The CERN Accelerator School

Advanced Accelerator Physics Course

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Low Emittance Machines

Part 1: Beam Dynamics with Synchrotron Radiation

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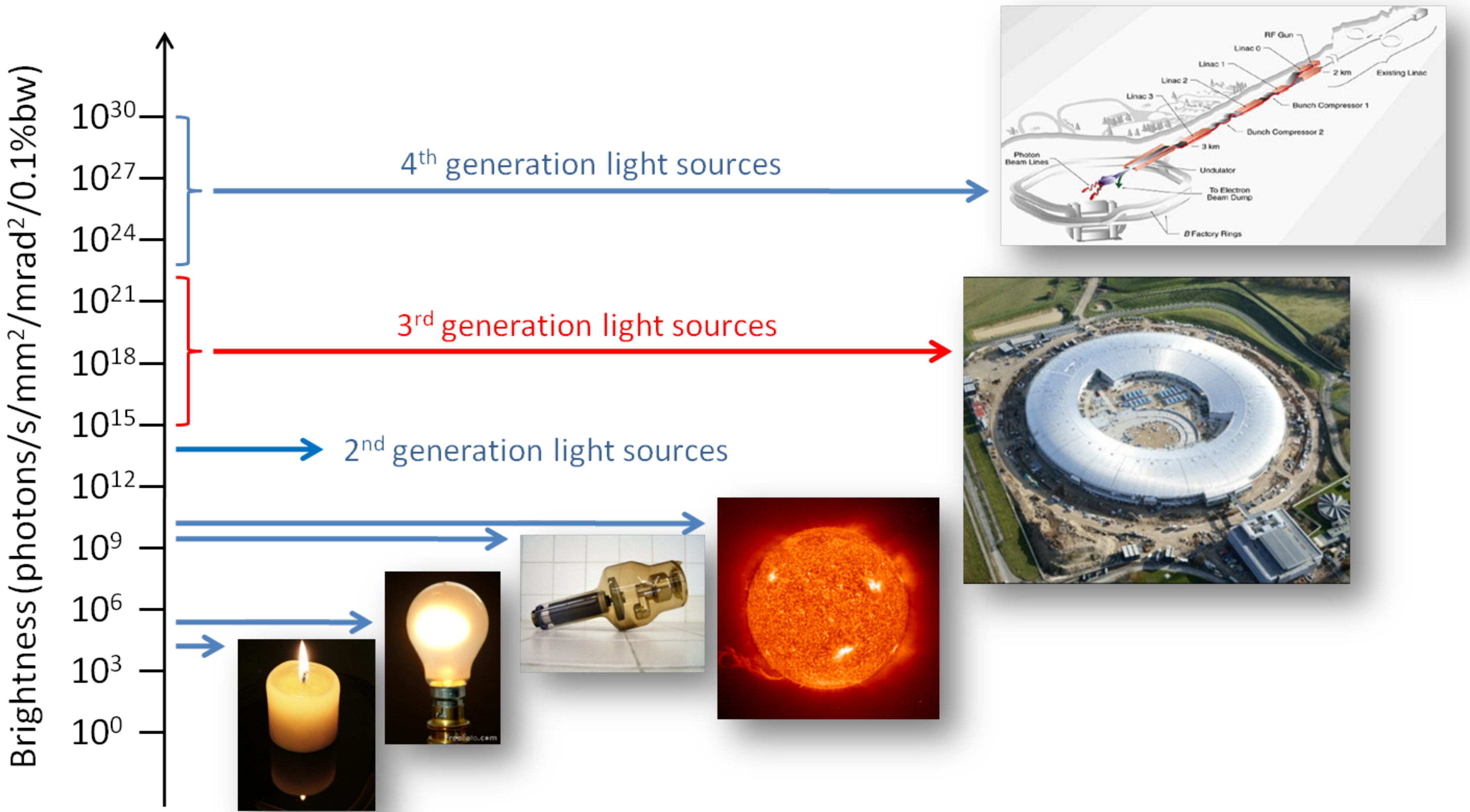
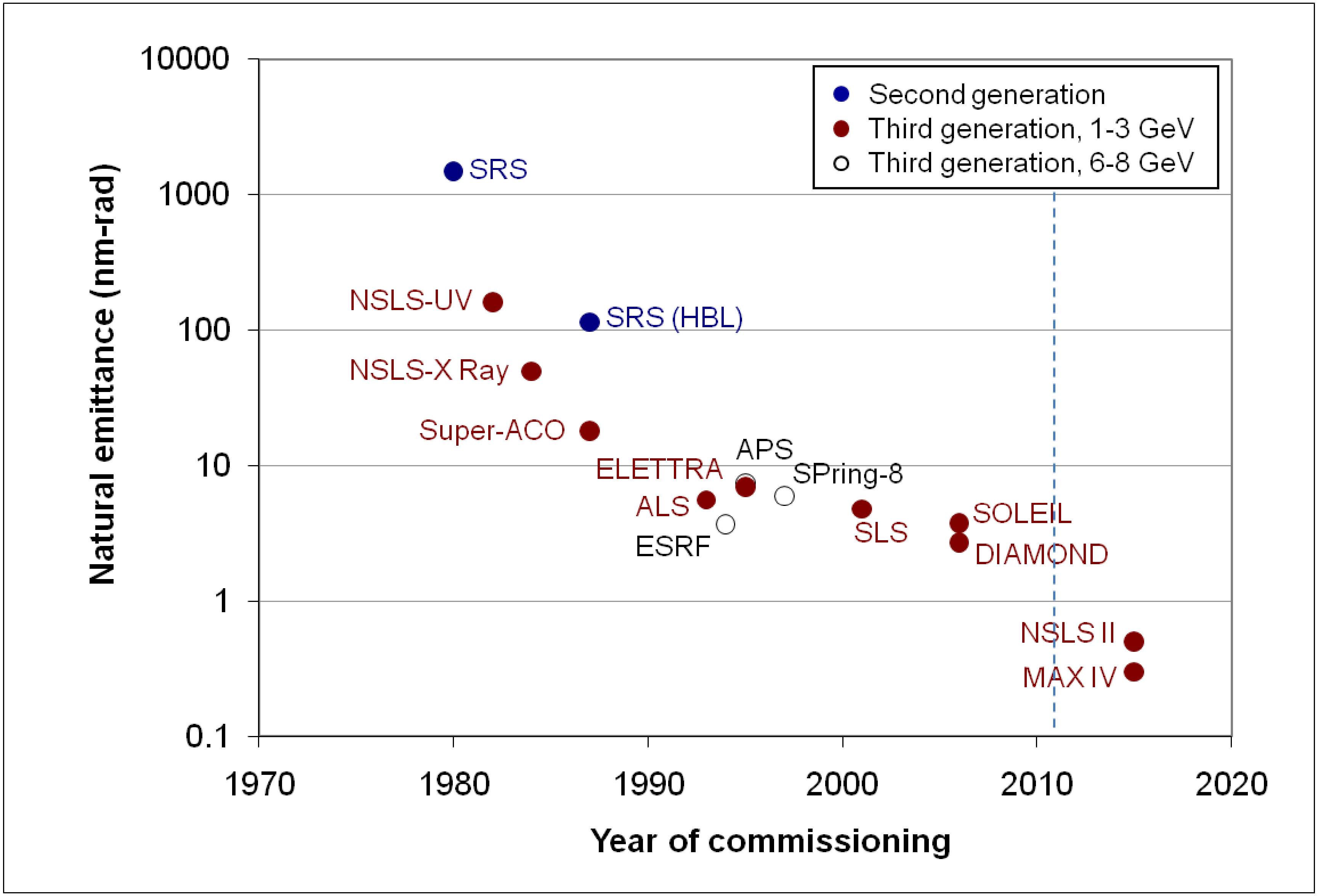
Brightness is a key figure of merit for SR sources



