Resonant Extraction

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Contents

Historical note (1)

- **Resonant extraction has its roots in cyclotrons.**
  - The first cyclotrons used scattering from an internal target to obtain an external proton beam. The activation and loss of intensity were incentives for change.
  - In 1951, James Tuck and Lee Teng (Chicago cyclotron) suggested a resonant scheme that increased the amplitude of radial oscillations sufficiently in a single turn to clear the mouth of a magnetic extraction septum.
  - A quantitative analysis, confirming the principle of what was then known as the 'Peeler-Regenerative Beam Extraction Method' was published by Kenneth Le Couteur (Liverpool cyclotron) in 1953.
  - The first successful extraction by this technique was in the Liverpool cyclotron in 1954 reported by Albert Crewe and Le Couteur in 1955.
- **At this stage, the emphasis was on inter-turn separation.**

James Tuck

Lee Teng

Kenneth Le Couteur

Albert Crewe

Historical note (2)

- In the 1950s, HEP synchrotrons were equipped with full-aperture, fast kickers. Single-turn extraction was efficient, but for fixed-target experiments it overloaded the physics counters leading to lost data.

- In 1961, Hugh Hereward addressed this problem in *'The possibility of resonant extraction from the CPS',* CERN-AR-Int-GS-61-5. This proposal cites Tuck , Teng and Le Couteur. At essentially the same time, C. L. Hammer and Lawrence Jackson Laslett published '*Resonant Beam Extraction from an AG Synchrotron',* Rev. Sci. Instr. 32, 144-149 (1962).

- **The emphasis here was on extending the spill time in synchrotrons. Hence a new name 'slow extraction'.**

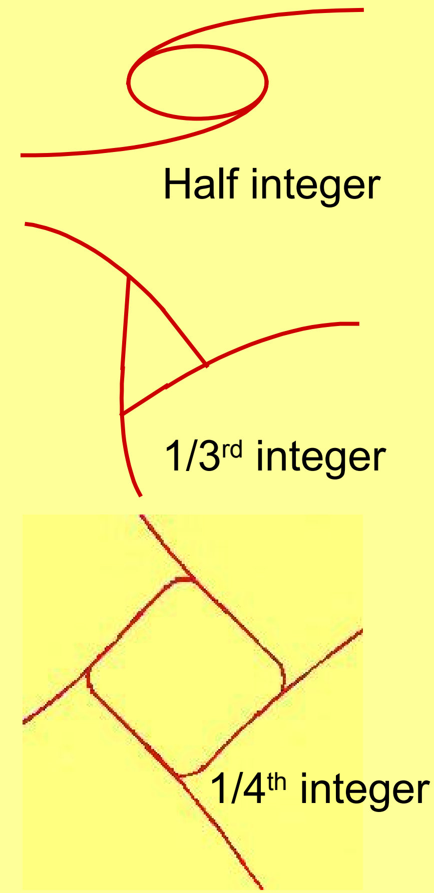
Hugh Hereward

Jackson Laslett

Overview

- For extraction, cyclotrons and synchro-cyclotrons work with the integer or half integer resonance. This is consistent with a need for a large inter-turn separation and a very fast extraction*.
- Synchrotrons work with the half integer or 1/3rd integer resonance.
  - **The half integer resonance is usually associated with fixed-target physics and spills of a few milliseconds.**
  - **The 1/3rd integer resonance is weaker and is used widely in medical machines for spills of a few seconds (a few 10^6 turns).**
  - **Higher-order resonances are not used because the extraction separatrices become too close together, the resonances get weaker and the transit time in the resonance gets longer.**
- This lecture will concentrate on the 1/3rd integer resonance.

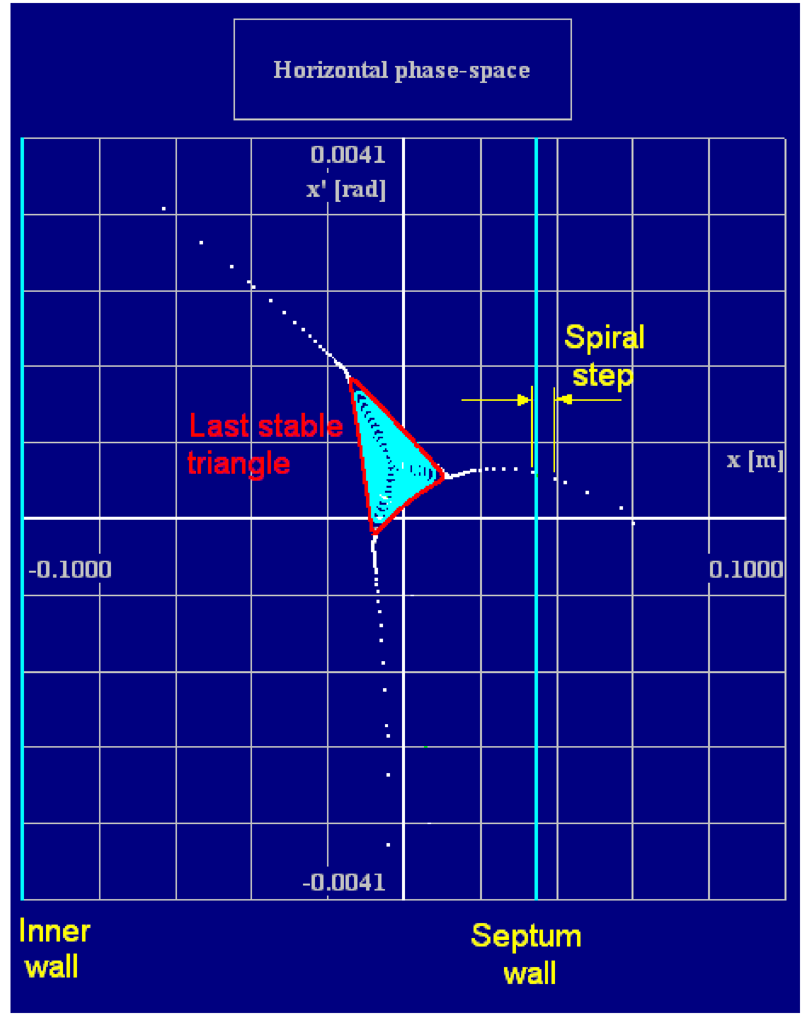
Phase space sketches



*** In the 1980s, H⁻ extraction replaced resonant extraction for high-intensity proton beams and the IBA Cyclone 30 became the de facto world standard cyclotron for isotope production.**

Phase space

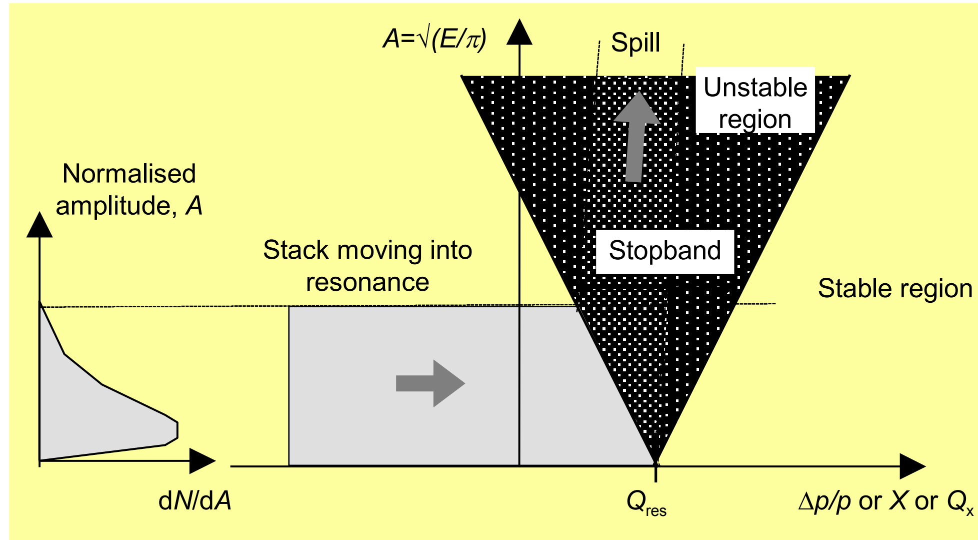
- The full beauty of resonant extraction is only visible in phase space.
- The figure shows a 1/3rd integer extraction.
  - **In the central region, the beam is stable. At the very centre, the invariants of the motion are unperturbed ellipses. At bigger amplitudes the ellipses take on a triangular shape.**
  - **In the outer region, the beam is unstable. The individual ions 'lock' onto the 3 separatices and gain amplitude turn after turn, while moving from one separatrix to the next.**
- The **spiral step** (inter-turn separation) and other geometric details determine the design, of the extraction elements and the interception losses.
- How the beam is moved across the **last stable triangle** into the unstable region is the job of the machine designer. This determines the length and quality of the spill.



Other descriptive views (1)

Steinbach diagram.

- This diagram shows the radial position versus the oscillation amplitude. In the example, a wide momentum stack is being accelerated into the resonance. The chromaticity is finite and positive.



- This diagram has the clear advantage of showing the inclined amplitude boundary between stability and instability.

