Introduction to Transverse Beam Dynamics

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1.) the basic ideas

„ … in the end and after all it should be a kind of circular machine“

→ need transverse deflecting force

Lorentz force

$$\vec{F} = q * (\cancel{\vec{E}} + \vec{v} \times \vec{B})$$

typical velocity in high energy machines:

$$v \approx c \approx 3 * 10^8 \, {}^{m}\!/_{s}$$

Example:

$$B = 1\,T \quad \rightarrow \quad F = q * 3 * 10^8 \frac{m}{s} * 1 \frac{Vs}{m^2}$$

$$F = q * \underbrace{300 \frac{MV}{m}}_{E}$$

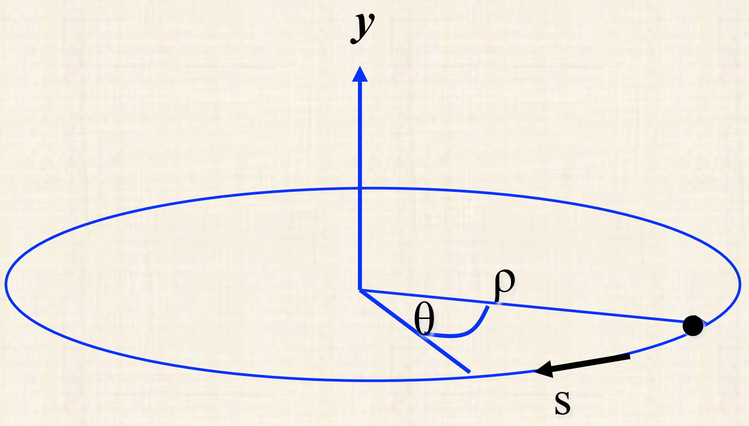
equivalent el. field …

technical limit for el. field

$$E \le 1 \frac{MV}{m}$$

old greek dictum of wisdom:
if you are clever, you use magnetic fields in an accelerator wherever it is possible.

The ideal circular orbit



circular coordinate system

condition for circular orbit:

Lorentz force $F_L = e\,v\,B$

centrifugal force $F_{centr} = \dfrac{\gamma\, m_0\, v^2}{\rho}$

$$\frac{\gamma\, m_0\, \cancel{v}^2}{\rho} = e\,\cancel{v}\,B$$

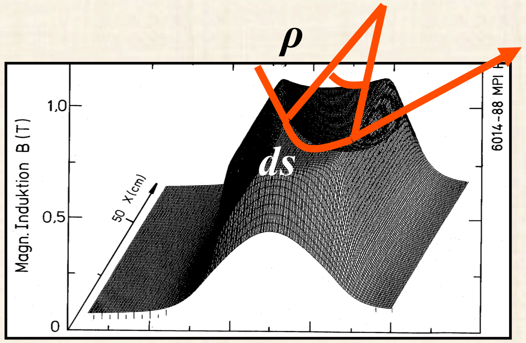
$$\boxed{\frac{p}{e} = B\,\rho}$$

$B\,\rho =$ ***"beam rigidity"***

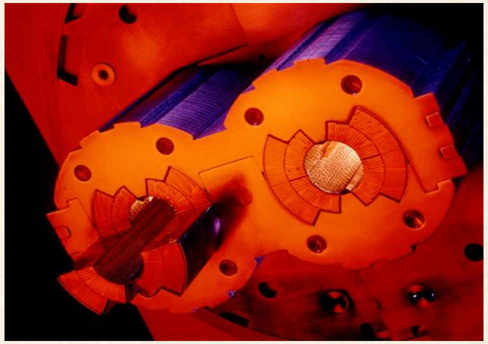
2.) The Magnetic Guide Field

Dipole Magnets:

define the ideal orbit
homogeneous field created
by two flat pole shoes



field map of a storage ring dipole magnet



s.c. LHC dipole

Normalise magnetic field to momentum:

$$\frac{p}{e} = B\,\rho \quad \longrightarrow \quad \frac{1}{\rho} = \frac{e\,B}{p}$$

convenient units:

$$B = [T] = \left[\frac{Vs}{m^2}\right] \qquad p = \left[\frac{GeV}{c}\right]$$

$B \approx 1 \ldots 8\ T$

Example LHC:

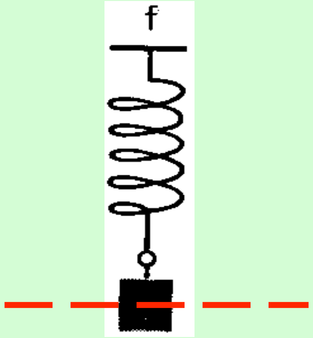
$$\left.\begin{aligned} B &= 8.3\,T \\ p &= 7000\,\frac{GeV}{c} \end{aligned}\right\}$$

$$\frac{1}{\rho} = e\,\frac{8.3\,{Vs}/{m^2}}{7000*10^9\,{eV}/{c}} = \frac{8.3\,s\,3*10^8\,{m}/{s}}{7000*10^9\,m^2}$$

$\rho = 2.81\ km$

Focusing Properties and Quadrupole Magnets

classical mechanics:
pendulum



there is a restoring force, proportional to the elongation x:

$$m * \frac{d^2 x}{dt^2} = -c * x$$

general solution: free harmonic oszillation

$$x(t) = A * \cos(\omega t + \varphi)$$

this is how grandma‘s Kuckuck‘s clock is working!!!

Storage Rings: *linear increasing Lorentz force to keep trajectories in vicinity of the ideal orbit*

linear increasing magnetic field

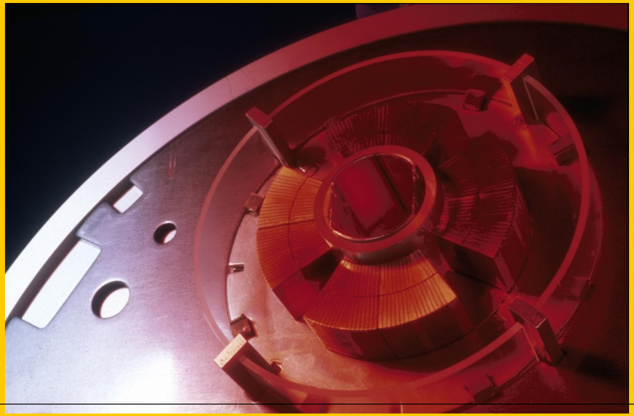
$$B_y = g\ x \qquad B_x = g\ y$$

$$F(x) = q * v * B(x)$$

as in the dipole case we normalise to the beam rigidity

$$k = \frac{g}{B\rho} = \frac{g}{p/q}$$

LHC main quadrupole magnet $g \approx 25 \ldots 220\ T/m$



4.) The equation of motion:

Linear approximation:

* ideal particle → design orbit

* any other particle → coordinates x, y small quantities
$x,y \ll \rho$

→ magnetic guide field: only linear terms in x & y of B have to be taken into account

Taylor Expansion of the B field ...

normalised to momentum $p/e = B\rho$
and only terms linear in x, y taken into account
dipole fields / quadrupole fields

$$\frac{B(x)}{p/e} = \frac{B_0}{B_0\rho} + \frac{g*x}{p/e} + \cancel{\frac{1}{2!}\frac{eg'}{p/e}} + \cancel{\frac{1}{3!}\frac{eg''}{p/e}} + \dots$$

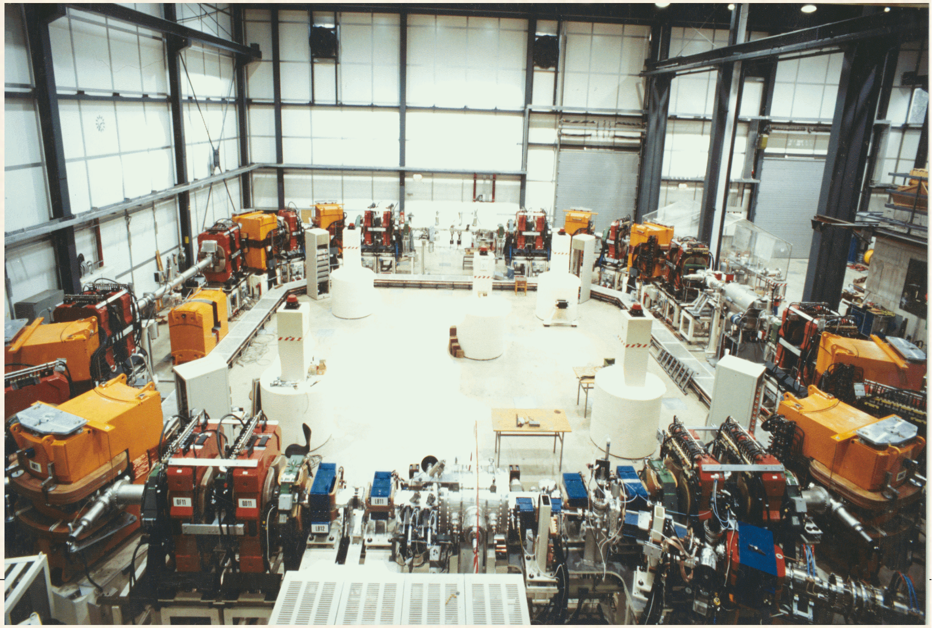
$$= \frac{1}{\rho} + k*x$$

Separate Function Machines:

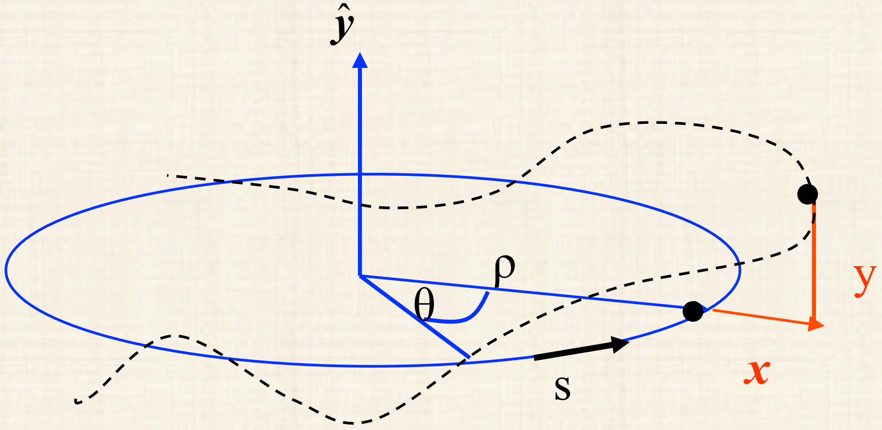
Split the magnets and optimise them according to their job:

bending, focusing etc

Example:
heavy ion storage ring TSR



Equation of Motion:



Remember:

Hamiltonian for ideal particle, δ = 0

$$H = \frac{p_x^2 + p_y^2}{2} - \frac{x^2}{2\rho(s)^2} + \frac{k_1(s)}{2}(x^2 - y^2)$$

with k and ρ representing the normalised quadrupole and dipole fields

putting into Hamiltonian equations

$$\frac{\partial H}{\partial x} = \frac{-dp_x}{ds}, \qquad \frac{\partial H}{\partial p_x} = \frac{dx}{ds}$$

... see e.g. Goldstein p 241

we get the equation of motion

$$\frac{d^2x}{ds^2} + \left\{\frac{1}{\rho(s)^2} - k_1(s)\right\} * x = 0, \qquad \frac{d^2y}{ds^2} + k_1(s) * y = 0$$

Equation of Motion:

*In linear aproximation (x, y << ρ and only dipole & quadruple fields)
we can derive a differential equation for the transverse motion of the particles*

- *Equation for the* ***horizontal motion:***

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = 0$$

Under the influence of the focusing fields from the quadrupoles „k" and dipoles 1/ ρ^2 the transverse movement of the particles inside looks like a harmonic oscillation

- *Equation for the* ***vertical motion:***

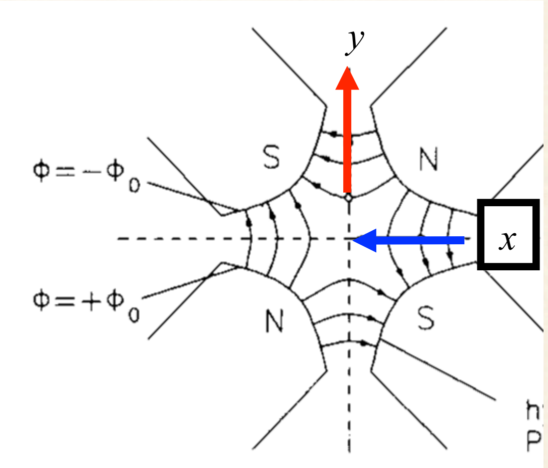
$$\frac{1}{\rho^2} = 0$$

no dipoles … in general …

$$k \quad \leftrightarrow \quad -k$$

quadrupole field changes sign

$$y'' + k\ y = 0$$



... mmmppfff ... just another differential equation .... but it does not look sooo comfortable.

5.) Solution of Trajectory Equations

$$\left.\begin{array}{ll}\textit{Define … hor. plane:} & K = 1/\rho^2 - k \\ \textit{… vert. Plane:} & K = k\end{array}\right\} \qquad \boxed{x'' + K\ x = 0}$$

Differential Equation of harmonic oscillator … with spring constant K > 0 → focusing case

Ansatz: $x(s) = a_1 \cdot \cos(\omega s) + a_2 \cdot \sin(\omega s)$

general solution: linear combination of two independent solutions

$$x'(s) = -a_1\omega \sin(\omega\, s) + a_2\omega \cos(\omega s)$$

$$x''(s) = -a_1\omega^2 \cos(\omega s) - a_2\omega^2 \sin(\omega s) = -\omega^2 x(s) \longrightarrow \omega = \sqrt{K}$$

general solution:

$$x(s) = a_1 \cos(\sqrt{K}s) + a_2 \sin(\sqrt{K}s)$$

determine a_1, a_2 by boundary conditions:

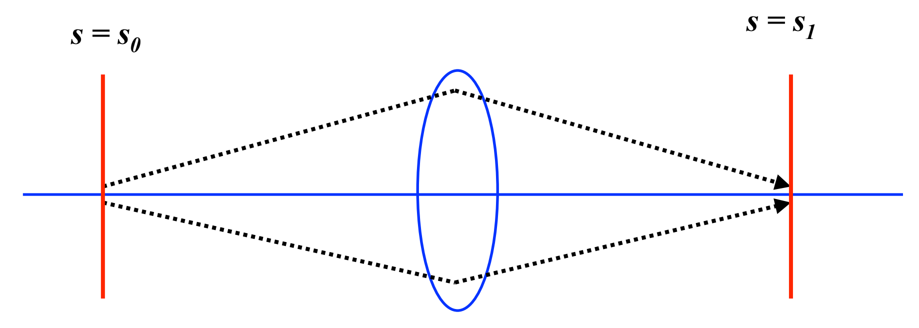
$$s = 0 \quad \longrightarrow \quad \left\{ \begin{array}{ll} x(0) = x_0 \quad , & a_1 = x_0 \\ x'(0) = x'_0 \quad , & a_2 = \dfrac{x'_0}{\sqrt{K}} \end{array} \right.$$

Hor. Focusing Quadrupole $K > 0$:

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x'_0 \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x'_0 \cdot \cos(\sqrt{|K|}s)$$

For convenience expressed in matrix formalism:

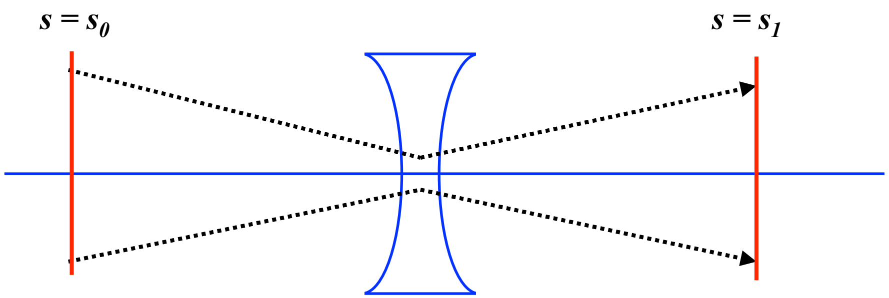


$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \dfrac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s \\ -\sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

hor. defocusing quadrupole:

$$x'' - K\ x = 0$$



Remember from school:

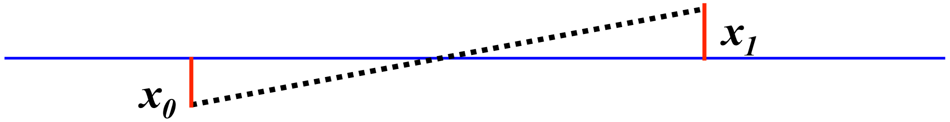
$$f(s) = \cosh(s) \quad , \qquad f'(s) = \sinh(s)$$

Ansatz: $$x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s)$$

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}l \\ \sqrt{|K|}\sinh\sqrt{|K|}l & \cosh\sqrt{|K|}l \end{pmatrix}$$

drift space:

K = 0



$$x_1 = x_0 + x_0' * l$$

$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

Combining the two planes:

Clear enough (hopefully ... ?): a quadrupole magnet that is focussing in one plane acts as defocusing lens in the other plane ... et vice versa.

hor foc. quadrupole lens

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}l \\ -\sqrt{|K|}\sin(\sqrt{|K|}l) & \cos(\sqrt{|K|}l) \end{pmatrix}_{f}$$

matrix of the same magnet in the vert. plane:

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}l \\ \sqrt{|K|}\sinh\sqrt{|K|}l & \cosh\sqrt{|K|}l \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{f} = \begin{pmatrix} \cos(\sqrt{|k|}l) & \frac{1}{\sqrt{|k|}}\sin(\sqrt{|k|}l) & 0 & 0 \\ -\sqrt{|k|}\sin(\sqrt{|k|}l) & \cos(\sqrt{|k|}l) & 0 & 0 \\ 0 & 0 & \cosh(\sqrt{|k|}l) & \frac{1}{\sqrt{|k|}}\sinh(\sqrt{|k|}l) \\ 0 & 0 & \sqrt{|k|}\sinh(\sqrt{|k|}l) & \cosh(\sqrt{|k|}l) \end{pmatrix} * \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{i}$$

