Recapitulation of Relativity and Space Charge

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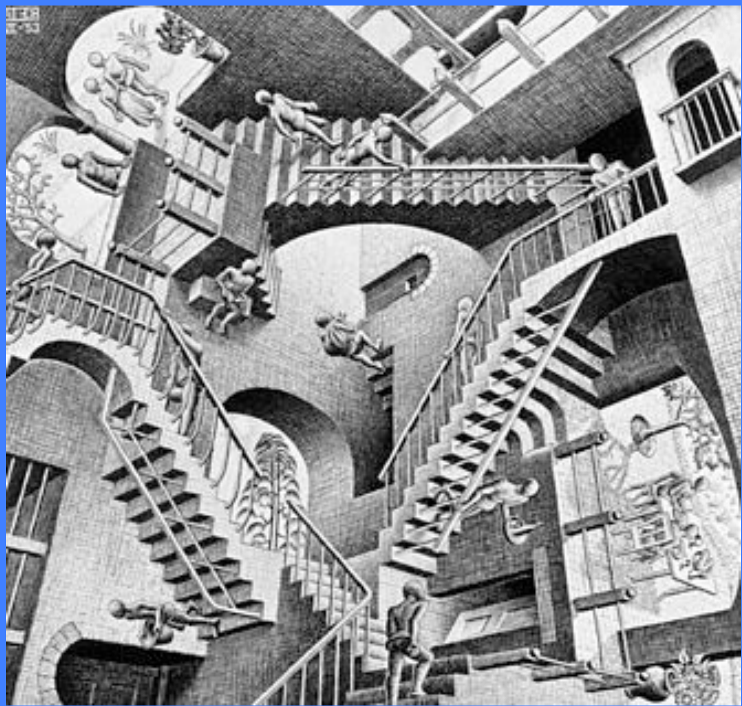


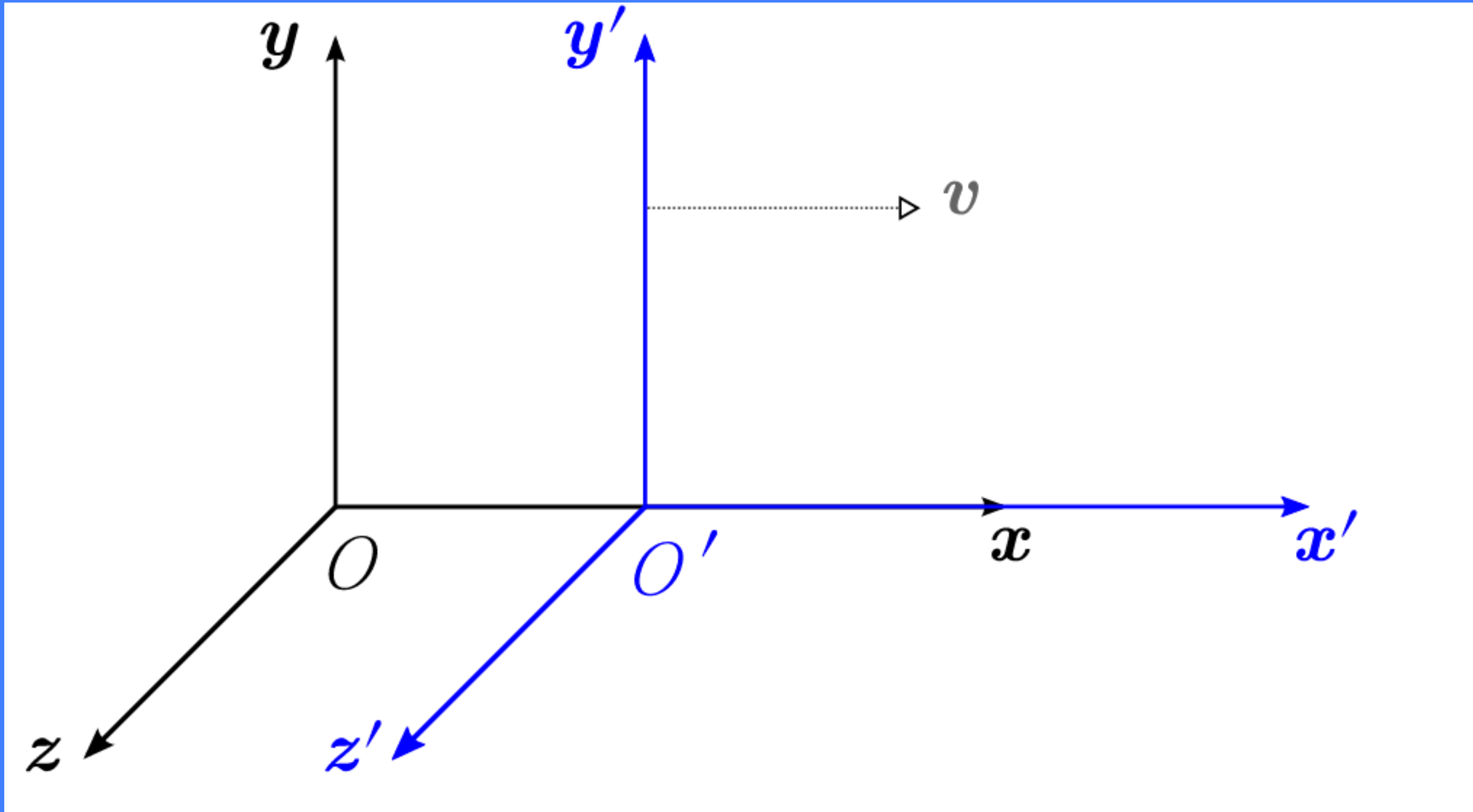
Erice – 11 March 2017

Relativity - Basic principles

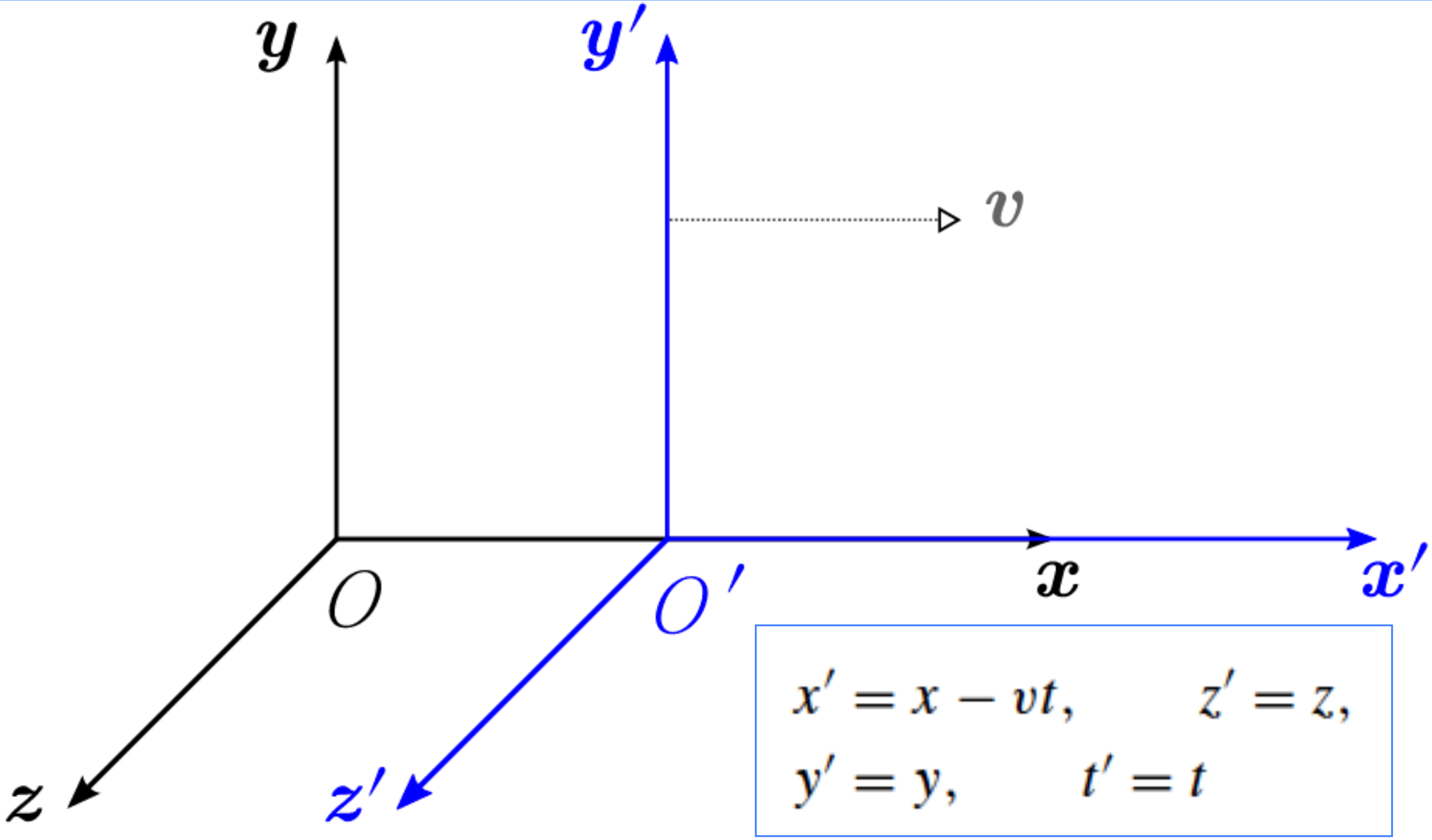
- **The Principle of Relativity** – The laws of physics are invariant (i.e. identical) in all inertial systems (non-accelerating frames of reference) =>
  - All experiments run the same in all inertial frames of reference

- **The Principle of Invariant Light Speed** – The speed of light in a vacuum is the same for all observers, regardless of the motion of the light source =>
  - c = 299792458 m/s

Inertial Systems

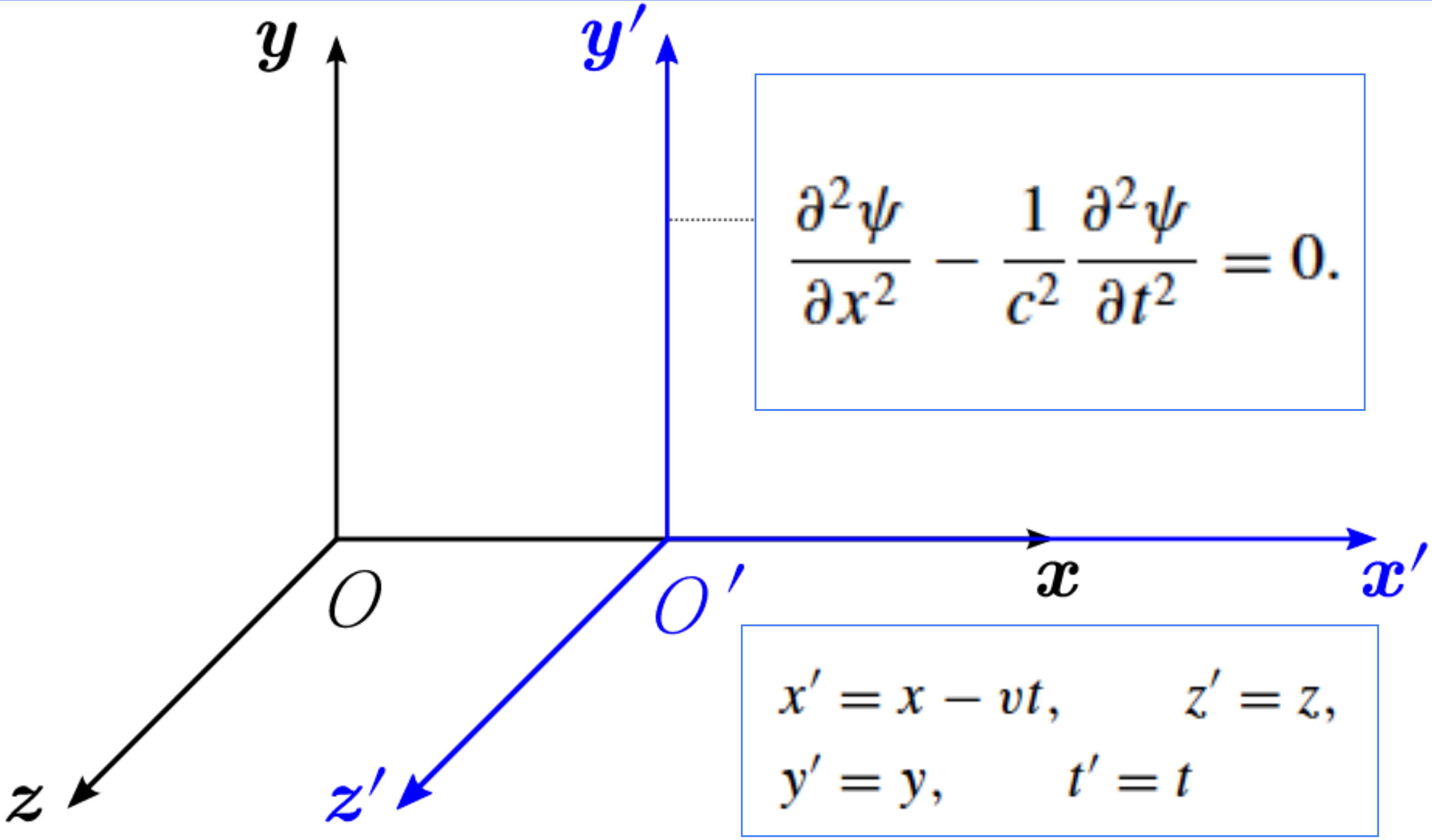


Galileo Transformations



$$x' = x - vt, \qquad z' = z,$$
$$y' = y, \qquad t' = t$$

Wave Equation ?