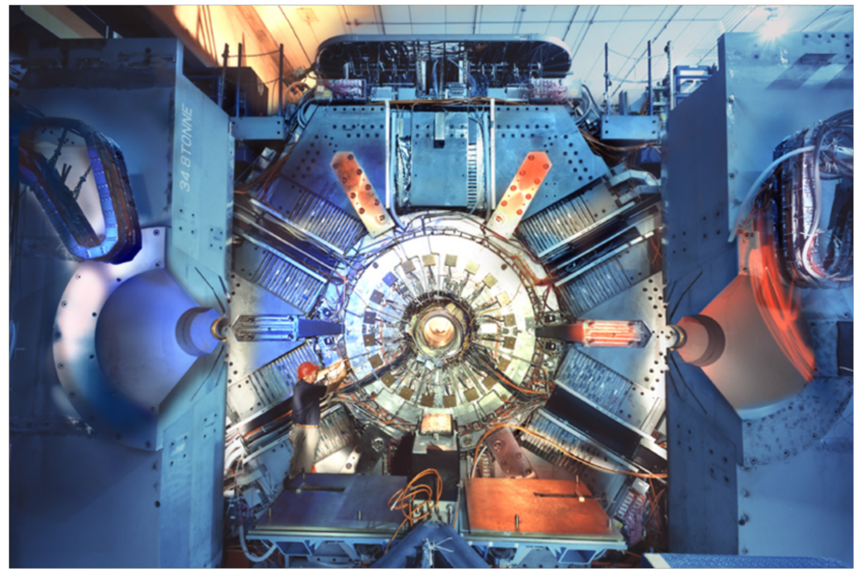
High photon brightness needs low electron beam emittance



Luminosity is a key figure of merit for colliders. The luminosity depends directly on the horizontal and vertical emittances.

$$\mathcal{L} = \frac{N_+ N_- f}{2\pi \Sigma_x \Sigma_y}$$

$$\Sigma_{x,y} = \sqrt{\sigma^{*2}_{x,y+} + \sigma^{*2}_{x,y-}}$$



Dynamical effects associated with the collisions mean that it is sometimes helpful to *increase* the horizontal emittance; but generally, reducing the vertical emittance as far as possible helps to increase the luminosity.

In these two lectures we shall discuss the effects of synchrotron radiation on beams in lepton (electron or positron) storage rings, with particular attention to low-emittance rings.

1. Beam dynamics with synchrotron radiation
   - A classical model of synchrotron radiation describes the "damping" of synchrotron and betatron oscillations.
   - The emission of radiation in discrete quanta (photons) leads to "quantum excitation" of synchrotron and betatron oscillations.
   - The balance between damping and excitation leads to equilibrium longitudinal and transverse emittances.
2. Equilibrium emittance and storage ring lattice design
   - The equilibrium longitudinal and horizontal emittances are determined by the design of the storage ring lattice.
   - In a planar storage ring, the vertical emittance is dominated by betatron coupling and vertical dispersion (from alignment and tuning errors).

Lecture 1 objectives: linear dynamics with synchrotron radiation

In this lecture, we shall:

- describe the damping of synchrotron and betatron oscillations by the emission of electromagnetic radiation;

- discuss how quantum excitation leads to equilibrium values for the longitudinal and transverse beam emittances;

- give expressions for the damping times and equilibrium emittances in terms of the *synchrotron radiation integrals*.

Our first goal is to understand how synchrotron radiation leads to the damping of synchrotron oscillations.

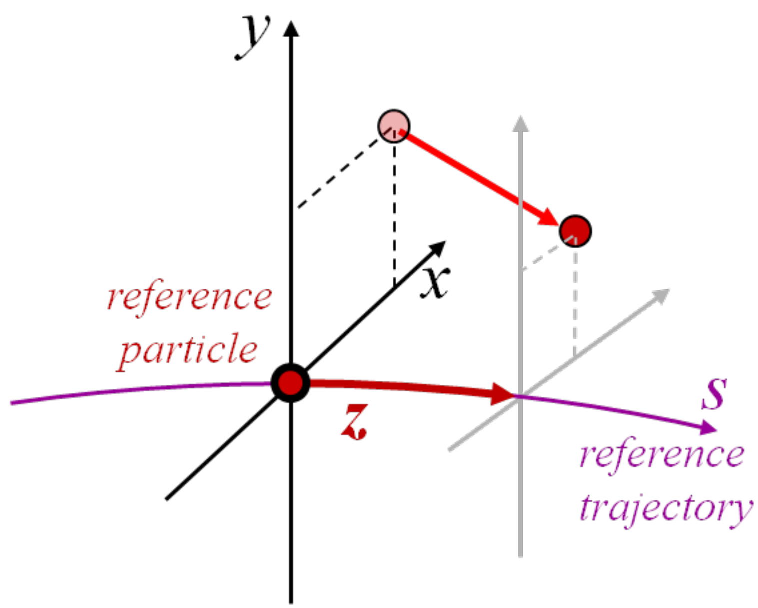
We shall proceed as follows:

- We write down the equations of motion for a particle performing synchrotron motion, including the radiation energy loss.
- We express the energy loss per turn as a function of the energy of the particle: this introduces a "damping term" into the equations of motion.
- Solving the equations of motion gives synchrotron oscillations (as expected) with amplitude that decays exponentially.

We describe the longitudinal motion of a particle in a storage ring in terms of the variables z and δ.

The co-ordinate z is the longitudinal position of a particle with respect to a *reference particle*.

The reference particle is moving round the ring on the reference trajectory, with the reference energy E_0.



δ is the energy deviation of a particle with energy E:

$$\delta = \frac{E - E_0}{E_0}. \tag{1}$$

As a particle moves around a storage ring, the energy changes because of the RF cavities and because of synchrotron radiation.

Averaged over one turn, the change in energy deviation δ is given by:

$$\Delta\delta = \frac{eV_{RF}}{E_0} \sin\left(\phi_s - \frac{\omega_{RF} z}{c}\right) - \frac{U}{E_0}, \tag{2}$$

where V_{RF} is the RF voltage, ω_{RF} the RF frequency, and U is the energy lost by the particle through synchrotron radiation.

The "synchronous phase" ϕ_s is defined by the condition that $\Delta\delta = 0$ when $z = 0$, that is:

$$\sin(\phi_s) = \frac{U}{eV_{RF}}. \tag{3}$$

The change in the longitudinal co-ordinate z as a particle makes one turn around a storage ring is given by the *momentum compaction factor* α_p:

$$\Delta z = -\alpha_p C_0 \delta, \tag{4}$$

where C_0 is the circumference of the storage ring.

The momentum compaction factor can be written (see the next slide):

$$\alpha_p = \frac{1}{C_0} \frac{dC}{d\delta}\bigg|_{\delta=0} = \frac{I_1}{C_0}, \tag{5}$$

where I_1 is the *first synchrotron radiation integral*:

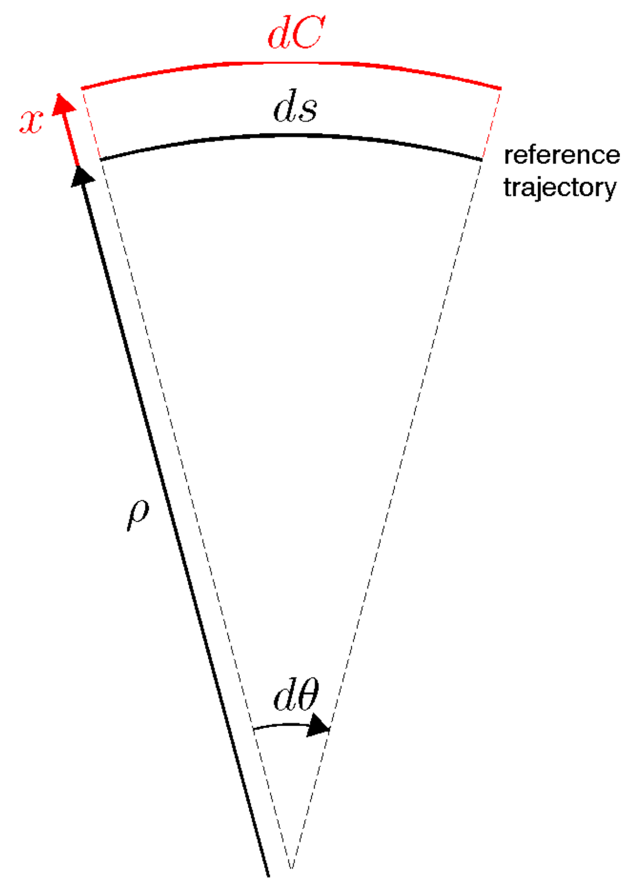
$$I_1 = \oint \frac{\eta_x}{\rho} \, ds. \tag{6}$$

The length of the closed orbit changes with energy because of dispersion in regions where the reference trajectory has some curvature.

$$dC = (\rho + x)\,d\theta = \left(1 + \frac{x}{\rho}\right) ds. \quad (7)$$

If $x = \eta_x \delta$, then:

$$dC = \left(1 + \frac{\eta_x \delta}{\rho}\right) ds. \quad (8)$$



Let us assume that over a single revolution, the change in the energy deviation $\Delta\delta$ and the change in the longitudinal co-ordinate Δz are both small.