There are other configurations for extraction, but this example illustrates the main features.

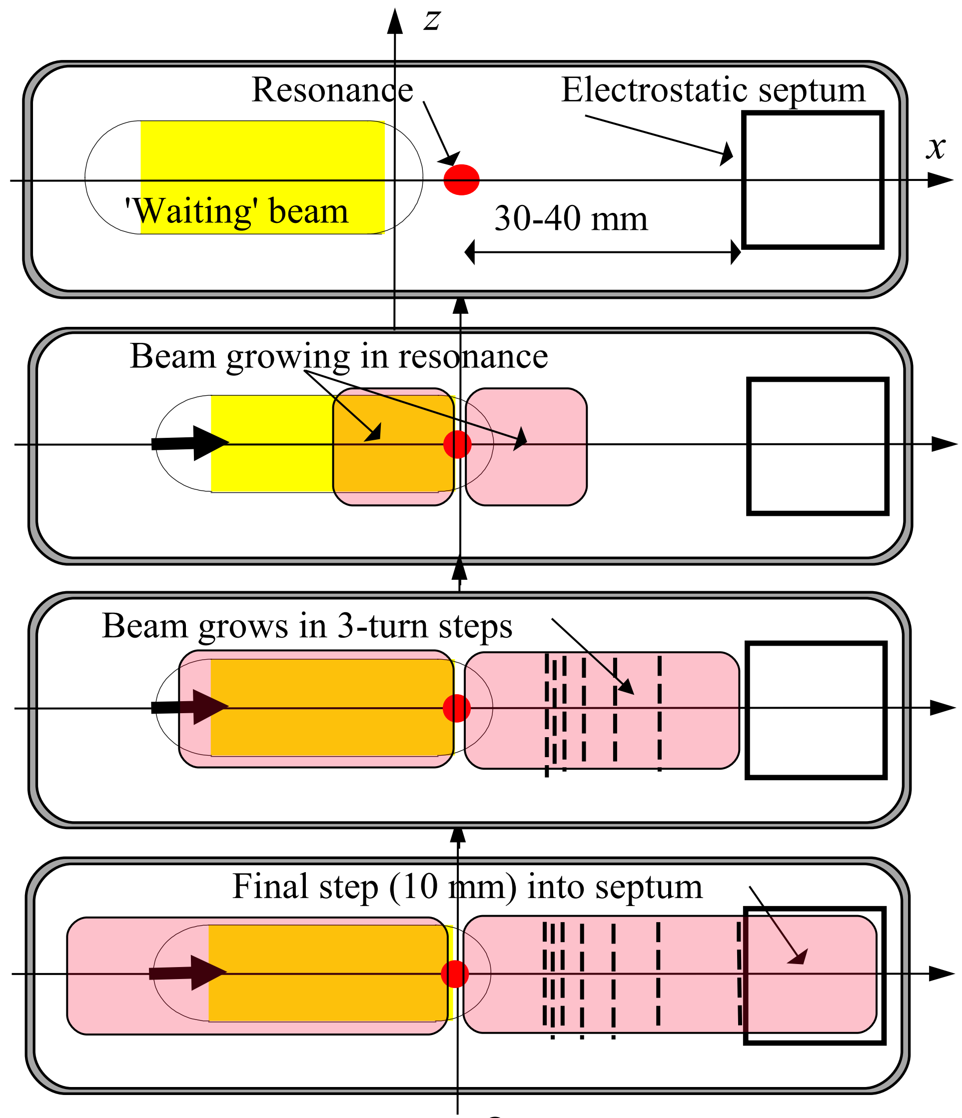
Charles Steinbach

Other descriptive views (2)

Real space view

- The 'Waiting' beam is created by multi-turn injection in the inner half of the aperture away from the 3rd order resonance.
- A betatron core is used to smoothly accelerate the 'Waiting' beam into the resonance.
- The particles 'locked' in the resonance grows in steps of 3-turns. Starting with small steps, they extents outwards over several hundred turns.
- The beam enters the electrostatic septum with a final step of ~10 mm and is kicked onto the extraction trajectory.

This configuration corresponds to the last side.



Sextupole fields

- The radial 1/3rd integer or third-order resonance is driven by normal sextupole fields and can be treated as a perturbation to the linear machine.
- The transverse fields in a normal sextupole magnet are well known

$$B_x(x,z) = -6B_3xz \qquad \text{and} \qquad B_z(x,z) = -3B_3(x^2 - z^2)$$

where $$B_3 = -\frac{1}{6}\left(\frac{d^2B_z}{dx^2}\right)_0$$

- For positively charged particles in an anticlockwise ring, the kicks given by a sextupole are,

$$\Delta x' = \frac{B_z\ell_S}{|B\rho|} = \frac{1}{2}\frac{\ell_S}{|B\rho|}\left(\frac{\mathrm{d}^2B_z}{\mathrm{d}x^2}\right)_0(x^2 - z^2) = \frac{1}{2}\ell_S k'(x^2 - z^2)$$

$$\Delta z' = -\frac{\ell_S}{|B\rho|}\left(\frac{\mathrm{d}^2B_z}{\mathrm{d}x^2}\right)_0 xz = -\ell_S k' xz$$ where $$k' = \frac{1}{|B\rho|}\left(\frac{\mathrm{d}^2B_z}{\mathrm{d}x^2}\right)_0$$

Normalised coordinates (1)

- The use of normalised coordinates (*X*, *Z*) clarifies diagrams. The correspondences are,

$$x \Rightarrow \sqrt{\beta_x}\,X, \quad \Delta x' \Rightarrow \frac{1}{\sqrt{\beta_x}}\Delta X', \quad z \Rightarrow \sqrt{\beta_z}\,Z \quad \text{and} \quad \Delta z' \Rightarrow \frac{1}{\sqrt{\beta_z}}\Delta Z'$$

- The effect of a thin-lens sextupole in normalised co-ordinates appears as,

$$\Delta X = 0 \text{ and } \Delta X' = \left[\frac{1}{2}\beta_x^{\,3/2}\frac{\ell_S}{|B\rho|}\left(\frac{d^2 B_z}{dx^2}\right)_0\right]\left(X^2 - \frac{\beta_z}{\beta_x}Z^2\right) = S\left(X^2 - \frac{\beta_z}{\beta_x}Z^2\right)$$

$$\Delta Z = 0 \text{ and } \Delta Z' = -2\left[\frac{1}{2}\beta_x^{\,3/2}\frac{\ell_S}{|B\rho|}\left(\frac{d^2 B_z}{dx^2}\right)_0\right]\frac{\beta_z}{\beta_x}XZ = -2S\frac{\beta_z}{\beta_x}XZ$$

- where *S* is the **normalised sextupole strength**

$$S = \frac{1}{2}\beta_x^{\,3/2}\frac{\ell_S}{|B\rho|}\left(\frac{d^2 B_z}{dx^2}\right)_0 = \frac{1}{2}\beta_x^{\,3/2}\ell_S k'$$

Normalised coordinates (2)

- Unless Z = 0, a sextupole couples the horizontal and vertical motions and the strength of the coupling is proportional to the ratio of the vertical and horizontal betatron amplitude functions (β_z/β_x) at the sextupole.
- For a horizontal extraction, *Z* is generally much smaller than *X* and, provided the vertical tune does not satisfy a resonance condition, the influence of the vertical motion can be neglected to first order.
- Accepting this approximation the motion in the thin-lens sextupole becomes,

$$\Delta X = \Delta Z = \Delta Z' = 0 \quad \text{and} \quad \Delta X' = SX^2$$

Basic theory of the resonance (1)

- The transfer matrix $\boldsymbol{M}_{\mathrm{n}}$ for normalised co-ordinates, describing *n* turns in the machine is given by:

$$\boldsymbol{M}_{\mathrm{n}} = \begin{pmatrix} \cos 2\pi(nQ_{\mathrm{x}}) & \sin 2\pi(nQ_{\mathrm{x}}) \\ -\sin 2\pi(nQ_{\mathrm{x}}) & \cos 2\pi(nQ_{\mathrm{x}}) \end{pmatrix}$$

- Consider a particle with a horizontal betatron tune close to a third-integer, i.e. $Q_x = m \pm 1/3 + \delta Q$, where *m* is integer and $|\delta Q| \ll 1/3$). The tune increment δQ is defined as the **tune distance** of the particle from the resonance,

$$\delta Q = Q_{\mathrm{particle}} - Q_{\mathrm{resonance}}$$

- The explicit transfer matrix for *n* turns **in the unperturbed machine** can then be written as:

$$\boldsymbol{M}_{n} = \begin{pmatrix} \cos[2n\pi(m \pm 1/3 + \delta Q)] & \sin[2n\pi(m \pm 1/3 + \delta Q)] \\ -\sin[2n\pi(m \pm 1/3 + \delta Q)] & \cos[2n\pi(m \pm 1/3 + \delta Q)] \end{pmatrix}$$

Basic theory of the resonance (2)

- We want to find what happens over 3 turns. Start by approximating the unperturbed machine as,
  - 1 turn, M_1, neglecting δQ :
    $$\begin{pmatrix} X \\ X' \end{pmatrix}_1 \cong \begin{pmatrix} -1/2 & \pm\sqrt{3}/2 \\ \mp\sqrt{3}/2 & -1/2 \end{pmatrix} \begin{pmatrix} X \\ X' \end{pmatrix}_0$$
  - 2 turns, M_2, neglecting δQ :
    $$\begin{pmatrix} X \\ X' \end{pmatrix}_2 \cong \begin{pmatrix} -1/2 & \mp\sqrt{3}/2 \\ \pm\sqrt{3}/2 & -1/2 \end{pmatrix} \begin{pmatrix} X \\ X' \end{pmatrix}_0$$
  - 3 turns, M_3, with $\varepsilon = 6\pi\delta Q$:
    $$\begin{pmatrix} X \\ X' \end{pmatrix}_3 \cong \begin{pmatrix} 1 & \varepsilon \\ -\varepsilon & 1 \end{pmatrix} \begin{pmatrix} X \\ X' \end{pmatrix}_0$$

    where ε is the **modified tune distance**.