$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0.$$

$$x' = x - vt, \qquad z' = z,$$
$$y' = y, \qquad t' = t$$

Galileo Transformations Fail !

$$x' = x - vt, \qquad t' = t.$$

The partial derivatives are related by

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'}, \qquad \frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'},$$

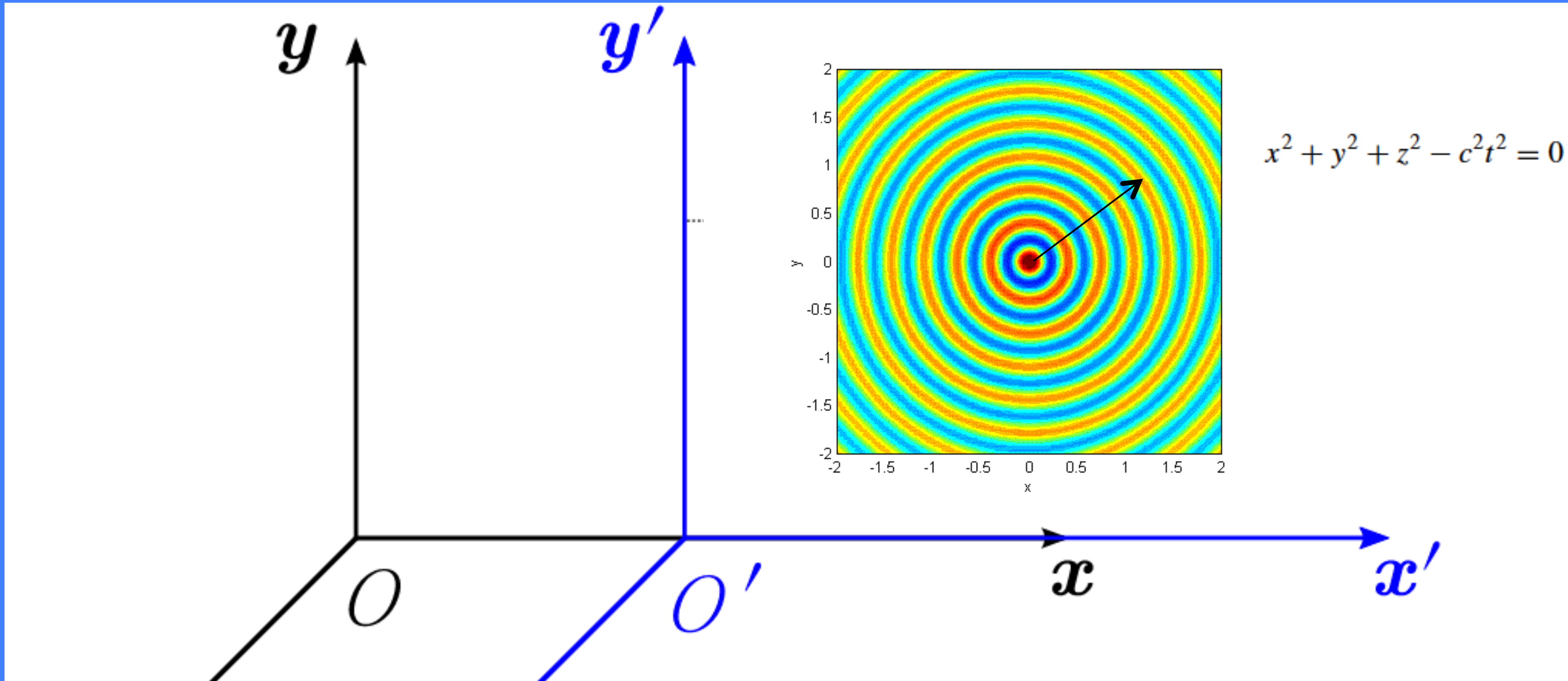
$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'},$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial x'^2}, \qquad \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial t'^2} + v^2\frac{\partial^2}{\partial x'^2} - 2v\frac{\partial}{\partial t'}\frac{\partial}{\partial x'}.$$

Insertion into the equation yields

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2} - \frac{v^2}{c^2}\frac{\partial^2\psi}{\partial x'^2} + \frac{2v}{c^2}\frac{\partial^2\psi}{\partial t'\partial x'} = 0$$

In both reference frames a spherical wave propagates with velocity c and must remain spherical



$$x^2 + y^2 + z^2 - c^2t^2 = 0 = x'^2 + y'^2 + z'^2 - c^2t'^2$$

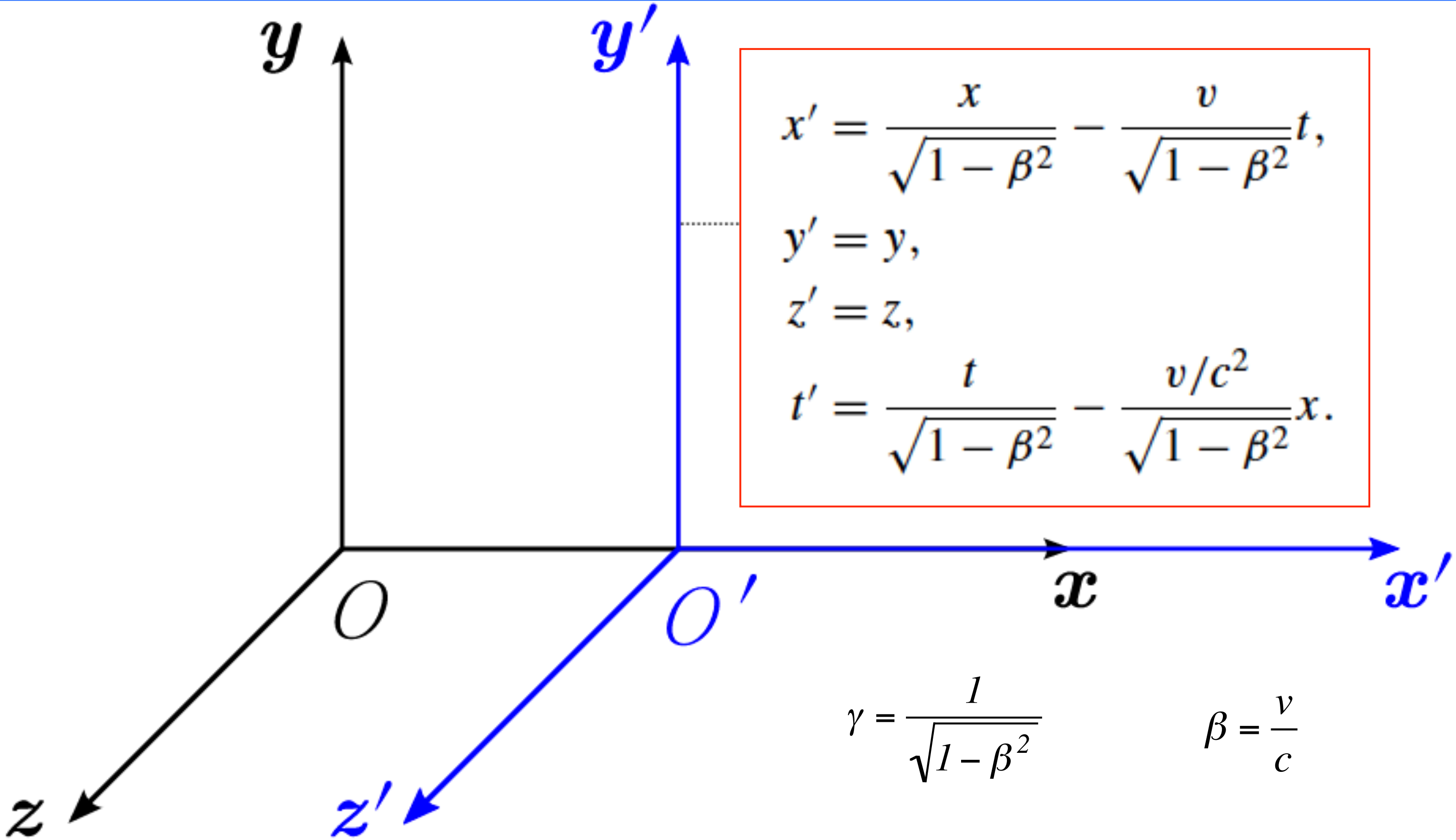
Derivation of Lorentz Transformations

$$\begin{cases} x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t \\ y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t \\ z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t \\ t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t \ . \end{cases}$$

+

$$x^2 + y^2 + z^2 - c^2t^2 = 0 = x'^2 + y'^2 + z'^2 - c^2t'^2$$

Lorentz Transformations



$$x' = \frac{x}{\sqrt{1-\beta^2}} - \frac{v}{\sqrt{1-\beta^2}}t,$$

$$y' = y,$$

$$z' = z,$$

$$t' = \frac{t}{\sqrt{1-\beta^2}} - \frac{v/c^2}{\sqrt{1-\beta^2}}x.$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{v}{c}$$

Lorentz Transformations and the invariance of light vector

$$
\begin{aligned}
& x'^2 + y'^2 + z'^2 - c^2 t'^2 \\
&= \frac{1}{1-\beta^2}(x - vt)^2 + y^2 + z^2 - \frac{c^2}{1-\beta^2}\left(t - \frac{v}{c^2}x\right)^2 \\
&= \left[\frac{1}{1-\beta^2} - \frac{v^2/c^2}{1-\beta^2}\right]x^2 + y^2 + z^2 - c^2 t^2 \left[\frac{1}{1-\beta^2} - \frac{v^2/c^2}{1-\beta^2}\right] \\
&\quad - tx\left[\frac{2v}{1-\beta^2} - \frac{2v}{1-\beta^2}\right] \\
&= x^2 + y^2 + z^2 - c^2 t^2
\end{aligned}
$$

Lorentz Transformations and the invariance of wave equation

$$\begin{aligned}\frac{\partial^2\psi}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}&=\frac{1}{1-\beta^2}\left(\frac{\partial^2\psi}{\partial x'^2}-\frac{2v}{c^2}\frac{\partial^2\psi}{\partial x'\partial t'}+\frac{v^2\partial^2\psi}{c^4\partial t'^2}-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2}+\frac{2v}{c^2}\frac{\partial^2\psi}{\partial x'\partial t'}-\frac{v^2}{c^2}\frac{\partial^2\psi}{\partial x'^2}\right)\\&=\frac{1}{1-\beta^2}\left[\frac{\partial^2\psi}{\partial x'^2}-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2}\right]-\frac{v^2/c^2}{1-\beta^2}\left[\frac{\partial^2\psi}{\partial x'^2}-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2}\right]\\&=\frac{\partial^2\psi}{\partial x'^2}-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2}=0.\end{aligned}$$

The consequences of Lorentz Transformations

- **Time Dilation:**

$$\boxed{\Delta t = \gamma \Delta t'}$$

- **Length Contraction:**

$$\boxed{\Delta x = \frac{\Delta x'}{\gamma}}$$

Relativistic dynamics

Fundamental relations of the relativistic dynamics

Rest Energy	Relativistic momentum	Relativistic γ-factor	Total Energy	Kinetic Energy
$W_0 = m_0 c^2$	$p = \gamma m_o v$, $\beta < 1 \ always!$	$\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$ $\gamma \geq 1 \ always!$ $m = \gamma\, m_0$	$W = \gamma\, m_0 c^2 = \gamma\, W_0$ $W^2 = W_0^2 + p^2 c^2$	$W_k = W - W_0 =$ $= (\gamma - 1) m_0 c^2 \approx$ $\approx \dfrac{1}{2} m_0 v^2 \quad se \quad \beta << 1$

Newton's 2nd Law	Lorentz Force
$\vec{F} = \dfrac{d}{dt}\vec{p} = \dfrac{d}{dt}(m\vec{v})$	$\vec{F} = q\ (\vec{E} + \vec{v} \times \vec{B})$

Relativistic equation of motion

$$\boldsymbol{P} = m\boldsymbol{v} = m_0\,\gamma(v)\,\boldsymbol{v} \qquad \boxed{\boldsymbol{f} = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t}} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{v}{c}$$

$$\boxed{1 + \beta^2\gamma^2 \equiv \gamma^2}$$

$$\boldsymbol{f} = m_0\,\frac{\mathrm{d}}{\mathrm{d}t}\,\boldsymbol{v}\,\gamma(v) = m_0\left[\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\cdot\gamma(v) + \boldsymbol{v}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\gamma(v)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\gamma(v) = \frac{\mathrm{d}}{\mathrm{d}t}\left(1-\frac{\boldsymbol{v}^2}{c^2}\right)^{-1/2} = -\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\cdot\left(-2\,\frac{\boldsymbol{v}}{c^2}\,\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right) = \gamma^3(v)\,\frac{\boldsymbol{a}\boldsymbol{v}}{c^2}$$

$$\boxed{\boldsymbol{f} = m_0\,\gamma(v)\left[\boldsymbol{a} + \gamma^2(v)\,\frac{\boldsymbol{a}\boldsymbol{v}}{c^2}\cdot\boldsymbol{v}\right]}$$

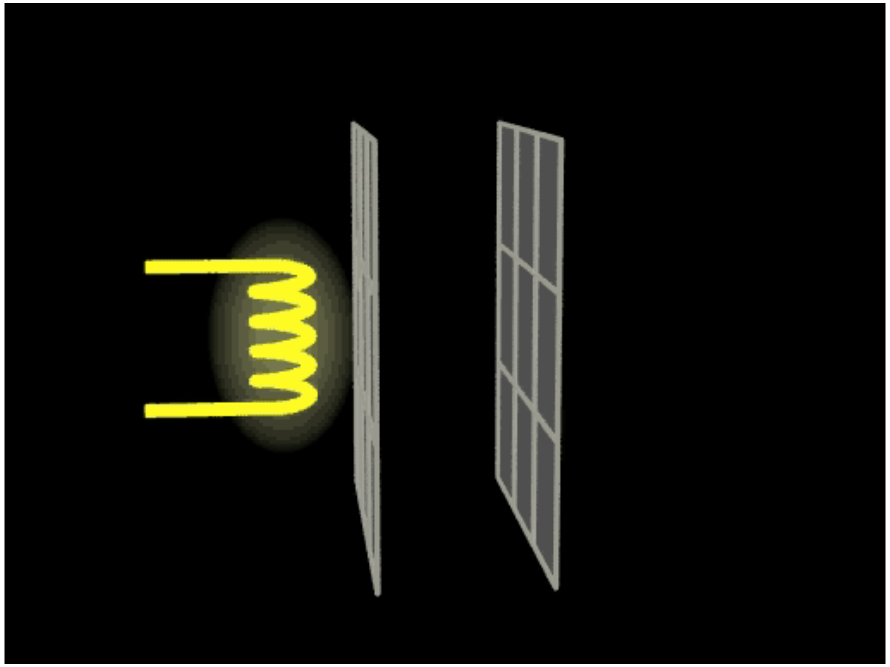
Acceleration does not generally point in the direction of velocity

$$\boxed{a \perp v} \qquad \boldsymbol{f} = m_0\,\gamma(v)\,\boldsymbol{a} \qquad \boxed{m_\perp = m_0\,\gamma(v)}$$

$$\boxed{a \,//\, v} \qquad \boldsymbol{f} = m_0\,\gamma(v)\left[\boldsymbol{a} + \gamma^2(v)\,\frac{v^2}{c^2}\cdot\boldsymbol{a}\right] = m_0\,\gamma^3(v)\,\boldsymbol{a} \qquad \boxed{m_{//} = m_o\gamma^3(v)}$$

A moving body is more inert in the longitudinal direction than in the transverse direction

Longitudinal motion in the laoratory frame
==> ex: beam dynamics in a relativistic capacitor



Consider longitudinal motion only :

$$\boxed{\gamma^3 \frac{d\beta}{dt} = \frac{a_o}{c}} \qquad a_o = \frac{eE_z}{m_o}$$

$$\int_{\beta_o}^{\beta} \frac{d\beta}{\left(1-\beta^2\right)^{3/2}} = \frac{a_o}{c}\int_{t_o}^{t} dt$$

$$\frac{\beta}{\sqrt{1-\beta^2}} - \beta_o\gamma_o = \frac{a_o}{c}\left(t - t_o\right)$$