In that case, we can write the longitudinal equations of motion for the particle:

$$\frac{d\delta}{dt} = \frac{eV_{RF}}{E_0 T_0} \sin\left(\phi_s - \frac{\omega_{RF} z}{c}\right) - \frac{U}{E_0 T_0}, \tag{9}$$

$$\frac{dz}{dt} = -\alpha_p c\delta, \tag{10}$$

where $T_0 = C_0/c$ is the revolution period.

To solve the longitudinal equations of motion, we have to make some assumptions...

First, we assume that the particle arrives at each RF cavity at a phase close to the synchronous phase, so that:

$$\sin\left(\phi_s - \frac{\omega_{RF} z}{c}\right) \approx \sin(\phi_s) - \cos(\phi_s)\frac{\omega_{RF} z}{c}. \tag{11}$$

Particles with higher energy radiate higher synchrotron radiation power. We assume $|\delta| \ll 1$, so we can work to first order in δ:

$$U = U_0 + \Delta E \left.\frac{dU}{dE}\right|_{E=E_0} = U_0 + E_0 \delta \left.\frac{dU}{dE}\right|_{E=E_0}. \tag{12}$$

With these assumptions, and using (3) for ϕ_s, the equation of motion (9) becomes:

$$\frac{d\delta}{dt} = -\frac{eV_{RF}}{E_0 T_0}\cos(\phi_s)\frac{\omega_{RF}}{c} z - \frac{1}{T_0}\delta \left.\frac{dU}{dE}\right|_{E=E_0}. \tag{13}$$

Combining the equations of motion (10) and (13) gives:

$$\frac{d^2\delta}{dt^2} + 2\alpha_E \frac{d\delta}{dt} + \omega_s^2 \delta = 0. \tag{14}$$

This is the equation of motion for a damped harmonic oscillator, with frequency ω_s given by:

$$\omega_s^2 = -\frac{eV_{RF}}{E_0} \cos(\phi_s) \frac{\omega_{RF}}{T_0} \alpha_p, \tag{15}$$

and damping constant α_E given by:

$$\alpha_E = \frac{1}{2T_0} \left. \frac{dU}{dE} \right|_{E=E_0}. \tag{16}$$

(Note that for stable oscillations, we require $\cos(\phi_s) < 0$...)

If $\alpha_E \ll \omega_s$, the energy deviation and longitudinal co-ordinate vary as:

$$\delta(t) = \delta_0 \exp(-\alpha_E t) \sin(\omega_s t - \theta_0), \tag{17}$$

$$z(t) = \frac{\alpha_p c}{\omega_s} \delta_0 \exp(-\alpha_E t) \cos(\omega_s t - \theta_0). \tag{18}$$

where δ_0 and θ_0 are constants (respectively, the amplitude and phase of the oscillation at $t = 0$).

To evaluate the damping constant α_E, we need to know how the energy loss per turn U depends on the energy deviation δ...

Classical electromagnetic theory gives the result that a particle with charge e, mass m and energy E following a curved trajectory with radius of curvature ρ radiates electromagnetic waves with power:

$$P_\gamma \approx \frac{C_\gamma}{2\pi} c \frac{E^4}{\rho^2}, \tag{19}$$

where:

$$C_\gamma = \frac{e^2}{3\varepsilon_0 (mc^2)^4}. \tag{20}$$

For electrons, $C_\gamma \approx 8.846 \times 10^{-5} \mathrm{m/GeV}^3$.

The energy loss per turn U for a particle is found by integrating the radiation power over one orbit of the ring:

$$U = \frac{1}{c} \oint P_\gamma \left(1 + \frac{\eta x}{\rho} \delta \right) ds. \tag{21}$$

For the reference particle ($\delta = 0$), the energy loss per turn is:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2, \tag{22}$$

where I_2 is the *second synchrotron radiation integral*:

$$I_2 = \oint \frac{1}{\rho^2} \, ds. \tag{23}$$

Recall that the damping constant α_E for synchrotron oscillations is given by (16):

$$\alpha_E = \frac{1}{2T_0} \left. \frac{dU}{dE} \right|_{E=E_0}. \tag{24}$$

We can find α_E by taking the derivative of U in equation (21), but we need to take into account two effects:

- The radius of curvature ρ of a particle trajectory through a fixed magnetic field will increase in proportion to an increase in energy.
- If a dipole magnet has a gradient (quadrupole field component), then the field strength seen by a particle will vary with the particle energy, because of dispersion.

Taking these effects into account, we find that:

$$\left.\frac{dU}{dE}\right|_{E=E_0} = j_E \frac{U_0}{E_0}, \tag{25}$$

where j_E is the *longitudinal damping partition number*:

$$j_E = 2 + \frac{I_4}{I_2}. \tag{26}$$

I_4 is the *fourth synchrotron radiation integral*, which accounts for dispersion and any quadrupole field component in the dipole magnets:

$$I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds, \qquad k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}. \tag{27}$$

If the dipoles in a storage ring have no quadrupole field component, then in most cases $I_4 \ll I_2$, so $j_E \approx 2$.

The longitudinal damping time τ_z is defined by:

$$\tau_z = \frac{1}{\alpha_E} = \frac{2}{j_z}\frac{E_0}{U_0}T_0. \tag{28}$$

The longitudinal emittance can be defined as:

$$\varepsilon_z = \sqrt{\langle z^2\rangle\langle\delta^2\rangle - \langle z\delta\rangle^2}. \tag{29}$$

Since the amplitudes of the synchrotron oscillations decay with time constant τ_z, the damping of the longitudinal emittance can be written:

$$\varepsilon_z(t) = \varepsilon_z(0)\exp\left(-2\frac{t}{\tau_z}\right). \tag{30}$$

Let us now consider the effect of synchrotron radiation on betatron oscillations.

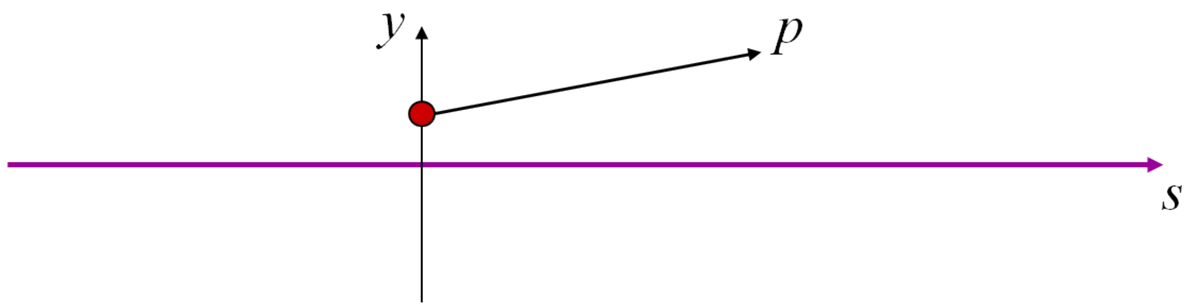
In the case of synchrotron oscillations, we assumed that the changes in the longitudinal variables were small over a single turn around the ring.