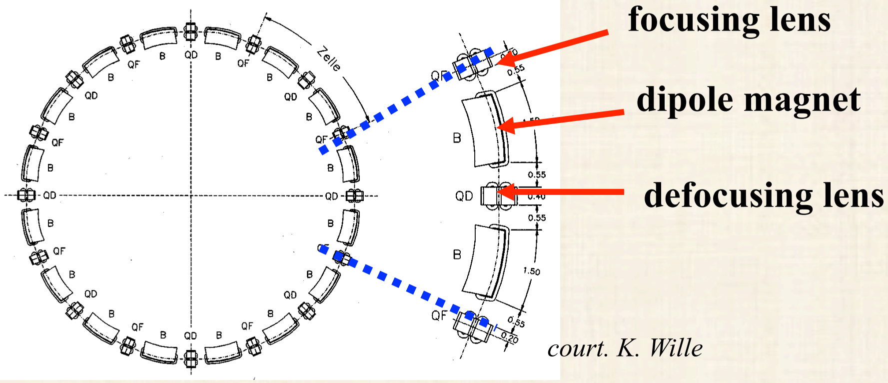
! with the assumptions made, the motion in the horizontal and vertical planes are independent „ ... the particle motion in x & y is uncoupled" !

Transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D^*}.....$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2,s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



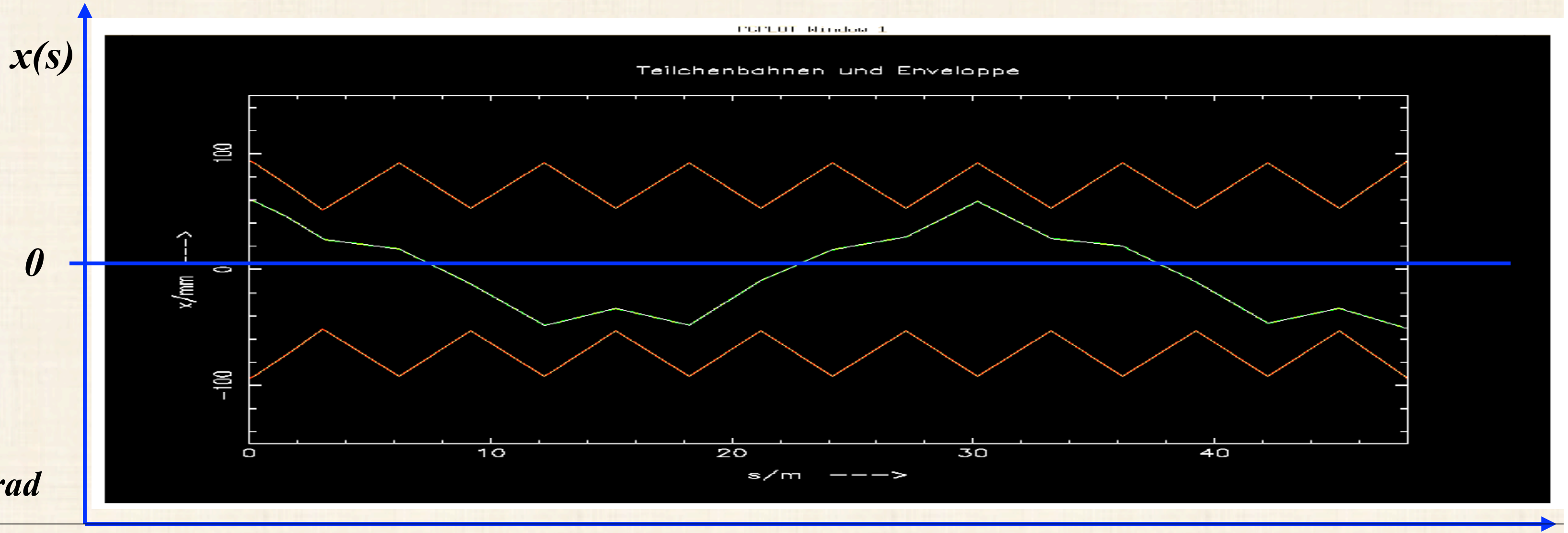
focusing lens

dipole magnet

defocusing lens

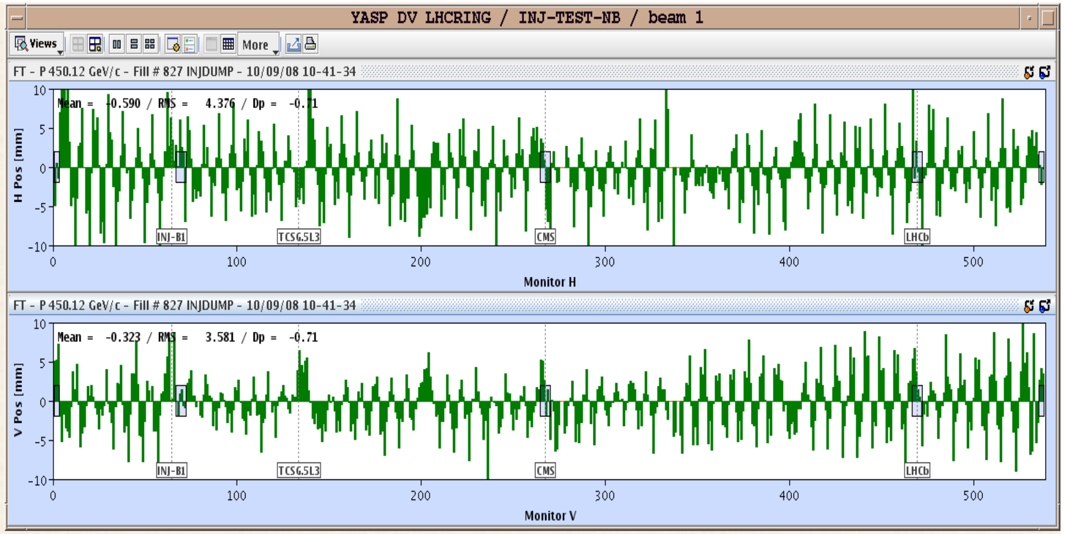
court. K. Wille

in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator „



typical values
in a strong
foc. machine:
x ≈ mm, x´ ≤ mrad

6.) Orbit & Tune:



Tune: number of oscillations per turn

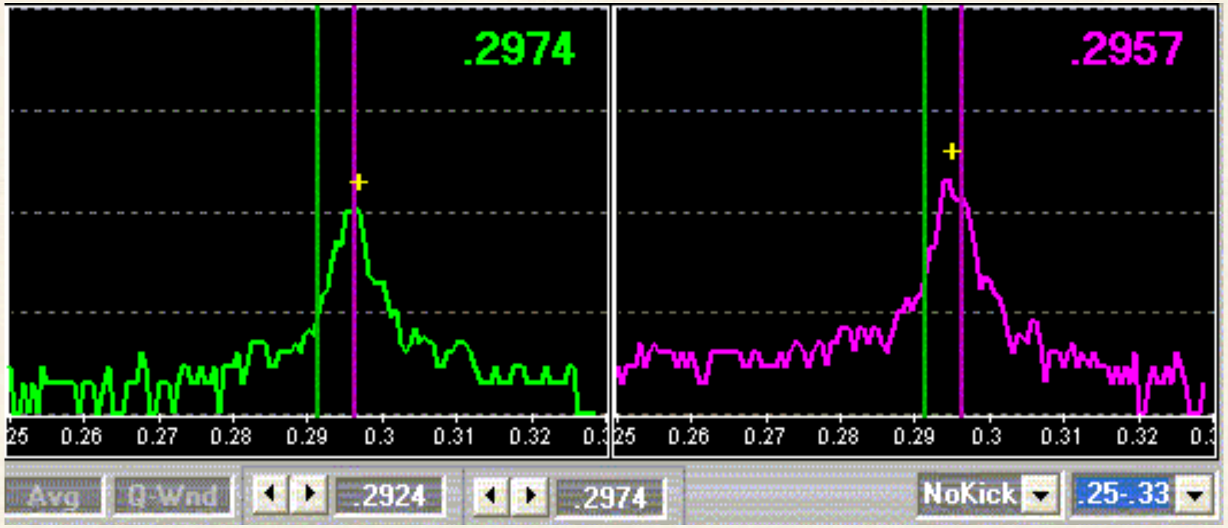
64.31
59.32

Relevant for beam stability:

non integer part

LHC revolution frequency: 11.3 kHz

$$f_q = 0.31 * 11.3 = 3.5 kHz$$

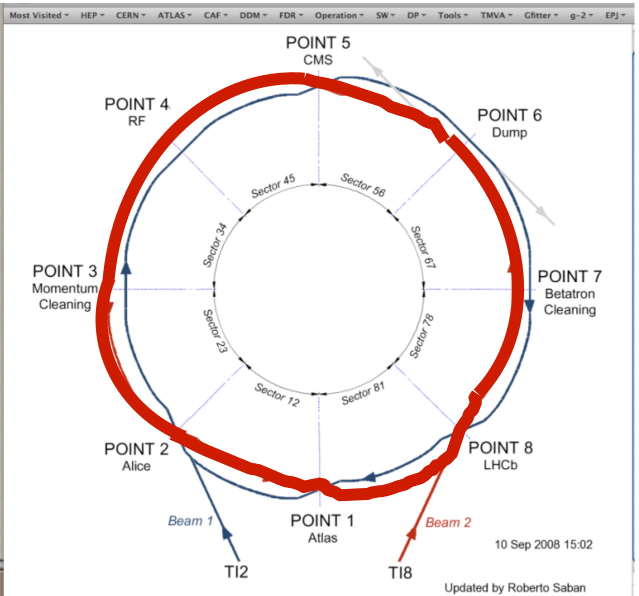


LHC Operation: Beam Commissioning

First turn steering "by sector:"