Solving explicitly for β one can find:

$$\boxed{\beta(t) = \frac{a_o(t - t_o) + c\beta_o\gamma_o}{\sqrt{c^2 + (c\beta_o\gamma_o + a_o(t - t_o))^2}}}$$

After separating the variables one can integrate once more to obtain the position as a function of time :

$$\boxed{z(t) - z_o = \frac{c^2}{a_o}\left(\sqrt{1 + \left(\beta_o\gamma_o + \frac{a_o}{c}(t - t_o)\right)^2} - \gamma_o\right) = h(t)}$$

In the non relativistic limit: $$z(t) - z_o = \beta_o c(t - t_o) + \frac{1}{2}a_o(t - t_o)^2$$

The previous solution can be written also in the form:

$$\boxed{\left(z(t) - z_o + \gamma_o\frac{c^2}{a_o}\right)^2 - \left(\frac{c^2}{a_o}\beta_o\gamma_o + c(t - t_o)\right)^2 = \left(\frac{c^2}{a_o}\right)^2}$$

the corresponding world line in the Minkowsky space-time (ct,z) is an hyperbola

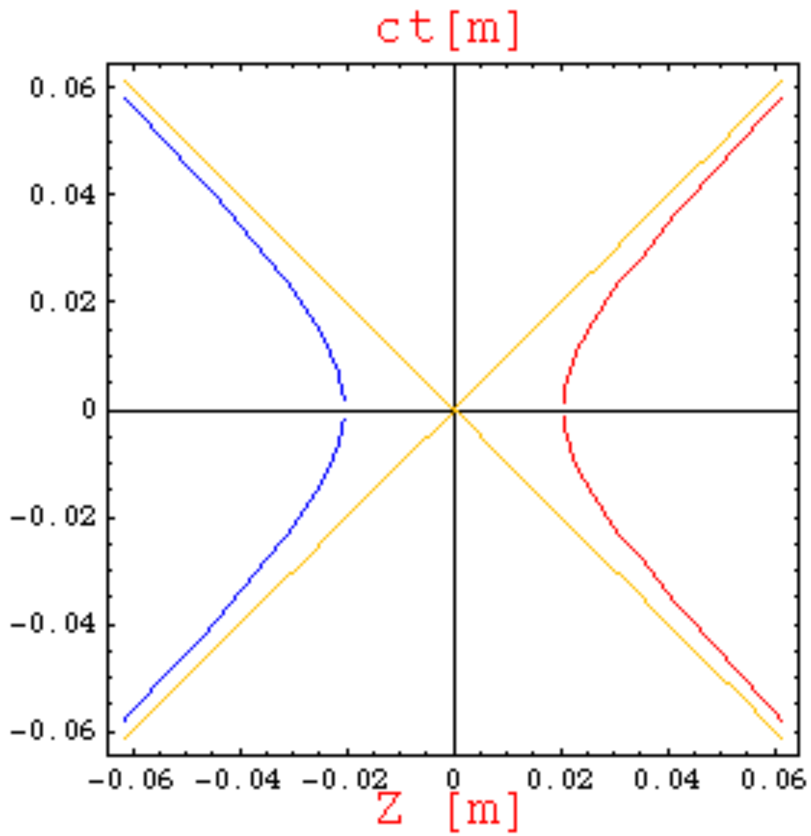
==> hyperbolic motion

in the simpler case with initial conditions:

$$\begin{cases} \beta_o = 0 \\ \gamma_o = 1 \\ z_o = 0 \end{cases}$$

and shifted variable: $Z(t) = z(t) + \dfrac{c^2}{a_o}$

$$\boxed{Z(t)^2 - (ct)^2 = \left(\frac{c^2}{a_o}\right)^2}$$



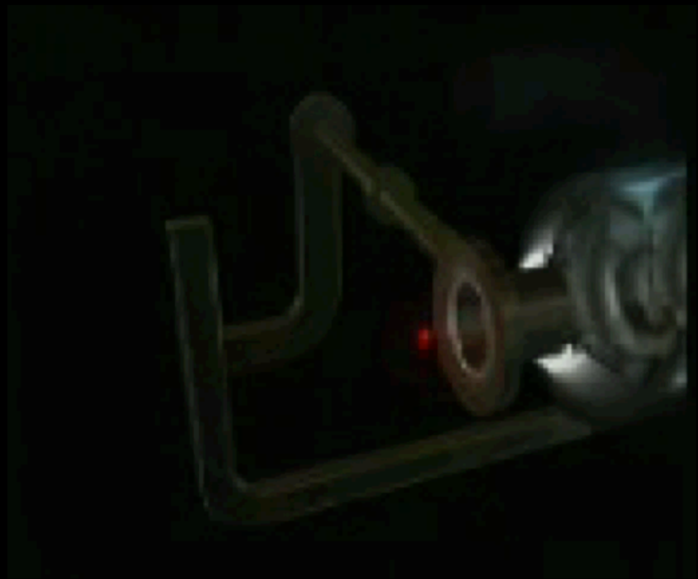
Therefore such motion is called hyperbolic motion.
It describes the motion of a particle that arrives from large positive z, slows down and stops at turning point $Z_t = c^2/a_o$ then it accelerates back up the z axis.
The world-line is asymptotic to the light cones, and obviously, it will never reach the speed of light.

The problem of relativistic bunch length

Low energy electron bunch injected in a linac:

$$\gamma \approx 1$$

$$L_b = 3mm \approx L_b'$$



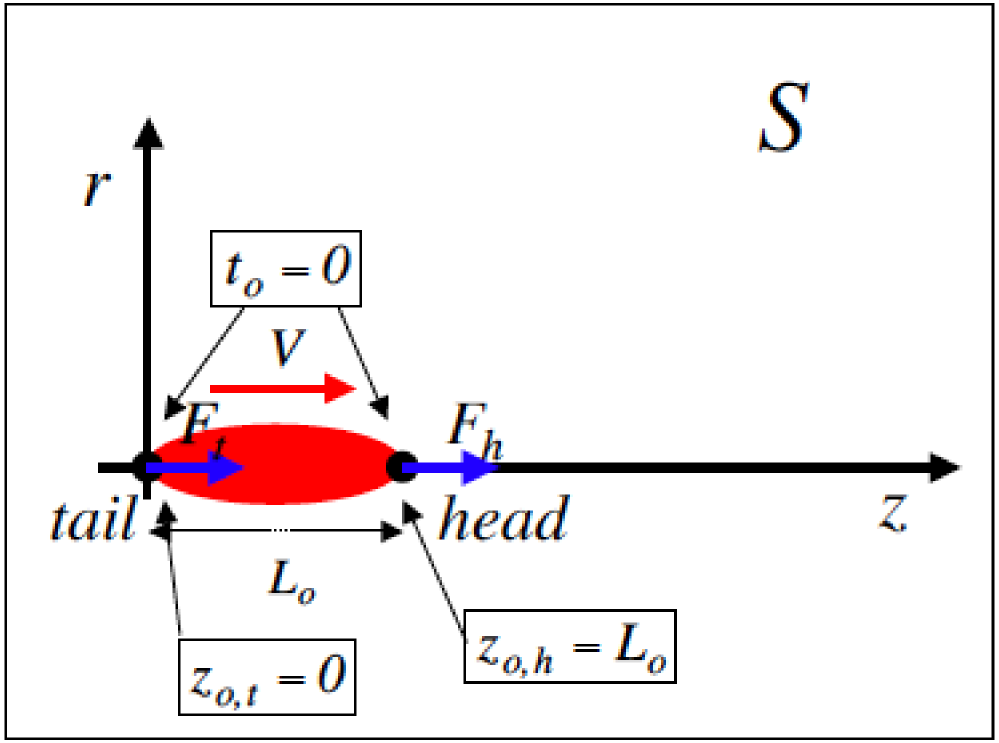
Length contraction?

$$\cancel{\gamma = 1000}$$

$$\cancel{L_b = \frac{L_b'}{\gamma} = 3\mu m}$$

Bunch length in the laboratory frame S

Let consider an electron bunch of initial length L_o inside a capacitor when the field is suddenly switched on at the time t_o.



$$L(t) = z_h(t) - z_t(t)$$

$$L(t) = \left(L_o + h(t)\right) - h(t) = L_o$$

Thus a simple computation show that no observable contraction occurs in the laboratory frame, as should be expected since both ends are subject to the same acceleration at the same time.

Bunch length in the moving frame S'

More interesting is the bunch dynamics as seen by a moving reference frame S', that we assume it has a relative velocity V with respect to S such that at the end of the process the accelerated bunch will be at rest in the moving frame S'. It is actually a deceleration process as seen by S'

Lorentz transformations:

$$\begin{cases} ct' = \gamma\left(ct - \frac{V}{c}z\right) \\ z' = \gamma(z - Vt) \end{cases}$$

leading for the tail particle to:

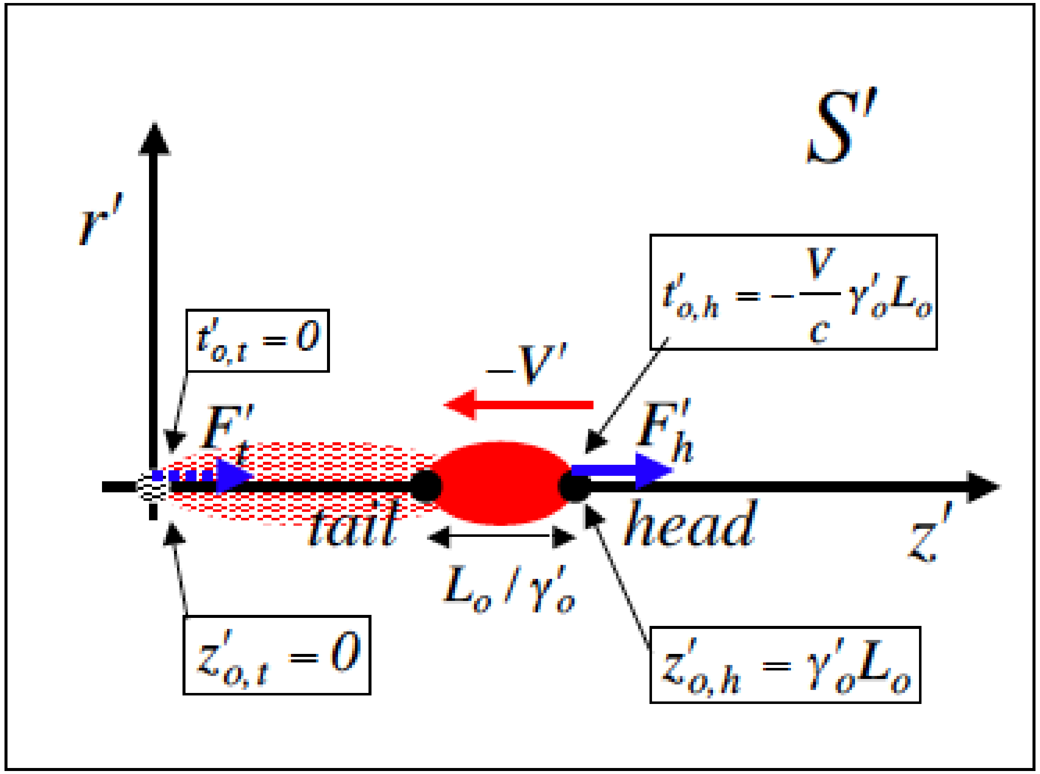
$$\begin{cases} t'_{o,t} = t_o = 0 \\ z'_{o,t} = z_{o,t} = 0 \end{cases}$$

and for the head particle to:

$$\begin{cases} t'_{o,h} = -\frac{V}{c}\gamma'_o L_o < t_o \\ z'_{o,h} = \gamma'_o L_o > z_{o,h} \end{cases}$$

The key point is that as seen from S' the decelerating force is not applied *simultaneously* along the bunch but with a *delay* given by:

$$\Delta t'_o = t'_{o,h} - t'_{o,t} = -\frac{V}{c}\gamma'_o L_o < 0$$



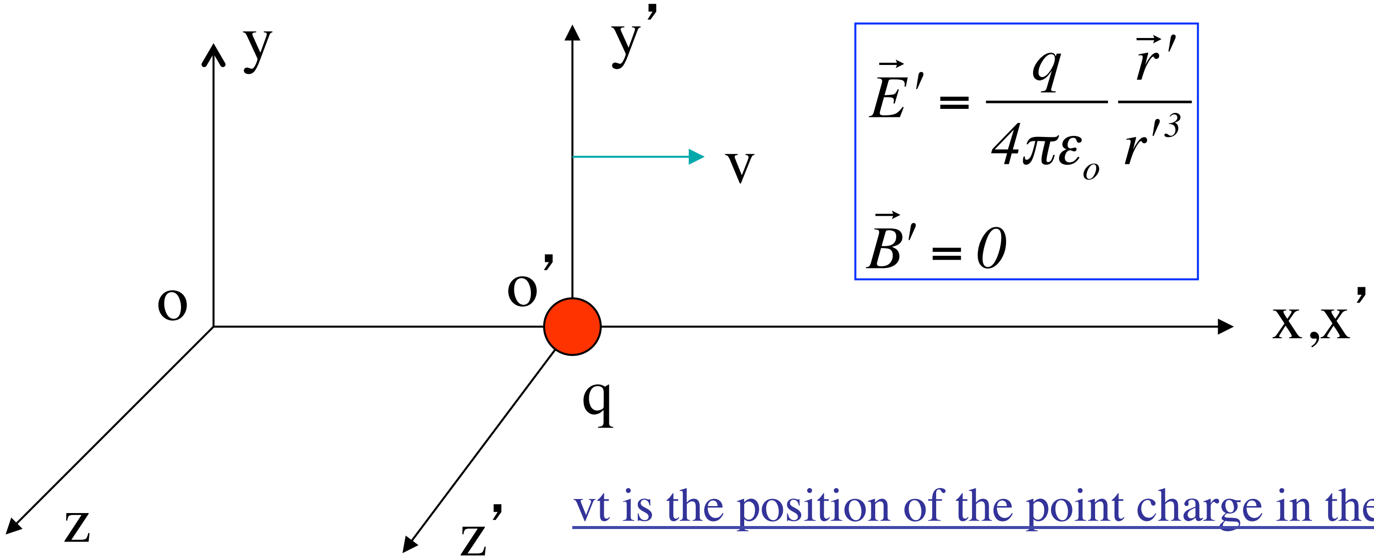
At the end of the process when both particle have been subject to the same decelerating field for the same amount of time the bunch length results to be:

$$L'(t') = \left(\gamma' L_o + h'(t')\right) - h'(t') = \gamma' L_o$$

$$z'(t') - z'_o = \frac{c^2}{a_o}\left(\sqrt{1 + \left(\beta'_o \gamma'_o + \frac{a_o}{c}\left(t' - t'_{o,h}\right)\right)^2} - \gamma'_o\right) = h'(t')$$

Electromagnetic Fields of a moving charge

Fields of a point charge with uniform motion



$$\vec{E}' = \frac{q}{4\pi\varepsilon_o}\frac{\vec{r}\,'}{r'^3}$$

$$\vec{B}' = 0$$

vt is the position of the point charge in the lab. frame O.

