole:

→ $\Delta x' = \frac{1}{2} \cdot lk_2 \cdot x^2$

→ sextupole kick changes the amplitude and the phase advance per turn!

$$\Delta Q_{turn} \propto x^2$$



Poincare Section: Stability?

instability can be fixed by ‘detuning’:

→ overall stability depends on the balance between amplitude increase per turn and tune change per turn:

$\Delta Q_{turn}(x)$ → motion moves eventually off resonance

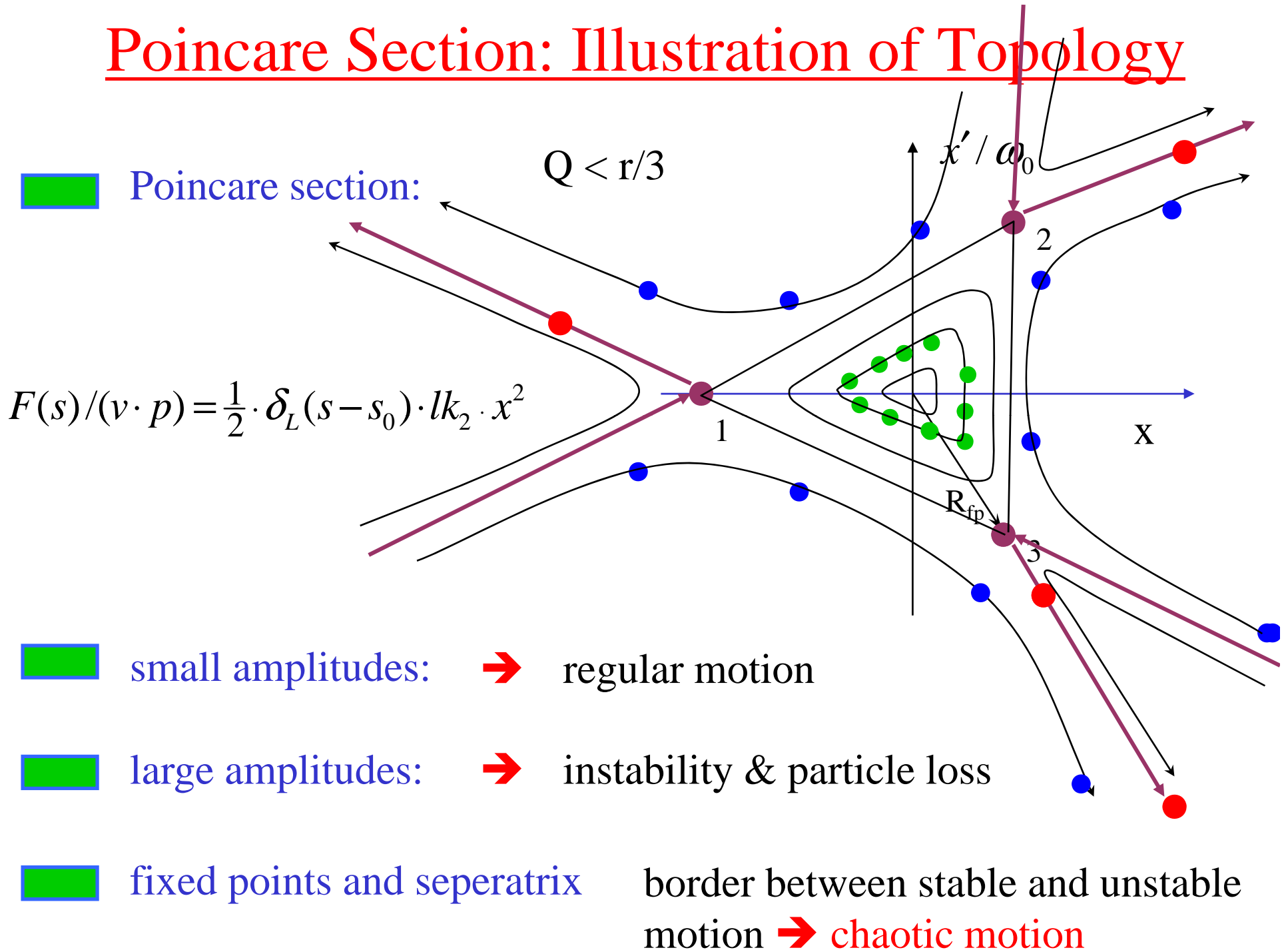
$\Delta R_{turn}(x)$ → motion becomes unstable