In other words, we assumed that the synchrotron frequency was small compared to the revolution frequency.

This is not a valid assumption in the case of betatron oscillations, so we shall have to take a different approach to the analysis.

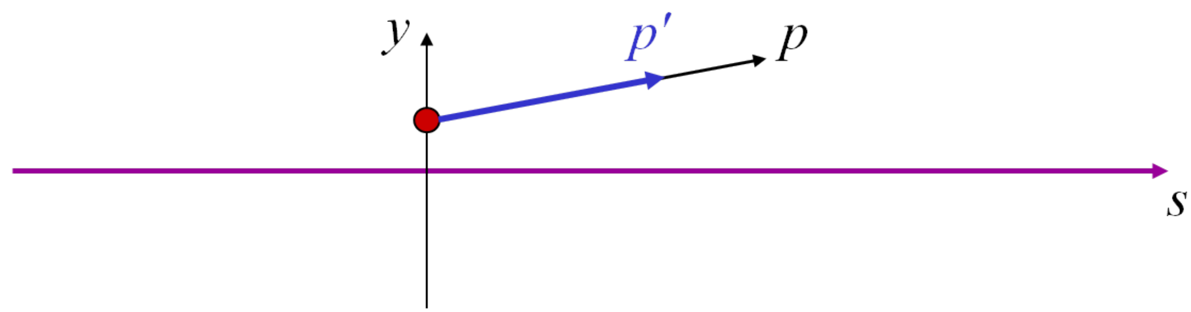
We shall first consider vertical betatron oscillations: this turns out to be a simpler case than horizontal betatron oscillations.

Radiation damping of betatron oscillations is a result of particles losing momentum through the emission of synchrotron radiation.



The radiation emitted by a relativistic particle has an opening angle of $1/\gamma$, where γ is the relativistic factor for the particle.

For an ultra-relativistic particle, $\gamma \gg 1$, and we can assume that the radiation is emitted directly along the instantaneous direction of motion of the particle.



The momentum of the particle after emitting radiation is:

$$p' = p - dp \approx p\left(1 - \frac{dp}{P_0}\right), \tag{31}$$

where dp is the momentum carried by the radiation, P_0 is the reference momentum (i.e. the momentum of the reference particle), and we assume that $p \approx P_0$.

Since there is no change in direction of the particle, the vertical momentum after the emission of the radiation is:

$$p'_y \approx p_y\left(1 - \frac{dp}{P_0}\right). \tag{32}$$

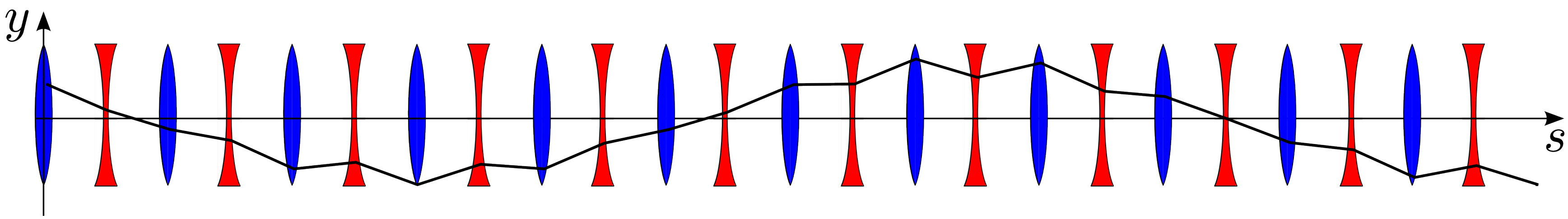
The amplitude of the betatron oscillations of a given particle can be characterised by the *betatron action*.

The vertical betatron action J_y is given by:

$$2J_y = \gamma_y y^2 + 2\alpha_y y p_y + \beta_y p_y^2, \tag{33}$$

where p_y is now the vertical momentum scaled by the reference momentum, $p_y = \gamma m v_y / P_0$.

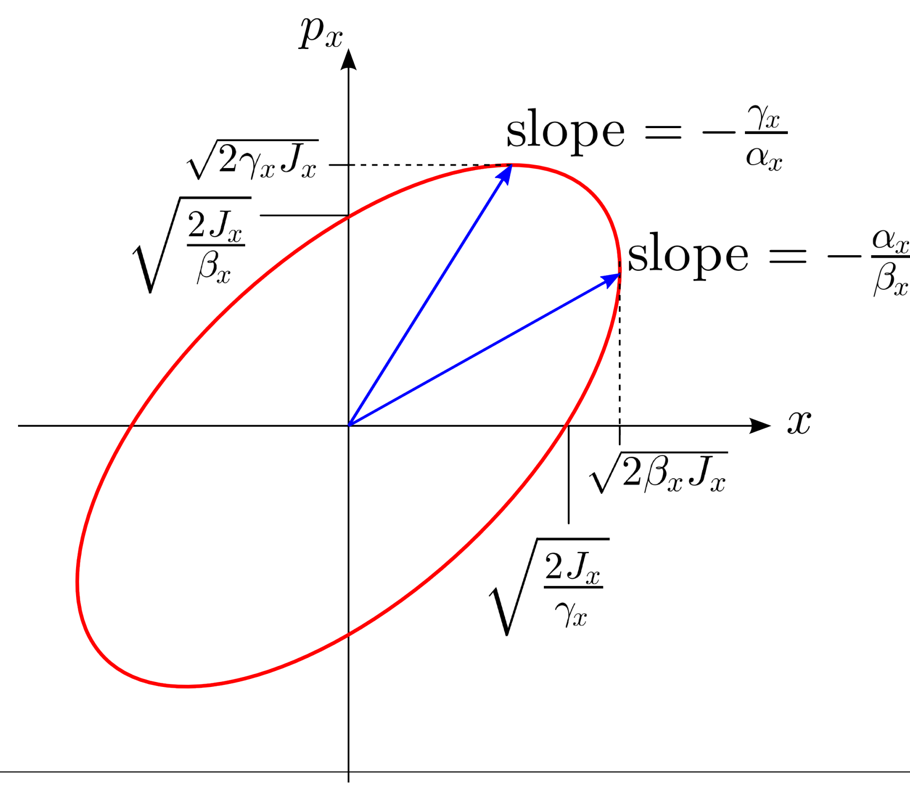
Suppose we record the co-ordinate y and momentum p_y of a particle at a given point in each cell in a periodic beamline:



The area of the ellipse mapped out by plotting p_y versus y is $2\pi J_y$.

The emittance of a bunch of particles is the average betatron action of all particles in the bunch:

$$\varepsilon_y = \langle J_y \rangle . \qquad (34)$$



In the absence of synchrotron radiation, the betatron action J_y is a constant of the motion.

The change in vertical momentum resulting from the emission of radiation leads to a change in the betatron action.

The change in betatron action dJ_y from emission of radiation with momentum dp can be found from equations (32) and (33).

If all particles lose an equal amount of momentum dp by synchrotron radiation, the change in the beam emittance is:

$$d\varepsilon_y = \langle dJ_y \rangle = -\varepsilon_y \frac{dp}{P_0}. \qquad (35)$$

Assuming that the rate of change of emittance is slow compared to the revolution frequency, we can find the rate of change of emittance by integrating the momentum loss around the ring:

$$\frac{d\varepsilon_y}{dt} = -\frac{\varepsilon_y}{T_0}\oint\frac{dp}{P_0} \approx -\frac{U_0}{E_0 T_0}\varepsilon_y = -\frac{2}{\tau_y}\varepsilon_y, \tag{36}$$

where T_0 is the revolution period, E_0 is the reference energy, and U_0 is the energy loss in one turn.

The approximation in the above formulae is valid for an ultra-relativistic particle, which has $E \approx pc$.