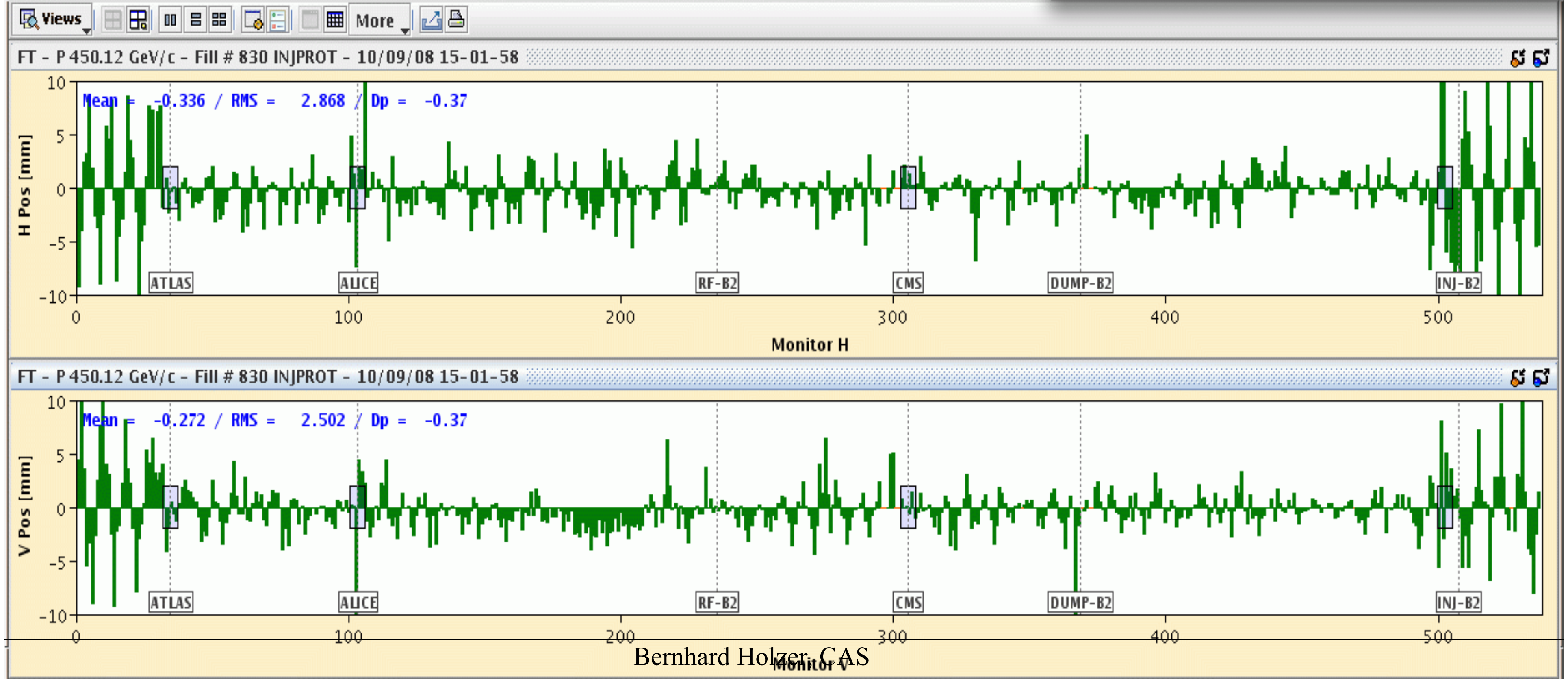
- ❑One beam at the time
- ❑Beam through 1 sector (1/8 ring),

correct trajectory, open collimator and move on.



10 Sep 2008 15:02

Updated by Roberto Saban



YASP DV LHCRING / INJ-TEST-NB / beam 2

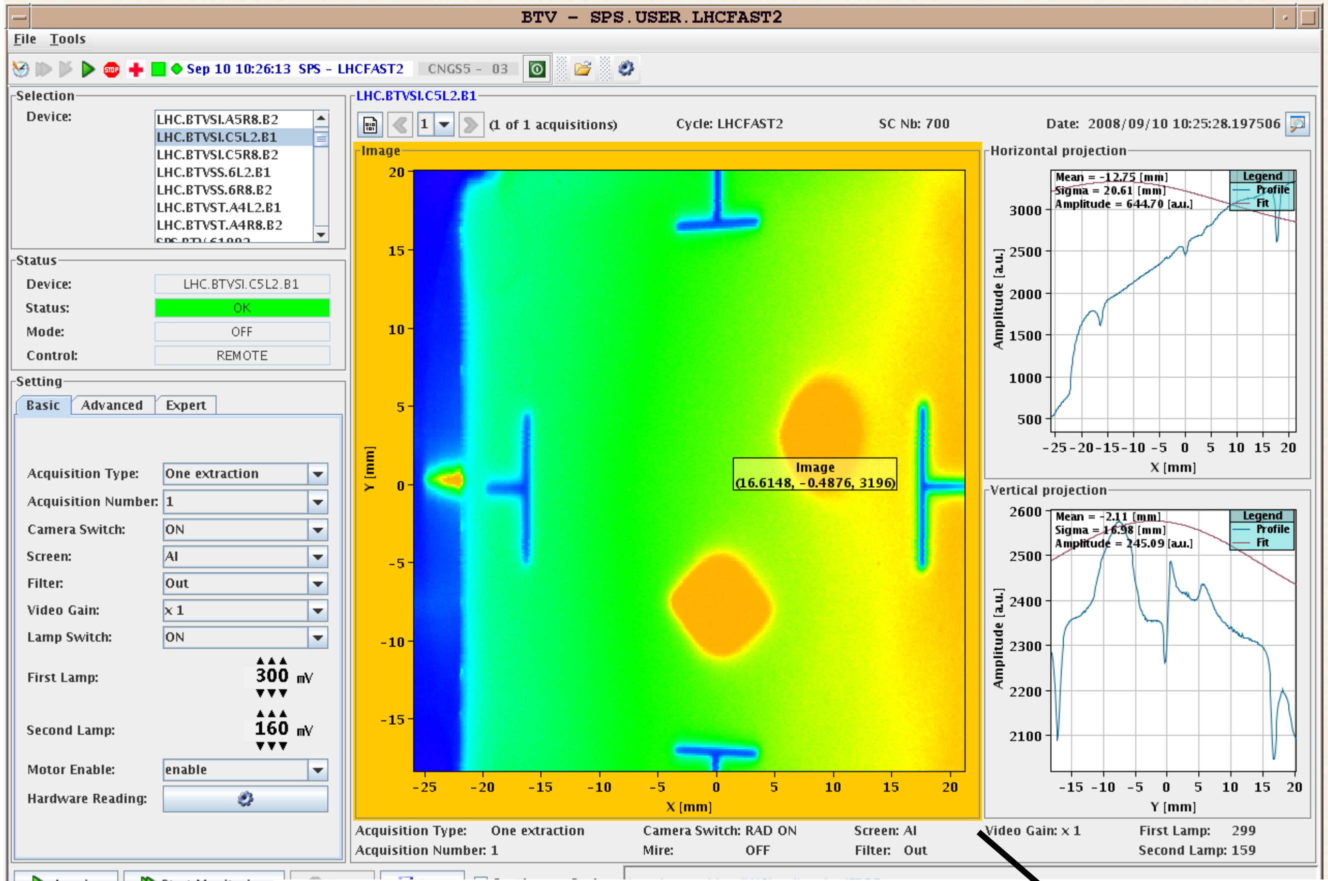
FT - P 450.12 GeV/c - Fill # 830 INJPROT - 10/09/08 15-01-58

Mean = -0.336 / RMS = 2.868 / Dp = -0.37

FT - P 450.12 GeV/c - Fill # 830 INJPROT - 10/09/08 15-01-58

Mean = -0.272 / RMS = 2.502 / Dp = -0.37

LHC Operation: the First Beam



BTV – SPS.USER.LHCFAST2

Sep 10 10:26:13 SPS - LHCFAST2 CNGS5 - 03

Selection

Device:
- LHC.BTVSI.A5R8.B2
- LHC.BTVSI.C5L2.B1
- LHC.BTVSI.C5R8.B2
- LHC.BTVSS.6L2.B1
- LHC.BTVSS.6R8.B2
- LHC.BTVST.A4L2.B1
- LHC.BTVST.A4R8.B2