sextupole kick:

amplitudes increases faster then the tune can change

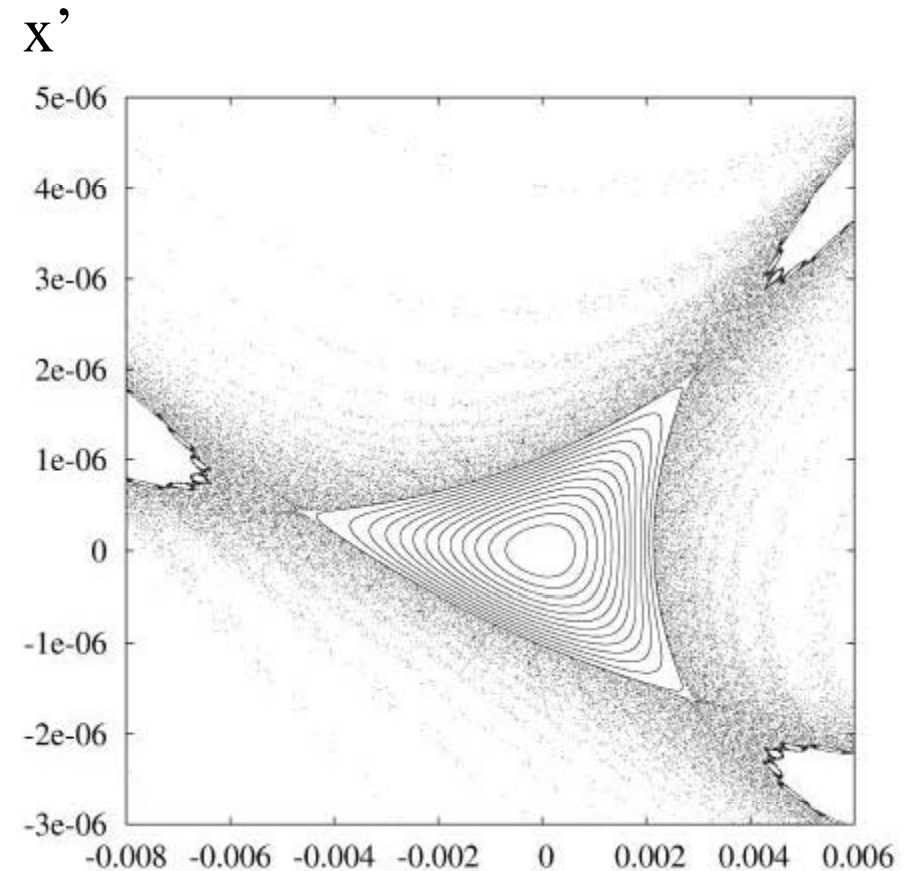
→ overall instability!

Poincare Section: Illustration of Topology



Poincare Section: Simulation for a Sextupole Perturbation

- Poincare Section right after the sextupole kick
- for small amplitudes the intersections still lie on closed curves → regular motion!
- separatrix location depends on the tune distance from the exact resonance condition ($Q < n/3$)



for large amplitudes and near the separatrix the intersections fill areas in the Poincare Section → chaotic motion;
→ no analytical solution exist!

Stabilization of Resonances

instability can be fixed by stronger ‘detuning’:

→ if the phase advance per turn changes uniformly with increasing R the motion moves off resonance and stabilizes

octupole perturbation:

$$F(s)/(v \cdot p) = \frac{1}{6} \cdot lk_3 x^3$$

perturbation treatment:

$$x(s) = x_0(s) + \varepsilon \cdot x_1(s) + \dots$$

→
$$\frac{d^2}{ds^2} x_1(s) + (2\pi Q_{x,0} / L)^2 \cdot x_1(s) = \frac{1}{6} \cdot lk_3 \cdot x_0^2 \cdot x_1$$

