CERN

Instabilities Part II: Longitudinal wake fields – impact on machine elements and beam dynamics

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Outline

We will close in into the description and the impact of **longitudinal wake fields**. We will discuss the **energy balance** and then show some examples of phenomena linked to **longitudinal wake fields** such as beam induced heating, potential well distortion, microwave and Robinson instabilities.

Part 2: Longitudinal wakefields –

impact on machine elements and beam dynamics

- Longitudinal wake function and impedance
- Energy loss – beam induced heating and stable phase shift
- Potential well distortion, bunch lengthning and microwave instability
- Robinson instability

Signpost

- We have learned about the concept of **particles**, **distributions** and **macroparticles** as well as some **peculiarities of multiparticle dynamics** in accelerators, decoherence, filamentation.
- We have learned about the basic **concept of wake fields** and how these can be characterized as a **collective effect** in that they depend on the particle distribution.
- We now have a basic understanding of multiparticle systems and wakefields and are now ready to look at the **impact of these** in the longitudinal and transverse planes.

Part 2: Longitudinal wakefields –

impact on machine elements and beam dynamics

- Longitudinal wake fields and the longitudinal wake function
- Energy loss – beam induced heating and stable phase shift
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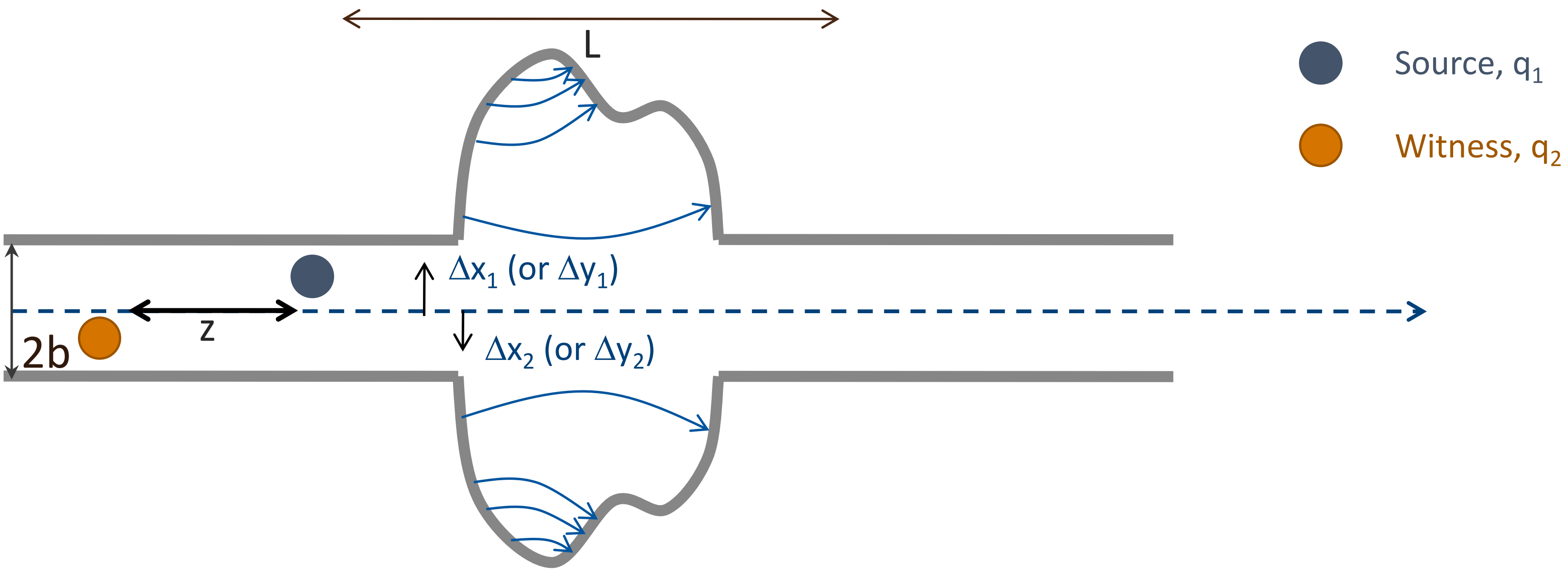
Recap: wake functions in general



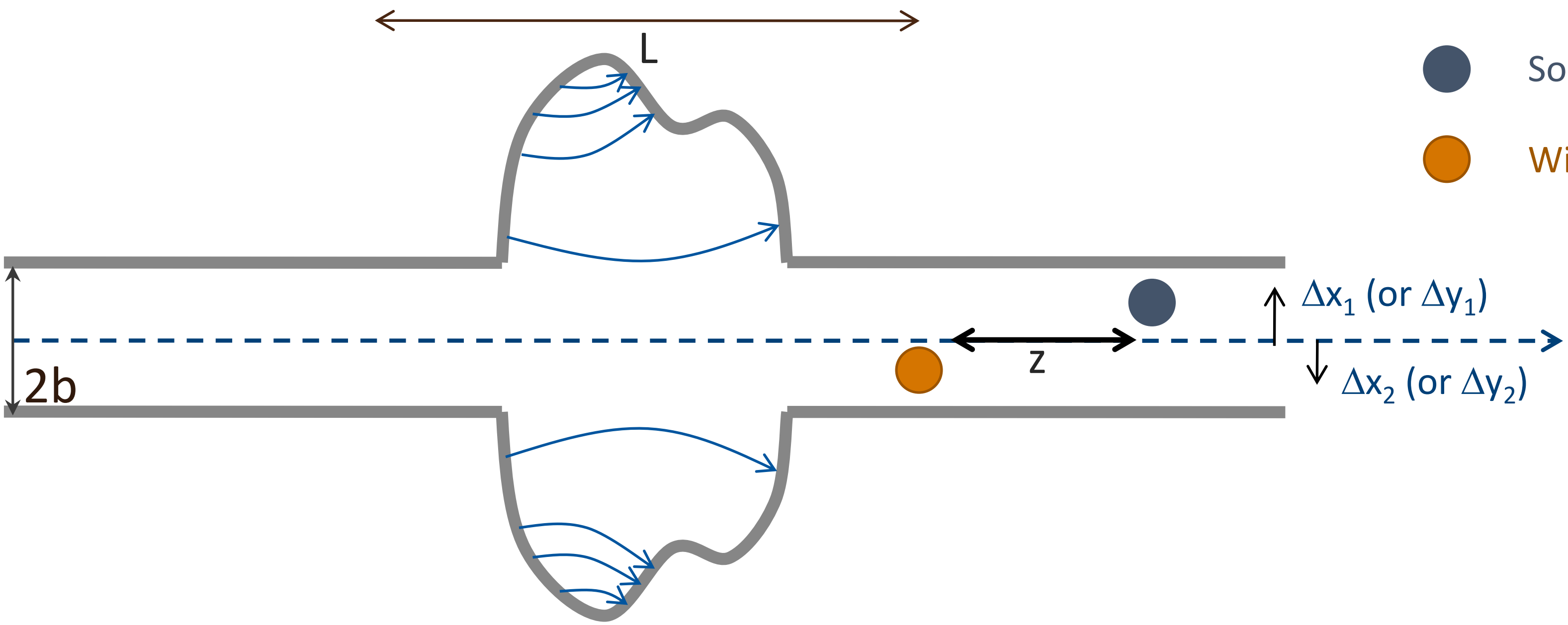
Definition as the **integrated force** associated to a change in energy:

- In general, for two point-like particles, we have

$$\Delta E_2 = \int F(x_1, x_2, z, s)\, ds = -q_1 q_2\, \boldsymbol{w(x_1, x_2, z)}$$

w is typically expanded in the transverse offsets of source and witness particles. This yields the different types of wake fields (dipole, quadrupole, coupling wakes)

Longitudinal wake function



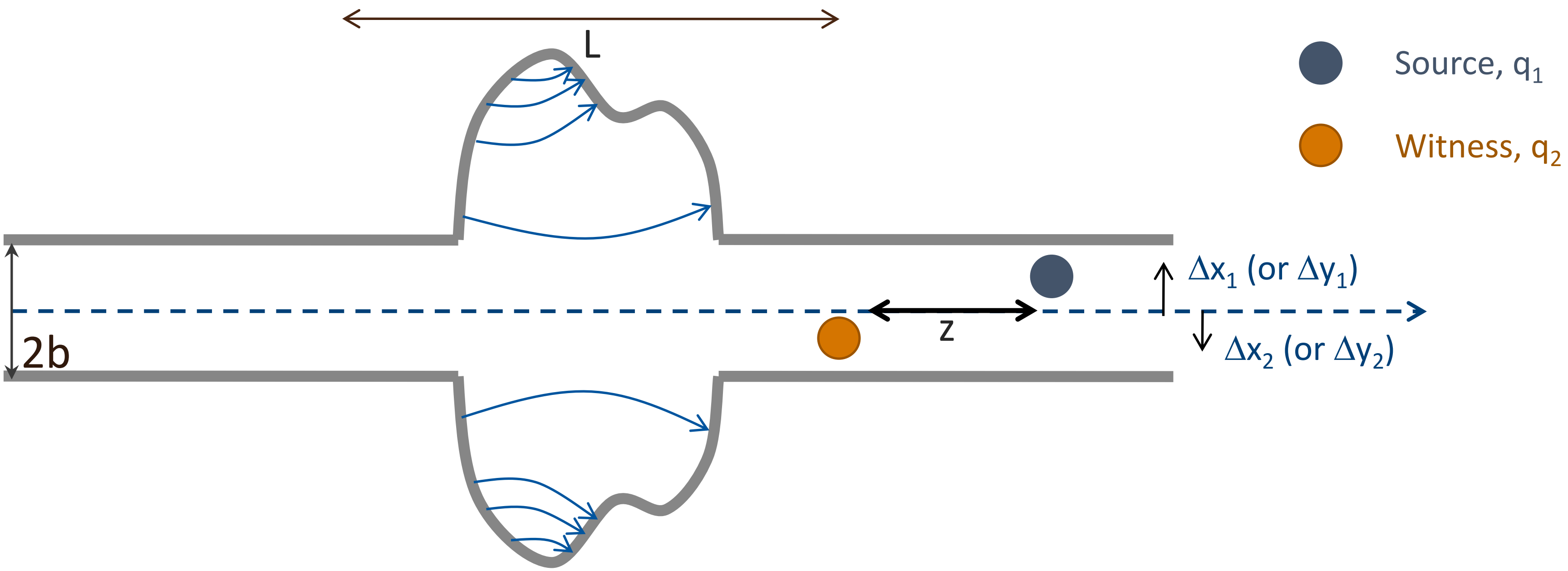
- Longitudinal wake fields

$$\int F_z(x_1, x_2, z, s)\, ds = -q_1 q_2 \left(\boxed{W_{||}(z)} + \boxed{O(\Delta x_1) + O(\Delta x_2)}\right)$$

Zeroth order with source and test centred
usually dominant

Higher order terms
Usually negligible for small offsets

Longitudinal wake function



- Longitudinal wake fields

$$\Delta E_2 = \int F_z(z,s)\,ds = -q_1 q_2 W_{||}(z)$$

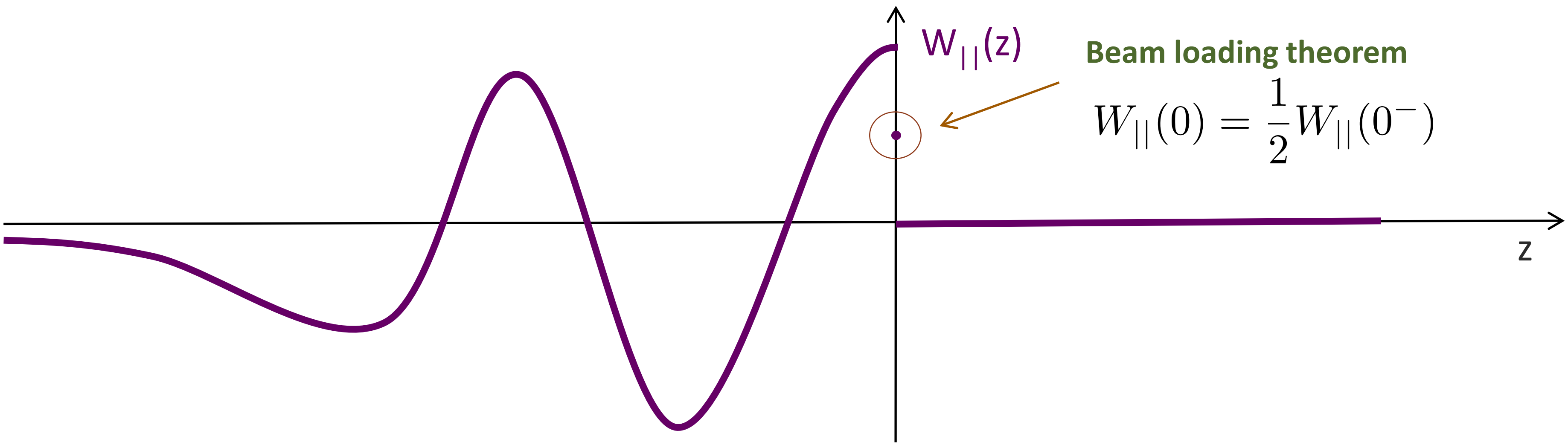
$$\longrightarrow \frac{\Delta E_2}{E_0} = \left(\frac{\gamma^2-1}{\gamma}\right)\frac{\Delta p_2}{p_0}$$

Energy kick of the witness particle from longitudinal wakes

Longitudinal wake function

$$W_{||}(z) = -\frac{\Delta E_2}{q_1 q_2} \quad \xrightarrow[q_2 \to q_1]{z \to 0} \quad W_{||}(0) = -\frac{\Delta E_1}{q_1^2}$$

- The value of the wake function in z=0 is related to the **energy lost by the source particle** in the creation of the wake
- ***$W_{||}(0)>0$*** since ***$\Delta E_1<0$***
- ***$W_{||}(z)$*** is discontinuous in z=0 and it vanishes for all z>0 because of the ultra-relativistic approximation