Status

Device: LHC.BTVSI.C5L2.B1
Status: OK
Mode: OFF
Control: REMOTE

Setting

Acquisition Type: One extraction
Acquisition Number: 1
Camera Switch: ON
Screen: Al
Filter: Out
Video Gain: x 1
Lamp Switch: ON
First Lamp: 300 mV
Second Lamp: 160 mV
Motor Enable: enable
Hardware Reading:

LHC.BTVSI.C5L2.B1

(1 of 1 acquisitions) Cycle: LHCFAST2 SC Nb: 700 Date: 2008/09/10 10:25:28.197506

Image (16.6148, -0.4876, 3196)

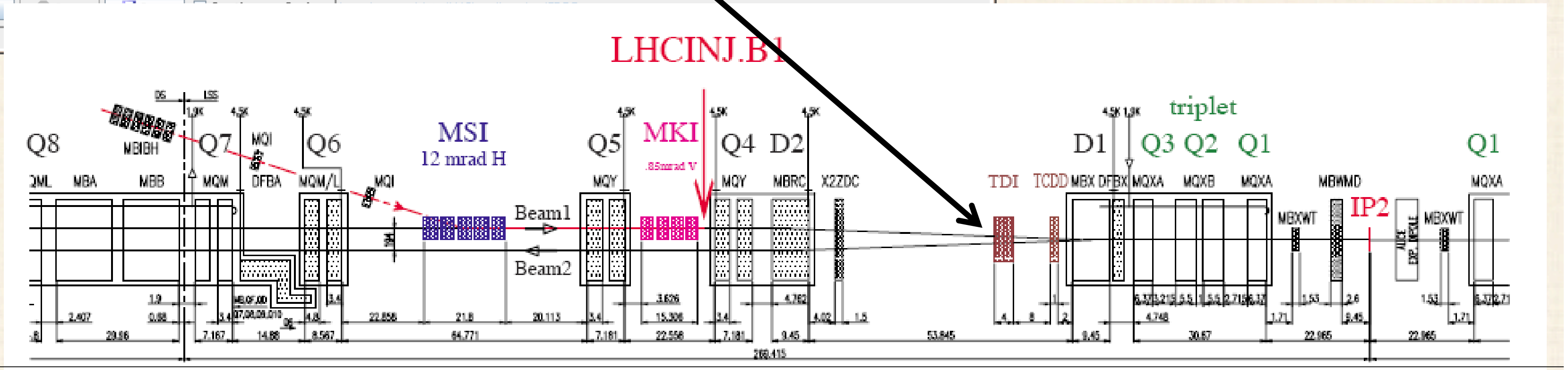
Horizontal projection: Mean = -12.75 [mm], Sigma = 20.61 [mm], Amplitude = 644.70 [a.u.]

Vertical projection: Mean = -2.11 [mm], Sigma = 16.98 [mm], Amplitude = 245.09 [a.u.]

Acquisition Type: One extraction
Acquisition Number: 1
Camera Switch: RAD ON
Mire: OFF
Screen: Al
Filter: Out
Video Gain: x 1
First Lamp: 299
Second Lamp: 159

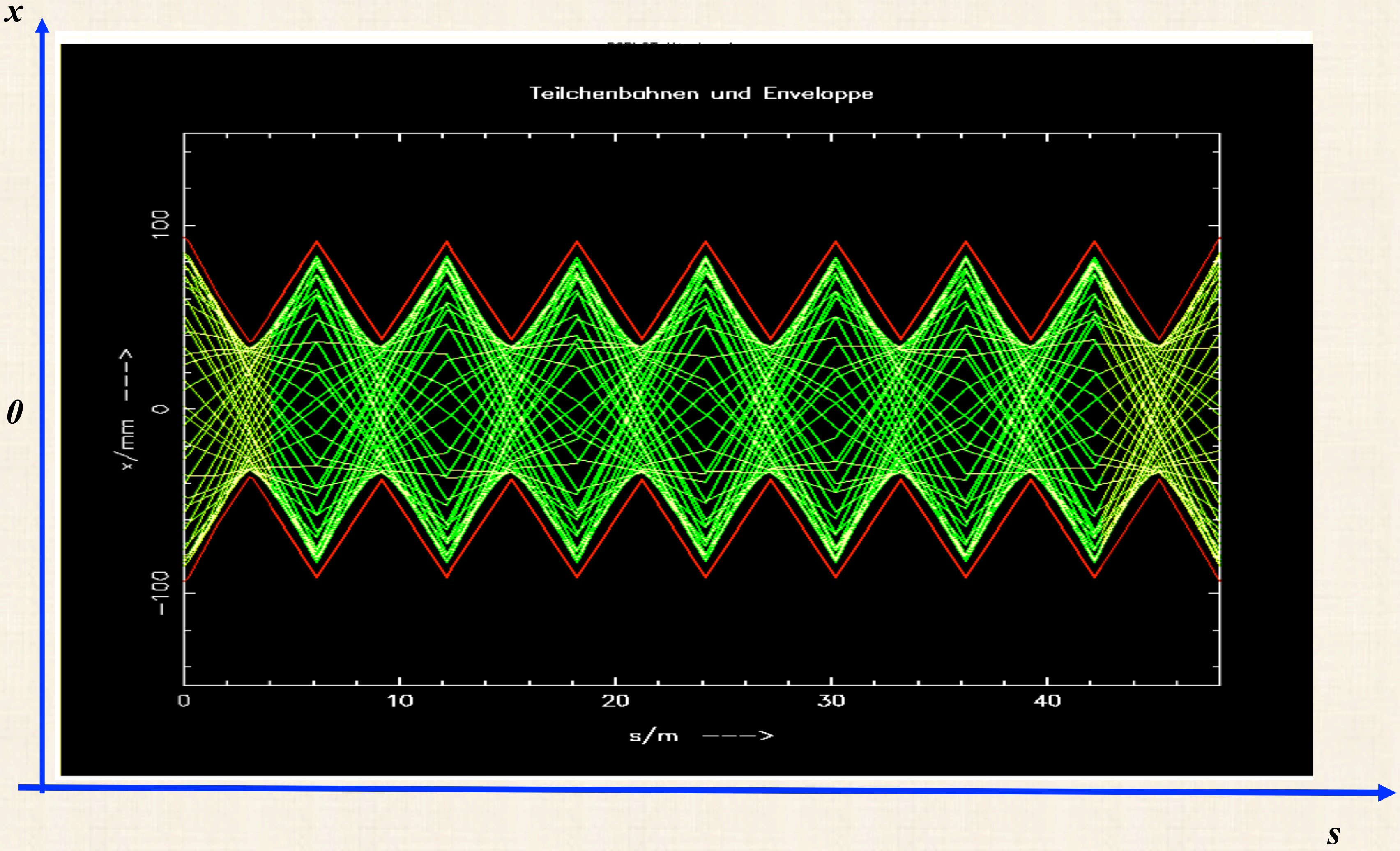
10:25:32 - Done.

Beam 1 on OTR screen
1st and 2nd turn



Question: what will happen, if the particle performs a second turn ?

... or a third one or ... 10^{10} turns



7.) The Beta Function

General solution of Hill´s equation:

(i) $$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$$

ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s+L) = \beta(s)$$

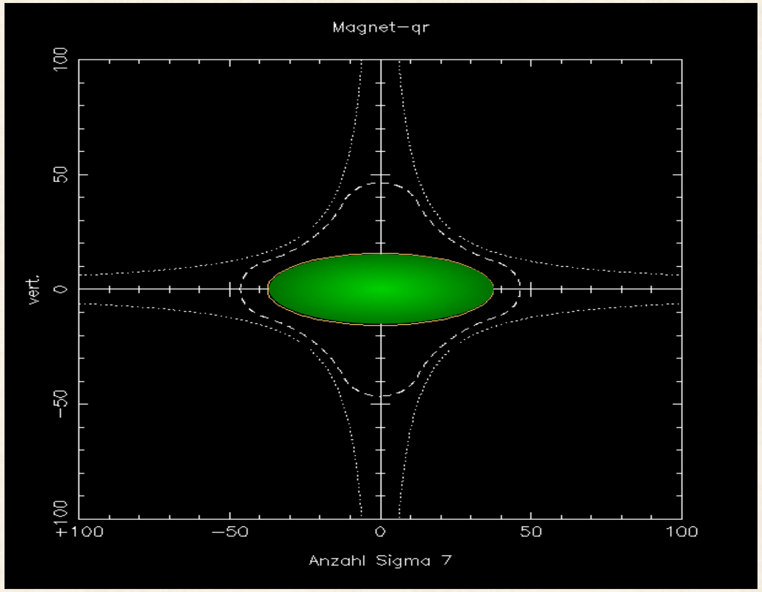
Inserting (i) into the equation of motion …