- In the moving frame O' the charge is at rest
- The electric field is radial with spherical symmetry
- The magnetic field is zero

$$E'_x = \frac{q}{4\pi\varepsilon_o}\frac{x'}{r'^3} \qquad E'_y = \frac{q}{4\pi\varepsilon_o}\frac{y'}{r'^3} \qquad E'_z = \frac{q}{4\pi\varepsilon_o}\frac{z'}{r'^3}$$

Relativistic transforms of the fields from O' to O

$$\begin{cases} E_x = E'_x \\ E_y = \gamma(E'_y + vB'_z) \\ E_z = \gamma(E'_z - vB'_y) \end{cases} \qquad \begin{cases} B_x = B'_x \\ B_y = \gamma(B'_y - vE'_z / c^2) \\ B_z = \gamma(B'_z + vE'_y / c^2) \end{cases}$$

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ ct' = \gamma\left(ct - \frac{v}{c}x\right) \end{cases}$$

$$r' = \left(x'^2 + y'^2 + z'^2\right)^{1/2}$$

$$r' = \left[\gamma^2(x - vt)^2 + y^2 + z^2\right]^{1/2}$$

$$E_x = E'_x = \frac{q}{4\pi\varepsilon_o}\frac{x'}{r'^3} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma(x-vt)}{\left[\gamma^2(x-vt)^2+y^2+z^2\right]^{3/2}}$$

$$E_y = \gamma E'_y = \frac{q}{4\pi\varepsilon_o}\frac{y'}{r'^3} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma y}{\left[\gamma^2(x-vt)^2+y^2+z^2\right]^{3/2}}$$

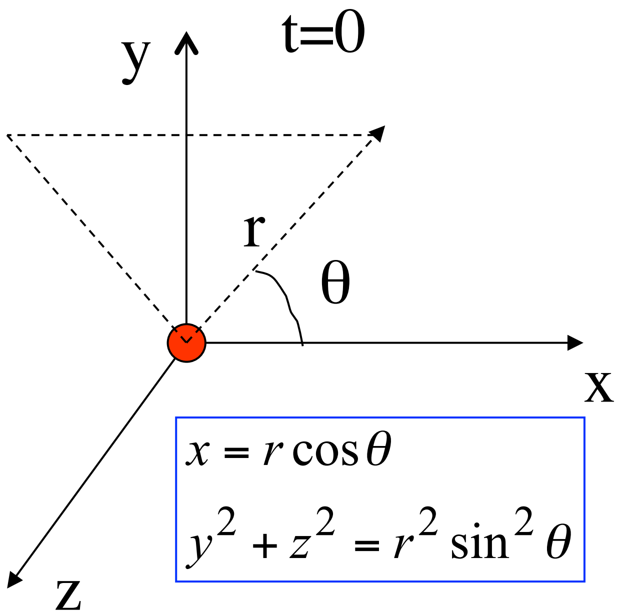
$$E_z = \gamma E'_z = \frac{q}{4\pi\varepsilon_o}\frac{z'}{r'^3} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma z}{\left[\gamma^2(x-vt)^2+y^2+z^2\right]^{3/2}}$$

The field pattern is moving with the charge and it can be observed at t=0:

$$\vec{E} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma\,\vec{r}}{\left[\gamma^2x^2+y^2+z^2\right]^{3/2}}$$

The fields have lost the spherical symmetry

$$\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma\,\vec{r}}{\left[\gamma^2 x^2 + y^2 + z^2\right]^{3/2}}}$$



$$\boxed{\begin{aligned} x &= r\cos\theta \\ y^2 + z^2 &= r^2\sin^2\theta \end{aligned}}$$

$$\gamma^2 x^2 + y^2 + z^2 = r^2\gamma^2\left(1-\beta^2\sin^2\theta\right)$$

$$\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_o}\frac{\left(1-\beta^2\right)}{r^2\left(1-\beta^2\sin^2\theta\right)^{3/2}}\frac{\vec{r}}{r}}$$

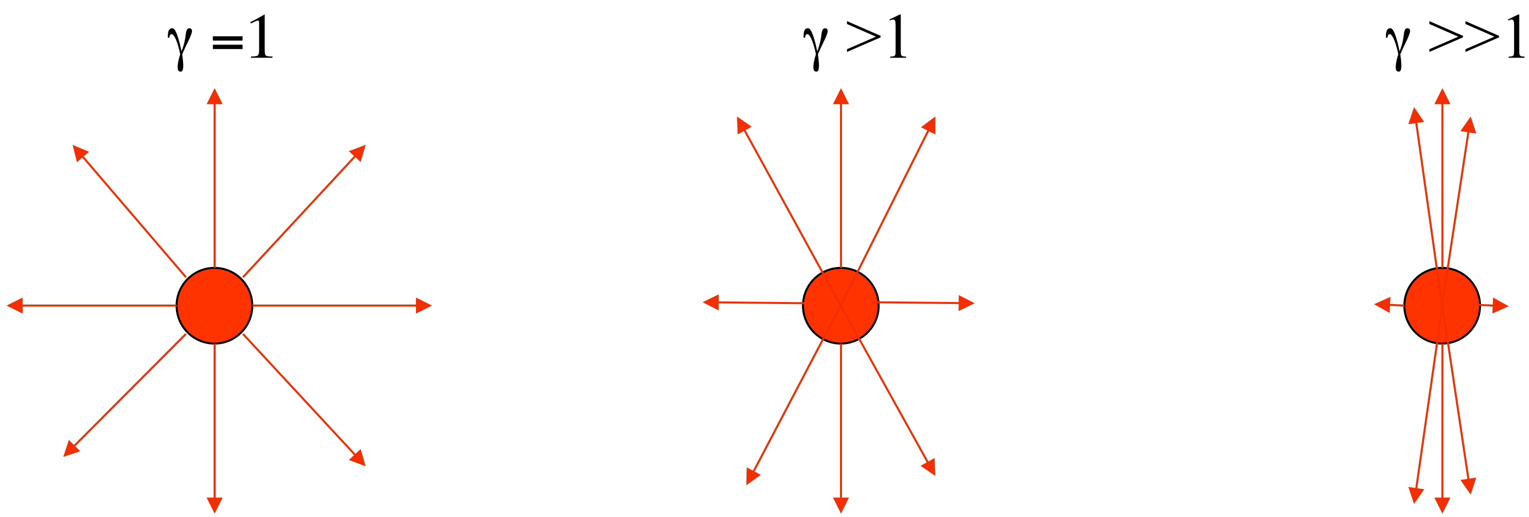
$$\vec{E} = \frac{q}{4\pi\varepsilon_o} \frac{(1-\beta^2)}{r^2(1-\beta^2 \sin^2\theta)^{3/2}} \frac{\vec{r}}{r}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\beta = 0 \Rightarrow \vec{E} = \frac{q}{4\pi\varepsilon_o} \frac{1}{r^2} \frac{\vec{r}}{r}$$

$$\theta = 0 \Rightarrow E_{//} = \frac{q}{4\pi\varepsilon_o} \frac{1}{\gamma^2 r^2} \frac{\vec{r}}{r} \xrightarrow{\gamma \to \infty} 0$$

$$\theta = \frac{\pi}{2} \Rightarrow E_{\perp} = \frac{q}{4\pi\varepsilon_o} \frac{\gamma}{r^2} \frac{\vec{r}}{r} \xrightarrow{\gamma \to \infty} \infty$$

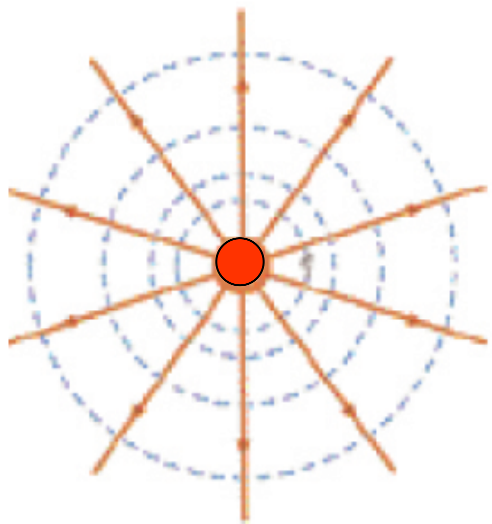


$\boxed{\vec{B}' = 0}$

B is transverse to the direction of motion

$$B_x = 0$$
$$B_y = -vE_z / c^2$$
$$B_z = vE_y / c^2$$

$$\vec{B}_\perp = \frac{\vec{v} \times \vec{E}}{c^2}$$



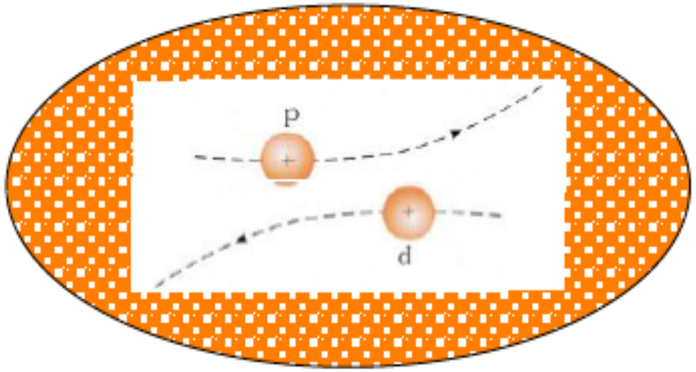
$\gamma \longrightarrow \infty$

Space Charge

The net effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**

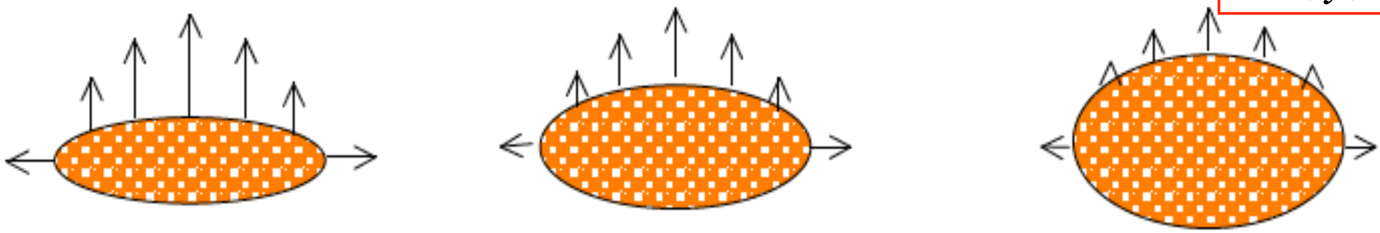
$$\boxed{\lambda_D = \sqrt{\frac{\varepsilon_o k_B T}{e^2 n}}}$$



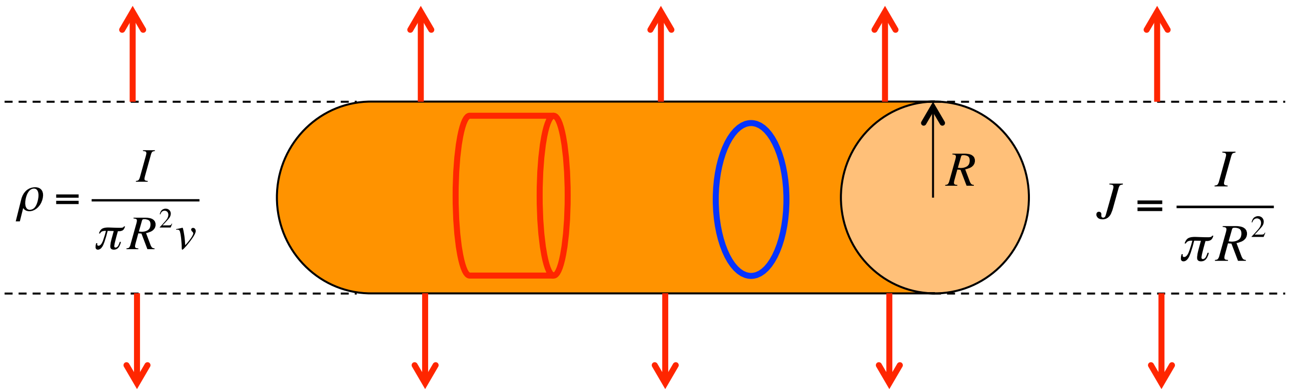
$$\boxed{\sigma_{x,y,z} << \lambda_D}$$

2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compare to the average separation of the particles ==> **Collective Effects, Single Component Cold Plasma**

$$\boxed{\sigma_{x,y,z} >> \lambda_D}$$



Continuous Uniform Cylindrical Beam Model



Gauss' s law

$$\int \varepsilon_o E \cdot dS = \int \rho dV$$

$$E_r = \frac{I}{2\pi\varepsilon_o R^2 v} r \quad \text{for} \quad r \le R$$

$$E_r = \frac{I}{2\pi\varepsilon_o v} \frac{1}{r} \quad \text{for} \quad r > R$$

$$B_\vartheta = \frac{\beta}{c} E_r$$

Ampere' s law

$$\int B \cdot dl = \mu_o \int J \cdot dS$$

$$B_\vartheta = \mu_o \frac{Ir}{2\pi R^2} \quad \text{for} \quad r \le R$$

$$B_\vartheta = \mu_o \frac{I}{2\pi r} \quad \text{for} \quad r > R$$

Lorentz Force

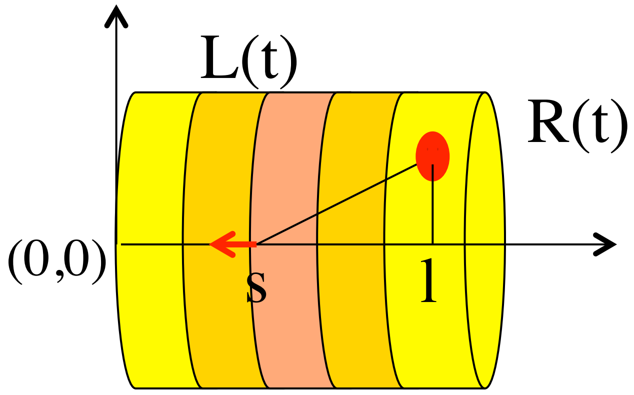
$$\boxed{F_r = e\left(E_r - \beta c B_\vartheta\right) = e\left(1 - \beta^2\right)E_r = \frac{eE_r}{\gamma^2}}$$

has only **radial** component and

is a **linear** function of the transverse coordinate

The attractive magnetic force , which becomes significant at high velocities, tends to compensate for the repulsive electric force.