- The effect of a sextupole over 3 turns can now calculated by the linear addition of :
  - 3 turns + a sextupole placed after the 3rd turn, *M_3 + Sextupole*
  - 3 turns with a sextupole placed after the 2nd turn, *M_2 + Sextupole + M_1*
  - 3 turns with a sextupole placed after the 1st turn, *M_1 + Sextupole + M_2*
  - Remembering :
    $$\Delta X = \Delta Z = \Delta Z' = 0 \quad \text{and} \quad \Delta X' = SX^2$$

Basic theory of the resonance (3)

- Skipping the algebra, the final expressions for the change of position and divergence of the particle over three turns, known as the **spiral step** and **spiral kick** are,

$$\Delta X_3 = \varepsilon X_0' + \frac{3}{2} S X_0 X_0'$$

$$\Delta X_3' = -\varepsilon X_0 + \frac{3}{4} S\left(X_0{}^2 - X_0'^2\right).$$

Note that the ± signs cancel and there is no fundamental difference between the 1/3rd and 2/3rd integer resonances.

- The time for three turns is short compared to the spill time and can be safely used as the basic time unit. The changes occurring in this time are the smallest that need to be resolved. Thus the subscripts are no longer needed and these results can be treated as continuous functions that are derived from a Hamiltonian ***H***,

$$\Delta X_3 \Rightarrow \left(\frac{\Delta X}{\Delta t}\right)_{\Delta t = 1\,(3\text{ turn})} = \frac{\partial \boldsymbol{H}}{\partial X'} = \varepsilon X' + \frac{3}{2} S X X'$$

$$\Delta X'_3 \Rightarrow \left(\frac{\Delta X'}{\Delta t}\right)_{\Delta t = 1\,(3\text{ turn})} = -\frac{\partial \boldsymbol{H}}{\partial X} = -\varepsilon X + \frac{3}{4} S\left(X^2 - X'^2\right)$$

$$\boldsymbol{H} = \frac{\varepsilon}{2}\left(X^2 + X'^2\right) + \frac{S}{4}\left(3XX'^2 - X^3\right)$$

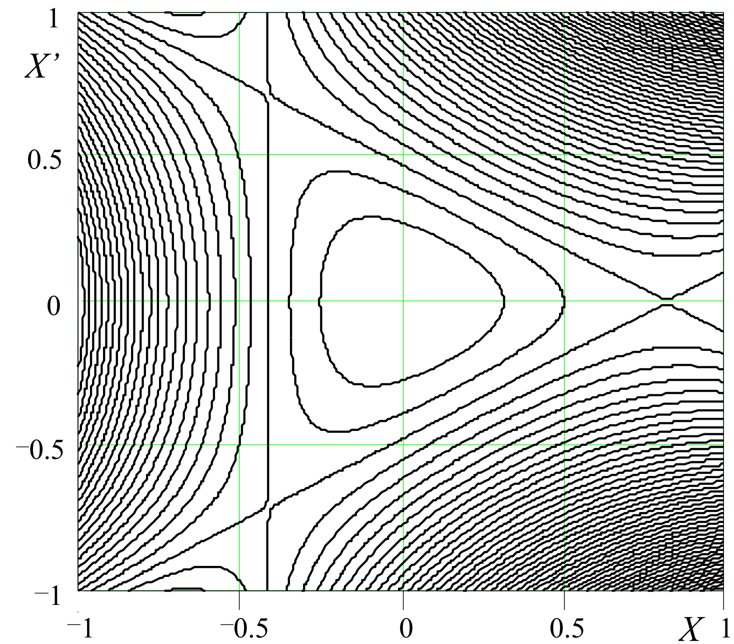
Kobayashi Hamiltonian

Here time is dimensionless

Kobayashi Hamiltonian (1)

- The Hamiltonian is time independent and a constant of the motion. Contours of constant ***H*** show the particle trajectories in normalized phase space **at the sextupole**.
- The first term describes the unperturbed particle motion in the linear machine (S = 0). These trajectories are circles of radius $\sqrt{(2\boldsymbol{H}/\varepsilon)}$ in normalized phase space. The second term contains the perturbation that distorts the circular phase-space trajectories into a triangular form. At a certain level of excitation, the triangle 'breaks' into open phase-space trajectories. A change in sign of either the modified tune distance ε or the normalized sextupole strength S is equivalent to a rotation of the phase-space trajectories by 180º.
- When ***H*** has the value $[(2\varepsilon/3)^3/S^2]$, it factorizes into three straight lines,

$$\boldsymbol{H} = \frac{\varepsilon}{2}\left(X^2 + X'^2\right) + \frac{S}{4}\left(3XX'^2 - X^3\right)$$



Normalized phase-space map

$\varepsilon / S > 0$

Separatrices, $\left(\frac{S}{4}X + \frac{\varepsilon}{6}\right)\left(\sqrt{3}X' + X - \frac{4\varepsilon}{3S}\right)\left(\sqrt{3}X' - X + \frac{4\varepsilon}{3S}\right) = 0$

Kobayashi Hamiltonian (2)

This is normalized phase space, the unperturbed trajectories are circles and *X* is equivalent to *X′* .

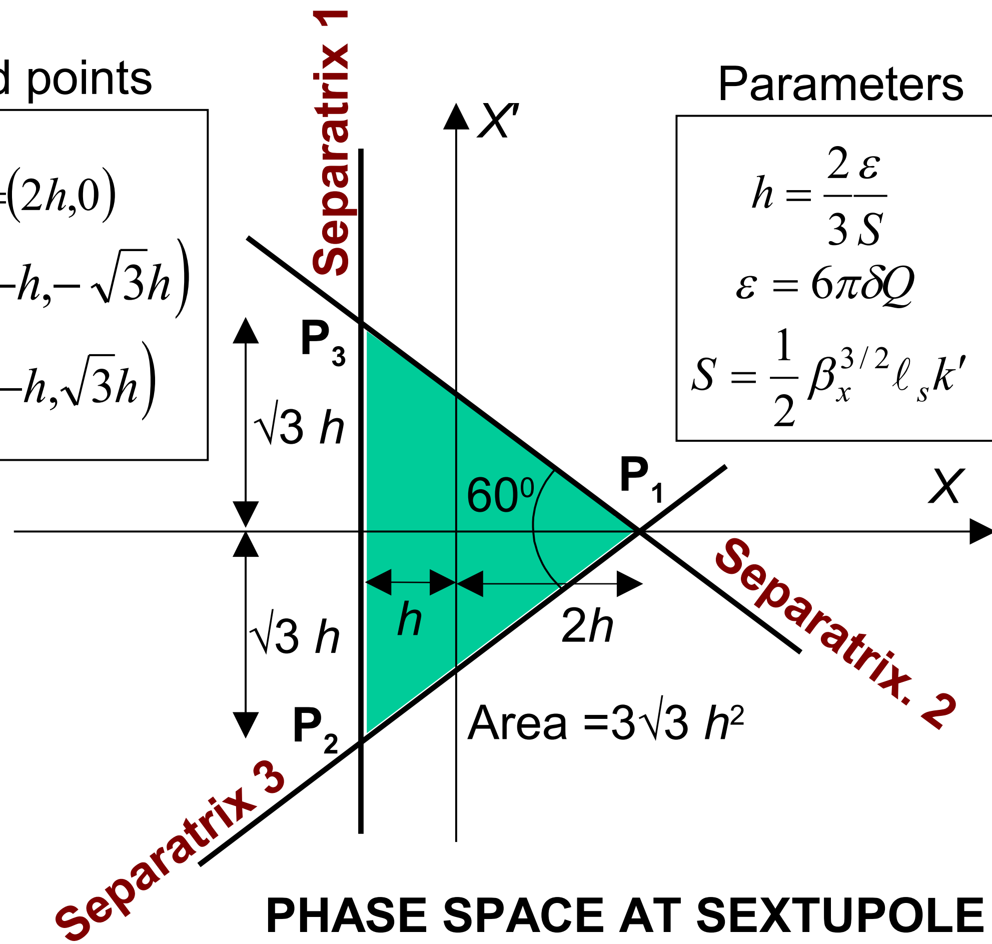
Hence, we can rotate the diagram to any position using the betatron phase advance and, in particular, a position suitable for extraction without altering its shape.