→
$$x_0 = A \cdot \cos(\omega_0 \cdot s + \phi_0) \Rightarrow x_0^2 = \frac{A^2}{2} \cdot [1 + \cos(2\omega_0 \cdot s + 2\phi_0)]$$

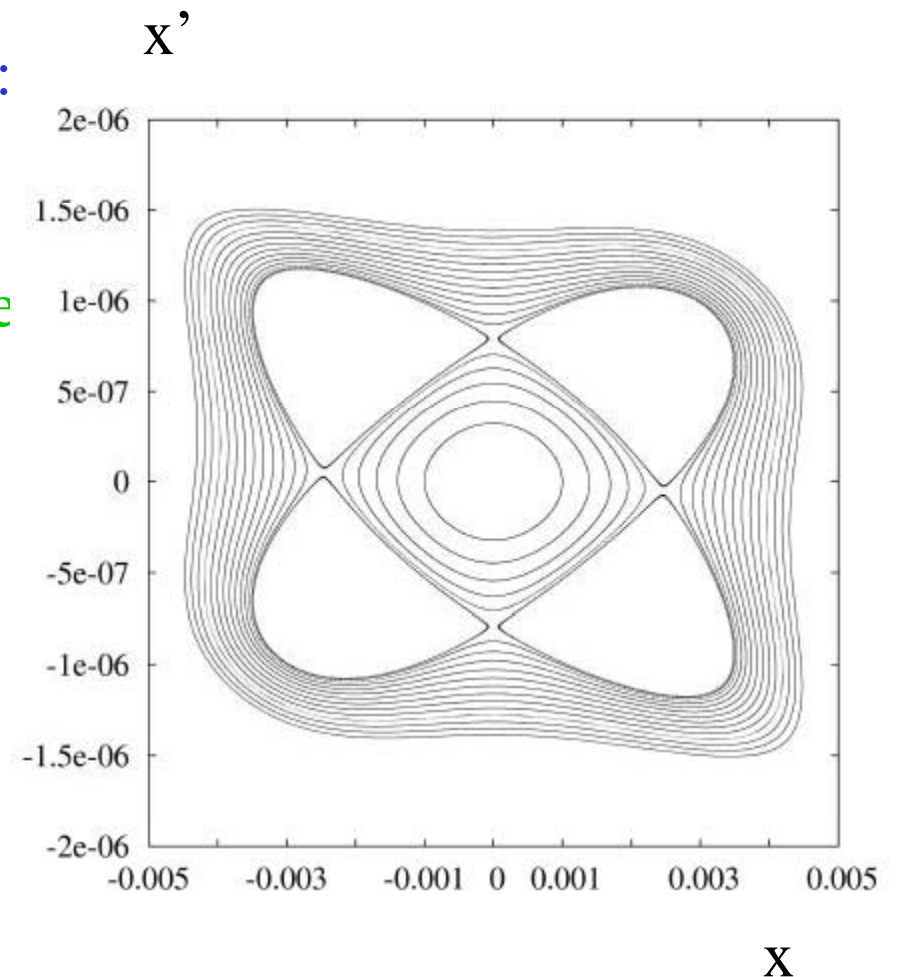
$$\frac{d^2}{ds^2} x_1(s) + \left[(2\pi Q_{x,0} / L)^2 - \frac{A^2 \cdot lk_3}{2 \cdot 6} \right] \cdot x_1(s) = \frac{A^2 \cdot lk_3}{2 \cdot 6} \cdot \cos(2\omega_0 \cdot s) \cdot x_1$$

Stabilization of Resonances

■ resonance stability for octupole:

→ an octupole perturbation generate phase independent detuning and amplitude growth of the same order

→ amplitude growth and detuning are balanced and the overall motion is stable!



→ this is not generally true in case of several resonance driving terms and coupling between the horizontal and vertical motion!