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

Ψ(s) = „phase advance“ of the oscillation between point „0“ and „s“ in the lattice.
For one complete revolution: number of oscillations per turn „Tune“

$$Q_y = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$$

The Beta Function

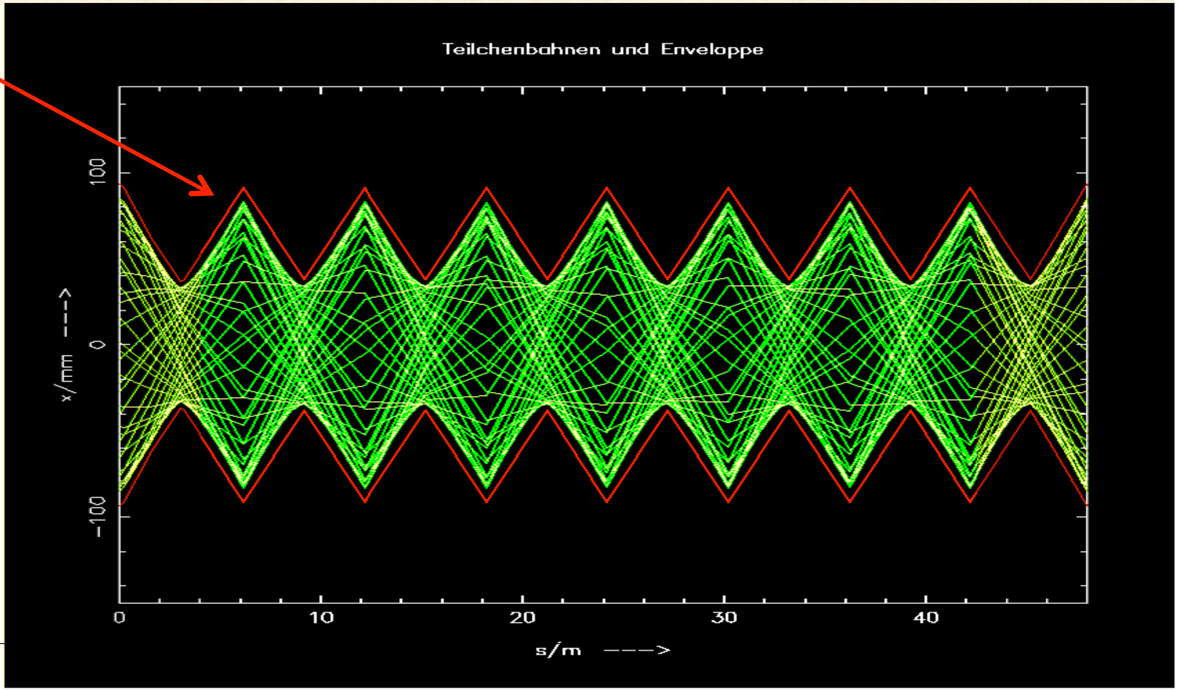


Amplitude of a particle trajectory:

$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \varphi)$$

Maximum size of a particle amplitude

$$\hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



β determines the beam size
(... the envelope of all particle trajectories at a given position "s" in the storage ring.

It reflects the periodicity of the magnet structure.

8.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

$$\begin{cases} (1) \quad x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\phi) \\ (2) \quad x'(s) = -\dfrac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\,\{\alpha(s)\cos(\psi(s)+\phi)+\sin(\psi(s)+\phi)\} \end{cases}$$

from (1) we get

$$\cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\varepsilon}\,\sqrt{\beta(s)}}$$

Insert into (2) and solve for ε

using the parameter definitions

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

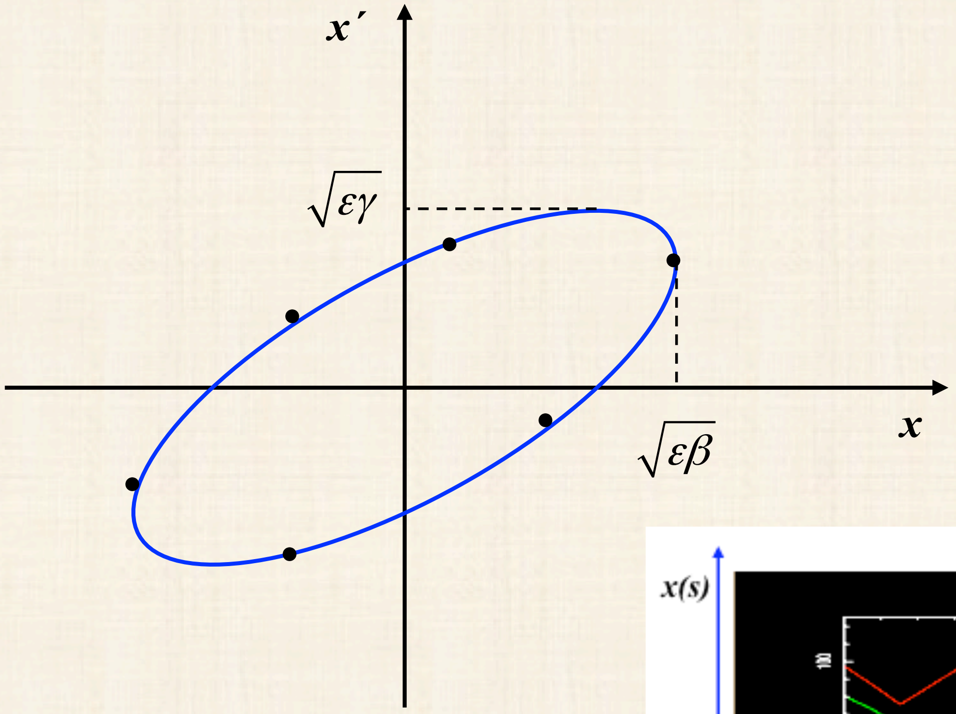
*** ε is a constant of the motion … it is independent of „s“**

*** parametric representation of an ellipse in the x x ‘ space**

*** shape and orientation of ellipse are given by α, β, γ**

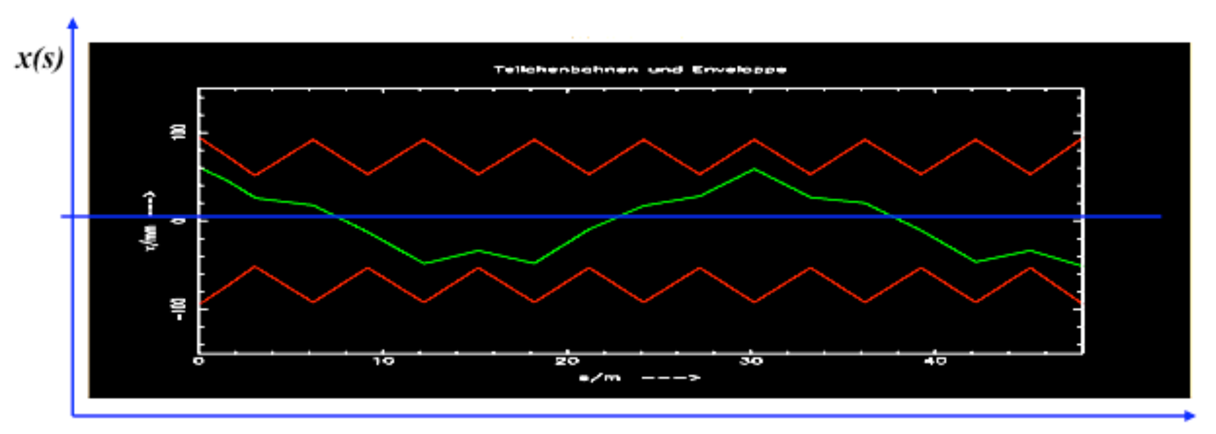
Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$