Fixed points

$$P_1 = (2h, 0)$$
$$P_2 = \left(-h, -\sqrt{3}h\right)$$
$$P_3 = \left(-h, \sqrt{3}h\right)$$

Parameters

$$h = \frac{2}{3}\frac{\varepsilon}{S}$$
$$\varepsilon = 6\pi\delta Q$$
$$S = \frac{1}{2}\beta_x^{3/2}\ell_s k'$$

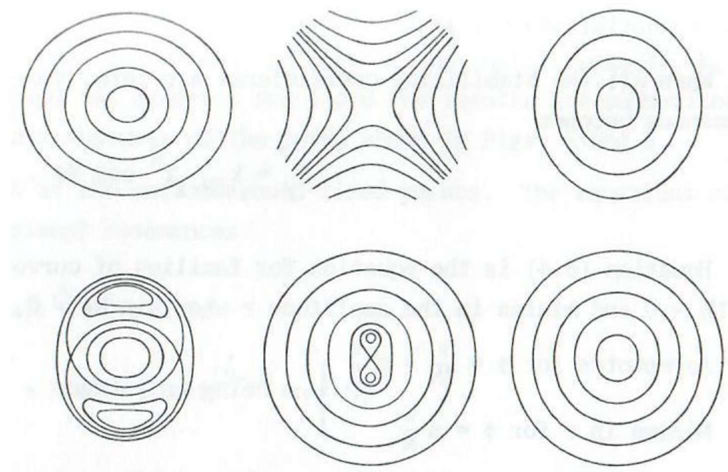


PHASE SPACE AT SEXTUPOLE

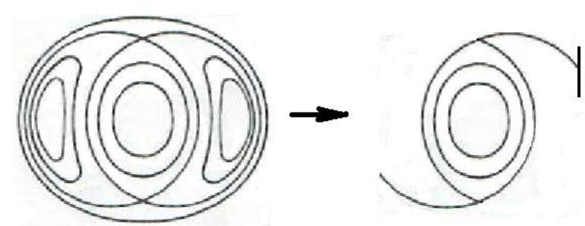
Note on resonances

- If this were a lecture on resonances, a more advanced Hamiltonian would be used to study mixed horizontal and vertical resonances driven by normal and skew fields over a larger aperture in phase space, for example the illustration shows phase-space views generated by half-integer resonances.
- We have used a simplified Hamiltonian, for a purely radial resonance. The trajectories are intercepted within the physical aperture before distant stable islands can be formed to return the particles. This truncated and simplified situation was used to get some basic parameters such as the spiral step.

Phase-space views half-integer resonance



Restricted view for extraction with pure radial resonance



The virtual sextupole (1)

- In most rings, sextupoles are mounted in dispersion regions for controlling the chromaticity. These sextupoles are usually positioned so as to cause as little resonance excitation as possible, but this internal compensation is never perfect.
- For resonant extraction, we will also need a dedicated sextupole for exciting the resonance and this sextupole is best positioned in a zero-dispersion region, so as not to disturb the chromaticity correction.
- The dedicated resonance sextupole, the residual excitation from the chromaticity sextupoles and any sextupole errors can be combined numerically into a **virtual sextupole** that fits the theory presented so far for a single magnet.
- For the third-order resonance $3Q_x = n$ the driving term is,

$$\kappa(s_0) = \frac{1}{24\sqrt{\pi C}} \int_{s_0}^{s_0+C} \beta_x^{3/2} \left[\frac{-1}{|B_0\rho|} \left(\frac{\mathrm{d}^2 B_z}{\mathrm{d}x^2} \right)_0 \right] \exp(3\mathrm{i}\mu_x)\,\mathrm{d}s$$

using the earlier definition for S this gives,

$$\kappa(s_0) = \frac{-1}{12\sqrt{\pi C}} \sum_{\mathrm{n}} S_{\mathrm{n}} \exp(3\mathrm{i}\mu_{\mathrm{x,n}})$$

The virtual sextupole (2)

- The virtual sextupole can be found by equating its driving term to the sum of all the driving terms around the ring.

$$S_{\mathrm{virt}} \exp\left(3 i \mu_{\mathrm{x,virt}}\right) = \sum_{n} S_{\mathrm{n}} \exp\left(3 i \mu_{\mathrm{x,n}}\right)$$

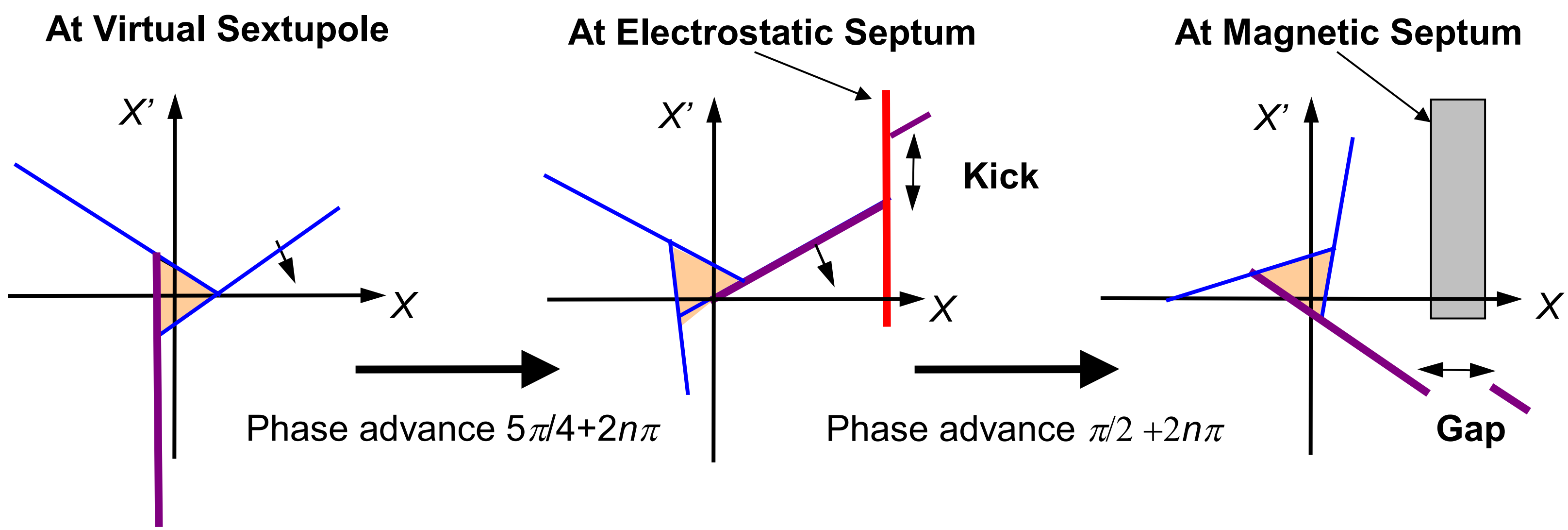
- By separating the real and imaginary parts, the betatron phase and strength of the virtual sextupole can be found as,

$$\tan\left(3\mu_{\mathrm{x,virt}}\right) = \frac{\sum_{\mathrm{n}} S_{\mathrm{n}} \sin\left(3\mu_{\mathrm{x,n}}\right)}{\sum_{n} S_{\mathrm{n}} \cos\left(3\mu_{\mathrm{x,n}}\right)}$$

$$S_{\mathrm{virt}}^{2} = \left[\sum_{\mathrm{n}} S_{\mathrm{n}} \cos\left(3\mu_{\mathrm{x,n}}\right)\right]^{2} + \left[\sum_{\mathrm{n}} S_{\mathrm{n}} \sin\left(3\mu_{\mathrm{x,n}}\right)\right]^{2}$$

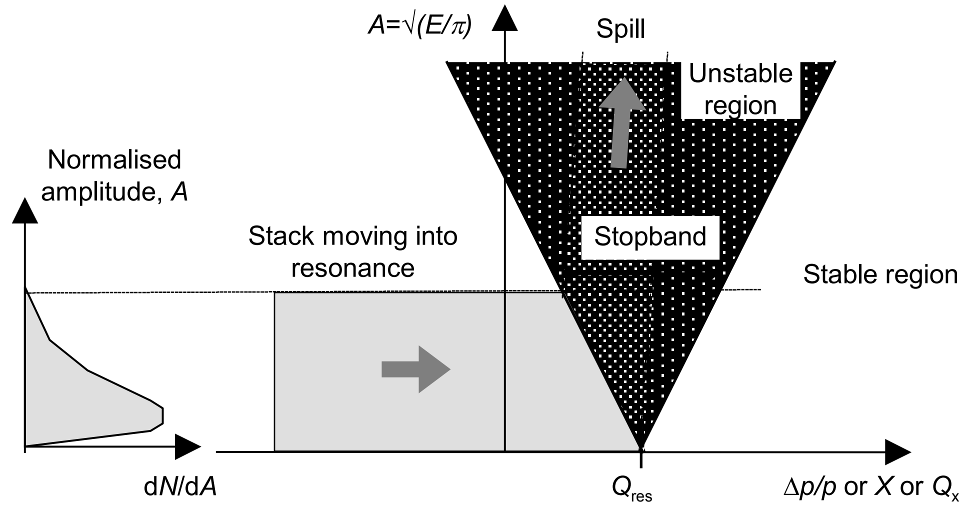
Configuring the extraction (1)