The evolution of the vertical emittance is given by:

$$\varepsilon_y(t) = \varepsilon_y(0) \exp\left(-2\frac{t}{\tau_y}\right), \tag{37}$$

where the vertical damping time τ_y is:

$$\tau_y = 2\frac{E_0}{U_0}T_0. \tag{38}$$

Note the similarity with the formula for the evolution of the longitudinal emittance (30):

$$\varepsilon_z(t) = \varepsilon_z(0) \exp\left(-2\frac{t}{\tau_z}\right). \tag{39}$$

where the longitudinal damping time is:

$$\tau_z = \frac{2}{j_z}\frac{E_0}{U_0}T_0. \tag{40}$$

Since (in many cases) $j_z \approx 2$, the longitudinal damping time is often about half the vertical damping time.

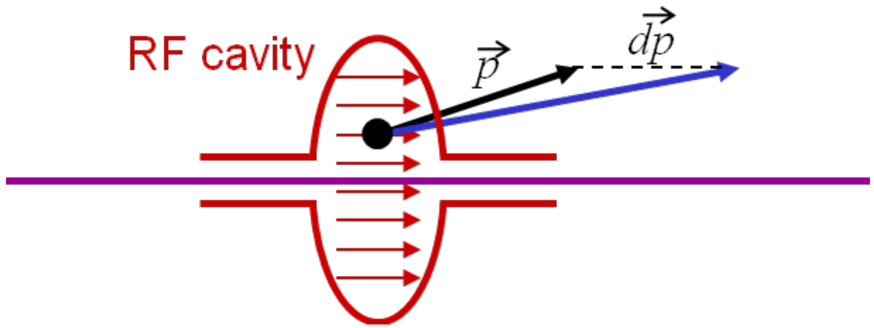
Typically, in an electron storage ring, the damping time is of order several tens of milliseconds, while the revolution period is of order of a microsecond.

Therefore, radiation effects are indeed “slow” compared to the revolution frequency.

But note that we made the assumption that the momentum of the particle was close to the reference momentum, i.e. $p \approx P_0$.

If the particle continues to radiate without any restoration of energy, eventually this assumption will no longer be valid.

However, electron storage rings contain RF cavities to restore the energy lost through synchrotron radiation. But then, we should consider the change in momentum of a particle as it moves through an RF cavity.



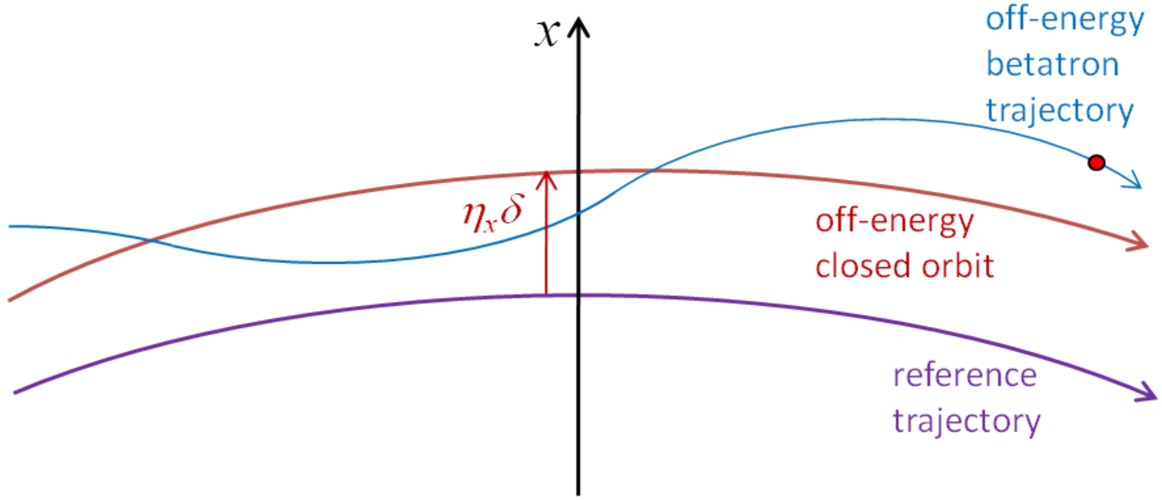
Fortunately, RF cavities are usually designed to provide a longitudinal electric field.

This means that particles experience a change in longitudinal momentum as they pass through a cavity, without any change in transverse momentum.

Therefore, we do not have to consider explicitly the effects of RF cavities on the emittance of the beam.

Analysis of radiation effects on the vertical emittance was relatively straightforward. When we consider the horizontal emittance, there are three complications that we need to address:

- The horizontal motion of a particle is often strongly coupled to the longitudinal motion (through the dispersion).
- Where the reference trajectory is curved (usually, in dipoles), the path length taken by a particle depends on the horizontal co-ordinate with respect to the reference trajectory.
- Dipole magnets are sometimes built with a gradient, so that the vertical field seen by a particle in a dipole depends on the horizontal co-ordinate of the particle.



Coupling between transverse and longitudinal planes in a beam line is usually represented by the dispersion, η_x.

In terms of the dispersion and betatron action, the horizontal co-ordinate and momentum of a particle are given by:

$$x = \sqrt{2\beta_x J_x}\cos\phi_x + \eta_x\delta \tag{41}$$

$$p_x = -\sqrt{\frac{2J_x}{\beta_x}}(\sin\phi_x + \alpha_x\cos\phi_x) + \eta_{px}\delta. \tag{42}$$

When a particle emits radiation, we have to take into account:

- the change in momentum of the particle;
- the changes in the co-ordinate x and in the momentum p_x that result from the change in the energy deviation δ.

When we analysed the vertical motion, we ignored the second effect, because we assumed that the vertical dispersion was zero.

Taking all the above effects into account, we can proceed along the same lines as for the analysis of the vertical emittance:

- Write down the changes in co-ordinate x and momentum p_x resulting from an emission of radiation with momentum dp (taking into account the additional effects of dispersion).
- Substitute expressions for the new co-ordinate and momentum into the expression for the horizontal betatron action, to find the change in the action resulting from the radiation emission.
- Average over all particles in the beam, to find the change in the emittance resulting from radiation emission from each particle.
- Integrate around the ring (taking account of changes in path length and field strength with x in the bends) to find the change in emittance over one turn.

The algebra gets somewhat cumbersome, and is not especially enlightening. See Appendix A for more details. Here, we just quote the result...

The horizontal emittance decays exponentially:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x, \tag{43}$$

where the horizontal damping time is given by:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0. \tag{44}$$

The horizontal damping partition number j_x is:

$$j_x = 1 - \frac{I_4}{I_2}, \tag{45}$$

where the fourth synchrotron radiation integral is given by (27):

$$I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds, \qquad k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}. \tag{46}$$

Summary: synchrotron radiation damping

The energy loss per turn is given by:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2, \qquad C_\gamma \approx 8.846 \times 10^{-5}\mathrm{m/GeV}^3. \tag{47}$$

The emittances damp exponentially:

$$\varepsilon(t) = \varepsilon(0) \exp\left(-2\frac{t}{\tau}\right). \tag{48}$$

The radiation damping times are given by:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0, \quad \tau_y = \frac{2}{j_y}\frac{E_0}{U_0}T_0, \quad \tau_z = \frac{2}{j_z}\frac{E_0}{U_0}T_0. \tag{49}$$

The damping partition numbers are:

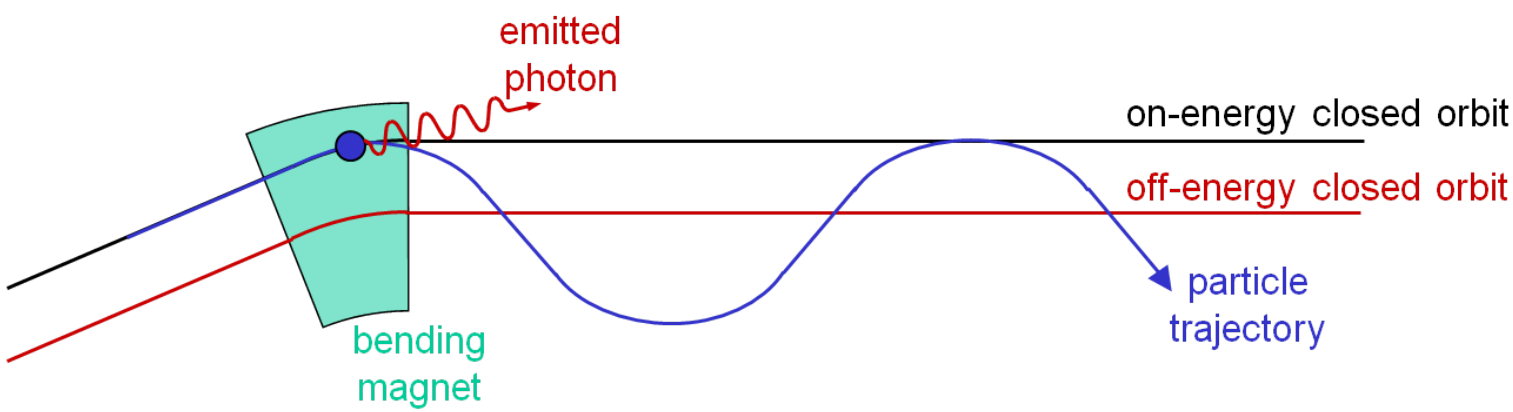
$$j_x = 1 - \frac{I_4}{I_2}, \quad j_y = 1, \quad j_z = 2 + \frac{I_4}{I_2}. \tag{50}$$

The second and fourth synchrotron radiation integrals are:

$$I_2 = \oint \frac{1}{\rho^2}\, ds, \qquad I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds. \tag{51}$$

If radiation were a purely classical process, the emittances would damp to (nearly) zero.

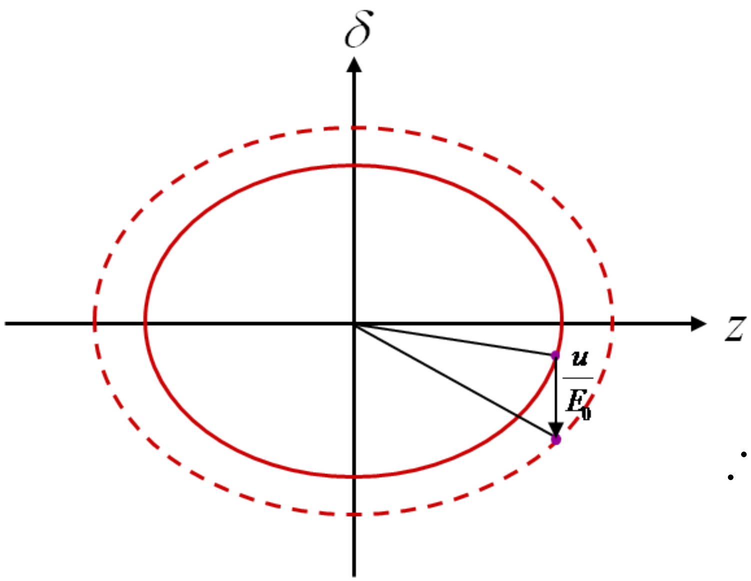
However radiation is emitted in discrete units (photons), which induces some "noise" on the beam. The effect of the noise is to increase the emittance.



The beam eventually reaches an equilibrium distribution determined by a balance between the radiation damping and the quantum excitation.

Let us first discuss the quantum excitation of longitudinal emittance.

Consider a particle with longitudinal co-ordinate z and energy deviation δ, which emits a photon of energy u.



$$\delta' = \delta_0' \sin\theta' = \delta_0 \sin\theta - \frac{u}{E_0}. \quad (52)$$

$$z' = \frac{\alpha_p c}{\omega_s}\delta_0' \cos\theta' = \frac{\alpha_p c}{\omega_s}\delta_0 \cos\theta. \quad (53)$$

$$\therefore \ \delta_0'^2 = \delta_0^2 - 2\delta_0\frac{u}{E_0}\sin\theta + \frac{u^2}{E_0^2}. \quad (54)$$

Averaging over the bunch gives:

$$\Delta\sigma_\delta^2 = \frac{\langle u^2\rangle}{2E_0^2} \quad \text{where} \quad \sigma_\delta^2 = \langle\delta^2\rangle = \frac{1}{2}\langle\delta_0^2\rangle. \quad (55)$$

Let us write the number of photons emitted per unit time with energy between u and $u + du$ as $\dot{N}(u)\,du$.

Then:

$$\frac{d\langle u^2\rangle}{dt} = \int_0^\infty \dot{N}(u)u^2\,du. \tag{56}$$

Including radiation damping, the energy spread evolves as:

$$\frac{d\sigma_\delta^2}{dt} = \frac{1}{2E_0^2}\left\langle \int_0^\infty \dot{N}(u)u^2\,du \right\rangle_C - \frac{2}{\tau_z}\sigma_\delta^2, \tag{57}$$

where the brackets $\langle\,\rangle_C$ represent an average around the ring.

Using Eq. (110) from Appendix B for $\int \dot{N}(u)u^2\,du$, we find:

$$\frac{d\sigma_\delta^2}{dt} = C_q\gamma^2\frac{2}{j_z\tau_z}\frac{I_3}{I_2} - \frac{2}{\tau_z}\sigma_\delta^2, \tag{58}$$

where the *third synchrotron radiation integral* I_3 is defined:

$$I_3 = \oint \frac{1}{|\rho^3|}\,ds, \tag{59}$$

and the "quantum radiation constant" is:

$$C_q = \frac{55}{32\sqrt{3}}\frac{\hbar}{mc}. \tag{60}$$

For electrons, $C_q \approx 3.832 \times 10^{-13}$m.

Quantum excitation gives a steady increase in the mean square energy spread, while damping gives an exponential decay.

There is, therefore, an equilibrium energy spread for which the quantum excitation is exactly balanced by the damping; the equilibrium can be found from $d\sigma_\delta^2/dt = 0$:

$$\sigma_{\delta 0}^2 = C_q \gamma^2 \frac{I_3}{j_z I_2}, \tag{61}$$

This is often referred to as the "natural" energy spread, since collective effects can often lead to an increase in the energy spread with increasing bunch charge.

The natural energy spread is determined essentially by the beam energy and by the bending radii of the dipoles.

Note that the natural energy spread *does not depend on the RF parameters (either voltage or frequency)*.

The bunch length σ_z in a *matched* distribution with energy spread σ_δ is:

$$\sigma_z = \frac{\alpha_p c}{\omega_s}\sigma_\delta. \tag{62}$$

We can increase the synchrotron frequency ω_s, and hence reduce the bunch length, by increasing the RF voltage, or by increasing the RF frequency.

Note: in a matched distribution, the shape of the distribution in phase space is the same as the path mapped out by a particle in phase space when observed on successive turns. Neglecting radiation effects, a matched distribution stays the same on successive turns of the bunch around the ring.

Let us now consider the quantum excitation of the horizontal emittance.

From the change in the co-ordinate and momentum when a particle emits radiation carrying momentum dp, we find that the betatron action changes as:

$$\frac{dJ_x}{dt} = -\frac{w_1}{P_0}\frac{dp}{dt} + \frac{w_2}{P_0^2}\frac{(dp)^2}{dt}, \tag{63}$$

where w_1 and w_2 are functions of the Courant–Snyder parameters, the dispersion, the co-ordinate x and the momentum p_x (see Appendix A).

In the classical approximation, we can take $dp \to 0$ in the limit of small time interval, $dt \to 0$.

In this approximation, the second term on the right hand side in the above equation vanishes, and we are left only with damping.

But since radiation is quantized, we cannot take $dp \to 0$.

To take account of the quantization of synchrotron radiation, we write:

$$\frac{dp}{dt} = \frac{1}{c}\int_0^\infty \dot{N}(u)\, u\, du, \tag{64}$$

and:

$$\frac{(dp)^2}{dt} = \frac{1}{c^2}\int_0^\infty \dot{N}(u)\, u^2\, du. \tag{65}$$

Here (as before) $\dot{N}(u)\,du$ is the number of photons emitted per unit time with energy between u and $u + du$.