Liouville: ***in reasonable storage rings area in phase space is constant.***

$$A = \pi^*\varepsilon = const$$

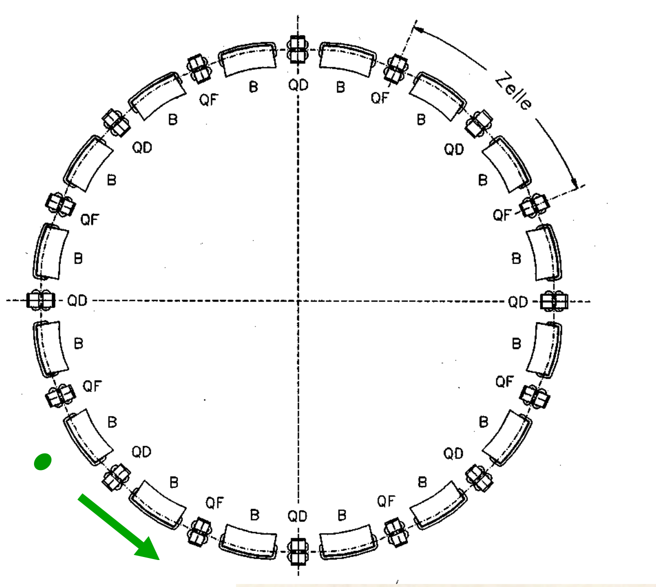


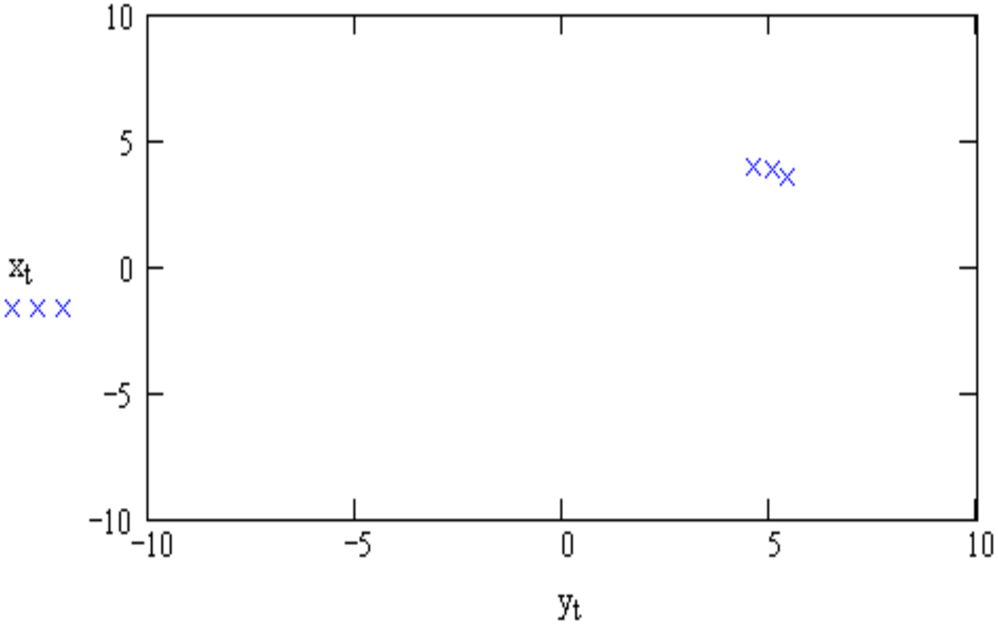
ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter,
cannot be changed by the foc. properties.
Scientifiquely speaking: it is the area covered in transverse x, x´ phase space … and it is constant !!!

Particle Tracking in a Storage Ring

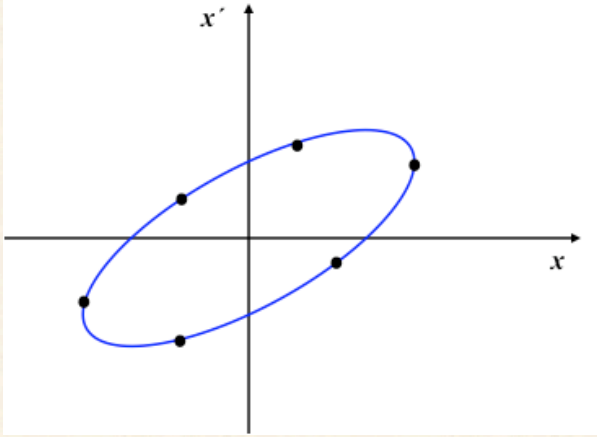
Calculate x, x´ for each linear accelerator element according to matrix formalism

plot x, x´as a function of „s"





Phase Space Ellipse



particel trajectory: $x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$

max. Amplitude: $\hat{x}(s) = \sqrt{\varepsilon\beta}$ → ***determine x´ at that position …***

… put $\hat{x}(s)$ ***into*** $\boldsymbol{\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)}$ ***and solve for x´***

$$\varepsilon = \gamma \cdot \varepsilon\beta + 2\alpha\sqrt{\varepsilon\beta} \cdot x' + \beta x'^2$$

$$\longrightarrow \quad x' = -\alpha \cdot \sqrt{\varepsilon / \beta}$$

* ***A high β-function means a large beam size and a small beam divergence.*** **!**
… et vice versa !!!

* ***In the middle of a quadrupole β = maximum, α = zero*** $\Big\}$ $x' = 0$

… and the ellipse is flat

Phase Space Ellipse

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\, x'^2(s)$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

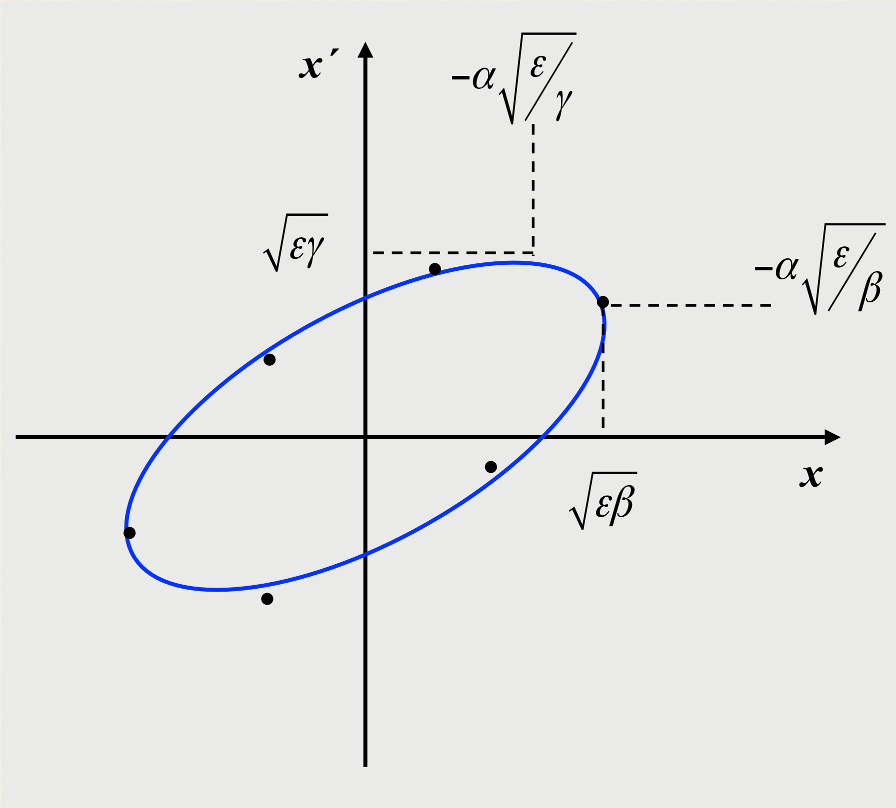
$$\longrightarrow \quad \varepsilon = \frac{x^2}{\beta} + \frac{\alpha^2 x^2}{\beta} + 2\alpha \cdot xx' + \beta \cdot x'^2$$

... solve for x´ $\quad x'_{1,2} = \dfrac{-\alpha \cdot x \pm \sqrt{\varepsilon\beta - x^2}}{\beta}$

... and determine $\hat{x}'$ *via:* $\quad \dfrac{dx'}{dx} = 0$

$$\longrightarrow \quad \hat{x}' = \sqrt{\varepsilon\gamma}$$

$$\longrightarrow \quad \hat{x} = \pm\alpha\sqrt{\varepsilon/\gamma}$$

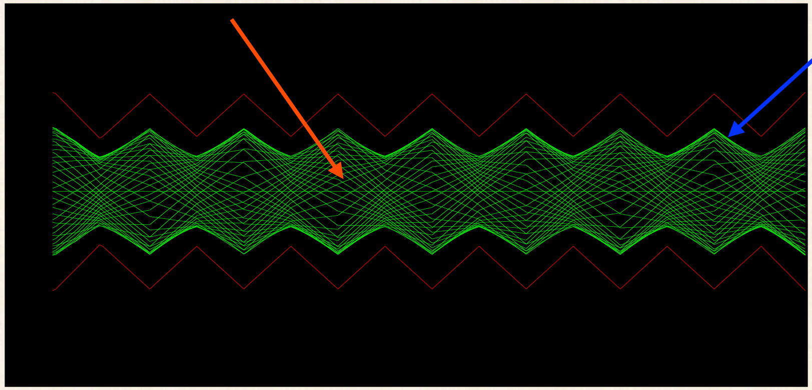


shape and orientation of the phase space ellipse depend on the Twiss parameters β α γ

Emittance of the Particle Ensemble:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cdot\cos(\Psi(s)+\phi)$$

$$\hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



single particle trajectories, N ≈ 10 ^{11} per bunch

Gauß
Particle Distribution:

$$\rho(x) = \frac{N\cdot e}{\sqrt{2\pi}\sigma_x}\cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x{}^2}}$$

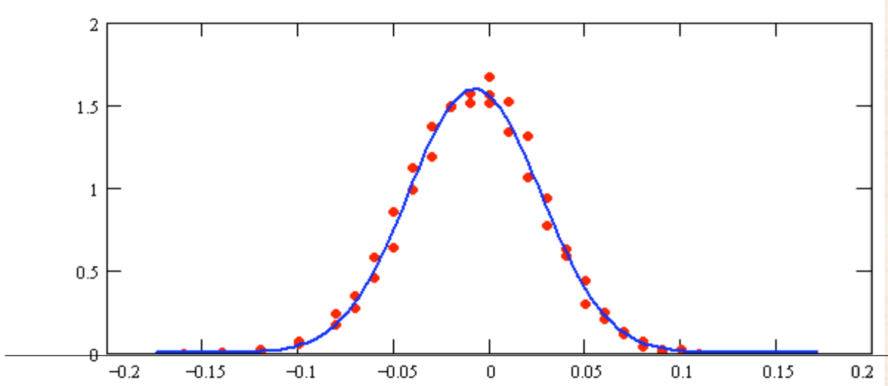
particle at distance 1 σ from centre
↔ 68.3 % of all beam particles