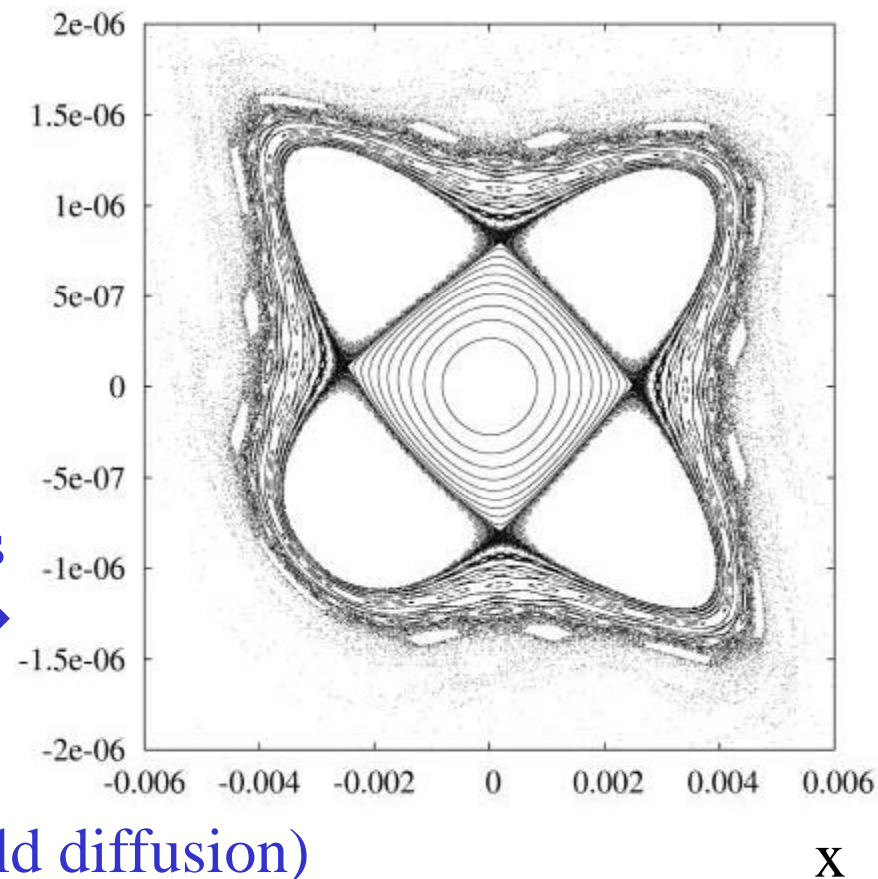
Chaotic Motion

■ octupole + sextupole perturbation: x'

→ the interference of the octupole and sextupole perturbations generate additional resonances
→ additional island chains in the Poincare Section!