In Appendix B, we show that with (63), these relations lead to the equation for the evolution of the emittance:

$$\frac{d\varepsilon_x}{dt} = \frac{2}{j_x \tau_x} C_q \gamma^2 \frac{I_5}{I_2} - \frac{2}{\tau_x} \varepsilon_x. \tag{66}$$

The *fifth synchrotron radiation integral* I_5 is given by:

$$I_5 = \oint \frac{\mathcal{H}}{|\rho^3|} \, ds, \tag{67}$$

where the "curly-H" function $\mathcal{H}$ is defined:

$$\mathcal{H} = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2. \tag{68}$$

C_q is the quantum radiation constant that we saw earlier (60).

Using Eq. (66) we see that there is an equilibrium horizontal emittance ε_0, for which the damping and excitation rates are equal:

$$\frac{d\varepsilon_x}{dt} = 0 \qquad \text{when} \qquad \varepsilon_x = \varepsilon_0 = C_q \frac{\gamma^2}{j_x} \frac{I_5}{I_2}. \tag{69}$$

Note that ε_0 is determined by the beam energy, the lattice functions (Courant–Snyder parameters and dispersion) in the dipoles, and the bending radius in the dipoles.

ε_0 is sometimes called the "natural emittance" of the lattice, since it includes only the most fundamental effects that contribute to the emittance: radiation damping and quantum excitation.

Typically, third generation synchrotron light sources have natural emittances of order a few nanometres. With beta functions of a few metres, this implies horizontal beam sizes of tens of microns (in the absence of dispersion).

As the current is increased, interactions between particles in a bunch can increase the emittance above the natural emittance.

Finally, let us consider the quantum excitation of the vertical emittance.

In principle, we can apply the formulae that we derived for the quantum excitation of the horizontal emittance, making appropriate substitutions of vertical quantities for horizontal ones.

In many storage rings, the vertical dispersion in the absence of alignment, steering and coupling errors is zero, so $\mathcal{H}_y = 0$.

However, the equilibrium vertical emittance is larger than zero, because the vertical opening angle of the radiation excites some vertical betatron oscillations.

The fundamental lower limit on the vertical emittance, from the opening angle of the synchrotron radiation, is given by*:

$$\varepsilon_y = \frac{13}{55} \frac{C_q}{j_y I_2} \oint \frac{\beta_y}{|\rho^3|} \, ds. \tag{70}$$

In most storage rings, this is an extremely small value, typically four orders of magnitude smaller than the natural (horizontal) emittance.

In practice, the vertical emittance is dominated by magnet alignment errors. Storage rings typically operate with a vertical emittance that is of order 1% of the horizontal emittance, but many can achieve emittance ratios somewhat smaller than this.

*T. Raubenheimer, SLAC Report 387, p.19 (1991).

Including the effects of radiation damping and quantum excitation, the emittances vary as:

$$\varepsilon(t) = \varepsilon(0)\exp\left(-2\frac{t}{\tau}\right) + \varepsilon(\infty)\left[1 - \exp\left(-2\frac{t}{\tau}\right)\right]. \tag{71}$$

The damping times are given by:

$$j_x\tau_x = j_y\tau_y = j_z\tau_z = 2\frac{E_0}{U_0}T_0. \tag{72}$$

The damping partition numbers are given by:

$$j_x = 1 - \frac{I_4}{I_2}, \qquad j_y = 1, \qquad j_z = 2 + \frac{I_4}{I_2}. \tag{73}$$

The energy loss per turn is given by:

$$U_0 = \frac{C_\gamma}{2\pi}E_0^4 I_2, \qquad C_\gamma = 9.846 \times 10^{-5}\ \text{m/GeV}^3. \tag{74}$$

The natural emittance is:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2}, \qquad C_q = 3.832 \times 10^{-13}\ \text{m}. \tag{75}$$

The natural energy spread and bunch length are given by:

$$\sigma_\delta^2 = C_q \gamma^2 \frac{I_3}{j_z I_2}, \qquad \sigma_z = \frac{\alpha_p c}{\omega_s} \sigma_\delta. \tag{76}$$

The momentum compaction factor is:

$$\alpha_p = \frac{I_1}{C_0}. \tag{77}$$

The synchrotron frequency and synchronous phase are given by:

$$\omega_s^2 = -\frac{eV_{RF}}{E_0} \frac{\omega_{RF}}{T_0} \alpha_p \cos\phi_s, \qquad \sin\phi_s = \frac{U_0}{eV_{RF}}. \tag{78}$$

The synchrotron radiation integrals are:

$$
\begin{aligned}
I_1 &= \oint \frac{\eta_x}{\rho}\, ds, && (79)\\
I_2 &= \oint \frac{1}{\rho^2}\, ds, && (80)\\
I_3 &= \oint \frac{1}{|\rho|^3}\, ds, && (81)\\
I_4 &= \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds, \qquad k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}, && (82)\\
I_5 &= \oint \frac{\mathcal{H}_x}{|\rho|^3}\, ds, \qquad \mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2. && (83)
\end{aligned}
$$

Appendix A: Damping of horizontal emittance

In this Appendix, we derive the expression for radiation damping of the horizontal emittance:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x, \tag{84}$$

where:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0, \qquad j_x = 1 - \frac{I_4}{I_2}. \tag{85}$$

To derive these formulae, we proceed as follows:

1. We find an expression for the change of horizontal action of a single particle when emitting radiation with momentum dp.

2. We integrate around the ring to find the change in action per revolution period.

3. We average the action over all the particles in the bunch, to find the change in emittance per revolution period.

To begin, we note that, in the presence of dispersion, the action J_x is written:

$$2J_x = \gamma_x \tilde{x}^2 + 2\alpha_x \tilde{x}\tilde{p}_x + \beta_x \tilde{p}_x^2, \tag{86}$$

where:

$$\tilde{x} = x - \eta_x \delta, \qquad \text{and} \qquad \tilde{p}_x = p_x - \eta_{px}\delta. \tag{87}$$

After emission of radiation carrying momentum dp, the variables change by:

$$\delta \mapsto \delta - \frac{dp}{P_0}, \quad \tilde{x} \mapsto \tilde{x} + \eta_x \frac{dp}{P_0}, \quad \tilde{p}_x \mapsto \tilde{p}_x \left(1 - \frac{dp}{P_0}\right) + \eta_{px}(1-\delta)\frac{dp}{P_0}. \tag{88}$$

We write the resulting change in the action as:

$$J_x \mapsto J_x + dJ_x. \tag{89}$$

The change in the horizontal action is:

$$dJ_x = -\frac{w_1}{P_0}dp + \frac{w_2}{P_0^2}dp^2 \qquad \therefore \qquad \frac{dJ_x}{dt} = -\frac{w_1}{P_0}\frac{dp}{dt} + \frac{w_2}{P_0^2}\frac{dp^2}{dt}, \tag{90}$$

where, in the limit $\delta \to 0$:

$$w_1 = \alpha_x x p_x + \beta_x p_x^2 - \eta_x(\gamma_x x + \alpha_x p_x) - \eta_{px}(\alpha_x x + \beta_x p_x), \tag{91}$$

and:

$$w_2 = \frac{1}{2}\left(\gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2\right) - \left(\alpha_x \eta_x + \beta_x \eta_{px}\right) p_x + \frac{1}{2}\beta_x p_x^2. \tag{92}$$

Treating radiation as a classical phenomenon, we can take the limit $dp \to 0$ in the limit of small time interval, $dt \to 0$.

In this approximation:

$$\frac{dJ_x}{dt} \approx -w_1 \frac{1}{P_0}\frac{dp}{dt} \approx -w_1 \frac{P_\gamma}{P_0 c}, \tag{93}$$

where P_γ is the *rate of energy loss* of the particle through synchrotron radiation.