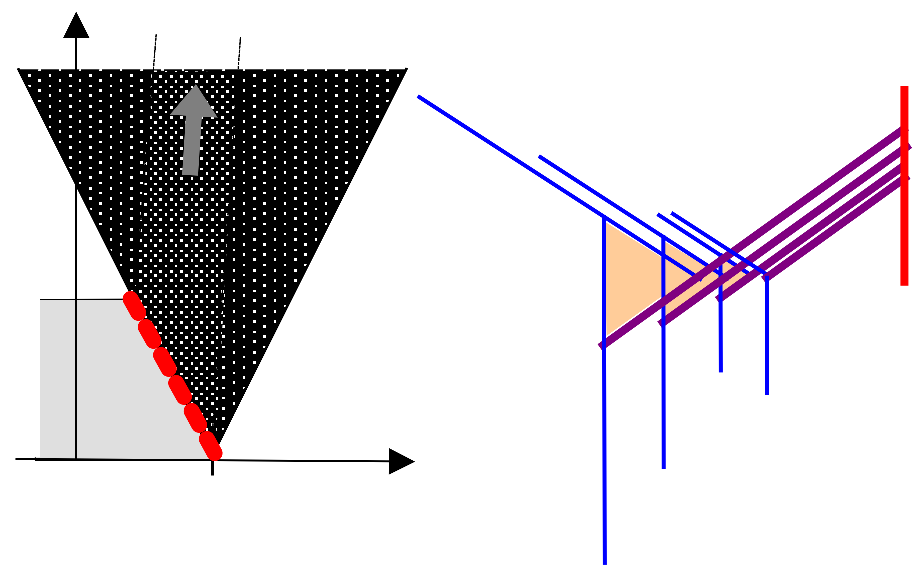
- The lattice must be built so as to bring the electrostatic septum (ES) into a suitable position with respect to the virtual sextupole.
- The magnetic septum (MS) is then placed so as to maximize the gap between the end of the extraction separatrix and the segment of extracted beam.
- Typically, the electrostatic septum wires are 0.1 mm diameter and the gap for the magnetic septum is 20 mm.



Configuring the extraction (2)

- Let us return to an earlier example showing a beam moving into a resonance.
- The large amplitude particles enter the resonance first. Particles with smaller amplitudes must wait until they move to a *Q* value closer to the resonance.
- This means that during the extraction there is a range of amplitudes being extracted and each amplitude follows the theory laid out so far.
- **All separatrices terminate at the same radial position, but with a wide range of angles. The septum can only be aligned longitudinally at one angle. Hence, this leads to large interception losses.**



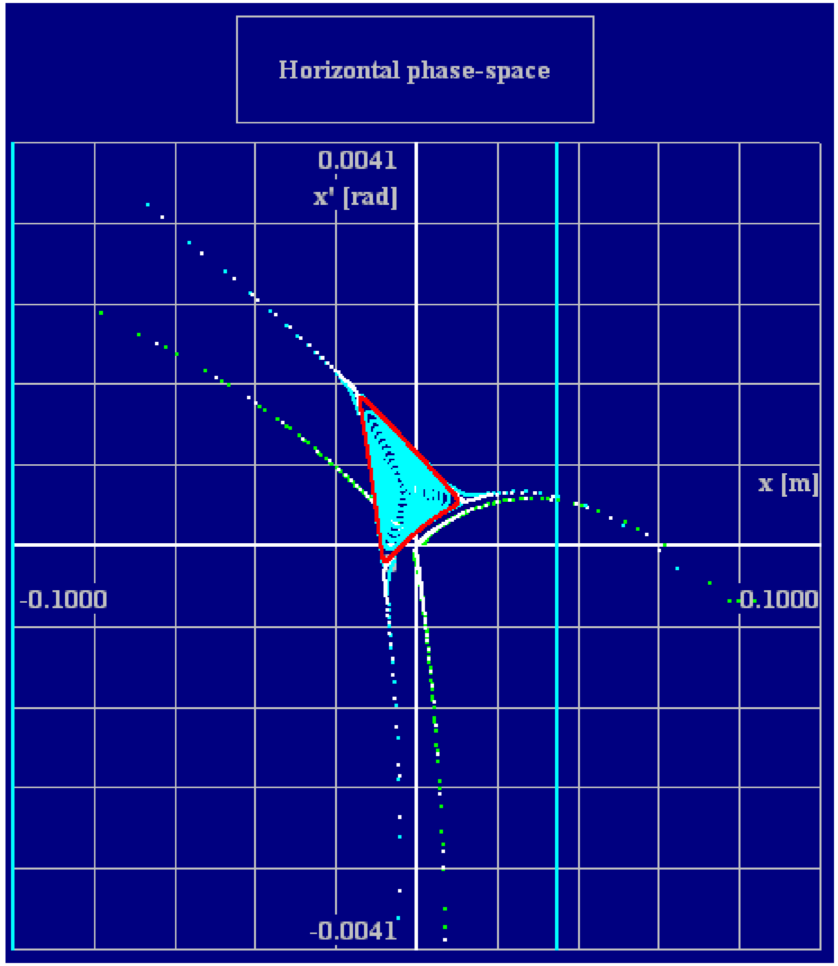


Hardt condition (1)

- The Hardt condition is constraint on the lattice optics that superimposes the extraction separatrices at a given point in the machine.
- The first step is to generalized the Kobayashi Hamiltonian for dispersion and then to derive the general separatrix equation. This would add too much theory to this lecture, so we will jump directly to the conclusion.

$$D_n \cos(\alpha - \Delta\mu) + D'_n \sin(\alpha - \Delta\mu) = -\frac{4\pi}{S} Q'$$

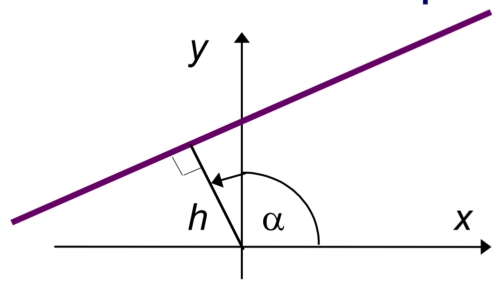
$\Delta\mu$ is the phase advance from a reference point to the ES, α is an angle that defines the separatrix at the reference point (see next slide) and (D_n, D'_n) is the normalized dispersion vector.



Extreme cases of the largest amplitude and the zero amplitude separatrices after the Hardt condition has been applied.

Hardt condition (2)

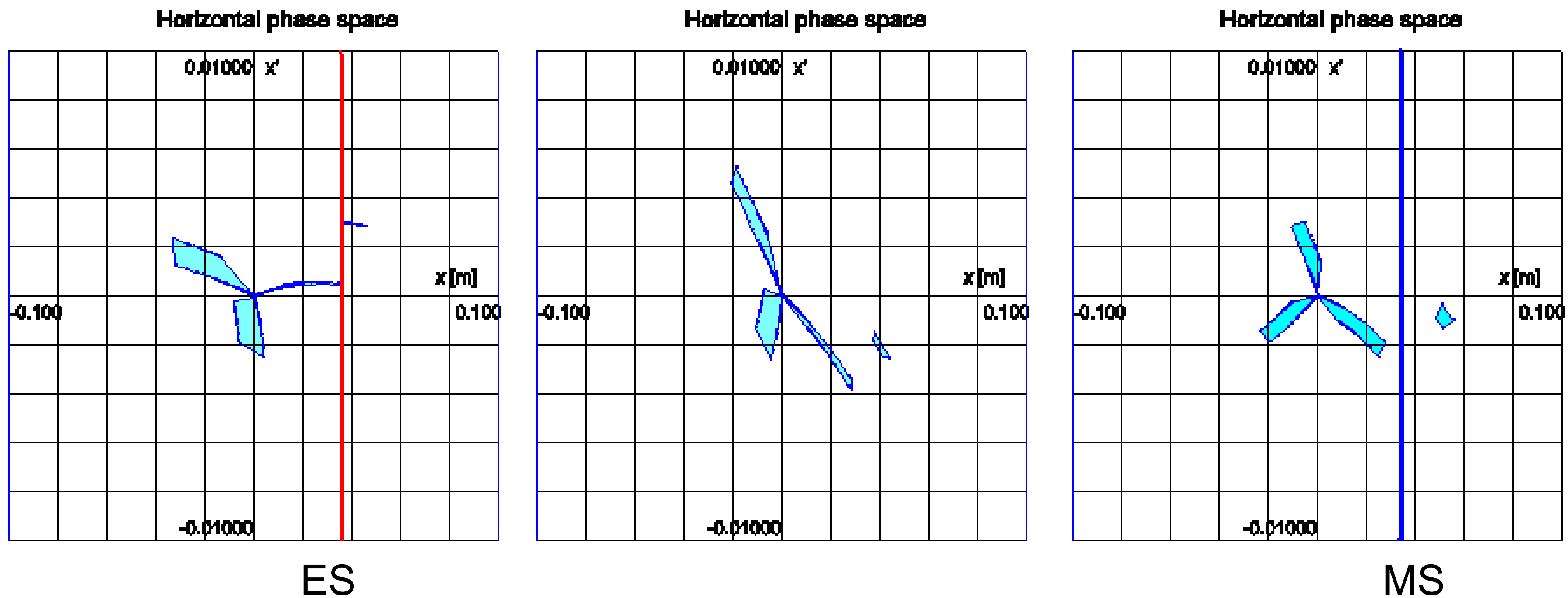
- The angle α is defined by the perpendicular form for a straight line. If the reference point is the virtual sextupole then $\alpha = \pi$.