Beam loading theorem

$$W_{||}(0) = \frac{1}{2} W_{||}(0^-)$$

Longitudinal impedance

$$W_{||}(z) = -\frac{\Delta E_2}{q_1 q_2} \quad \xrightarrow[q_2 \to q_1]{z \to 0} \quad W_{||}(0) = -\frac{\Delta E_1}{q_1^2}$$

- The **wake function** of an accelerator component is basically its **Green function in time domain** (i.e., its response to a pulse excitation)
  → Very useful for macroparticle models and simulations, because it can be used to describe the driving terms in the single particle equations of motion!
- We can also describe it as a **transfer function in frequency domain**

→ This is the definition of **longitudinal beam coupling impedance** of the element under study

$$\boxed{Z_{||}(\omega)} = \int_{-\infty}^{\infty} \boxed{W_{||}(z)} \exp\left(-\frac{i\omega z}{c}\right) \frac{dz}{c}$$

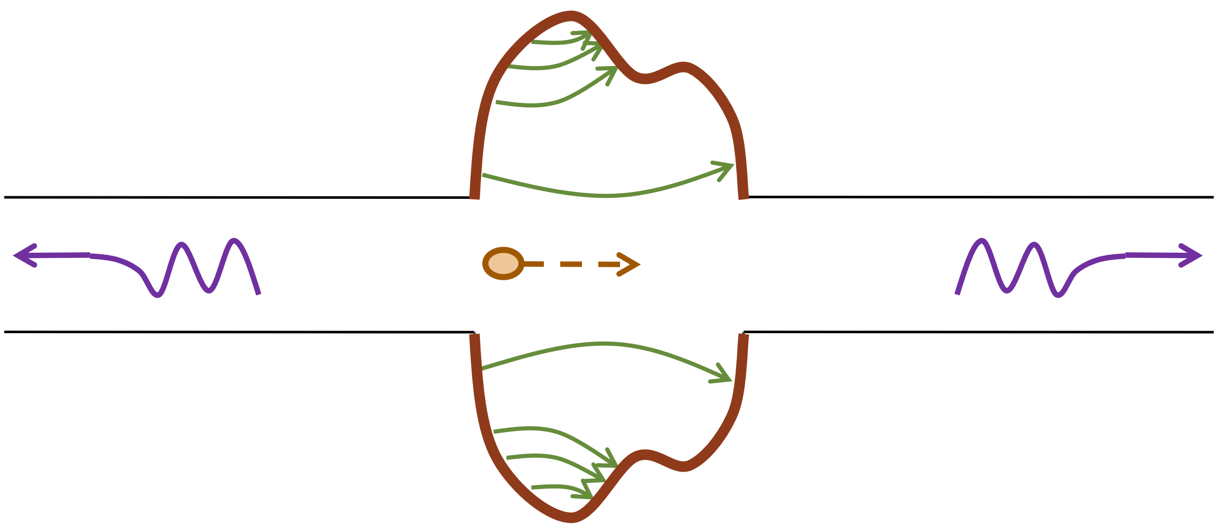
$[\Omega]$ $\qquad$ $[\Omega/s]$

The energy balance

$$W_{\|}(0) = \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left(Z_{\|}(\omega)\right)\, d\omega = -\frac{\Delta E_1}{q_1^2}$$

What happens to the energy lost by the source?

- In the global energy balance, the energy lost by the source splits into:
  - Electromagnetic energy of the **modes that remain trapped** in the object
    - → Partly dissipated on **lossy walls** or into purposely designed inserts or HOM absorbers
    - → Partly transferred to **following particles** (or the same particle over successive turns), possibly feeding into an instability!
  - Electromagnetic energy of **modes that propagate** down the beam chamber (above cut-off), eventually lost on surrounding lossy materials



The energy balance

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What happens to the energy lost by the source?

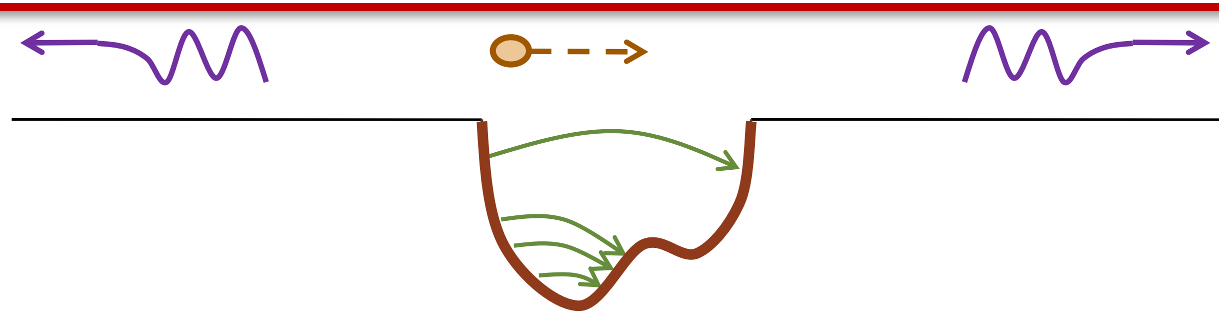
- In the global energy balance, the energy lost by the source splits into
  - ... ct ... absorbers ... ve turns),
  - ... chamber

The energy loss of a particle bunch

⇒ causes **beam induced heating** of the machine elements (damage, outgassing)

⇒ feeds into both **longitudinal and transverse instabilities** through the associated EM fields

⇒ is compensated by the RF system determining a **stable phase shift**



Signpost

- We have specialized the general definition of the wake function to the specific case of the **purely longitudinal wake function**.
- We have seen how longitudinal wake functions are related to the **energy loss** of the source particles.
- We have discussed the **energy balance** which containes all the **fundamental underlying mechanisms** for collective effects related to wake fields and impedances.

Part 2: Longitudinal wakefields –

impact on machine elements and beam dynamics

- Longitudinal wake function and impedance
- Energy loss – beam induced heating and stable phase shift
- Potential well distortion, bunch lengthning and microwave instability
- Robinson instability

Bunch energy loss per turn

- We remember the energy loss for two particles due to a longitudinal wake field:

$$\Delta E_2 = -\, q_1 q_2 \, W_{\|}(z)$$

- This can be generalized to an energy loss for a multi particle distribution for a single passage:

$$\Delta E_{\text{total}} = -e^2 \int \lambda(z) \underbrace{\int \lambda(z') \, W_{\|}(z - z') \, dz'}_{\propto \Delta E(z)} \, dz$$

- which in frequency domain becomes

$$\Delta E = -\frac{e^2}{2\pi} \int \left| \hat{\lambda}(\omega) \right|^2 \operatorname{Re}\left[Z_{\|}(\omega) \right] \, d\omega$$

- If instead, we consider a multi particle distribution over multiple passages spaced by $2\pi/\omega_0$, we arrive at

$$\Delta E = -\frac{e^2 \omega_0}{2\pi} \sum_{p=-\infty}^{\infty} \left| \hat{\lambda}(p\omega_0) \right|^2 \operatorname{Re}\left[Z_{\|}(p\omega_0) \right]$$

Beam energy loss per turn

The bunch energy loss is given by the **bunch/beam spectrum** and the real part of the machine **longitudinal impedance**

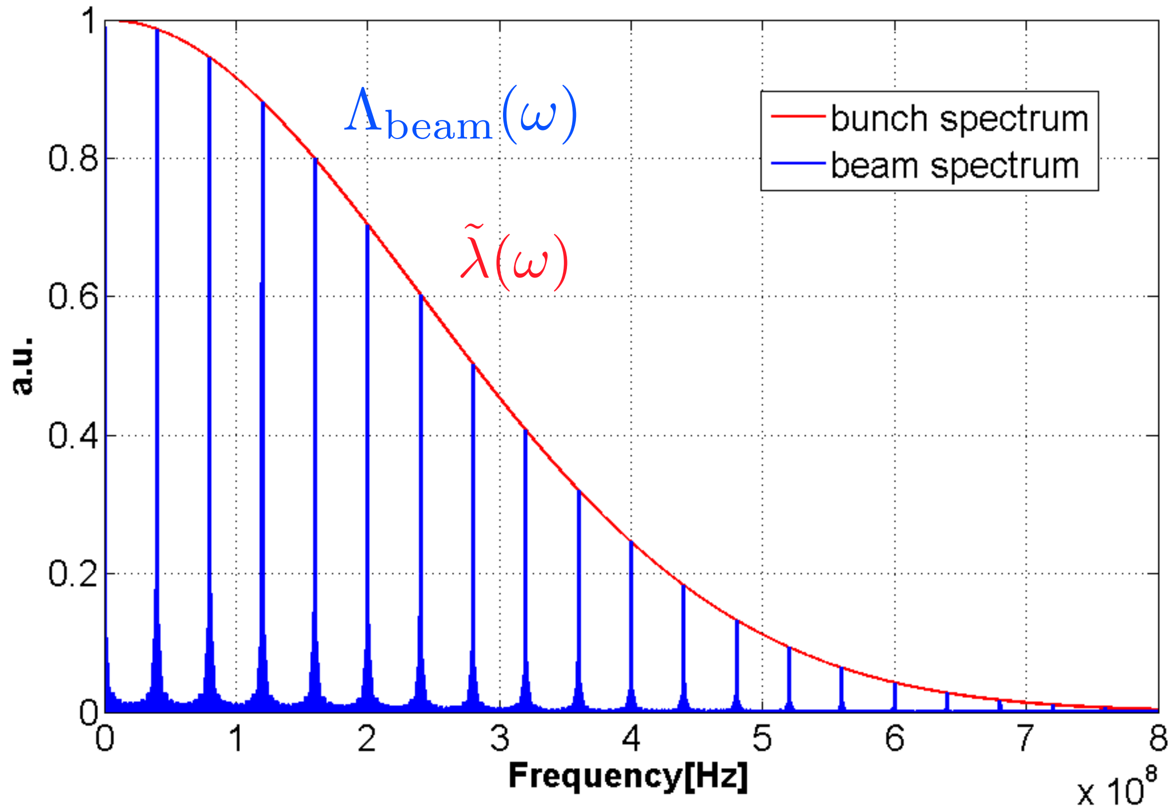
$$\Delta E = -\frac{e^2\omega_0}{2\pi}\sum_{p=-\infty}^{\infty}\left|\hat{\lambda}(p\omega_0)\right|^2 \mathrm{Re}\left[Z_{\parallel}(p\omega_0)\right]$$

Bunch spectrum

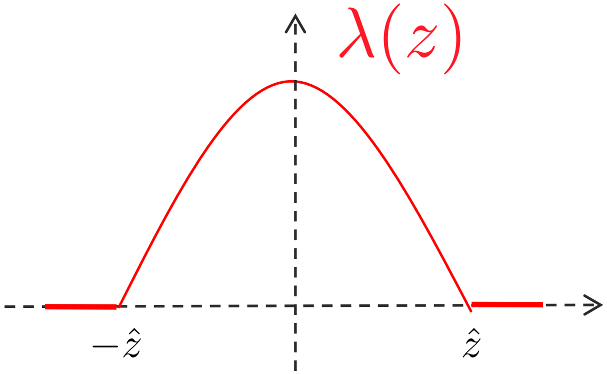
$\tilde{\lambda}(\omega)$

Beam spectrum

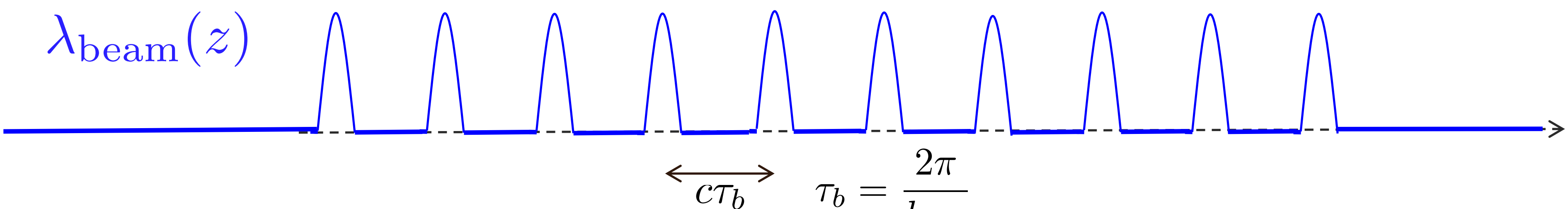
$\Lambda_{\mathrm{beam}}(\omega)$



Energy loss of a train of bunches



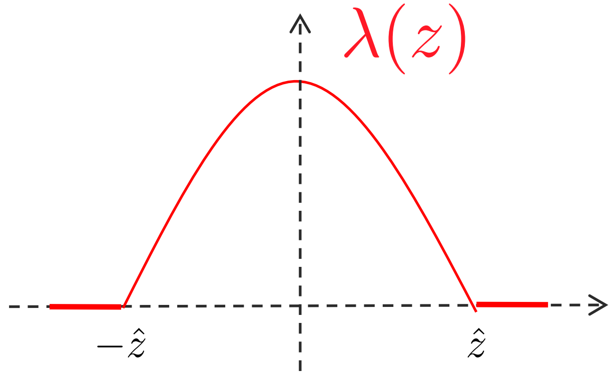
A train of *M* identical equally spaced bunches circulating in a ring



$$\tau_b = \frac{2\pi}{h\omega_0}$$

$$\lambda_{\rm beam}(z) = \sum_{n=0}^{M-1} \lambda(z - nc\tau_b) \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad \Lambda_{\rm beam}(\omega) = \tilde{\lambda}(\omega) \sum_{n=0}^{M-1} \exp(-in\omega\tau_b)$$