To find the *average* rate of change of horizontal action, we integrate over one revolution period:

$$\frac{dJ_x}{dt} = -\frac{1}{T_0}\oint w_1 \frac{P_\gamma}{P_0 c}\, dt. \qquad (94)$$

We have to be careful changing the variable of integration where the reference trajectory is curved:

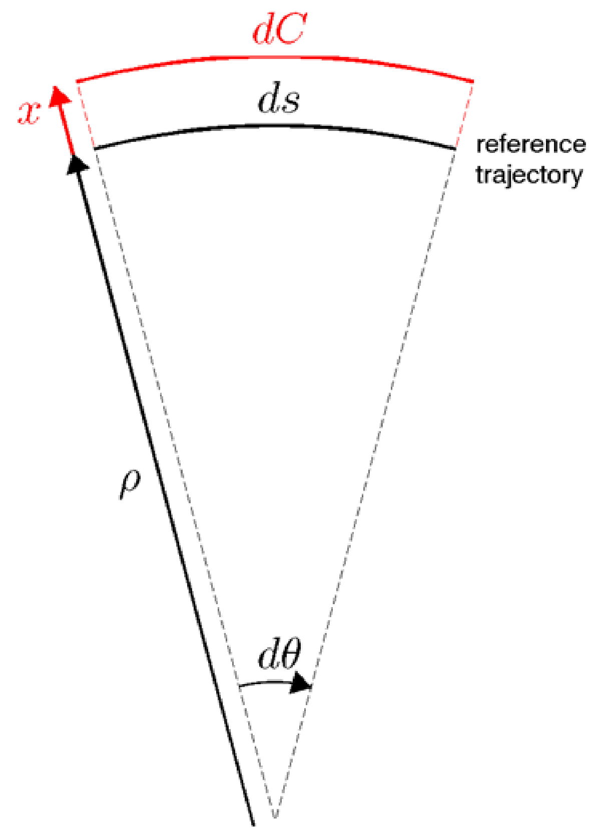
$$dt = \frac{dC}{c} = \left(1 + \frac{x}{\rho}\right)\frac{ds}{c}. \qquad (95)$$

So:

$$\frac{dJ_x}{dt} = -\frac{1}{T_0 P_0 c^2}\oint w_1 P_\gamma \left(1 + \frac{x}{\rho}\right) ds, \qquad (96)$$

where the rate of energy loss is:

$$P_\gamma = \frac{C_\gamma}{2\pi} c^3 e^2 B^2 E^2. \qquad (97)$$



We have to take into account the fact that the field strength in a dipole can vary with position. To first order in x we can write:

$$B = B_0 + x\frac{\partial B_y}{\partial x}. \tag{98}$$

Substituting Eq. (98) into (97), and with the use of (91), we find (after some algebra!) that, averaging over all particles in the beam:

$$\oint \left\langle w_1 P_\gamma \left(1 + \frac{x}{\rho}\right)\right\rangle ds = cU_0 \left(1 - \frac{I_4}{I_2}\right) \varepsilon_x, \tag{99}$$

where:

$$U_0 = \frac{C_\gamma}{2\pi} c E_0^4 I_2, \qquad I_2 = \oint \frac{1}{\rho^2} ds, \qquad I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds, \tag{100}$$

and k_1 is the normalised quadrupole gradient in the dipole field:

$$k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}. \tag{101}$$

Combining Eqs. (96) and (99) we have:

$$\frac{d\varepsilon_x}{dt} = -\frac{1}{T_0}\frac{U_0}{E_0}\left(1 - \frac{I_4}{I_2}\right)\varepsilon_x. \tag{102}$$

Defining the horizontal damping time τ_x:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0, \qquad j_x = 1 - \frac{I_4}{I_2}, \tag{103}$$

the evolution of the horizontal emittance can be written:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x. \tag{104}$$

The quantity j_x is called the *horizontal damping partition number*.

For most synchrotron storage ring lattices, if there is no gradient in the dipoles then j_x is very close to 1.

In deriving the equation of motion (96) for the action of a particle emitting synchrotron radiation, we made the classical approximation that in a time interval dt, the momentum dp of the radiation emitted goes to zero as dt goes to zero.

In reality, emission of radiation is quantized, so writing "$dp \to 0$" actually makes no sense.

Taking into account the quantization of radiation, the equation of motion for the action (90) should be written:

$$\frac{dJ_x}{dt} = -\frac{w_1}{P_0 c}\int_0^\infty \dot{N}(u)\, u\, du + \frac{w_2}{P_0^2 c^2}\int_0^\infty \dot{N}(u)\, u^2\, du, \tag{105}$$

where $\dot{N}(u)$ is the number of photons emitted per unit time in the energy range from u to $u + du$.

The first term on the right hand side of Eq. (105) just gives the same radiation damping as in the classical approximation.

The second term on the right hand side of Eq. (105) is an excitation term that we previously neglected.

To proceed, we find expressions for the integrals $\int \dot{N}(u)\, u\, du$ and $\int \dot{N}(u)\, u^2\, du$.

The required expressions can be found from the spectral distribution of synchrotron radiation from a dipole magnet. This is given by:

$$\frac{d\mathcal{P}}{d\vartheta} = \frac{9\sqrt{3}}{8\pi} P_\gamma \vartheta \int_\vartheta^\infty K_{5/3}(x)\, dx, \tag{106}$$

where $d\mathcal{P}/d\vartheta$ is the energy radiated per unit time per unit frequency range, and $\vartheta = \omega/\omega_c$ is the radiation frequency ω divided by the critical frequency ω_c:

$$\omega_c = \frac{3}{2}\frac{\gamma^3 c}{\rho}. \tag{107}$$

P_γ is the total energy radiated per unit time, and $K_{5/3}(x)$ is a modified Bessel function.

Since the energy of a photon of frequency ω is $u = \hbar\omega$, it follows that:

$$\dot{N}(u)\,du = \frac{1}{\hbar\omega}\frac{d\mathcal{P}}{d\vartheta}\,d\vartheta. \tag{108}$$

Using (106) and (108), we find:

$$\int_0^\infty \dot{N}(u)\,u\,du = P_\gamma, \tag{109}$$

and:

$$\int_0^\infty \dot{N}(u)\,u^2\,du = 2C_q\gamma^2\frac{E_0}{\rho}P_\gamma. \tag{110}$$

C_q is a constant given by:

$$C_q = \frac{55}{32\sqrt{3}}\frac{\hbar}{mc} \approx 3.832 \times 10^{-13}\,\mathrm{m}. \tag{111}$$

The final step is to substitute for the integrals in (105) from (109) and (110), substitute for w_1 and w_2 from (91) and (92), average over the circumference of the ring, and average also over all particles in the beam.

Then, since $\varepsilon_x = \langle J_x \rangle$, we find (for $x \ll \eta_x$ and $p_x \ll \eta_{px}$):

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x + \frac{2}{j_x\tau_x}C_q\gamma^2\frac{I_5}{I_2} \tag{112}$$

where the fifth synchrotron radiation integral I_5 is given by:

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho^3|}\,ds, \tag{113}$$

The "curly-H" function $\mathcal{H}_x$ is given by:

$$\mathcal{H}_x = \gamma_x\eta_x^2 + 2\alpha_x\eta_x\eta_{px} + \beta_x\eta_{px}^2. \tag{114}$$

The damping time and horizontal damping partition number are given by:

$$j_x\tau_x = 2\frac{E_0}{U_0}T_0, \qquad U_0 = \frac{C_\gamma}{2\pi}cE_0^4 I_2, \tag{115}$$

(U_0 is the energy loss per turn) and:

$$j_x = 1 - \frac{I_4}{I_2}. \tag{116}$$

Note that the excitation term is independent of the emittance.

The quantum excitation does not simply modify the damping time, but leads to a non-zero equilibrium emittance.

The equilibrium emittance ε_0 is determined by the condition:

$$\left.\frac{d\varepsilon_x}{dt}\right|_{\varepsilon_x=\varepsilon_0} = 0. \tag{117}$$

From (112), we see that the equilibrium emittance is given by:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2}. \tag{118}$$