Bunched Uniform Cylindrical Beam Model



$$v_z = \beta c$$

Longitudinal Space Charge field in the bunch moving frame:

$$\tilde{\rho} = \frac{Q}{\pi R^2 \tilde{L}}$$

$$\tilde{E}_z(\tilde{s}, r=0) = \frac{\tilde{\rho}}{4\pi\varepsilon_o} \int_0^R \int_0^{2\pi} \int_0^{\tilde{L}} \frac{\left(\tilde{l} - \tilde{s}\right)}{\left[\left(\tilde{l} - \tilde{s}\right)^2 + r^2\right]^{3/2}} \, r dr d\varphi d\tilde{l}$$

$$\tilde{E}_z(\tilde{s}, r=0) = \frac{\tilde{\rho}}{2\varepsilon_0}\left[\sqrt{R^2 + (\tilde{L} - \tilde{s})^2} - \sqrt{R^2 + \tilde{s}^2} + \left(2\tilde{s} - \tilde{L}\right)\right]$$

Radial Space Charge field in the bunch moving frame

by series representation of axisymmetric field:

$$\tilde{E}_r(r,\tilde{s}) \cong \left[\frac{\tilde{\rho}}{\varepsilon_0} - \frac{\partial}{\partial \tilde{s}}\tilde{E}_z(0,\tilde{s})\right]\frac{r}{2} + \left[\cdots\right]\frac{r^3}{16} +$$

$$\tilde{E}_r(r,\tilde{s}) = \frac{\tilde{\rho}}{2\varepsilon_0}\left[\frac{(\tilde{L}-\tilde{s})}{\sqrt{R^2+(\tilde{L}-\tilde{s})^2}} + \frac{\tilde{s}}{\sqrt{R^2+\tilde{s}^2}}\right]\frac{r}{2}$$

Lorentz Transformation back to the Lab frame

$$E_z = \tilde{E}_z$$

$$E_r = \gamma \tilde{E}_r$$

$$\tilde{L} = \gamma L \quad \Rightarrow \quad \tilde{\rho} = \frac{\rho}{\gamma}$$

$$\tilde{s} = \gamma s$$

$$E_z(0,s) = \frac{\rho}{\gamma 2\varepsilon_0}\left[\sqrt{R^2+\gamma^2(L-s)^2} - \sqrt{R^2+\gamma^2 s^2} + \gamma\left(2s-L\right)\right]$$

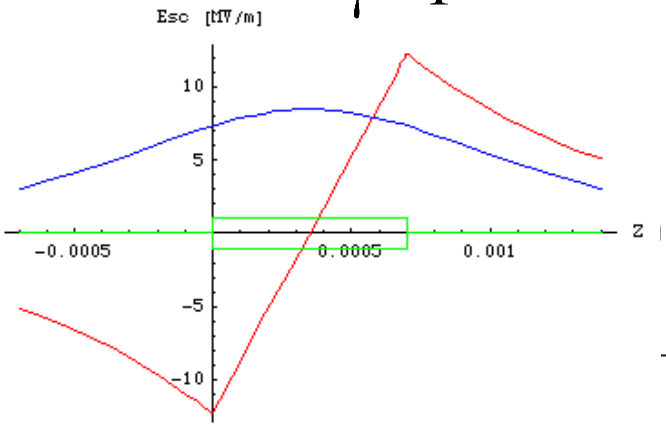
$$E_r(r,s) = \frac{\gamma\rho}{2\varepsilon_0}\left[\frac{(L-s)}{\sqrt{R^2+\gamma^2(L-s)^2}} + \frac{s}{\sqrt{R^2+\gamma^2 s^2}}\right]\frac{r}{2}$$

It is still a linear field with r but with a longitudinal correlation s

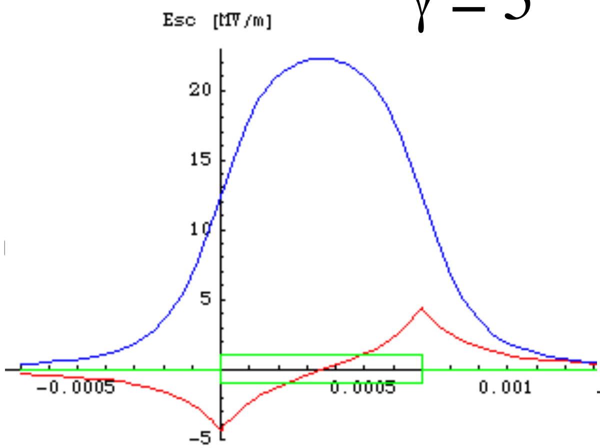
$$\boxed{E_z(0,s,\gamma) = \frac{I}{2\pi\gamma\varepsilon_0 R^2 \beta c} h(s,\gamma)}$$

$$\boxed{E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)}$$

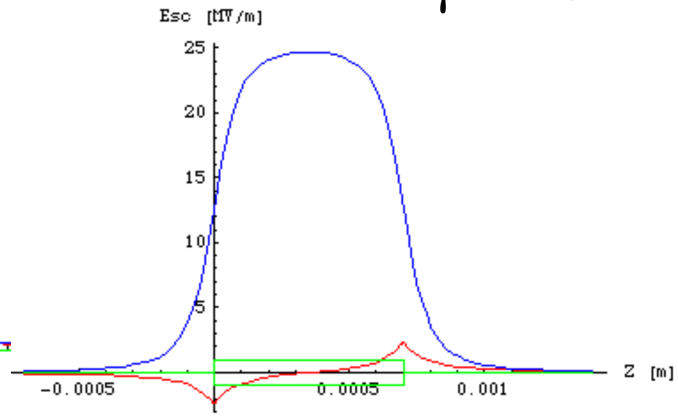
γ= 1



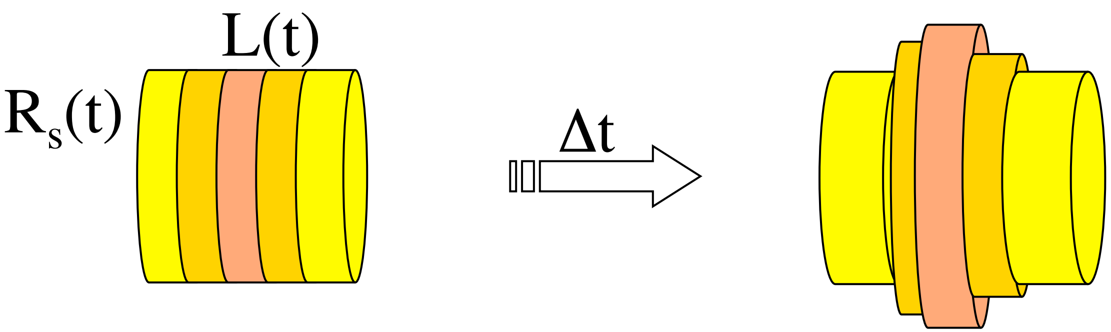
γ = 5



γ = 10

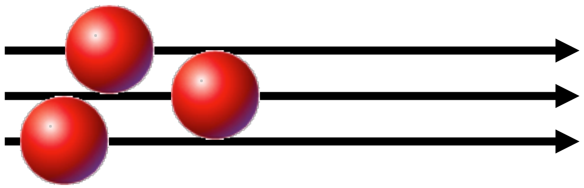


$$\boxed{F_r = \frac{eE_r}{\gamma^2} = \frac{eIr}{2\pi\gamma^2\varepsilon_0 R^2 \beta c} g(s,\gamma)}$$



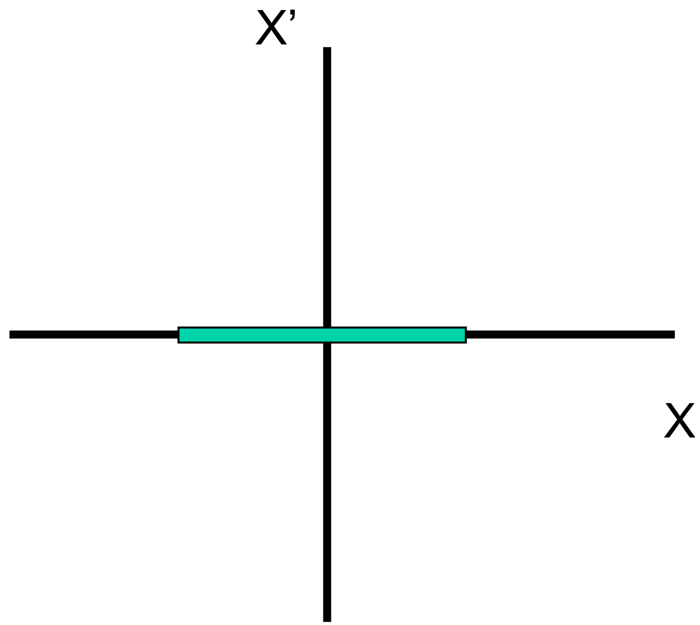
The Laminar beam

Trace space of an ideal laminar beam

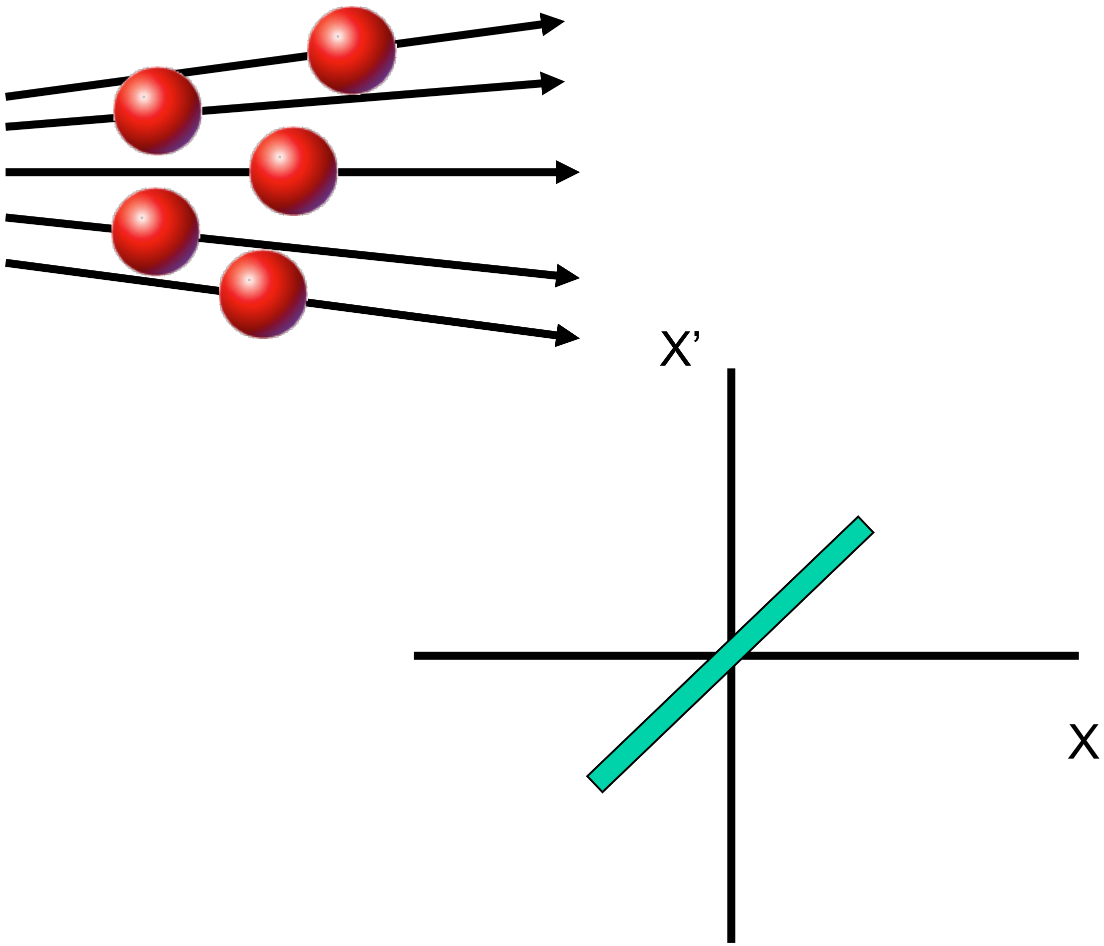


$$\begin{cases} x \\ x' = \dfrac{dx}{dz} = \dfrac{p_x}{p_z} \end{cases} \qquad p_x << p_z$$

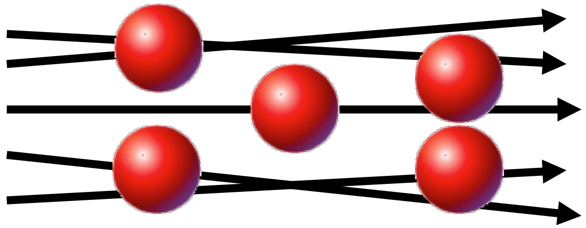
Paraxial approx.