Energy loss of a train of bunches



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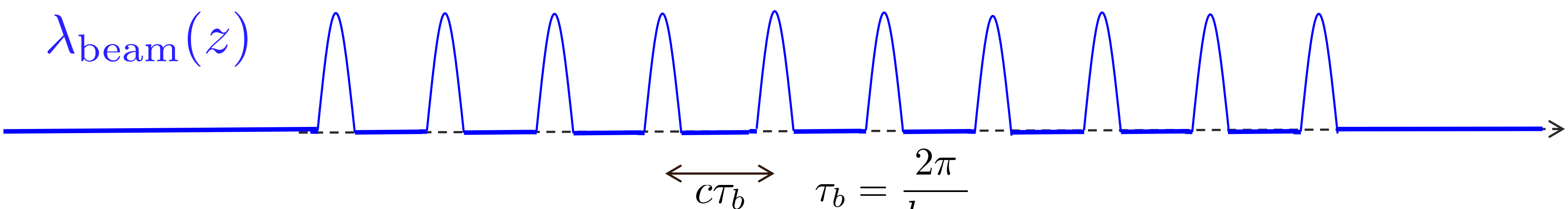
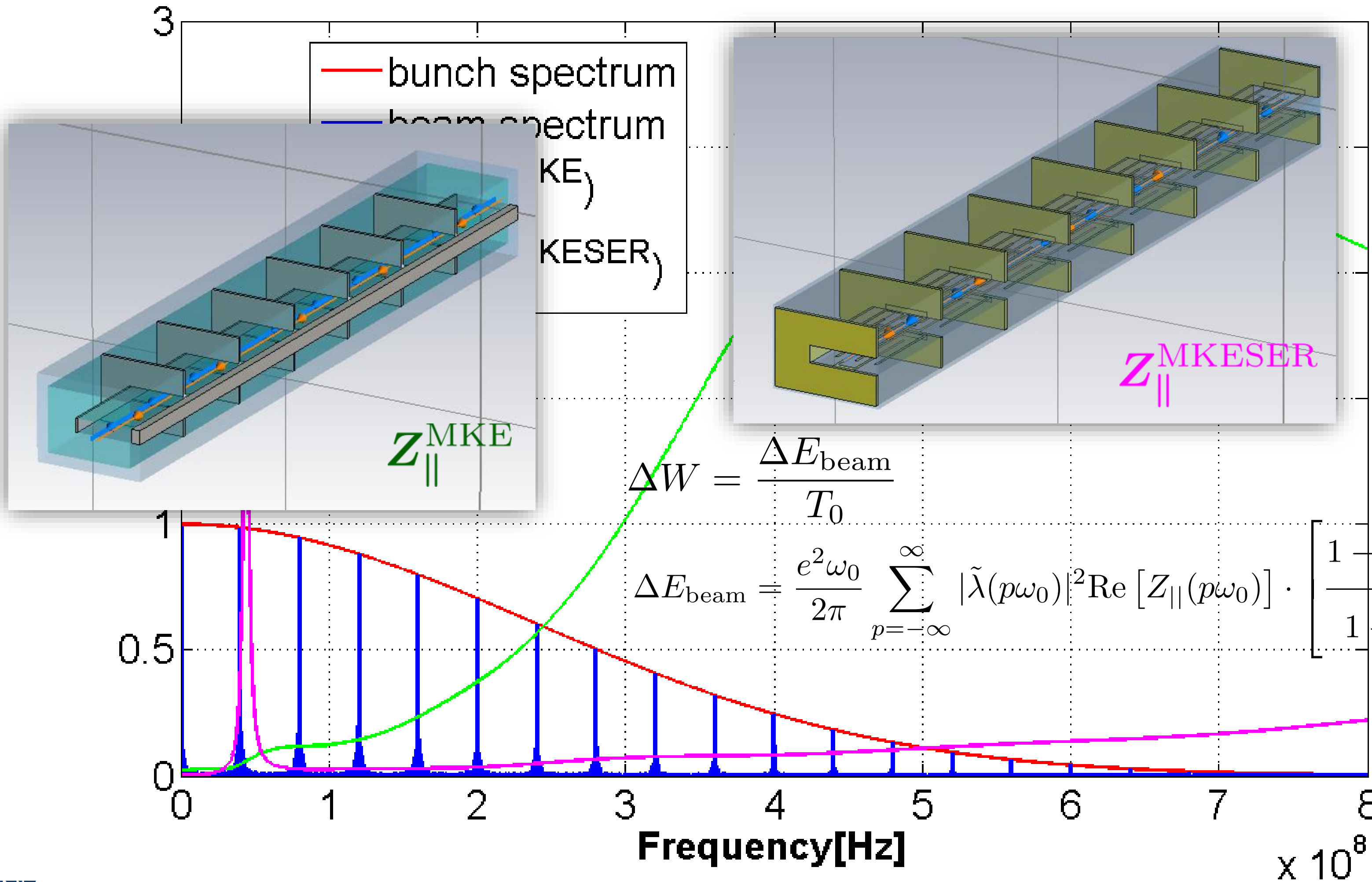


$$\tau_b = \frac{2\pi}{h\omega_0}$$

$$\Delta E_{\text{beam}} = \frac{e^2\omega_0}{2\pi} \sum_{p=-\infty}^{\infty} |\tilde{\lambda}(p\omega_0)|^2 \text{Re}\left[Z_{||}(p\omega_0)\right] \cdot \left[\frac{1-\cos(\frac{2\pi M p}{h})}{1-\cos(\frac{2\pi p}{h})}\right]$$

Application to the SPS extraction kickers



$$\Delta W = \frac{\Delta E_{\mathrm{beam}}}{T_0}$$

$$\Delta E_{\mathrm{beam}} = \frac{e^2 \omega_0}{2\pi} \sum_{p=-\infty}^{\infty} |\tilde{\lambda}(p\omega_0)|^2 \mathrm{Re}\left[Z_{||}(p\omega_0)\right] \cdot \left[\frac{1 - \cos(\frac{2\pi M p}{h})}{1 - \cos(\frac{2\pi p}{h})}\right]$$

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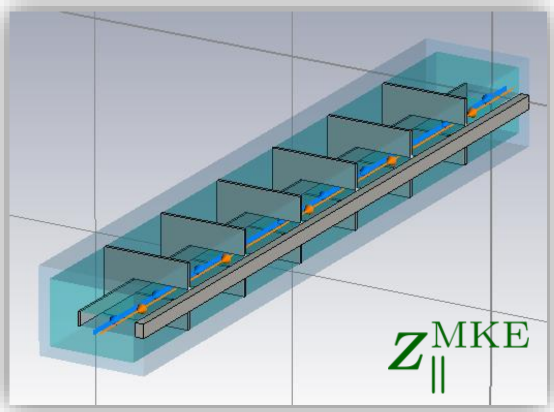

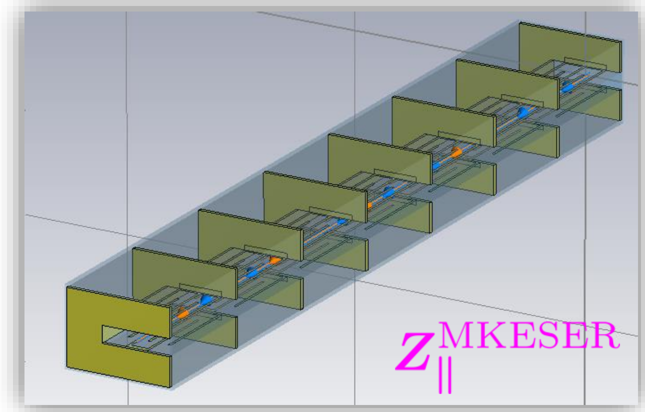

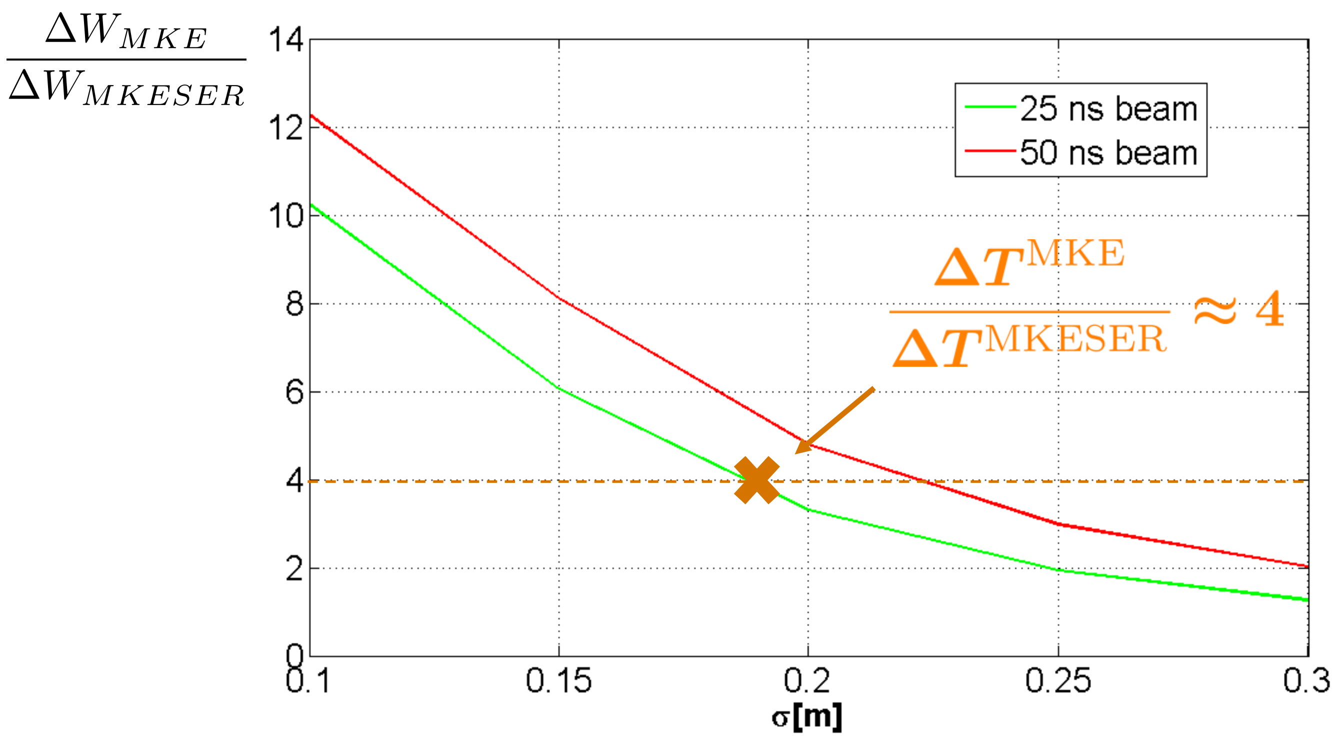

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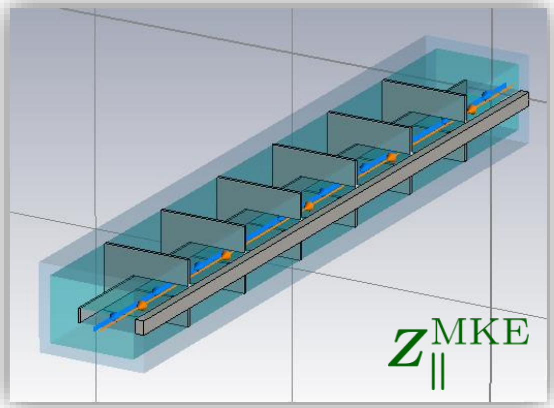
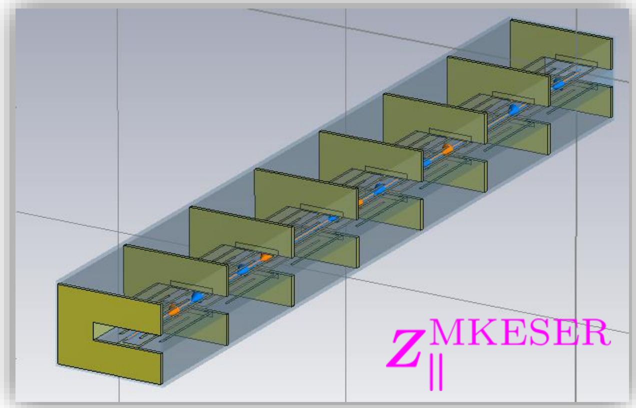
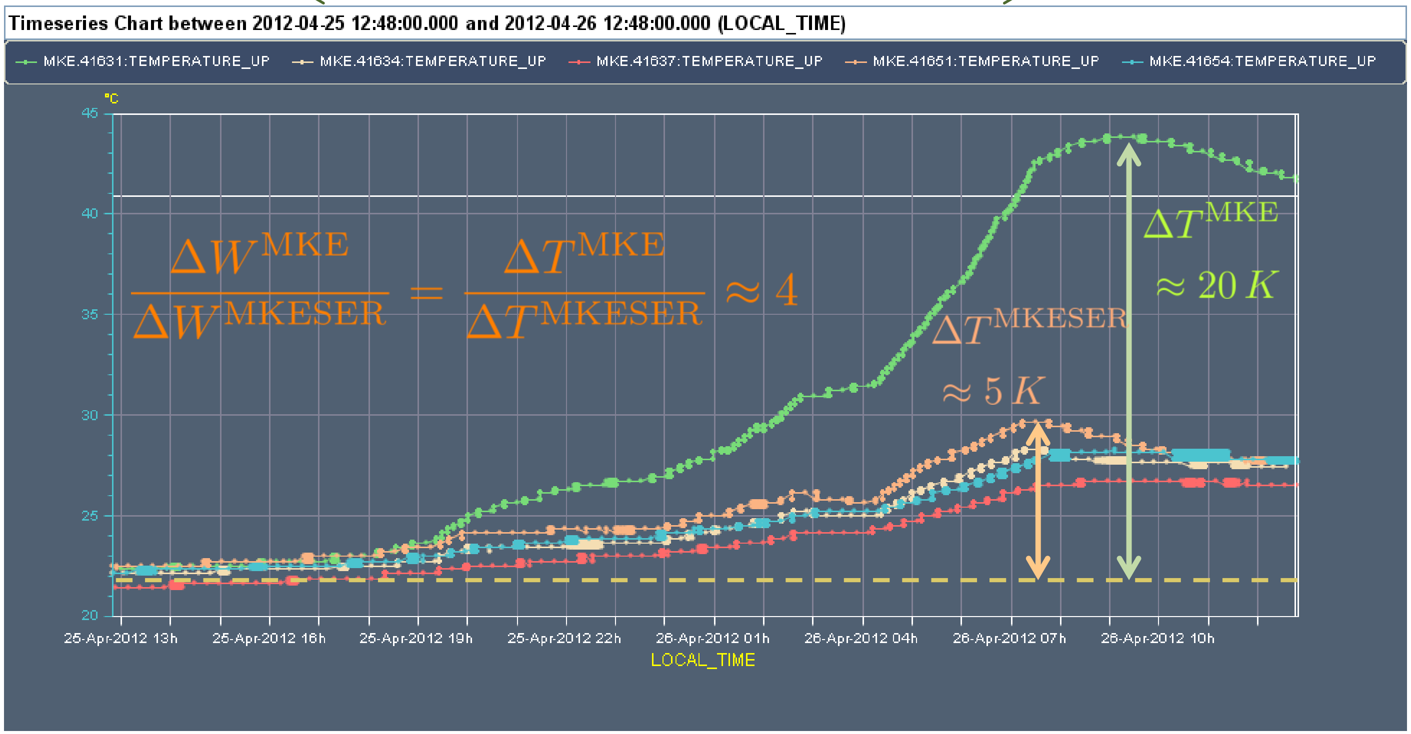


~ 17h run with 25 ns beams at 26 GeV after technical stop





Signpost

- We have further dived into the mechanism of energy loss and have seen the **impact of longitudinal impedances on machine elements** as these lead to **beam induced heating**.
- We have found that beam induced heating depends on the overlap of the **beam power spectrum** and the **impedance** of a given object.
- We have seen a **real world example** of the impact of an objects impedance on the beam induced heating.