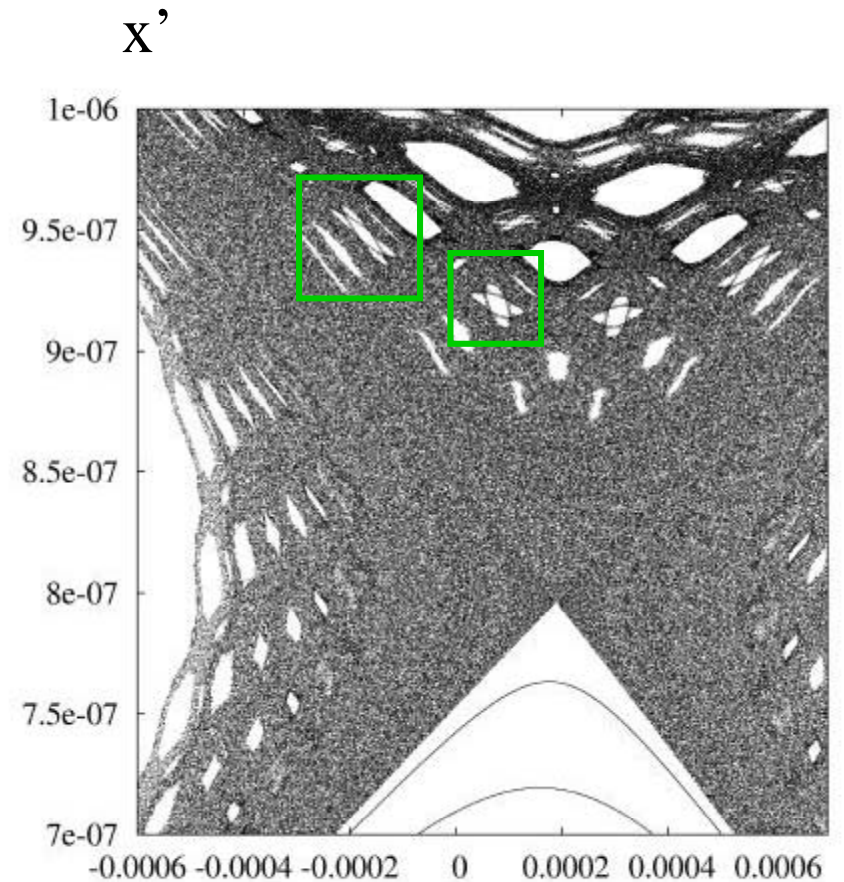
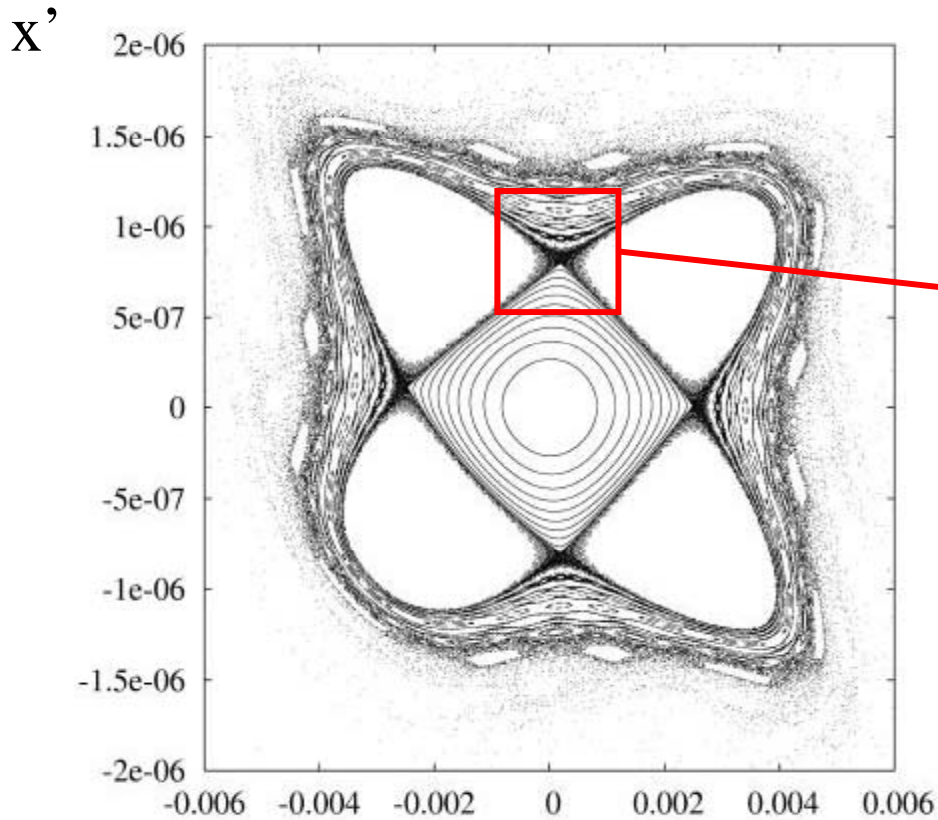
→ intersections near the resonances lie no longer on closed curves → local chaotic motion around the separatrix & instabilities
→ slow amplitude growth (Arnold diffusion)

→ neighboring resonance islands start to 'overlap' for large amplitudes → global chaos & fast instabilities



Chaotic Motion

 'Russian Doll' effect:



→ magnifying sections of the Poincaré Section reveals always the same pattern on a finer scale → renormalization theory!

Summary

- field imperfections drive resonances
- higher order than quadrupole field imperfections generate non-linear equations of motion (no closed analytical solution)
 - (three body problem of Sun, Earth and Jupiter)
 - solutions only via perturbation treatment
- Poincare Section as a graphical tool for analyzing the stability
- slow extraction as example of resonance application in accelerator
- island chains as signature for non-linear resonances
- island overlap as indicator for globally chaotic & unstable motion