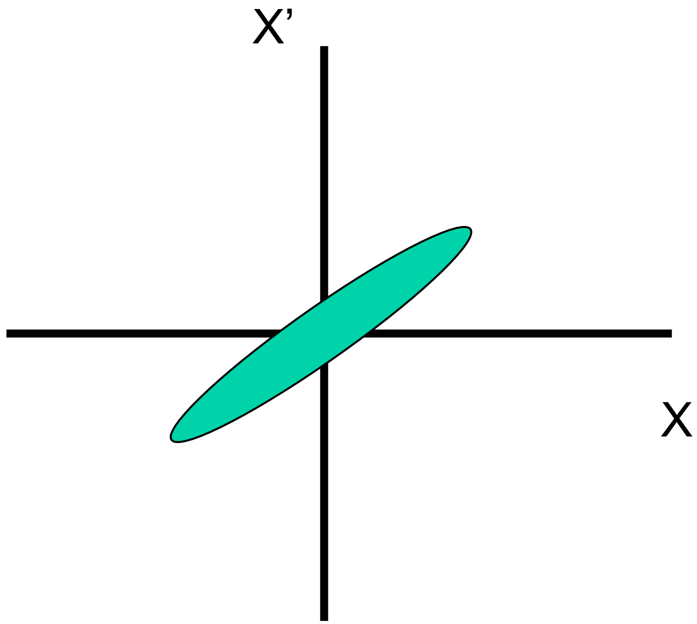


Trace space of a laminar beam



Trace space of non laminar beam



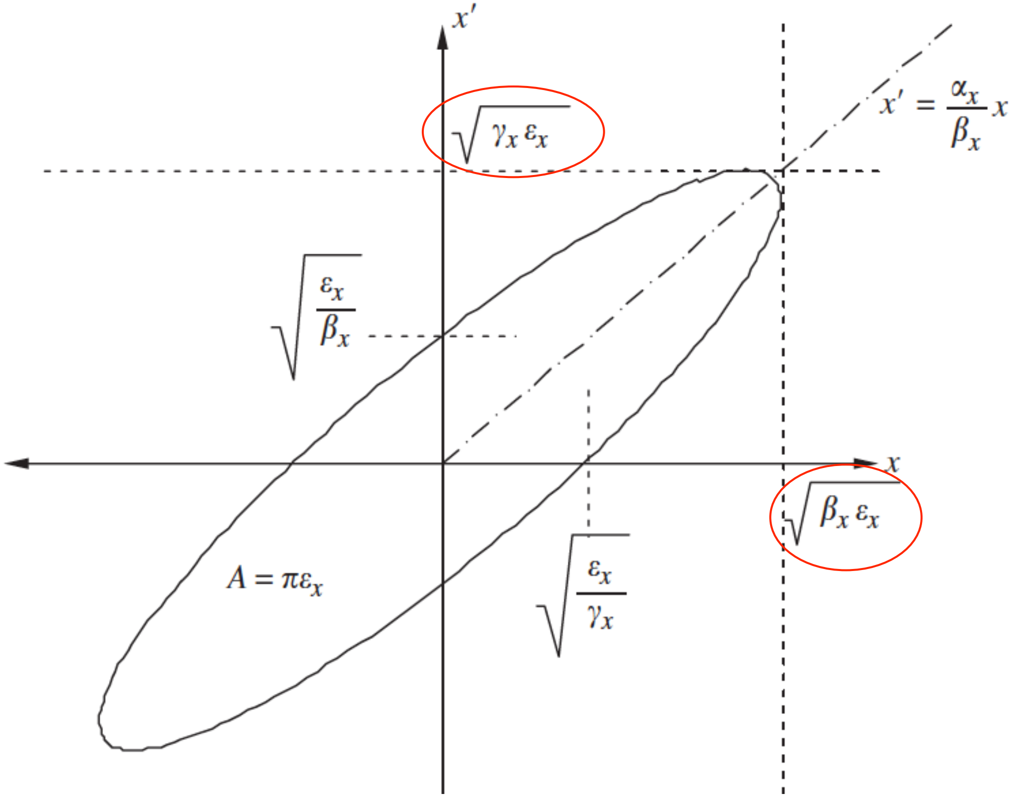


Geometric emittance: $\boxed{\varepsilon_g}$

Ellipse equation: $\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_g$

Twiss parameters: $\beta\gamma - \alpha^2 = 1$ $\quad\quad \beta' = -2\alpha$

Ellipse area: $A = \pi\varepsilon_g$



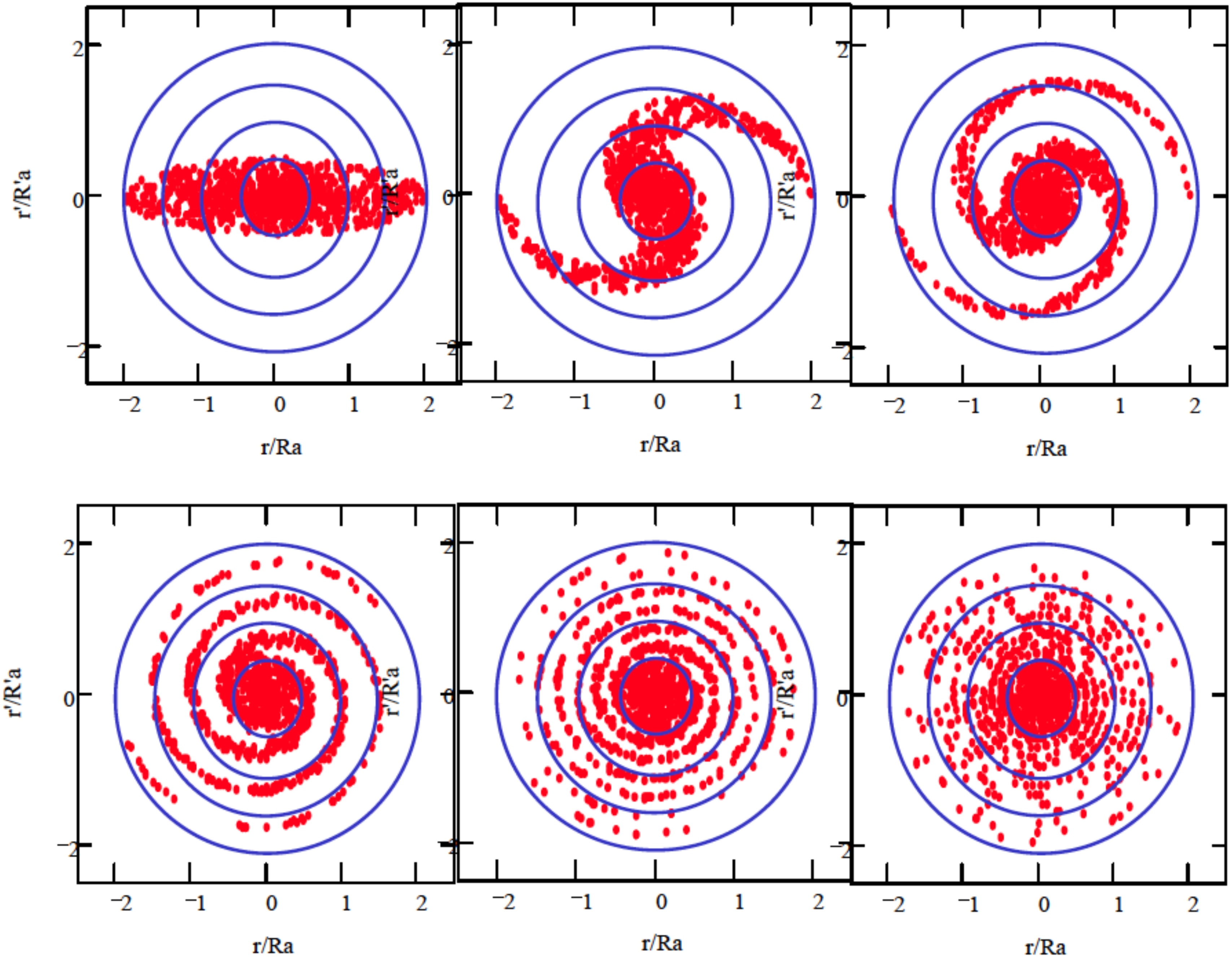
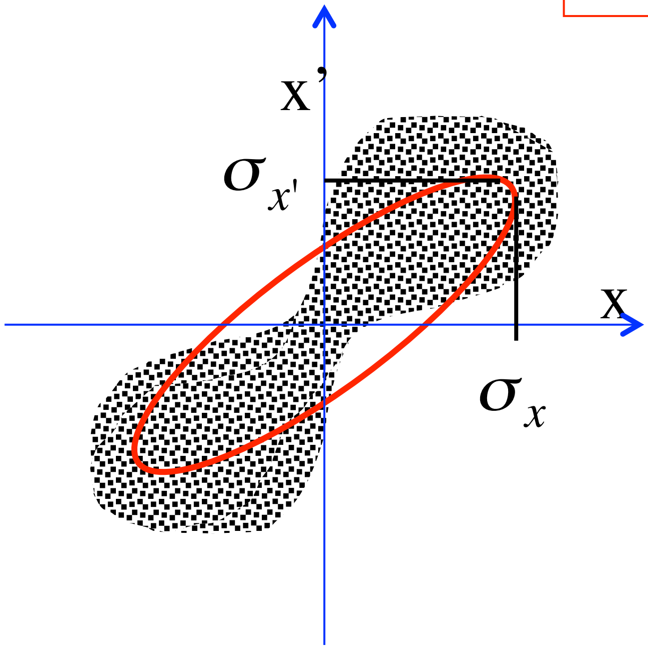


Fig. 17: Filamentation of mismatched beam in non-linear force

rms emittance $\boxed{\varepsilon_{rms}}$



$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,x')\,dx\,dx' = 1 \qquad \boxed{f'(x,x') = 0}$$

rms beam envelope:

$$\sigma_x^2 = \langle x^2 \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f(x,x')\,dx\,dx'$$

Define rms emittance:

$$\boxed{\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_{rms}}$$

such that:

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$

$$\sigma_{x'} = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

Since: $\quad \alpha = -\dfrac{\beta'}{2} \qquad \beta = \dfrac{\langle x^2 \rangle}{\varepsilon_{rms}}$

it follows: $\quad \alpha = -\dfrac{1}{2\varepsilon_{rms}} \dfrac{d}{dz}\langle x^2 \rangle = -\dfrac{\langle x x' \rangle}{\varepsilon_{rms}} = -\dfrac{\sigma_{xx'}}{\varepsilon_{rms}}$

$$\boxed{\begin{aligned}\sigma_x &= \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}} \\ \sigma_{x'} &= \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}} \\ \sigma_{xx'} &= \langle x x' \rangle = -\alpha \varepsilon_{rms}\end{aligned}}$$

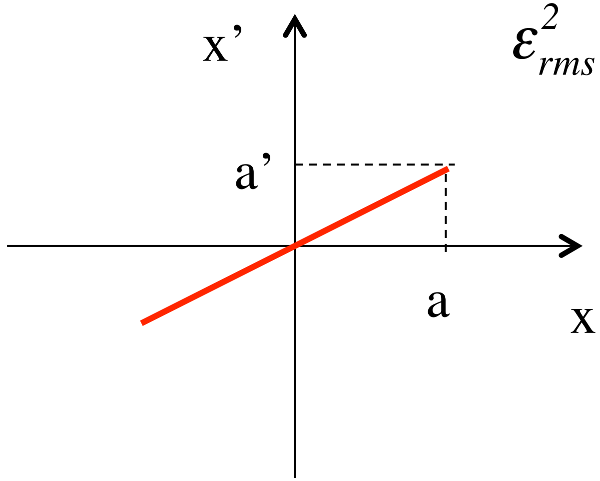
It holds also the relation: $\gamma\beta - \alpha^2 = 1$

Substituting α, β, γ we get $\dfrac{\sigma_{x'}^2}{\varepsilon_{rms}} \dfrac{\sigma_x^2}{\varepsilon_{rms}} - \left(\dfrac{\sigma_{xx'}}{\varepsilon_{rms}}\right)^2 = 1$

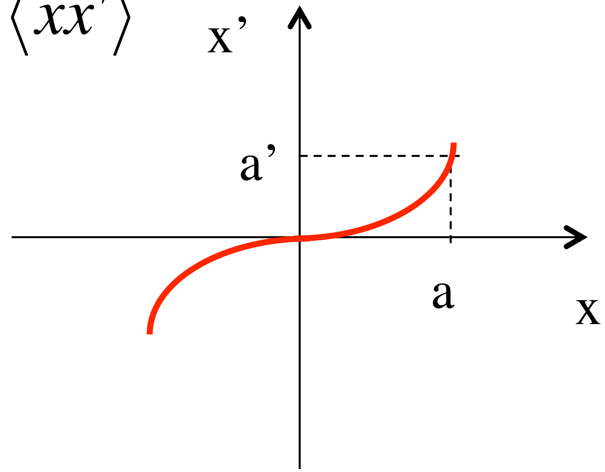
We end up with the definition of rms emittance in terms of the second moments of the distribution:

$$\boxed{\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2\right)}}$$

What does rms emittance tell us about trace space distributions under linear or non-linear forces acting on the beam?



$$\varepsilon_{rms}^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2$$



Assuming a generic x, x' correlation of the type: $x' = C x^n$

$$\varepsilon_{rms}^2 = C^2 \left(\langle x^2 \rangle \langle x^{2n} \rangle - \langle x^{n+1} \rangle^2 \right)$$

When $n = 1$ ==> $\varepsilon_{rms} = 0$

When $n \neq 1$ ==> $\varepsilon_{rms} \neq 0$

Envelope Equation without Acceleration

Now take the derivatives:

$$\boxed{\frac{d\sigma_x}{dz}} = \frac{d}{dz}\sqrt{\langle x^2 \rangle} = \frac{1}{2\sigma_x}\frac{d}{dz}\langle x^2 \rangle = \frac{1}{2\sigma_x}2\langle xx' \rangle = \boxed{\frac{\sigma_{xx'}}{\sigma_x}}$$

$$\boxed{\frac{d^2\sigma_x}{dz^2}} = \frac{d}{dz}\frac{\sigma_{xx'}}{\sigma_x} = \frac{1}{\sigma_x}\frac{d\sigma_{xx'}}{dz} - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{1}{\sigma_x}\left(\langle x'^2 \rangle - \langle xx'' \rangle\right) - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \boxed{\frac{\sigma_{x'}^2 + \langle xx'' \rangle}{\sigma_x} - \frac{\sigma_{xx'}^2}{\sigma_x^3}}$$

And simplify: $\boxed{\sigma_x''} = \frac{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2}{\sigma_x^3} - \frac{\langle xx'' \rangle}{\sigma_x} = \boxed{\frac{\varepsilon_{rms}^2}{\sigma_x^3} + \frac{\langle xx'' \rangle}{\sigma_x}}$

We obtain the rms envelope equation in which the rms emittance enters as defocusing pressure like term.

$$\boxed{\sigma_x'' - \frac{\langle xx'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}}$$

$$\boxed{\sigma_x'' - \frac{\langle xx'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}}$$

Assuming that each particle is subject only to a linear focusing force, without acceleration: $x'' + k_x^2 x = 0$

take the average over the entire particle ensemble $\langle xx'' \rangle = -k_x^2 \langle x^2 \rangle$

$$\boxed{\sigma_x'' + k_x^2 \sigma_x = \frac{\varepsilon_{rms}^2}{\sigma_x^3}}$$

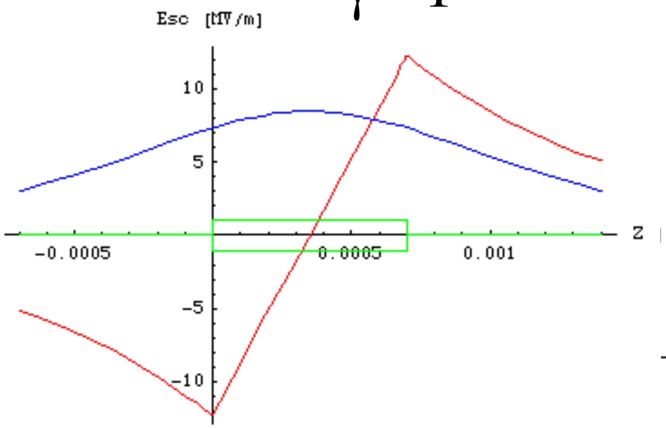
We obtain the rms envelope equation with a linear focusing force in which the rms emittance enters as defocusing pressure like term.