Angle α is measured anticlockwise from x-axis to perpendicular h

- The dispersion function is a property of the lattice. If the lattice already exists, or is determined by other factors, this could be a severe disadvantage.
- For optimized operation the extraction separatrix should be at $\pi/4$ in the 1st quadrant that is $(\alpha-\Delta\mu)=3\pi/4$.
- The sextupole strength cannot be used as a variable, since it determines the spiral step and therefore the horizontal size of the extracted beam.
- For small, low-energy machines working below transition, the chromaticity should be negative to ensure the stability of the coasting beam. However, the chromaticity can still be varied over a wide range.

Getting to the magnetic septum

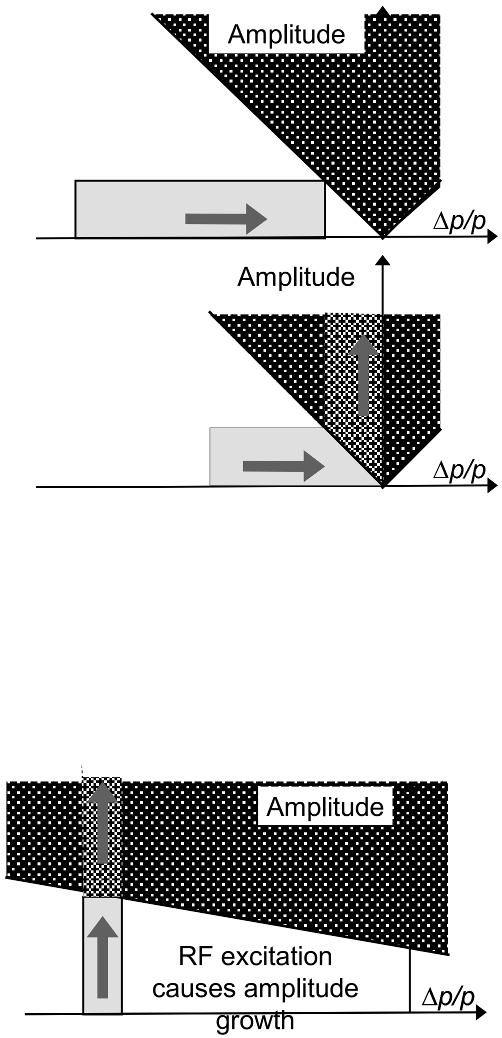


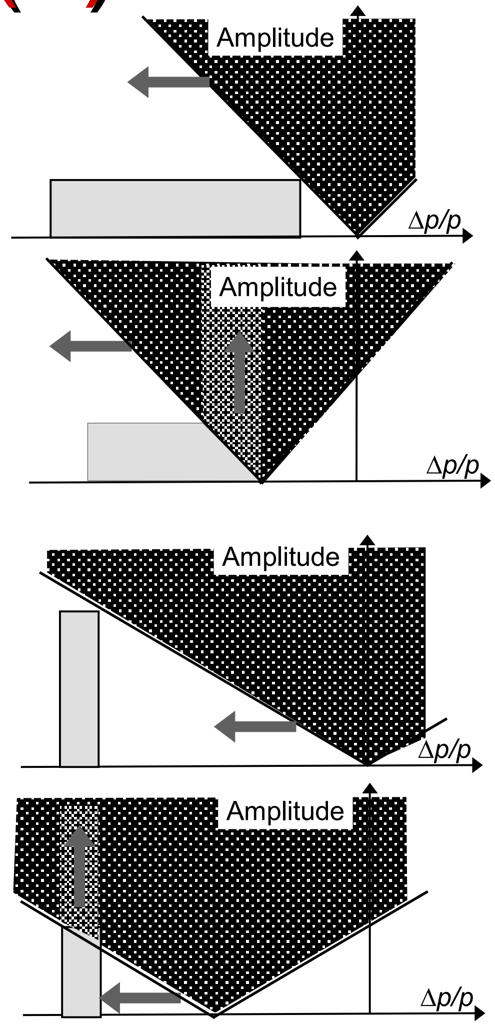
- The aperture between the electrostatic and magnetic septa is checked visually using a graphical method developed by C. Steinbach.
- The phase-space areas between the maximum and the zero-amplitude separatrices are marked in blue as well as the extracted beam segment.

Widening the scope (1)

- **Moving the beam.** This method has the advantage of leaving the optical parameters of the machine constant, and hence also those of the resonance:
  - **Acceleration-Driven.** The beam is accelerated towards the stationary resonance by a betatron core, or by stochastic noise, or possibly by a phase displacement or a RF micro-bucket acceleration system.
  - **RF 'knockout'.** The beam is excited by transverse stochastic noise or RF excitation at the revolution frequency, so that its betatron amplitudes grow. The chromaticity is set to zero, or a low value, so that the resonance line acts as a threshold in amplitude above which the ions become unstable [1].
- **Moving the resonance.** **This method is not recommended as it alters the optics of the machine:**
  - **Quadrupole-driven**. The tune of the machine is changed so that the resonance region moves towards the beam.
  - **Sextupole-driven**. The resonance excitation is changed by increasing the sextupole strength. This method is included only for academic completeness.

Widening the scope (2)



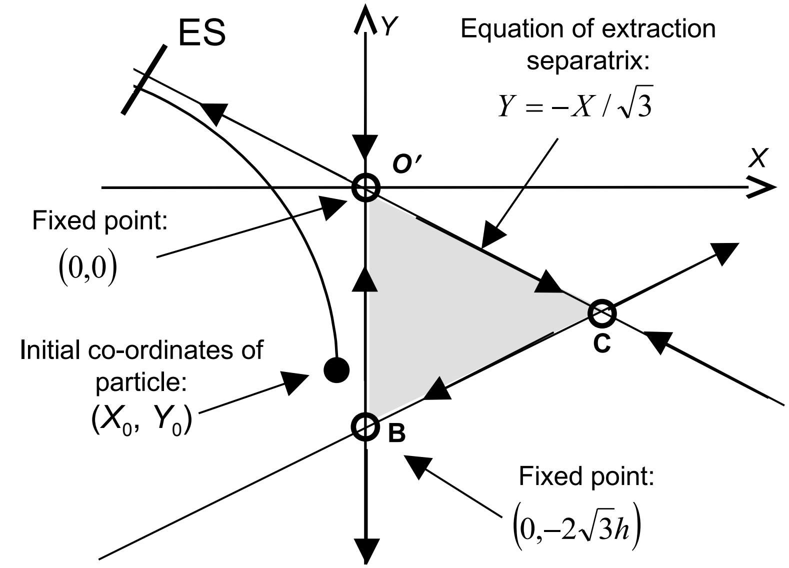


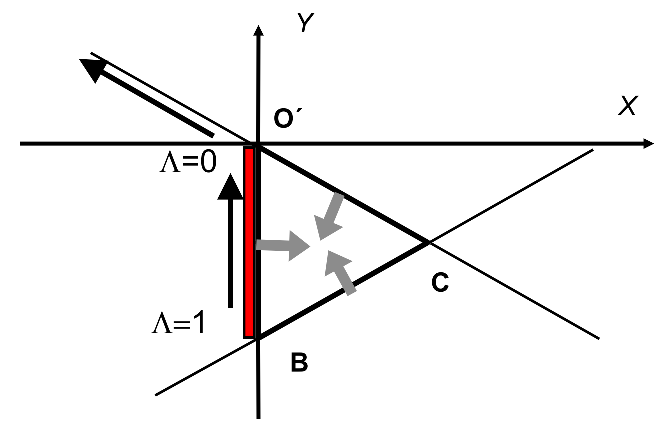
A 'strip spill' (1)

- For mathematical convenience, the origin of the Hamiltonian is shifted to the upper fixed point and the extraction is considered via this fixed point.

$$H_{\text{Shifted}} = \frac{S}{4}\left(4h^3 + 6hX^2 + 6\sqrt{3}hX\,X' + 3X\,X'^2 - X^3\right) \quad (1)$$

- The extraction process can be studied by considering what happens when the 'last stable triangle' shrinks by a small step.
- **When this happens a thin strip of particles that were previously stable now find themselves outside, in the unstable region.**
- In the lower figure, only one of the three sides of the triangle is considered. The black arrows show the general direction of movement of the particles. The grey arrows show the movement of the separatrices.