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Recap: formal description of a beam instability

- The mode and thus the instability is fully characterized by a single number defined by an eigenvalue problem:

  the complex tune shift Ω

- For the case of longitudinal wake fields, two regimes can be found:
  - Regime of potential well distortion (i.e. perturbations to **equilibrium solutions are damped**)
    - Stable phase shift
    - Synchrotron frequency shift
    - Different matching (→ bunch lengthening for lepton machines)
  - Regime of longitudinal instability (i.e. perturbations to **equilibrium solutions grow exponentially**):
    - Dipole mode instabilities
    - Coupled bunch instabilities
    - Microwave instability (longitudinal mode coupling)

Potential well distortion and Haissinki equation

- The **equilibrium distribution** in the presence of a longitudinal wake field can be found analytically. The (linearized) **longitudinal Hamiltonian** with longitudinal wake fields is given as:

$$H = -\frac{1}{2}\eta\,\delta^2 - \frac{1}{2\eta}\left(\frac{\omega_s}{\beta c}\right)^2 z^2 + \frac{e^2}{\beta^2 EC}\sum_k \int_0^z dz'' \int_{z''}^{\infty} dz'\,\lambda(z'+kC)W_{\|}(z''-z'-kC)$$

- We assume a Gaussian beam distribution:

$$\psi(z,\delta) = \exp\left(-\frac{\delta^2}{2\sigma_\delta^2}\right)\lambda(z)$$

- The equilibrium (matched) line charge density is then given by the self-consistency equation (**Haissinski equation**):

$$\lambda(z) = \exp\left(\left(-\frac{\omega_s z}{2\eta\sigma_\delta\beta c}\right)^2 + \frac{e^2}{\eta\sigma_\delta^2\beta^2 EC}\sum_k \int_0^z dz'' \int_{z''}^{\infty} dz'\,\lambda(z'+kC)W_{\|}(z''-z'-kC)\right)$$

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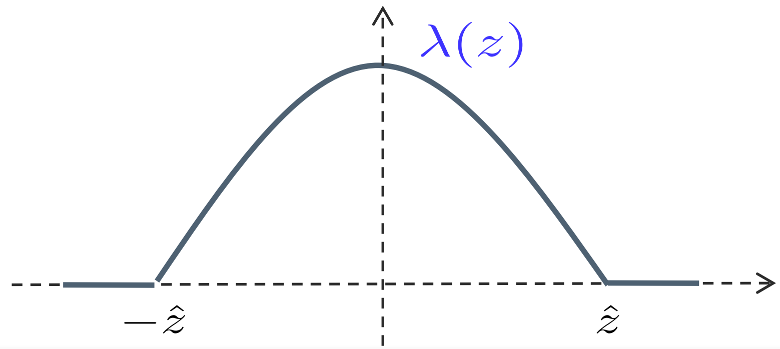
A simple Taylor expansion in z already qualitatively reveals some of the effects of the longitudinal wake fields onto the beam:

1. First order:
   shift in the mean position (**stable phase shift**)
2. Second order:
   change in bunch length accompanied by an (incoherent) **synchrotron tune shift**

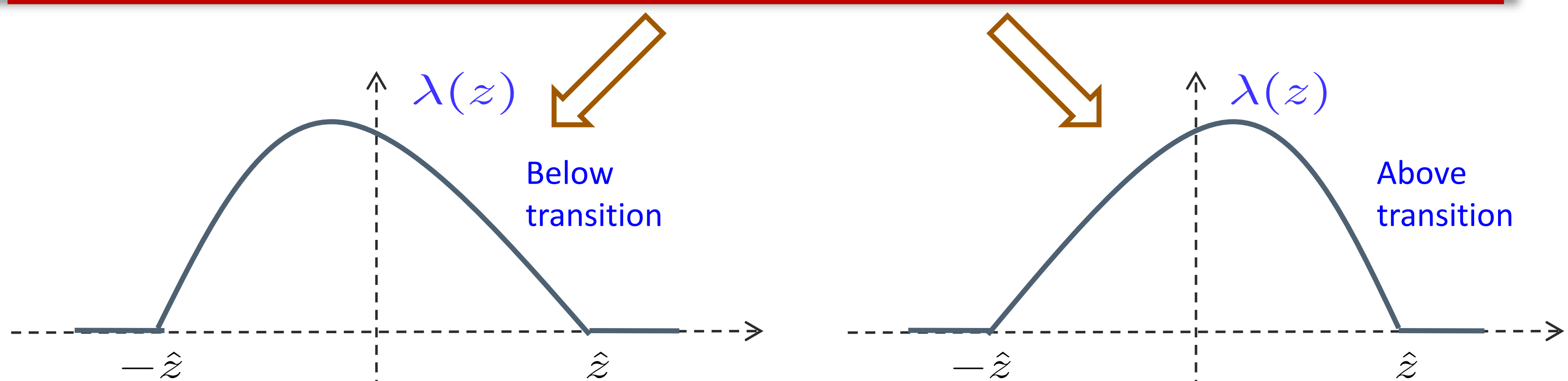
$$\lambda(z) = \exp\left(\left(-\frac{\omega_s z}{2\eta\sigma_\delta\beta c}\right)^2 + \frac{e^2}{\eta\sigma_\delta^2\beta^2 EC}\sum_k \int_0^z dz'' \int_{z''}^{\infty} dz'\,\lambda(z'+kC)W_{\|}(z''-z'-kC)\right)$$

Bunch energy loss per turn and stable phase

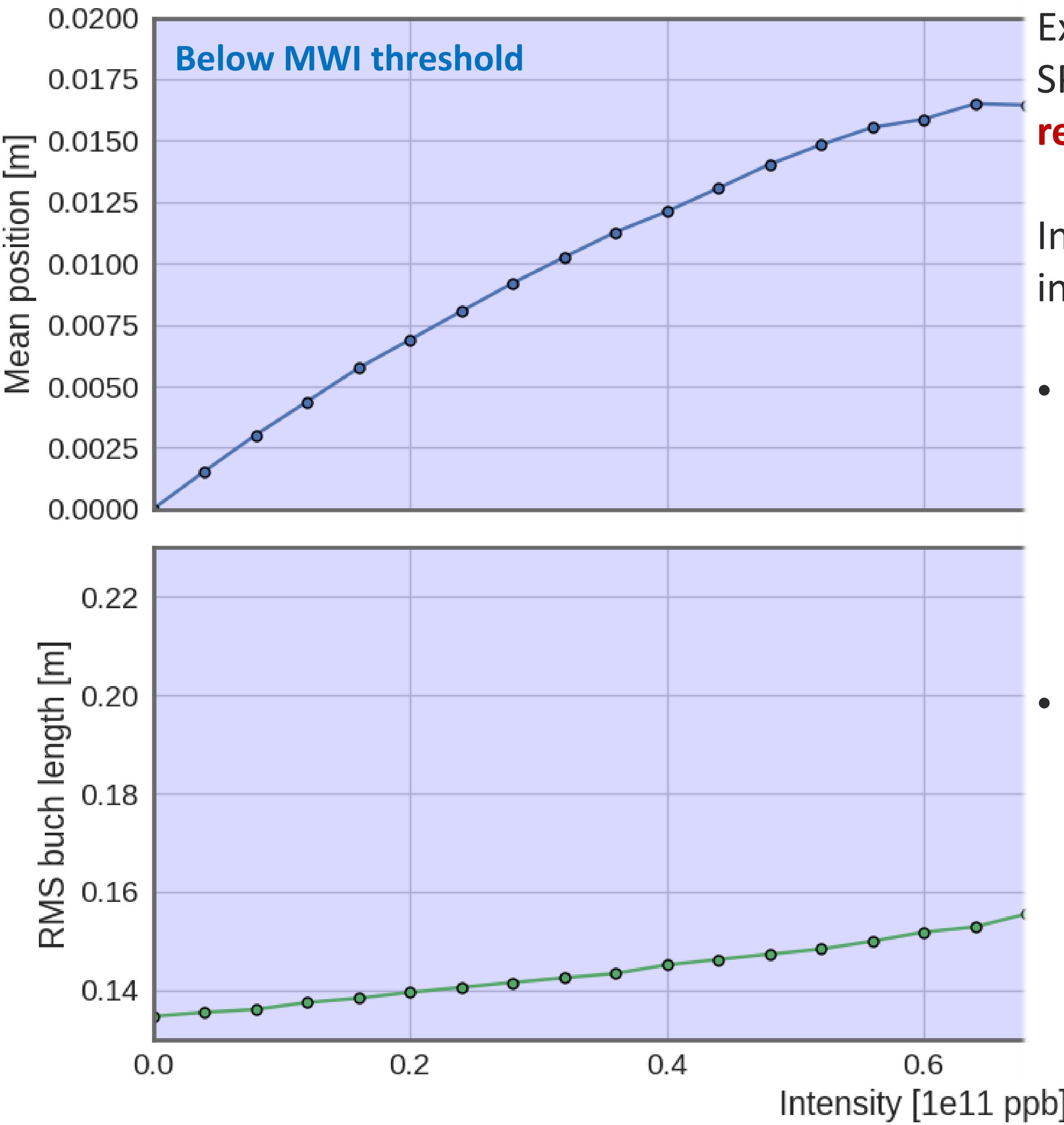
- The **RF system compensates** for the energy loss by imparting a net acceleration to the bunch
- Therefore, the bunch readjusts to a **new equilibrium distribution** in the bucket and moves to an average synchronous angle $\Delta\Phi_s$



$$\boxed{\sin\Delta\Phi_s = \frac{\Delta E_{\mathrm{turn}}}{NeV_m} = -\frac{e\omega_0}{2\pi N V_m}\sum_{p=-\infty}^{\infty}\left|\hat{\lambda}(p\omega_0)\right|^2 \mathrm{Re}\left(Z_{\parallel}(p\omega_0)\right)}$$



Bunch lengthening and MW instability



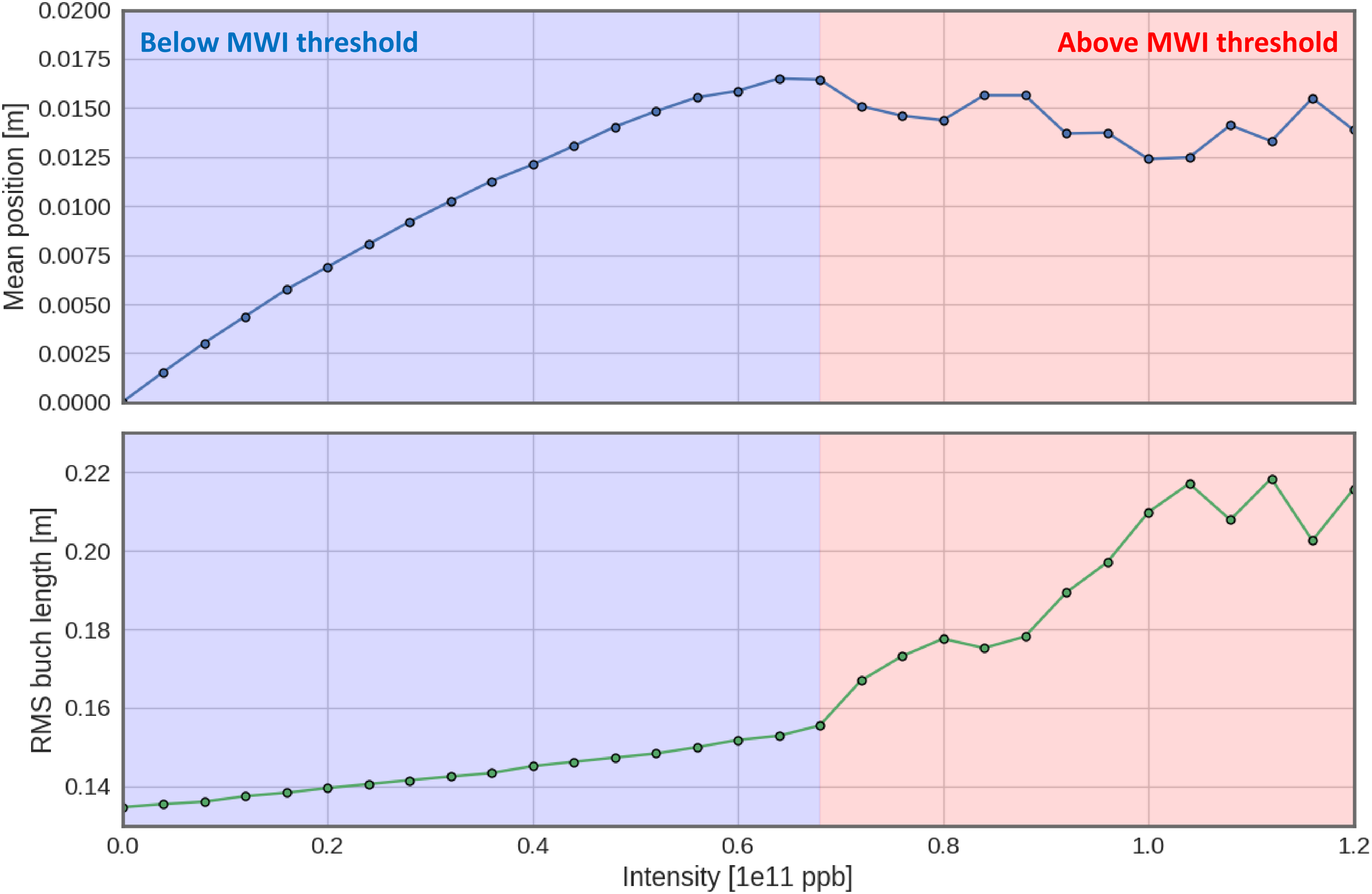
Examples of numerical simulations – SPS bunch with **single broad-band resonator** wake:

Initializing a matched bunch at low intensity, **two regimes are found**:

- Bunch lengthening/emittance blow up regime with roughly linear increase of the **synchronous phase** and **bunch length** with intensity

- Unstable regime (**turbulent bunch lengthening**)

Bunch lengthening and MW instability



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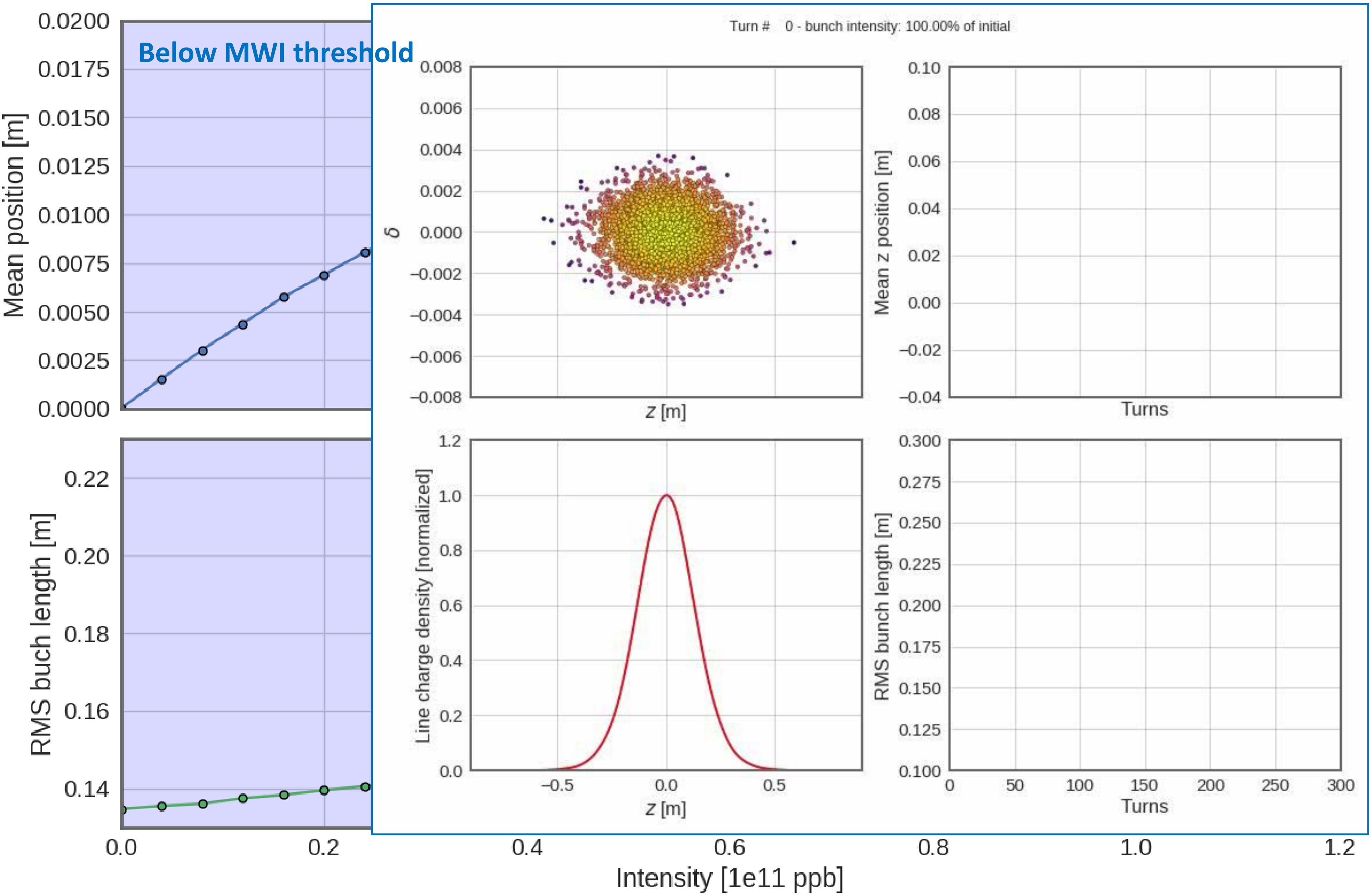
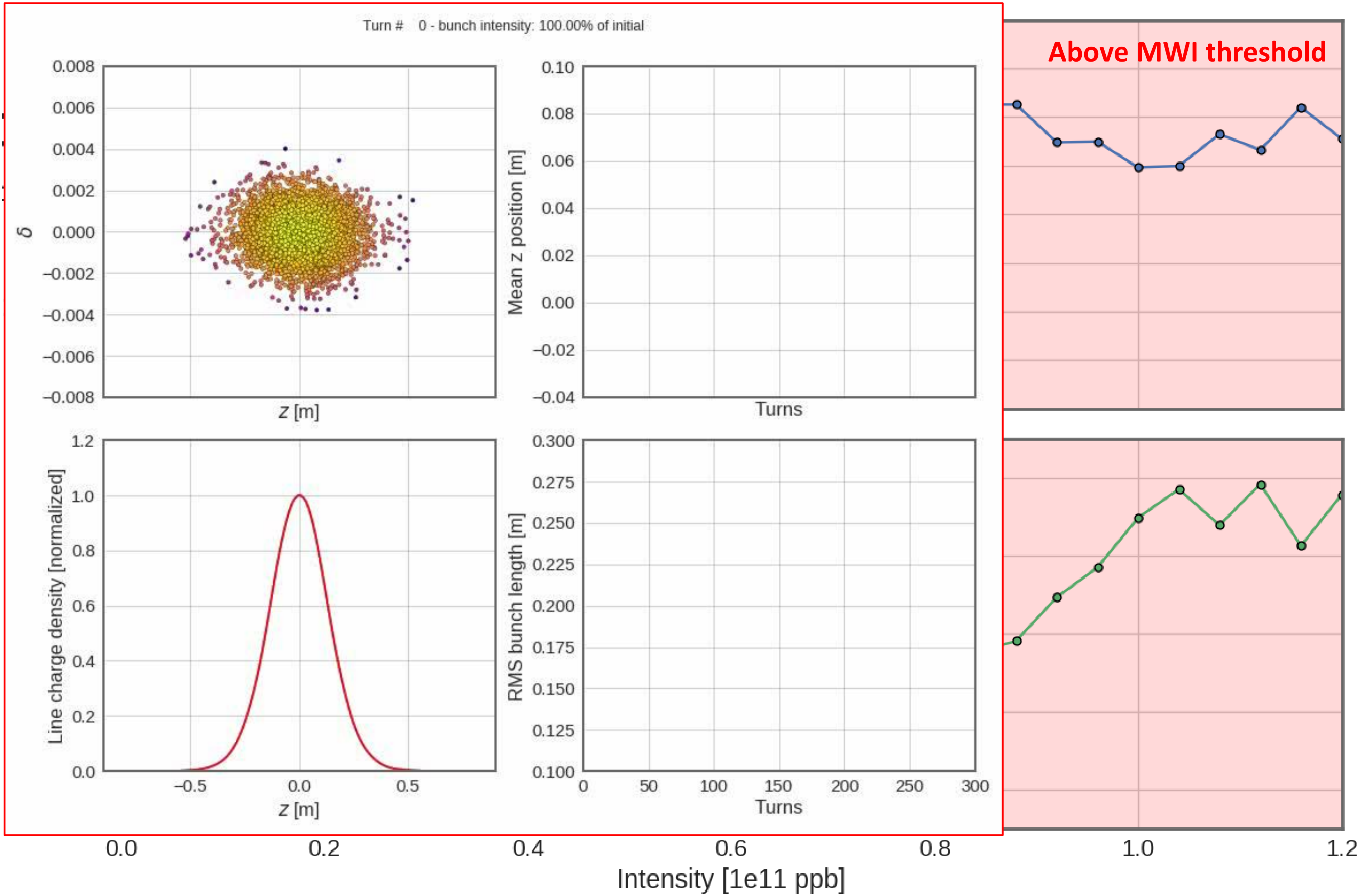


Bunch lengthening and MW instability



Signpost

- We have learned about the **impact of the longitudinal impedance on the beam**.
- We found the **Haissinki equation** and discussed the **potential well distortion** along with the **stable phase shift** and **synchrotron tune shift**.
- We looked at some generic wake fields and the **two regimes** of potential well distortion with **bunch lenthening** and its transition to the **microwave instability**.
- We will now look specifically at multi-turn wake fields and the phenomenon of **Robinson instability** and damping.

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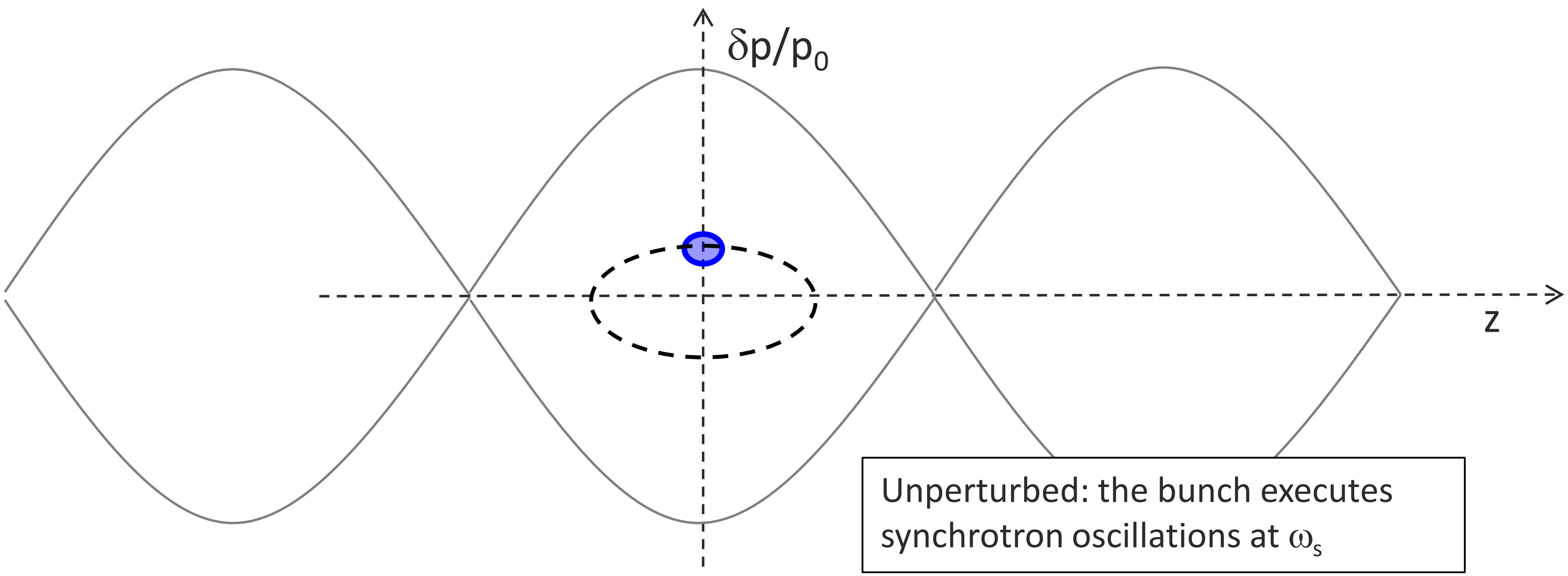
The Robinson instability

- To illustrate the Robinson instability we will use some simplifications:
  - The bunch is **point-like** and feels an external linear force (i.e. it would execute linear synchrotron oscillations in absence of the wake forces)
  - The bunch additionally feels the effect of a **multi-turn wake**