Bunched Uniform Cylindrical Beam Model

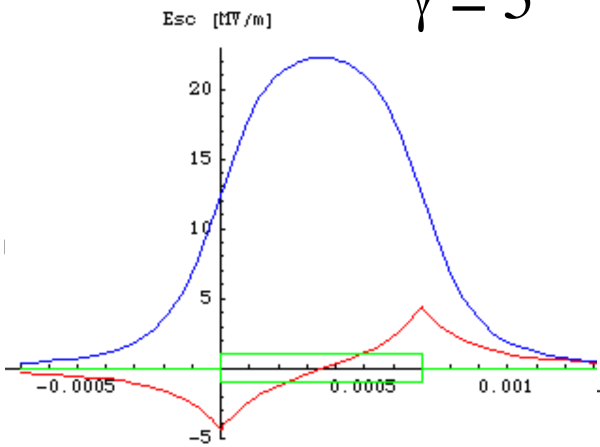
$$E_z(0,s,\gamma) = \frac{I}{2\pi\gamma\varepsilon_0 R^2 \beta c} h(s,\gamma)$$

$$E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)$$

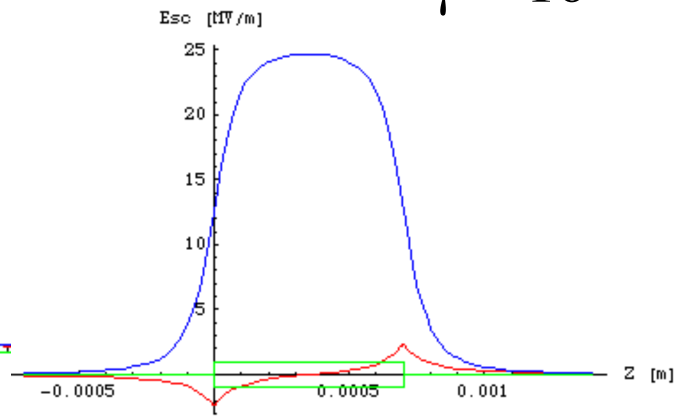
γ= 1

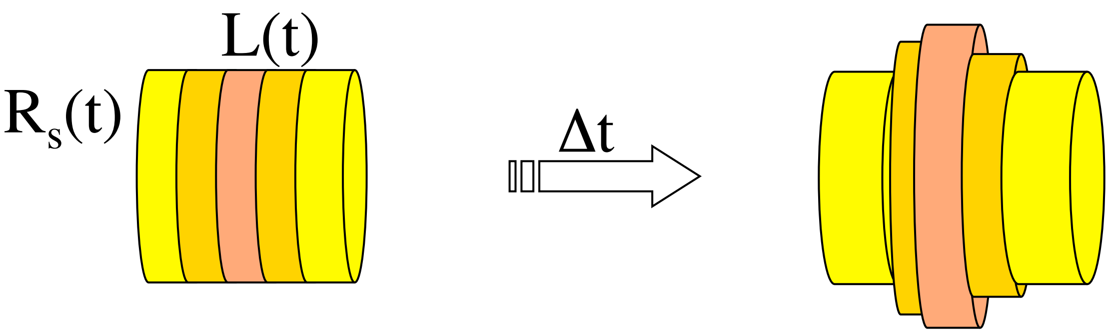


γ = 5



γ = 10





Lorentz Force

$$\boxed{F_r = e\left(E_r - \beta c B_\vartheta\right) = e\left(1-\beta^2\right)E_r = \frac{eE_r}{\gamma^2}}$$

is a **linear** function of the transverse coordinate

$$\boxed{\frac{dp_r}{dt} = F_r = \frac{eE_r}{\gamma^2} = \frac{eIr}{2\pi\gamma^2\varepsilon_0 R^2\beta c}\,g(s,\gamma)}$$

The attractive magnetic force , which becomes significant at high velocities, tends to compensate for the repulsive electric force. Therefore space charge defocusing is primarily a non-relativistic effect.

$$\boxed{F_x = \frac{eIx}{2\pi\gamma^2\varepsilon_0\sigma_x^2\beta c}\,g(s,\gamma)}$$

Envelope Equation with Space Charge

Single particle transverse motion:

$$\frac{dp_x}{dt} = F_x \qquad p_x = p\,x' = \beta\gamma m_o c x'$$

$$\frac{d}{dt}(px') = \beta c \frac{d}{dz}(p\,x') = F_x$$

$$\boxed{x'' = \frac{F_x}{\beta c p}}$$

$$\boxed{F_x = \frac{eIx}{2\pi\gamma^2\varepsilon_0\sigma_x^2\beta c} g(s,\gamma)}$$

$$\boxed{x'' = \frac{k_{sc}(s,\gamma)}{\sigma_x^2} x}$$

Now we can calculate the term $\langle xx'' \rangle$ that enters in the envelope equation

$$\boxed{\sigma_x'' = \frac{\varepsilon_{rms}^2}{\sigma_x^3} - \frac{\langle xx'' \rangle}{\sigma_x}}$$

$$\boxed{x'' = \frac{k_{sc}}{\sigma_x^2} x}$$

$$\langle xx'' \rangle = \frac{k_{sc}}{\sigma_x^2} \langle x^2 \rangle = k_{sc}$$

Including all the other terms the envelope equation reads:

Space Charge De-focusing Force

$$\boxed{\sigma_x'' + k^2 \sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2 \sigma_x^3} + \frac{k_{sc}}{\sigma_x}}$$

Emittance Pressure

External Focusing Forces

Laminarity Parameter: $\boxed{\rho = \frac{(\beta\gamma)^2 k_{sc} \sigma_x^2}{\varepsilon_n^2}}$

The beam undergoes two regimes along the accelerator

$$\sigma_x'' + k^2\sigma_x = \cancel{\frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3}} + \frac{k_{sc}}{\sigma_x}$$

ρ>>1 | Laminar Beam

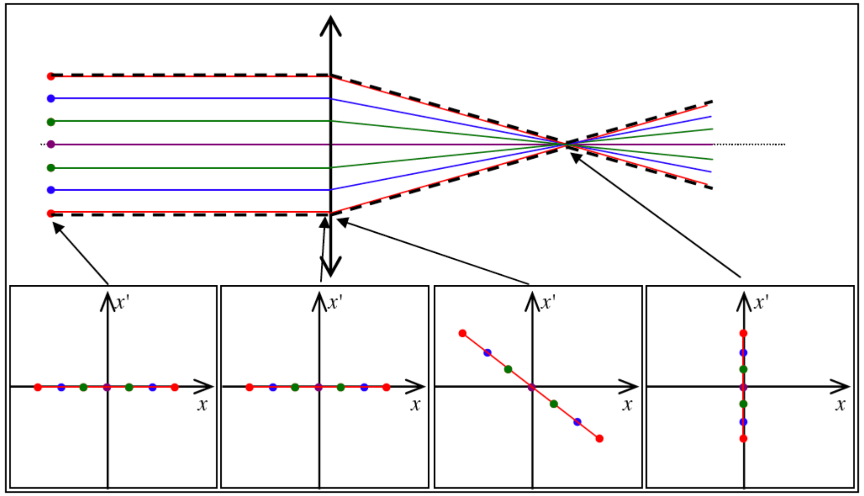


Fig. 10: Particle trajectories in laminar beam

$$\sigma_x'' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} \cancel{+ \frac{k_{sc}}{\sigma_x}}$$

ρ<<1 | Thermal Beam

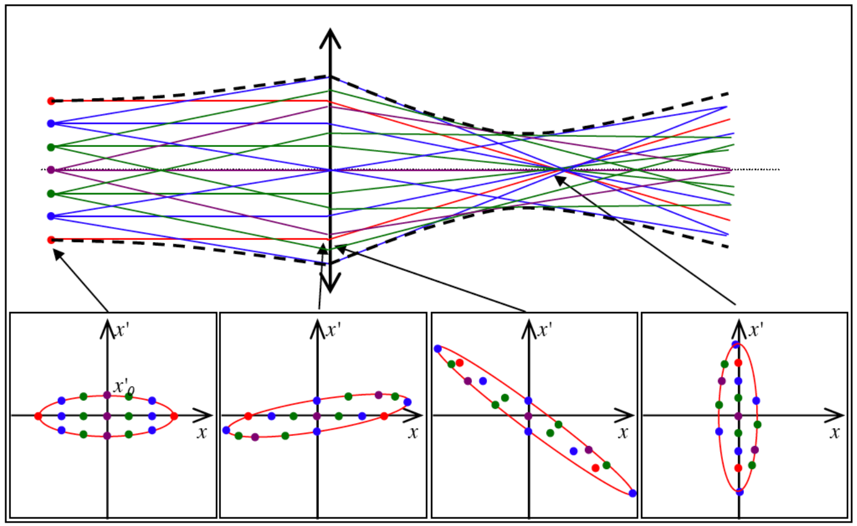


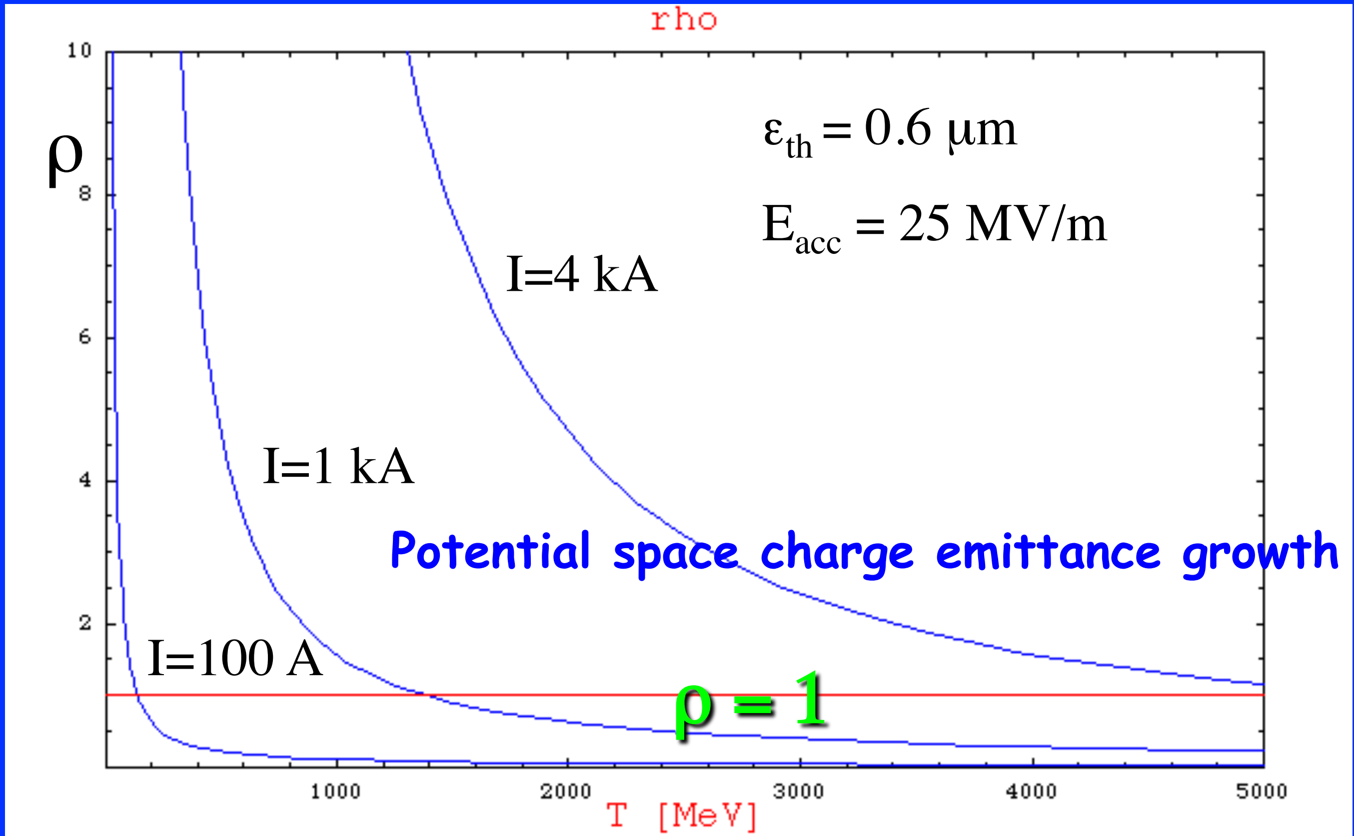
Fig. 11: Particle trajectories in non-zero emittance beam

Laminarity parameter

$$\rho = \frac{2I\sigma^2}{\gamma I_A \varepsilon_n^2} \equiv \frac{2I\sigma_q^2}{\gamma I_A \varepsilon_n^2} = \frac{4I^2}{\gamma'^2 I_A^2 \varepsilon_n^2 \gamma^2}$$

Transition Energy (ρ=1)

$$\gamma_{tr} = \frac{2I}{\gamma' I_A \varepsilon_n}$$



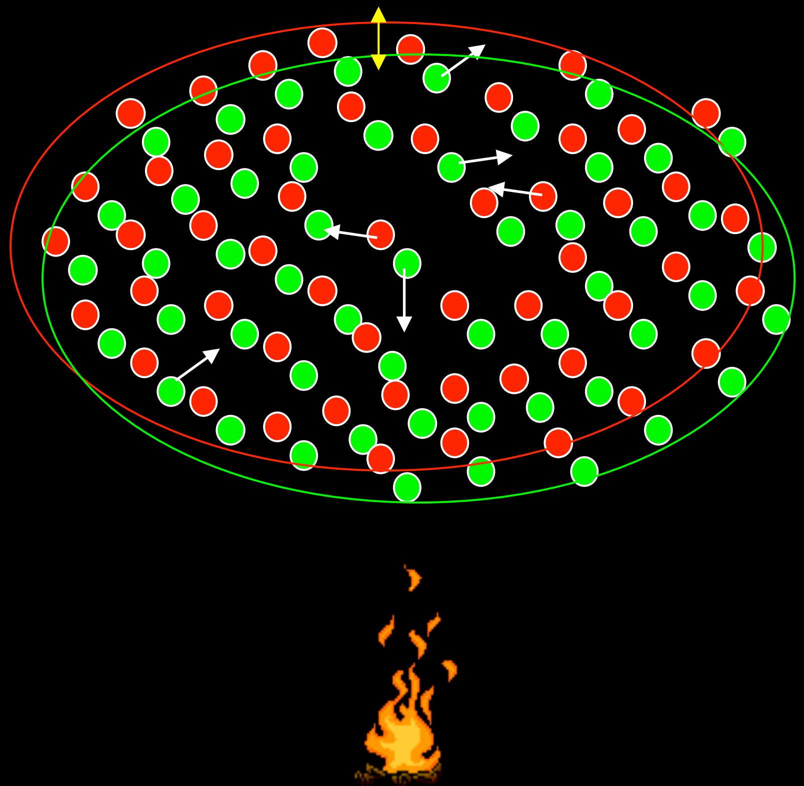
Space charge induced emittance oscillations in a laminar beam