Note: *Y* now replaces *X′* for brevity

A 'strip spill' (2)

- In the translated frame of reference, the equations of motion become:

$$\frac{dX}{dt} = \frac{\partial H}{\partial Y} = \frac{S}{4}\left(6\sqrt{3}hX + 6XY\right) \qquad (2)$$

$$\frac{dY}{dt} = -\frac{\partial H}{\partial X} = -\frac{S}{4}\left(12hX + 6\sqrt{3}hY + 3Y^2 - 3X^2\right) \qquad (3)$$

Time is dimensionless and is measured as the number of sets of three turns. Thus, an extraction time of 100 equals 300 revolutions.

- Extracted particles will follow paths of constant *H* close to the separatrices. The trajectories come directly from the initial conditions (X_0, Y_0) substituted into the Hamiltonian (1).

$$\frac{S}{4}(4h^3 + 6hX_0^2 + 6\sqrt{3}hX_0Y_0 + 3X_0Y_0^2 - X_0^3) = \frac{S}{4}(4h^3 + 6hX^2 + 6\sqrt{3}hXY + 3XY^2 - X^3) \qquad (4)$$

- **Close to the fixed point O′ (|*X*|, |*Y*| << *h*) the third-order terms in *X* and *Y* can be neglected. Thus:**

$$Y = \frac{X_0^2 + \sqrt{3}X_0Y_0 - X^2}{\sqrt{3}X} \qquad (5)$$

A 'strip spill' (3)

- Substituting (5) into the earlier expression for dX/dt (2) gives,

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{S}{4}\left(6\sqrt{3}hX + \frac{6}{\sqrt{3}}X_0^2 + 6X_0Y_0 - \frac{6}{\sqrt{3}}X^2\right) \qquad (6)$$

Fortunately, Eqn (6) is a standard form and can be integrated.

$$\int \frac{\mathrm{d}x}{ax^2 + bx + c} = \frac{1}{\sqrt{b^2 - 4ac}} \ln\left|\frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}}\right|$$

- Within the strict assumptions made earlier (|X|, |Y| << h) , the result is only valid close to O′, but, since the particle approaches the separatrix asymptotically the third-order terms in the Hamiltonian cancel out, so they can also be neglected far from O′ along the outgoing separatrix and the integration can be extended right up to the electrostatic septum.
- **This covers the essential mathematics for calculating extracted particles. The finer details can be found in Ref. CERN/PS 99-010 (DI). We will now go directly to the results.**

Transit time (1)

- The mathematics of the ‘strip spill’ can tell us how long it takes for a particle to leave the machine – the **transit time**.
- The transit time is split into two parts:
  - (i) The time to travel from O′ to the electrostatic septum, and
  - (ii) The time to move along the side of the stable triangle from an arbitrary point towards O′.
- In the first instance this done under static conditions i.e. while the stable triangle remains constant in size and position and then these expressions must be modified to take into account the dynamic conditions of a shrinking stable triangle.
- For convenience, the motion in *X* is measured in units of *h*, the motion in *Y* is measured in units of the length of the side of the stable triangle and the position of the ES is expressed in units of *h*,

$$X_0 = -\lambda_0 h; \qquad Y_0 = -\Lambda_0 Y_B = -2\sqrt{3}\, h\Lambda_0; \qquad ES = -nh$$

Transit time (2)

- The final result for the **Transit time** from a point on the side of the triangle to the ES under static conditions is:

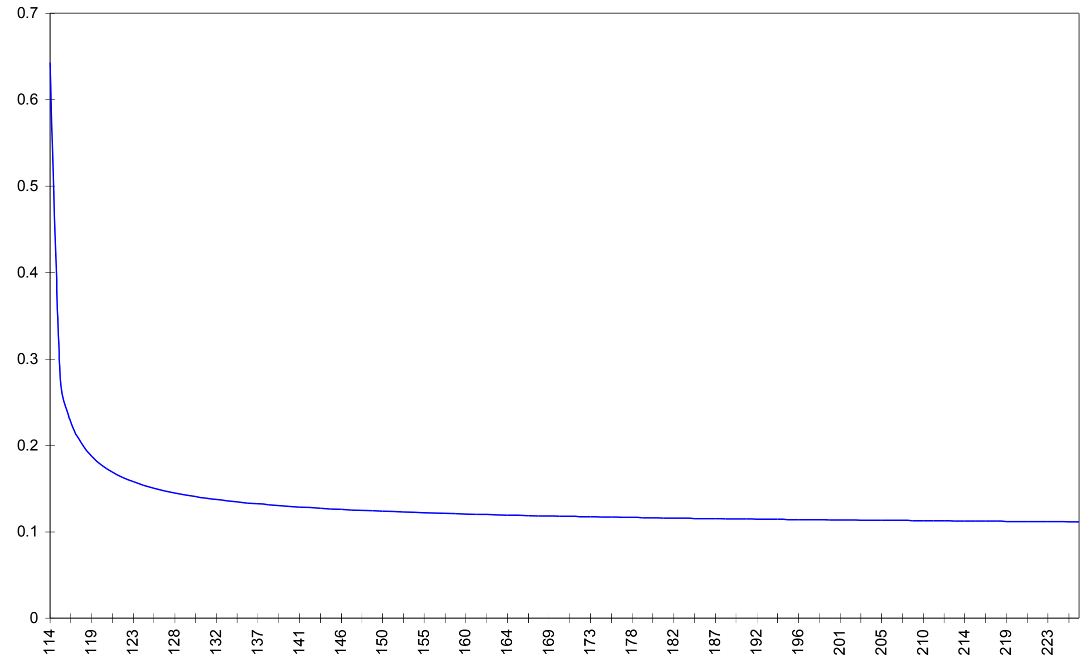
$$T_{\text{static}} \approx \frac{1}{\varepsilon\sqrt{3}} \ln \left| \underbrace{\frac{1}{\left(1-\Lambda_0\right)^2}}_{\substack{\text{Position on}\\ \text{side of}\\ \text{triangle}}} \underbrace{\left(\frac{n}{n+3}\right)}_{\substack{\text{Position of}\\ \text{electrostatic}\\ \text{septum}}} \underbrace{\frac{3}{\lambda_0}}_{\substack{\text{Distance}\\ \text{to stable}\\ \text{triangle}}} \right|$$

- If the shrink rate of the last stable triangle is taken into account we have,

$$T_{\text{dynamic}} \approx \frac{1}{\varepsilon\sqrt{3}} \ln \left| \underbrace{\left(\frac{n}{n+3}\right)}_{\substack{\text{Position of}\\ \text{electrostatic}\\ \text{septum}}} \underbrace{\left(\frac{1-\Lambda_F}{\Lambda_F}\right)}_{\substack{\text{Handover}\\ \text{point in } Y}} \underbrace{\left(\frac{\Lambda_0}{1-\Lambda_0}\right)}_{\substack{\text{Starting point}\\ \text{on stable}\\ \text{triangle}}} \underbrace{\left(\frac{3}{\lambda_{\text{F, dynamic}} - \frac{2\pi\sqrt{3}}{\varepsilon^2}\dot{Q}}\right)}_{\substack{\text{Effect of separatrix}\\ \text{motion}}} \right|$$

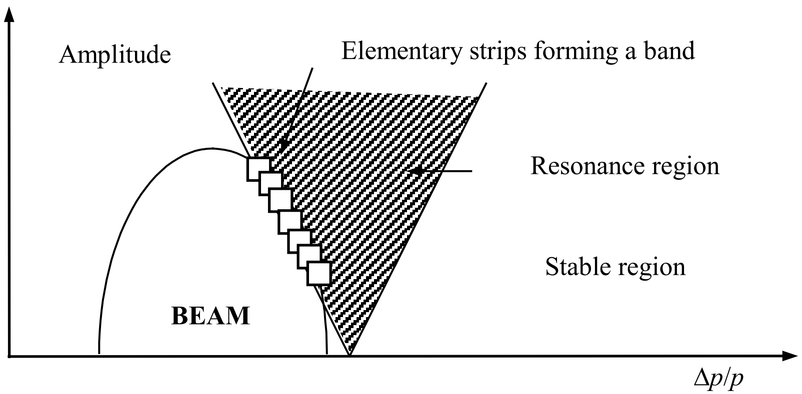
Time profile of a 'strip spill'