Unperturbed: the bunch executes synchrotron oscillations at ω_s

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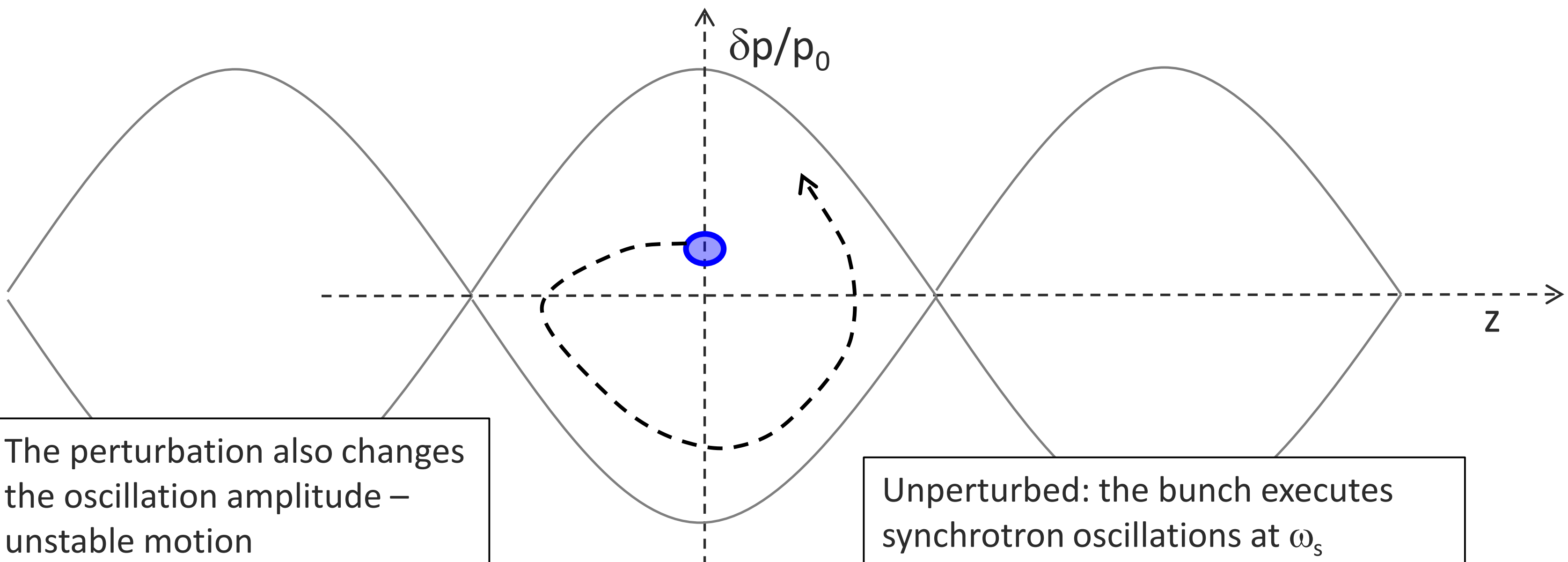

The perturbation also changes the oscillation amplitude – unstable motion

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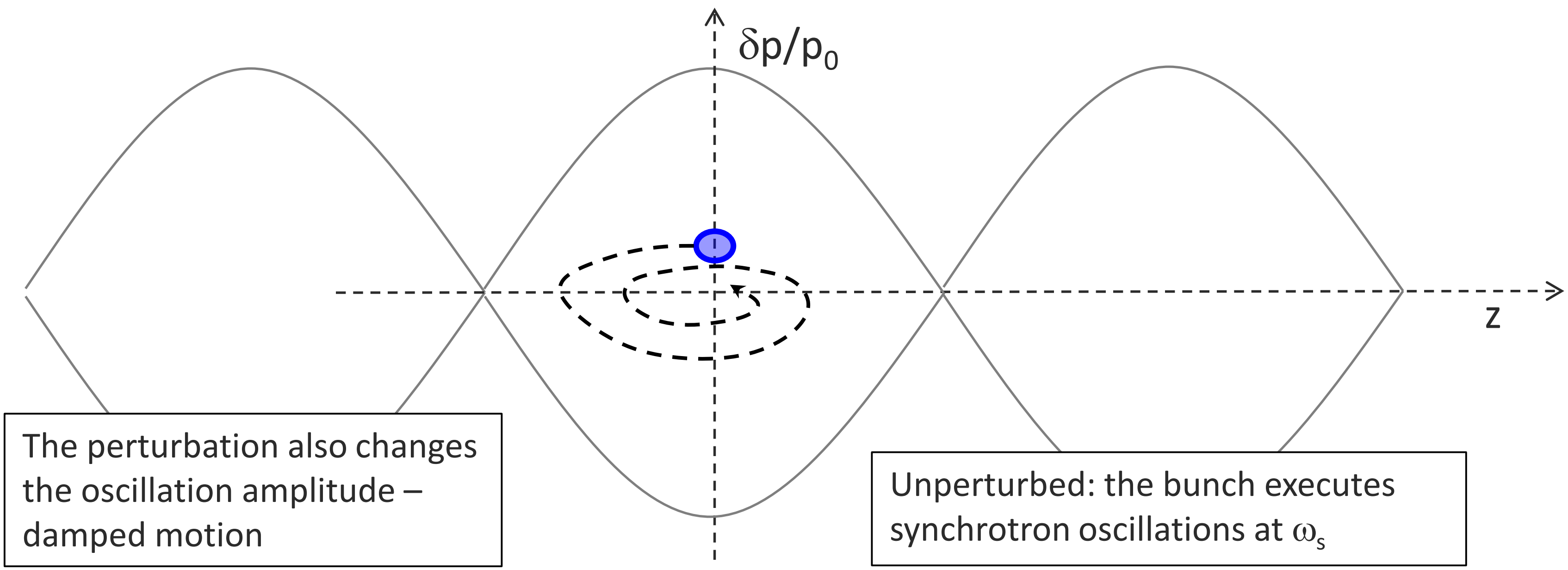

The perturbation also changes the oscillation amplitude – damped motion

Unperturbed: the bunch executes synchrotron oscillations at ω_s

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- Longitudinal Hamiltonian

$$H = -\frac{1}{2}\eta\,\delta^2 - \frac{1}{2\eta}\left(\frac{\omega_s}{\beta c}\right)^2 z^2 + \frac{e^2}{\beta^2 EC}\sum_k \int_0^z dz'' \int_{z''}^{\infty} dz'\,\lambda(z'+kC)\,W_{||}(z''-z'-kC)$$

$$= -\frac{1}{2}\eta\,\delta^2 - \frac{1}{2\eta}\left(\frac{\omega_s}{\beta c}\right)^2 z^2 + \frac{Ne^2}{\beta^2 EC}\sum_k \int_0^z dz''\,W_{||}\Big(z(t)-z(t-kT_0)-kC\Big)$$

- Expansion of wake field (we assume that the wake can be linearized on the scale of a synchrotron oscillation)

$$W_{||}(z(t)-z(t-kT_0)-kC) \approx W_{||}(kC) + W_{||}'(kC)\Big(z(t)-z(t-kT_0)\Big)$$

$$\approx W_{||}(kC) + W_{||}'(kC)\,kT_0\,\frac{dz(t)}{dt}$$

The Robinson instability

- The **first term** only contributes as a constant term in the solution of the equation of motion, i.e. the synchrotron oscillation will be executed around a certain z0 and not around 0. This term represents the **stable phase shift** that compensates for the energy loss
- The **second term** is a dynamic term introduced as a **"friction" term** in the equation of the oscillator, which can **lead to instability**!

- Equations of motion

$$\frac{d^2 z}{dt^2} + \omega_s^2\, z^2 = \frac{Ne^2\eta}{Cm_0\gamma} \sum_{k=-\infty}^{\infty} \cancel{W_{\parallel}(kC)} + W_{\parallel}'(kC)\, kT_0\, \frac{dz}{dt}$$

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- Ansatz

$$z(t) \propto \exp\left(-i\Omega t\right)$$

$$\boxed{\frac{i}{C} \sum_{p=-\infty}^{\infty} \Big(p\omega_0\, Z_{\|}\left(p\omega_0\right) - \left(p\omega_0 + \Omega\right)\, Z_{\|}\left(p\omega_0 + \Omega\right)\Big)}$$

Expressed in terms of impedance

- Solution

$$\left(\Omega^2 - \omega_s^2\right) = -\frac{Ne^2\eta}{Cm_0\gamma} \sum_{k=-\infty}^{\infty} \boxed{\Big(1 - \exp\left(-ik\Omega T_0\right)\Big)\, W_{\|}'(kC)}$$

The Robinson instability

- We assume a small deviation from the synchrotron tune:
  - Re(Ω - ω_s) → **Synchrotron tune shift**
  - Im(Ω - ω_s) → **Growth/damping rate**, only depends on the dynamic term, if it is positive there is an instability!

- **Solution:**

$$\left(\Omega^2-\omega_s^2\right)=-\frac{iNe^2\eta}{C^2m_0\gamma}\sum_{p=-\infty}^{\infty}\Big(p\omega_0\,Z_{\|}\left(p\omega_0\right)-\left(p\omega_0+\Omega\right)\,Z_{\|}\left(p\omega_0+\Omega\right)\Big)$$
$$\approx 2\omega_s\,\left(\Omega-\omega_s\right)$$

- **Tune shift:**

$$\Delta\omega_s=\mathrm{Re}\left(\Omega-\omega_s\right)=\frac{e^2}{m_0c^2}\frac{N\eta}{2\omega_s\gamma T_0^2}$$
$$\sum_{p=-\infty}^{\infty}\Big(p\omega_0\;\mathrm{Im}\left[Z_{\|}\right]\left(p\omega_0\right)-\left(p\omega_0+\omega_s\right)\,\mathrm{Im}\left[Z_{\|}\right]\left(p\omega_0+\omega_s\right)\Big)$$

- **Growth rate:**

$$\tau^{-1}=\mathrm{Im}\left[\Omega-\omega_s\right]=\frac{e^2}{m_0c^2}\frac{N\eta}{2\omega_s\gamma T_0^2}\sum_{p=-\infty}^{\infty}\Big(\left(p\omega_0+\omega_s\right)\,\mathrm{Re}\left[Z_{\|}\right]\left(p\omega_0+\omega_s\right)\Big)$$

The Robinson instability

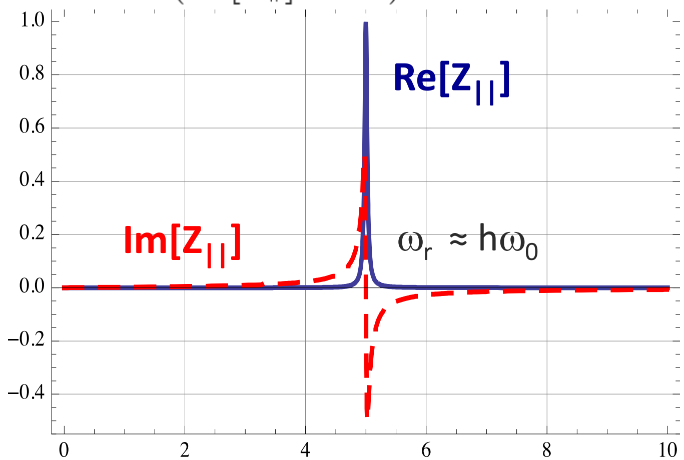
- We assume the impedance to be peaked at a frequency ω_r close to $h\omega_0 \gg \omega_s$ (e.g. RF cavity fundamental mode or HOM)
- Only two dominant terms are left in the summation at the RHS of the equation for the growth rate
- Stability requires that η and $\Delta\,\mathrm{Re}\left[Z_{\|}\right](p\omega_0)$ have different signs
- **Solution:**

$$\tau^{-1} = \mathrm{Im}\left(\Omega - \omega_s\right) = \frac{e^2}{m_0 c^2}\frac{N\eta}{2\omega_s\gamma T_0^2}\sum_{p=-\infty}^{\infty}\Big(\left(p\omega_0+\omega_s\right)\,\mathrm{Re}(Z)_{\|}(p\omega_0+\omega_s)\Big)$$

$$= \frac{e^2}{m_0 c^2}\frac{N\eta h\omega_0}{2\omega_s\gamma T_0^2}\underbrace{\Big(\mathrm{Re}\left[Z_{\|}\right](h\omega_0+\omega_s) - \mathrm{Re}\left[Z_{\|}\right](h\omega_0-\omega_s)\Big)}_{\Delta\left(\mathrm{Re}\left[Z_{\|}\right](h\omega_0)\right)}$$

- **Stability criterion:**

$$\eta\cdot\Delta\Big(\mathrm{Re}\left[Z_{\|}\right](h\omega_0)\Big) < 0$$



The Robinson instability

- **Stability criterion:** $\eta \cdot \Delta\Big(\mathrm{Re}\left[Z_{\parallel}\right](h\omega_0)\Big) < 0$