Neutral Plasma

Surface charge density

$$\sigma = e\,n\,\delta x$$

$$\delta x$$



Surface electric field

$$E_x = -\sigma/\epsilon_0 = -e\,n\,\delta x/\epsilon_0$$

Restoring force

$$m\,\frac{d^2\delta x}{dt^2} = e\,E_x = -m\,\omega_p{}^2\,\delta x$$

Plasma frequency

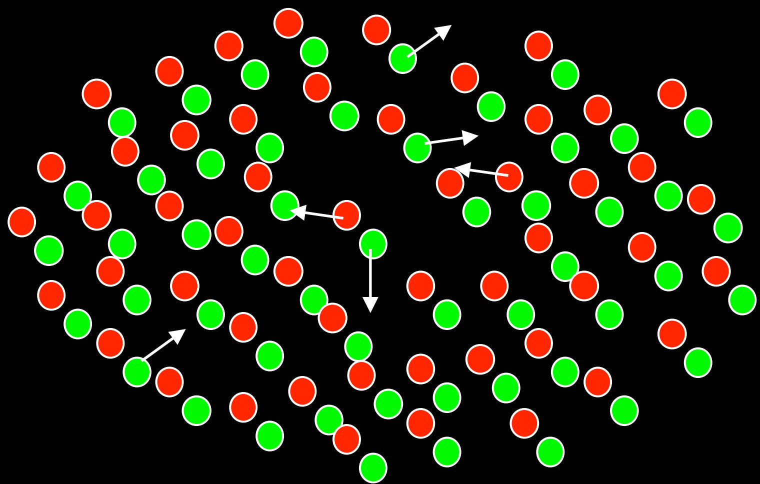
$$\omega_p{}^2 = \frac{n\,e^2}{\epsilon_0\,m}$$

Plasma oscillations

$$\delta x = (\delta x)_0\,\cos(\omega_p\,t)$$

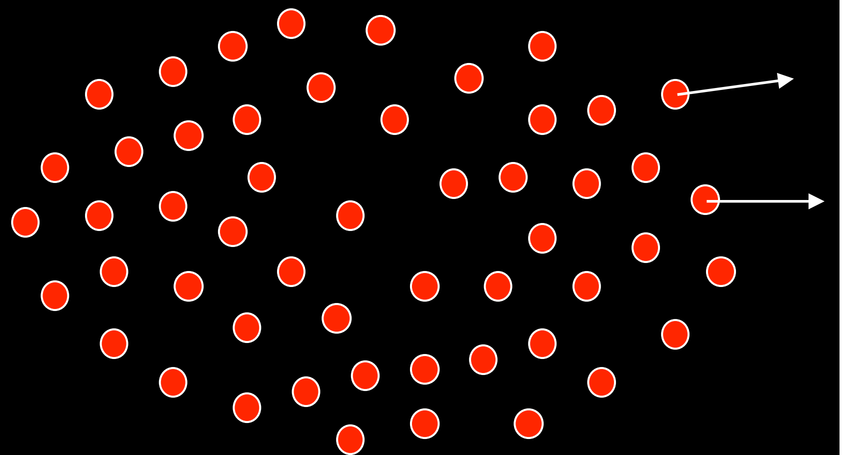
Neutral Plasma

- Oscillations
- Instabilities
- EM Wave propagation



Single Component Cold Relativistic Plasma

Magnetic focusing



Magnetic focusing

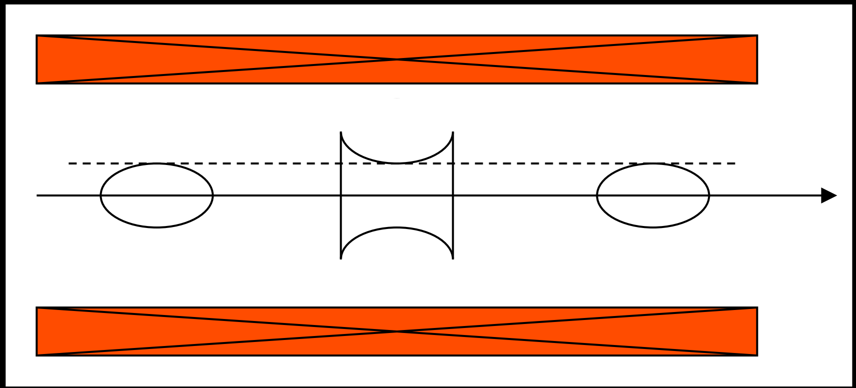
Single Component Relativistic Plasma

$$\sigma'' + k_s^2 \sigma = \frac{k_{sc}(s,\gamma)}{\sigma}$$

$$k_s = \frac{qB}{2mc\beta\gamma}$$

Equilibrium solution:

$$\sigma_{eq}(s,\gamma) = \frac{\sqrt{k_{sc}(s,\gamma)}}{k_s}$$



Small perturbation:

$$\sigma(\zeta) = \sigma_{eq}(s) + \delta\sigma(s)$$

$$\delta\sigma''(s) + 2k_s^2 \delta\sigma(s) = 0$$

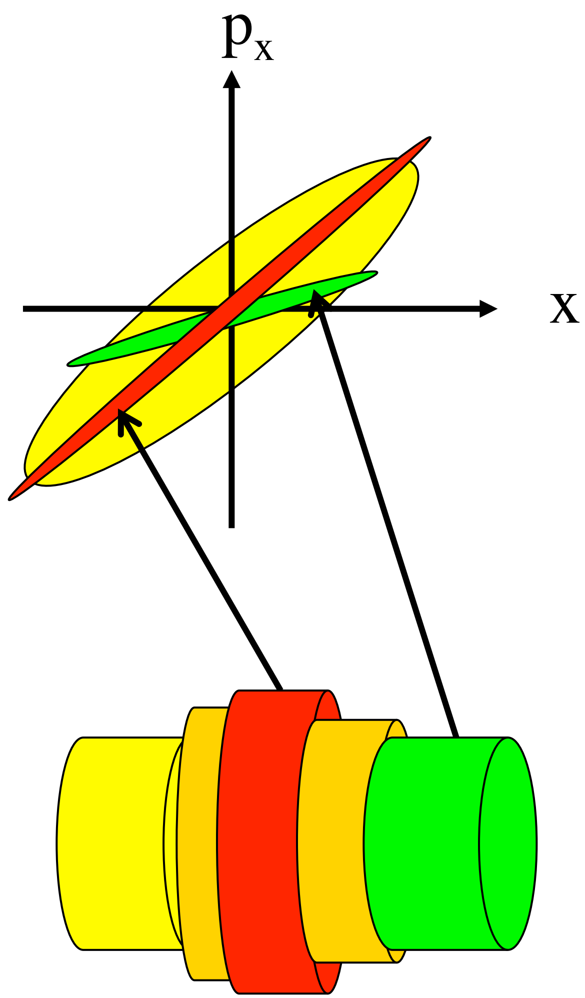
$$\delta\sigma(s) = \delta\sigma_o(s)\cos\left(\sqrt{2}k_s z\right)$$

Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes:

$$\sigma(s) = \sigma_{eq}(s) + \delta\sigma_o(s)\cos\left(\sqrt{2}k_s z\right)$$

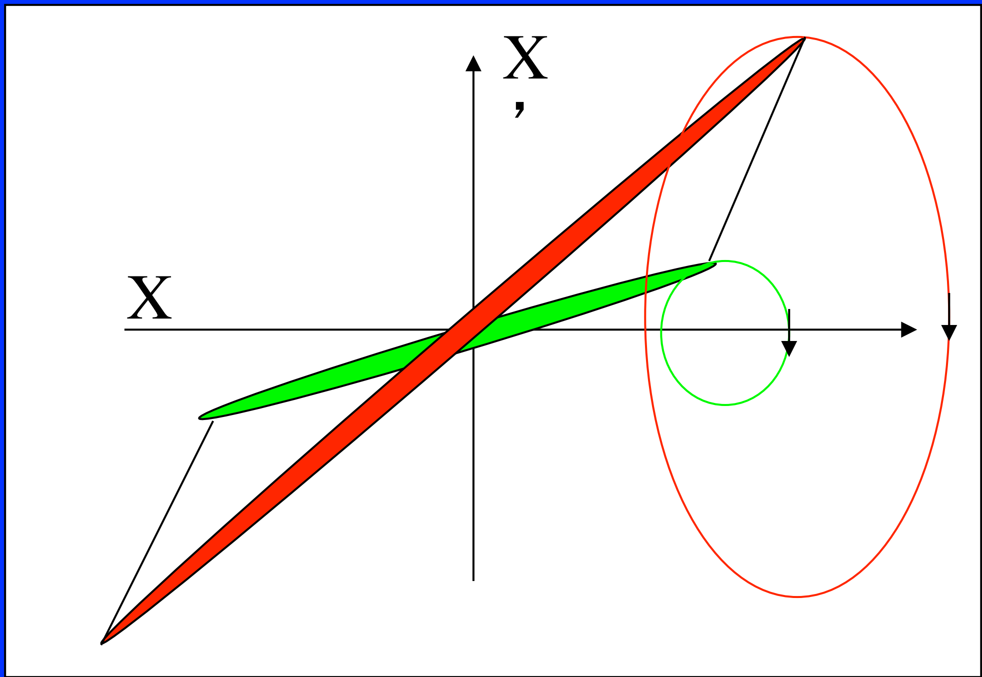
Emittance Oscillations are driven by space charge differential defocusing in core and tails of the beam

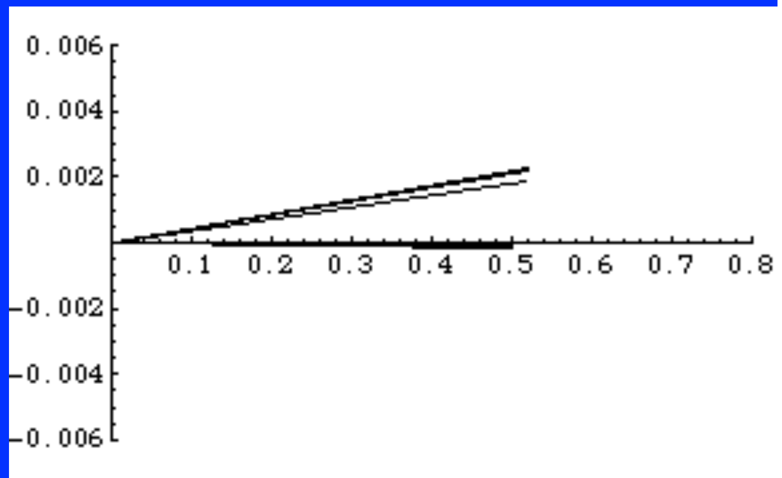
Projected Phase Space



Slice Phase Spaces

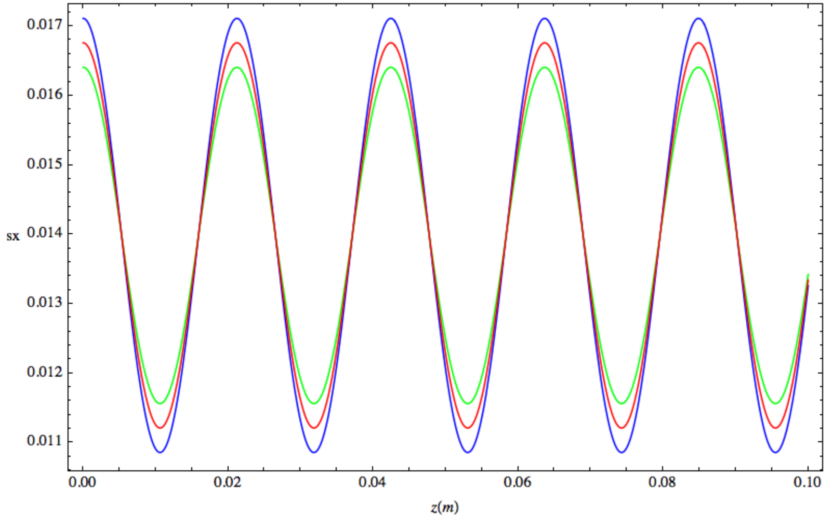
Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes



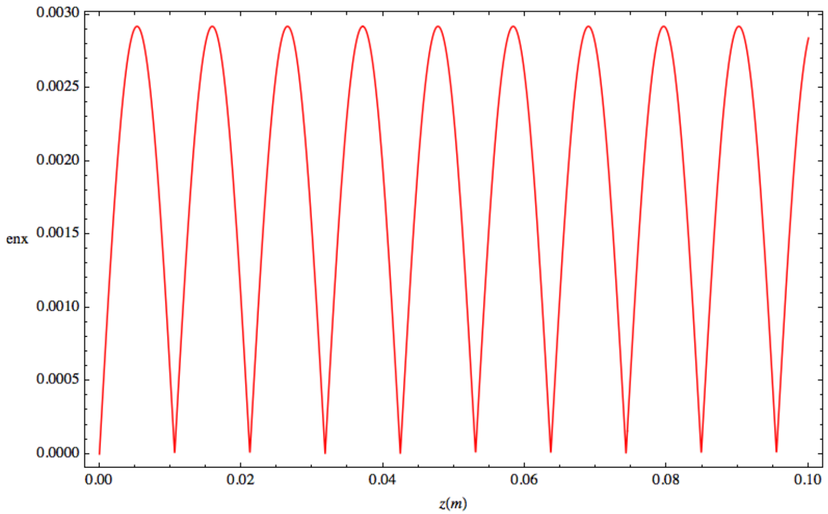


Envelope oscillations drive Emittance oscillations

$\sigma(z)$



$\varepsilon(z)$



$$\boxed{\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}{}^2} = \sqrt{\left(\left\langle x^2 \right\rangle \left\langle x'^2 \right\rangle - \left\langle xx' \right\rangle^2\right)} \approx \left| sin\left(\sqrt{2} k_s z\right) \right|}$$

References:

[1] T. Shintake, Proc. of the 22nd Particle Accelerator Conference, June 25-29, 2007, Albuquerque, NM (IEEE, New York, 2007), p. 89.
[2] L. Serafini, J. B. Rosenzweig, PR E55 (1997) 7565
[3] M. Reiser, "Theory and Design of Charged Particle Beams" , Wiley, New York, 1994
[4] J. B. Rosenzweig, "Fundamentals of beam physics", Oxford University Press, New York, 2003
[5] T. Wangler, "Principles of RF linear accelerators", Wiley, New York, 1998
[6] S. Humphries, "Charged particle beams", Wiley, New York, 2002
[7] F. J. Sacherer, F. J., IEEE Trans. Nucl. Sci. NS-18, 1105 (1971).
[8] M. Ferrario et al., Int. Journal of Modern Physics A, Vol 22, No. 23, 4214 (2007)
[9] J. Buon, "Beam phase space and emittance", in CERN 94-01
[10] C. Prior, "Special Relativity", CAS Proceedings, CERN 2006-012
[11] W. Greiner "CLASSICAL MECHANICS - Point Particles and Relativity", Springer