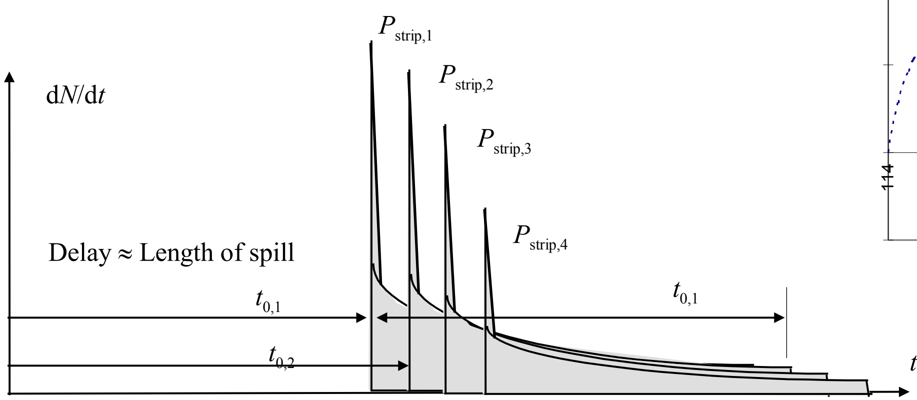
- The basic results for the transit time with an estimate of the particle distribution can be used to predict the time profile of the spill from an elementary 'strip' of particles sitting along the side of the last stable triangle.
- Assuming all the details in Ref. CERN/PS 99-010 (DI), a typical **strip spill** has a **time profile** as shown i.e.
  - **A narrow spike**
  - **Followed by a flat tail.**
  - **The spike arrives at** $t_0 \approx \dfrac{1}{\varepsilon\sqrt{3}}$
  - **The length of the flat tail is t_0.**

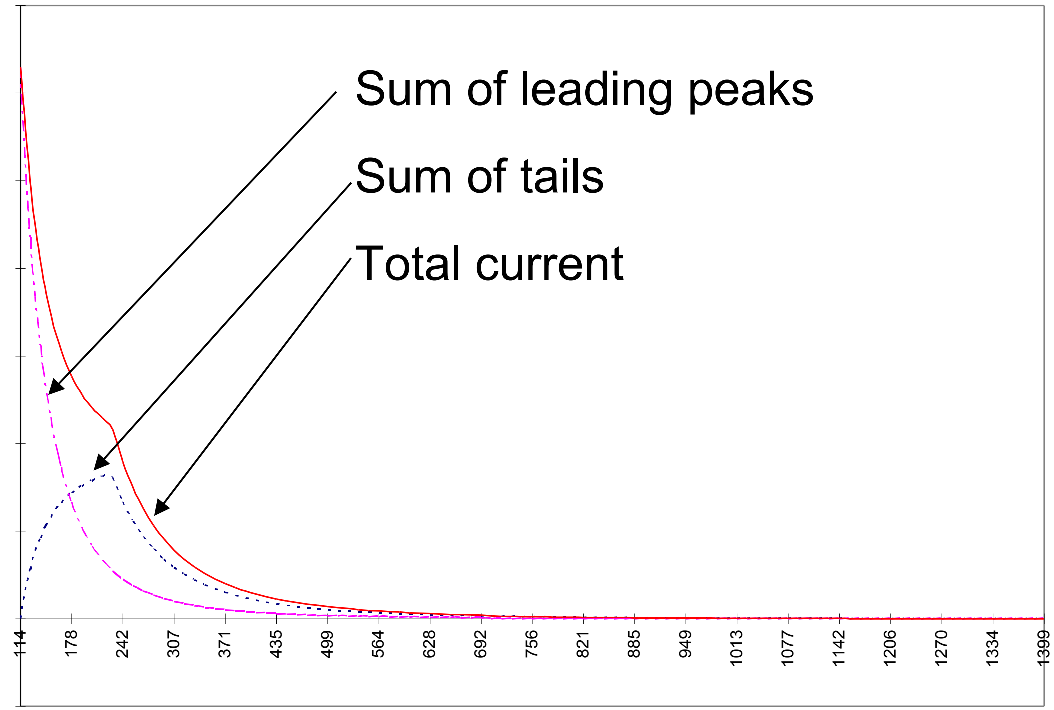


Ultimately, it will be possible to integrate over the elementary strips from all the different momenta that become unstable at any one time to form what is known as the 'band spill'.

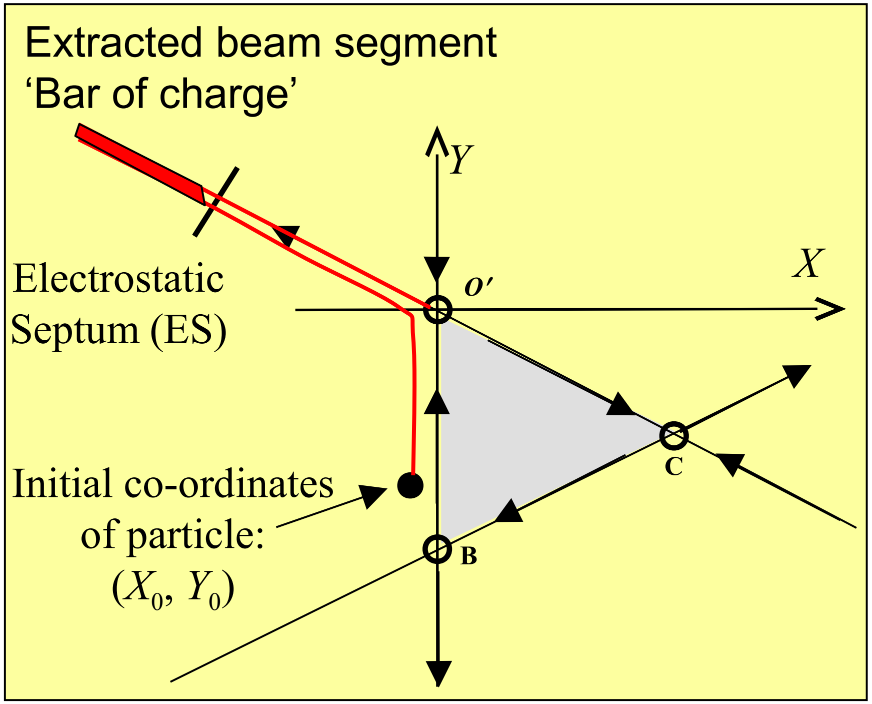
Time profile of a ‘band spill’



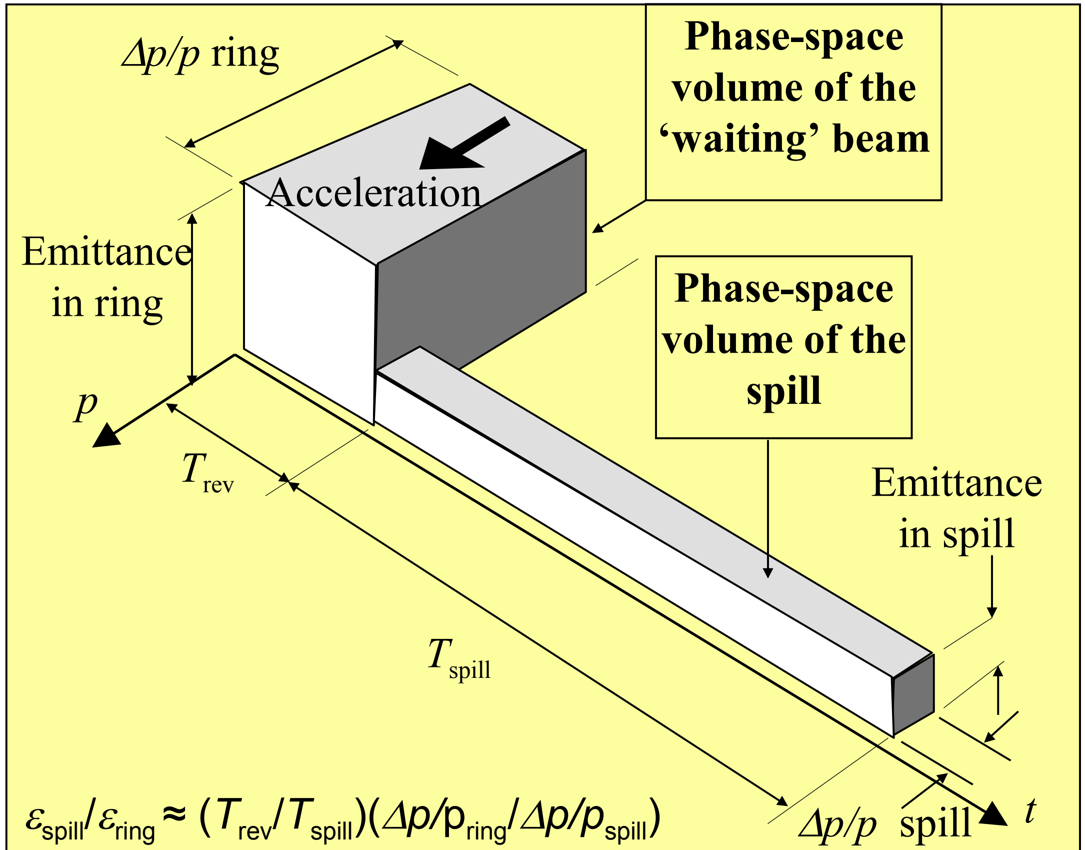




Emittance of the spill



The extracted beam segment known as the **'bar of charge'** has a small phase-space area (emittance) compared to the original beam.



$\varepsilon_{spill}/\varepsilon_{ring} \approx (T_{rev}/T_{spill})(\Delta p/p_{ring}/\Delta p/p_{spill})$

Typically *Trev* ≈ 1ms and *Tspill* ≈ 1s.

Comparison of the phase-space volumes of the **'waiting' beam** and the **spill**.

Thank you for your attention.