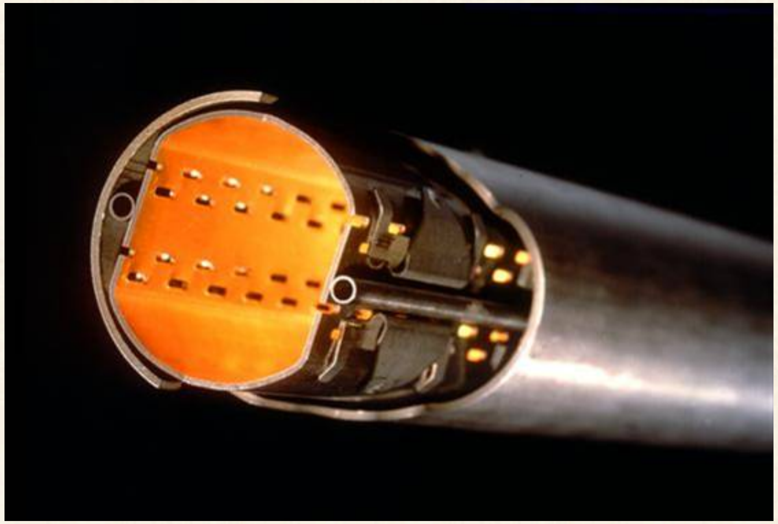
LHC:

$$\beta = 180\,m$$

$$\varepsilon = 5*10^{-10}\,m\,rad$$

$$\sigma = \sqrt{\varepsilon*\beta} = \sqrt{5*10^{-10}\,m*180\,m} = 0.3\,mm$$

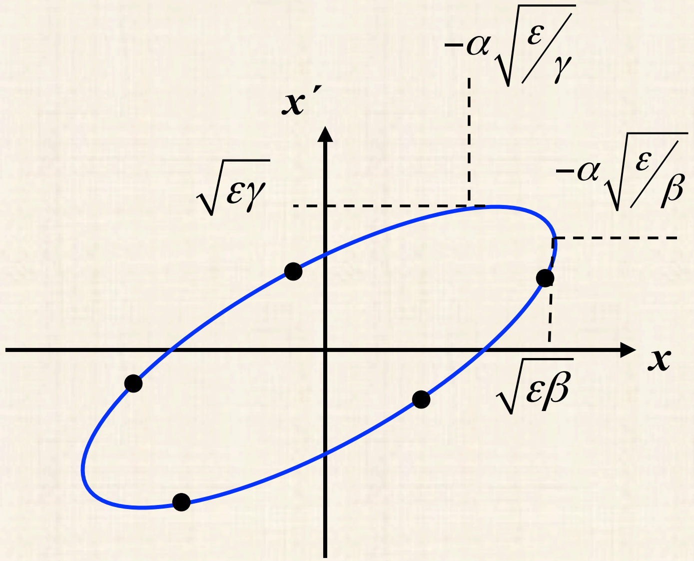




aperture requirements: r_0 = 12 * σ

13.) Liouville during Acceleration

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$



***Beam Emittance* corresponds to the area covered in the x, x´ Phase Space Ellipse**

***Liouville:* Area in phase space is constant.**

But so sorry ... $\varepsilon \neq$ const !

Classical Mechanics:

***phase space* = diagram of the two canonical variables**

position* & *momentum

x p_x

According to Hamiltonian mechanics:
phase space diagram relates the variables x and p_x

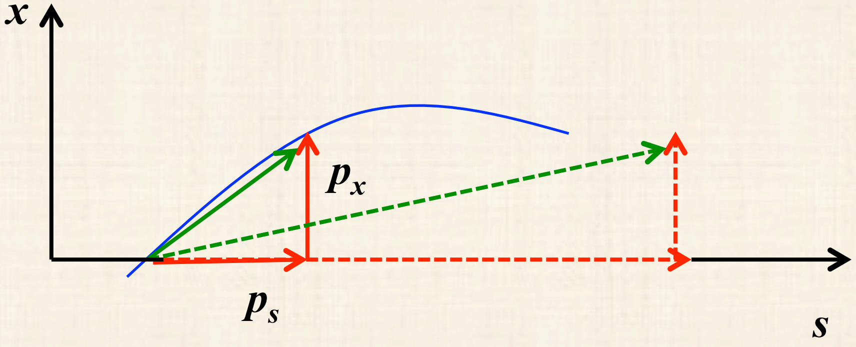
Liouvilles Theorem: $$\int p_x\, dx = const$$

for convenience (i.e. because we are lazy bones) we use in accelerator theory x ‘ instead of p_x

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta} = \frac{p_x}{p}$$ ***where p ~ p_s***

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}$$

$$\underbrace{\int x'\, dx}_{\varepsilon} = \frac{\int p_x\, dx}{p} \propto \frac{const}{m_0 c \cdot \gamma\beta}$$



$$\Rightarrow \quad \varepsilon = \int x' dx \propto \frac{1}{\beta\gamma}$$

the beam emittance shrinks during acceleration
$\varepsilon \sim 1 / \gamma$

Nota bene:

1.) A proton machine … or an electron linac … needs the highest aperture at injection energy !!!
as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.

$$\sigma = \sqrt{\varepsilon\beta}$$

2.) To confuse the students we introduce often a "normalized" emittance ε_n
… which is energy independent

$$\varepsilon_n = \varepsilon_0 * \beta\gamma$$

Example: HERA proton ring

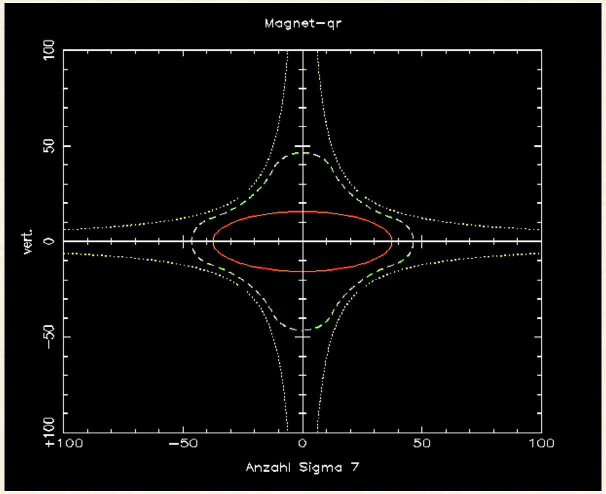
$\varepsilon_n = 5.0 * 10^{-6}$ *mrad*

injection energy: 40 GeV $\gamma = 43$

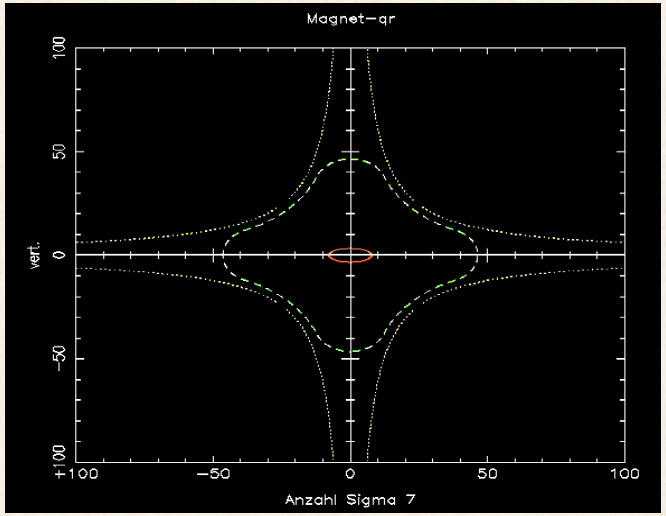
ε_0 *(40GeV)* $= 1.2 * 10^{-7}$ *mrad*

flat top energy: 920 GeV $\gamma = 980$

ε_0 *(920GeV)* $= 5.1 * 10^{-9}$ *mrad*



7 σ beam envelope at E = 40 GeV



… and at E = 920 GeV

11.) Résumé

1.) Beam rigidity

$$\frac{p}{e} = B\rho$$

2.) Equation of motion

$$\boldsymbol{x}'' + \boldsymbol{x}\,(\frac{1}{\boldsymbol{\rho}^2} - \boldsymbol{k}) = 0 \qquad y'' + k\,y = 0$$

3.) Transfer matrix foc. quadrupole

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}s \\ -\sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

defoc. quadrupole

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}l \\ \sqrt{|K|}\sinh\sqrt{|K|}l & \cosh\sqrt{|K|}l \end{pmatrix}$$

drift

$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

4.) general solution of Hill's equation

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cdot\cos(\psi(s)+\phi)$$

5.) Tune

$$Q_y = \frac{1}{2\pi}\oint\frac{ds}{\beta(s)}$$

6.) Emittance as phase space ellipse

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

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