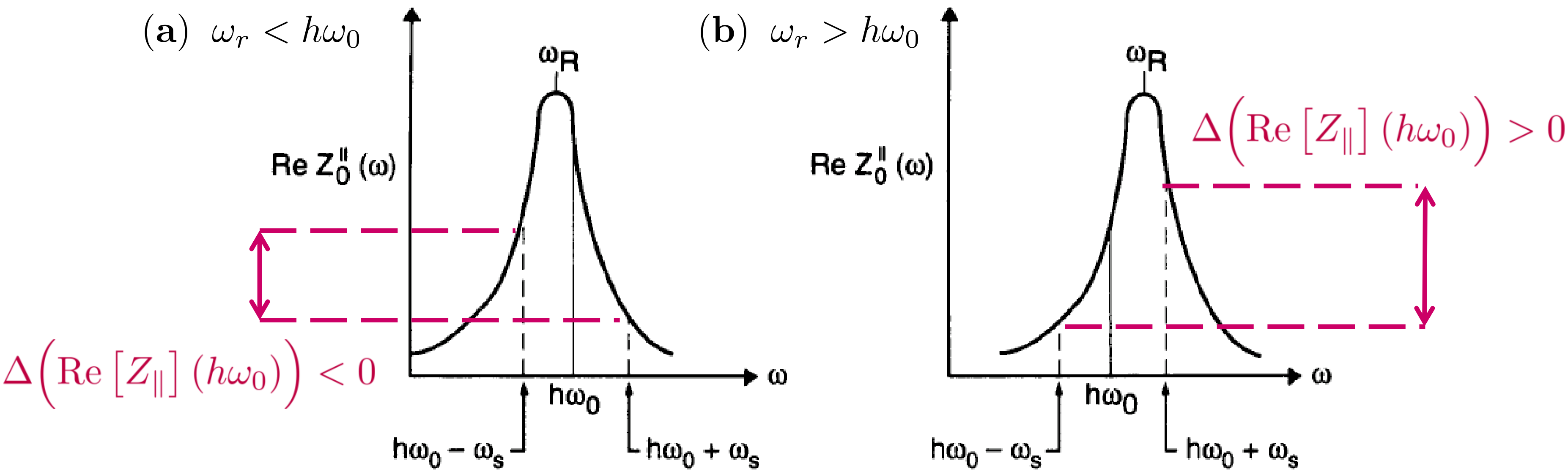
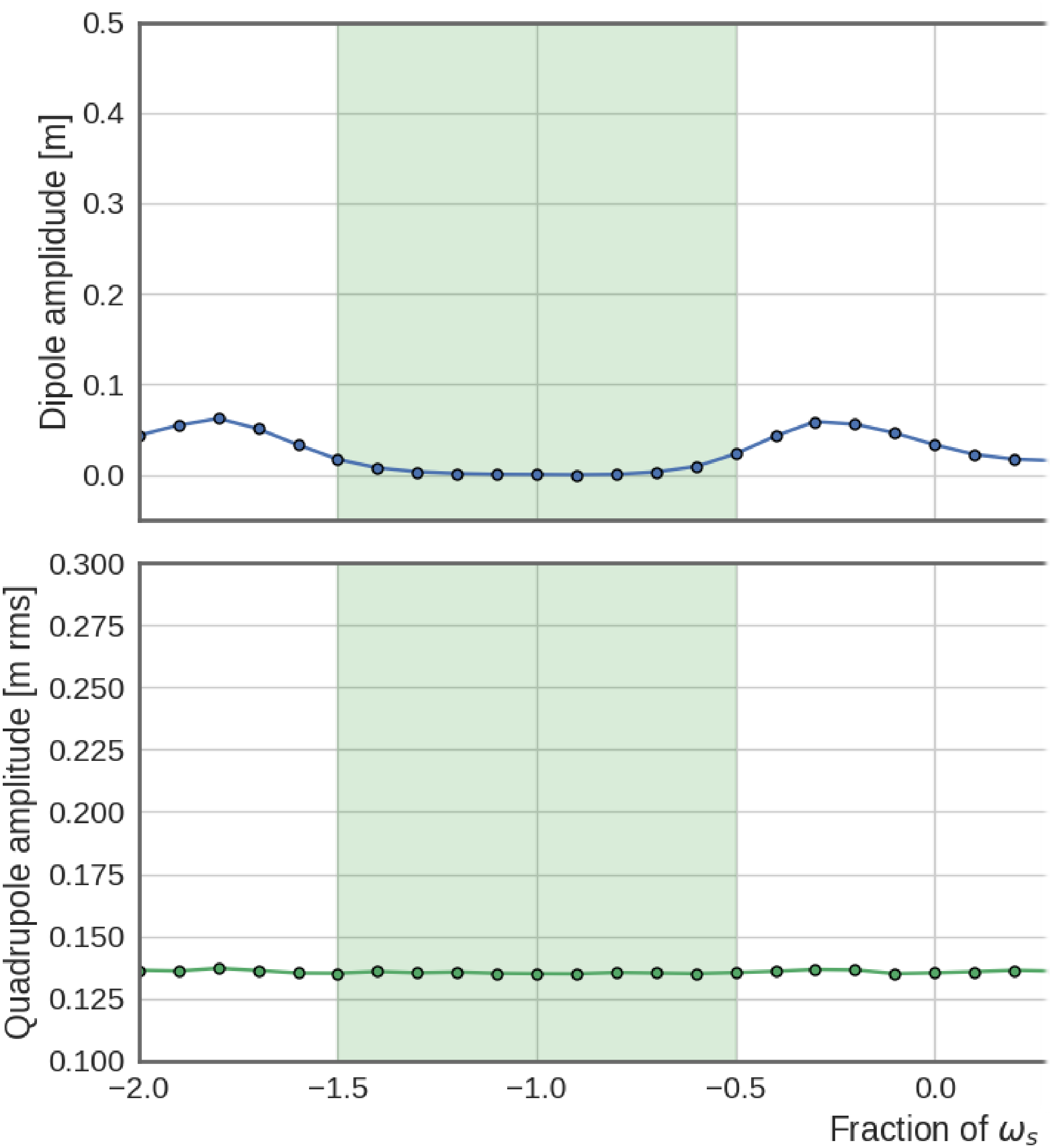


Figure 4.4. Illustration of the Robinson stability criterion. The rf fundamental mode is detuned so that ω_R is (a) slightly below $h\omega_0$ and (b) slightly above $h\omega_0$. (a) is Robinson damped above transition and antidamped below transition. (b) is antidamped above transition and damped below transition.

	$\omega_r < h\omega_0$	$\omega_r > h\omega_0$
Above transition (η > 0)	stable	unstable
Below transition (η < 0)	unstable	stable

Robinson damping and instability

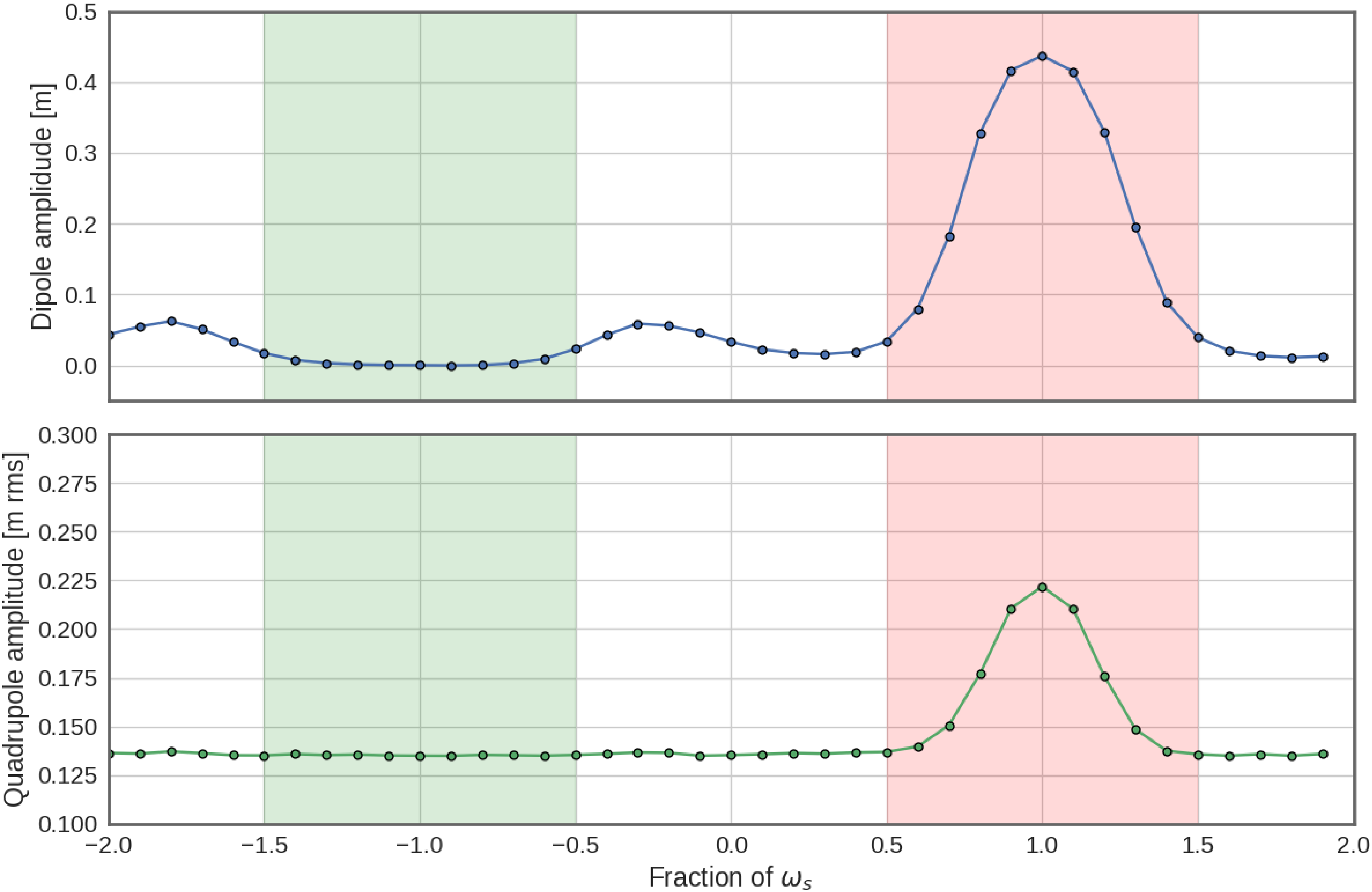


Examples of numerical simulations – SPS bunch with **single narrow-band resonator** wake:

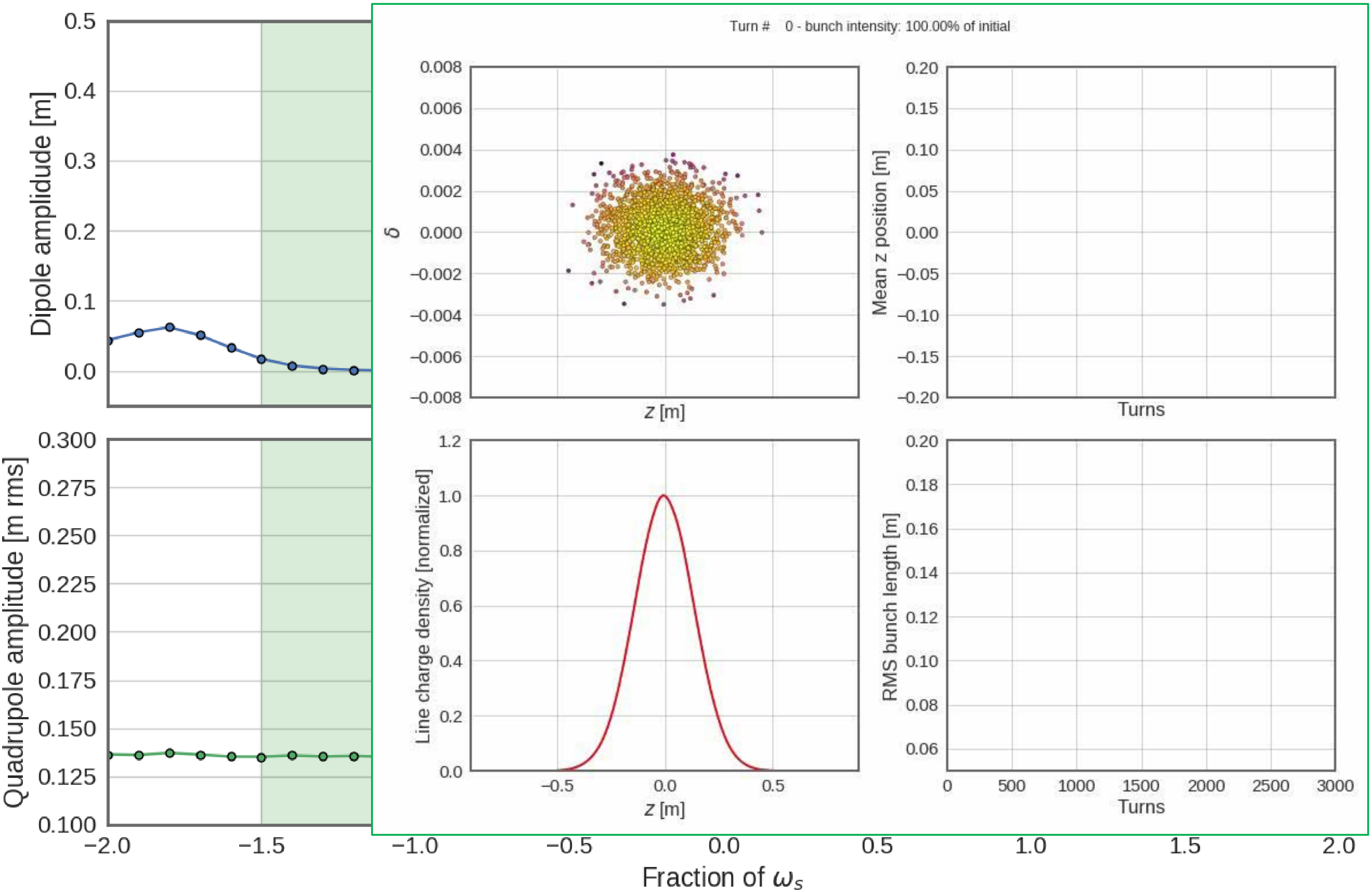
Initializing an otherwise matched bunch with a slight momentum error, **two regimes are found**:

- Regime of **Robinson damping** when the resonator is **detuned to $h\omega_0 - \omega_s$**. Initial dipole oscillations are damped.
- Regime of **Robinson instability** when the resonator is **detuned to $h\omega_0 + \omega_s$**. Initial dipole oscillations start to are grow exponentially.

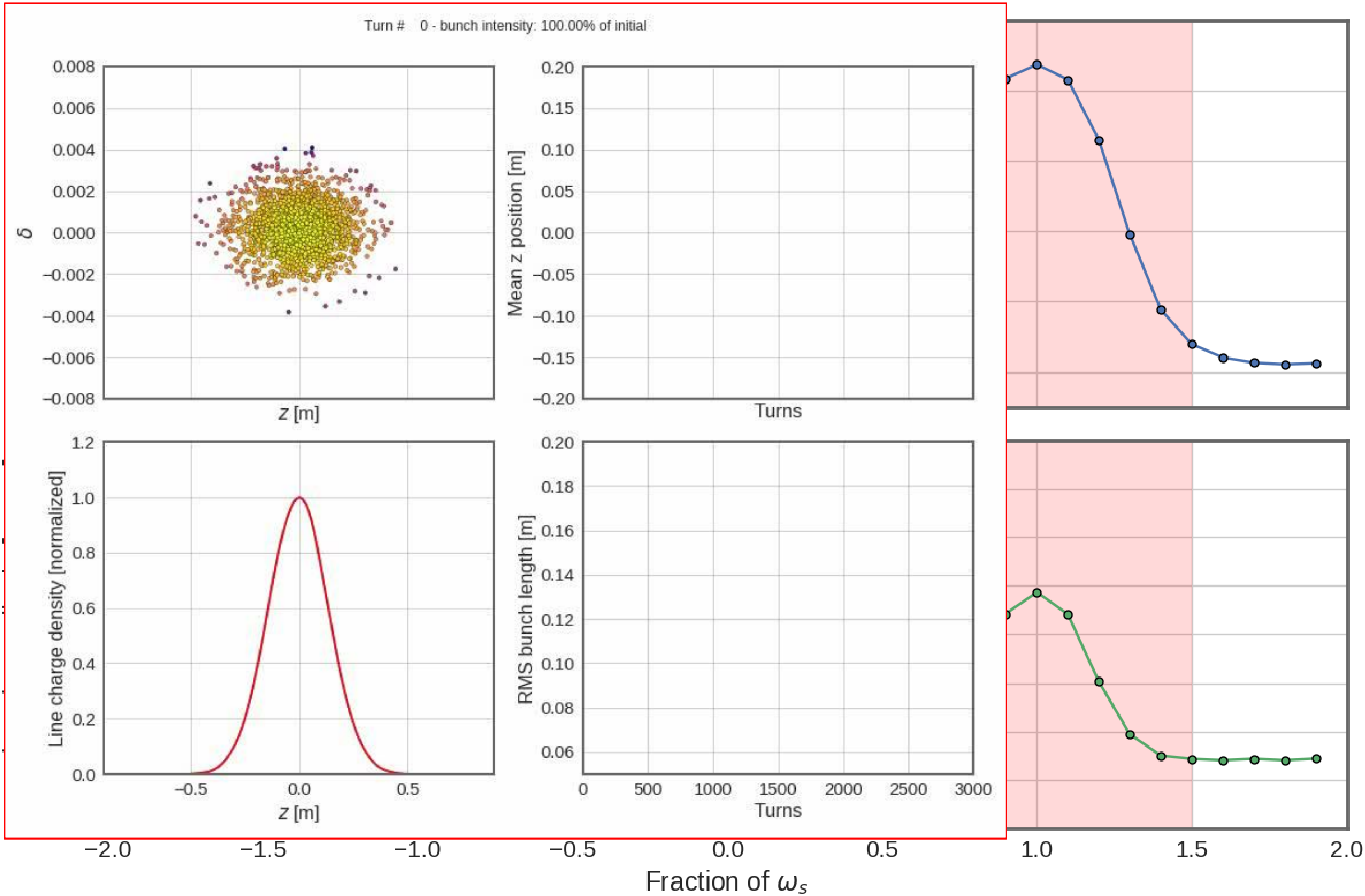
Robinson damping and instability



Robinson damping and instability



Robinson damping and instability



Other longitudinal instabilities

- The **Robinson instability** occurs for a single bunch under the action of a **multi-turn wake field**
  - It contains a term of coherent synchrotron tune shift which depends only on the imaginary part of the longitudinal impedance
  - It results into an unstable rigid bunch dipole oscillation where the growth rate depends on the real part of the longitudinal impedance

- Other **important collective effects** can affect a bunch in a beam – some of them of which we have also seen
  - Potential well distortion (resulting in synchronous phase shift, bunch lengthening or shortening, synchrotron tune shift/spread)
  - High intensity single bunch instabilities (e.g. microwave instability)
  - Coasting beam instabilities (e.g. negative mass instability)
  - Coupled bunch instabilities

- To be able to study these effects we would need to resort to a **more detailed description** of the bunch(es)
  - Vlasov equation (kinetic model)
  - Macroparticle simulations

Signpost

- We have **discussed longitudinal wake fields** and impedances and their impact on both the machine as well as the beam.
- We have learned about **beam induced heating** and how it is related to the beam power spectrum and the machine impedance.
- We have discussed the effects of **potential well distortion** (stable phase and synchrotron tune shifts, bunch lengthening and shortening).
- We have seen some examples of **longitudinal instabilities** (Microwave, Robinson).

Part 2: Longitudinal wakefields –

impact on machine elements and beam dynamics

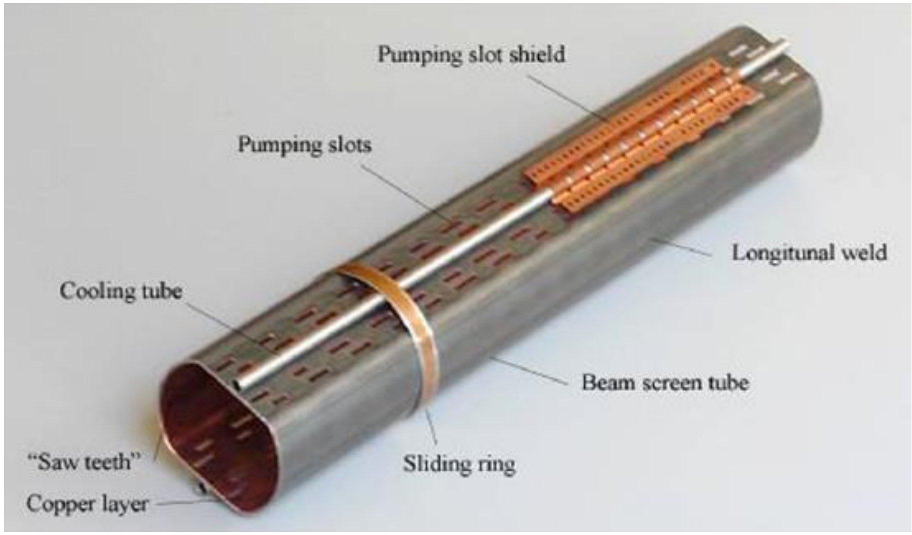
- Longitudinal wake fields and the longitudinal wake function
- Energy loss – beam induced heating and stable phase shift
- Potential well distortion, bunch lengthning and microwave instability
- Robinson instability

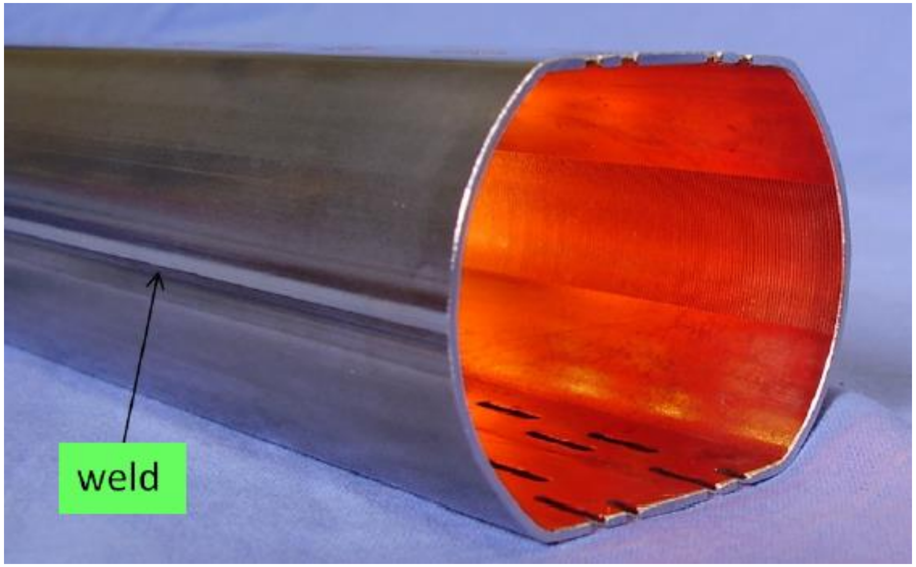
End part 2

Backup - wakefields

More heat loads and heat loads with exotic bunch spacings

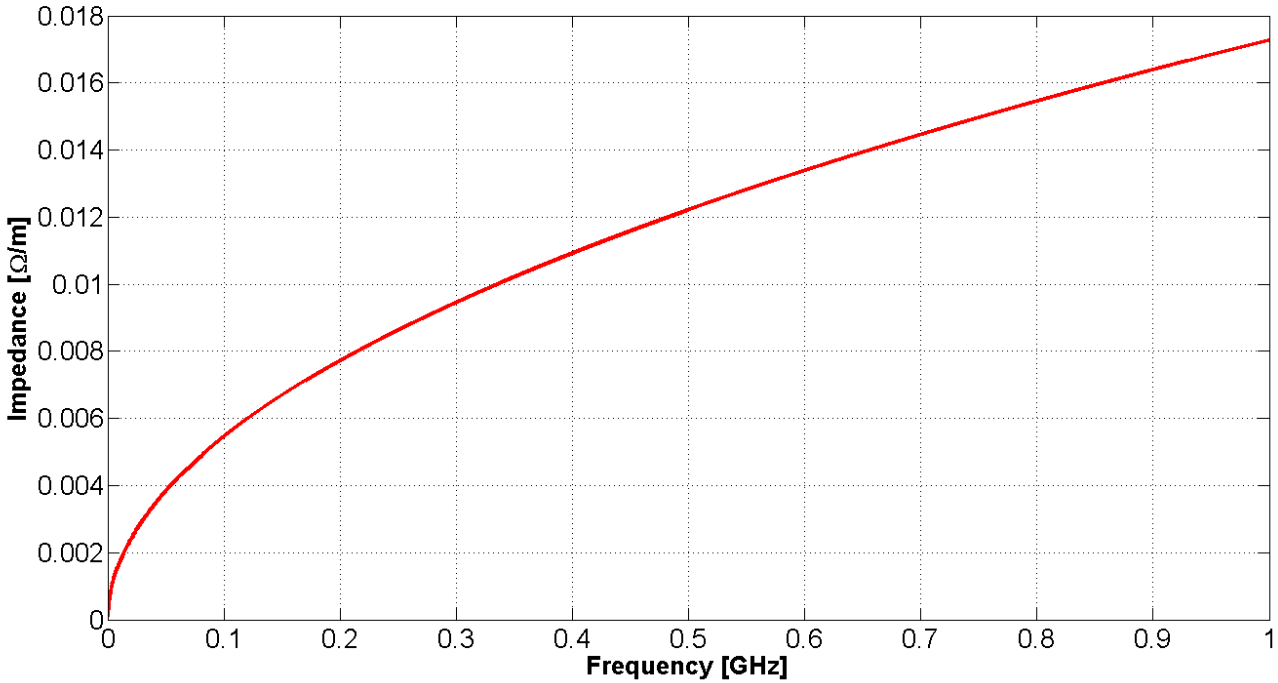
Application to the LHC beam screen





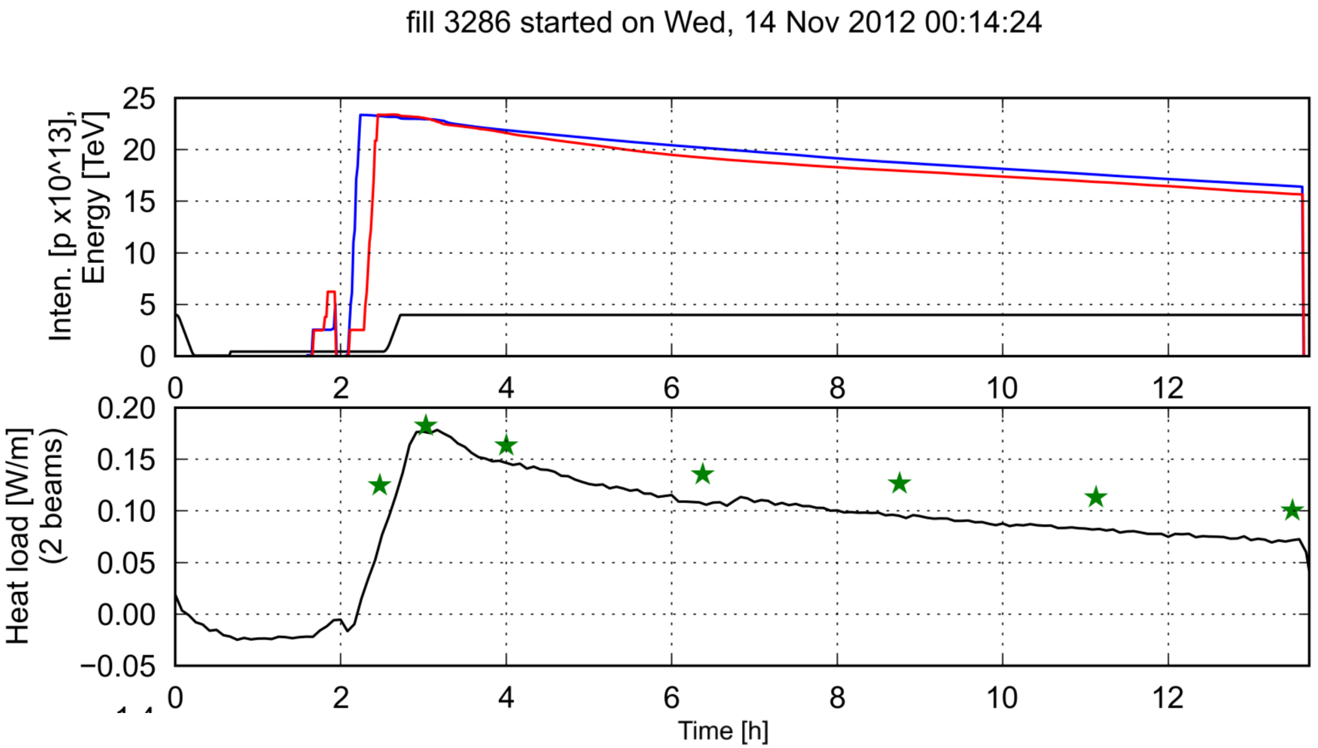
- All along the arcs and in other cold regions of the LHC, a beam screen is interposed between the beam and the cold bore
- The LHC beam screen is made of stainless steel with a layer of **few mm of co-laminated copper**
- Due to the production procedure, there is **a stainless steel weld** on one side of the beam screen that remains exposed to the beam.
- The screen has **holes for pumping** on top and bottom

Application to the LHC beam screen



- The impedance model includes the **weld on one side of the beam screen**, which means a small longitudinal stripe of exposed StSt, as well as **the pumping holes**

Application to the LHC beam screen



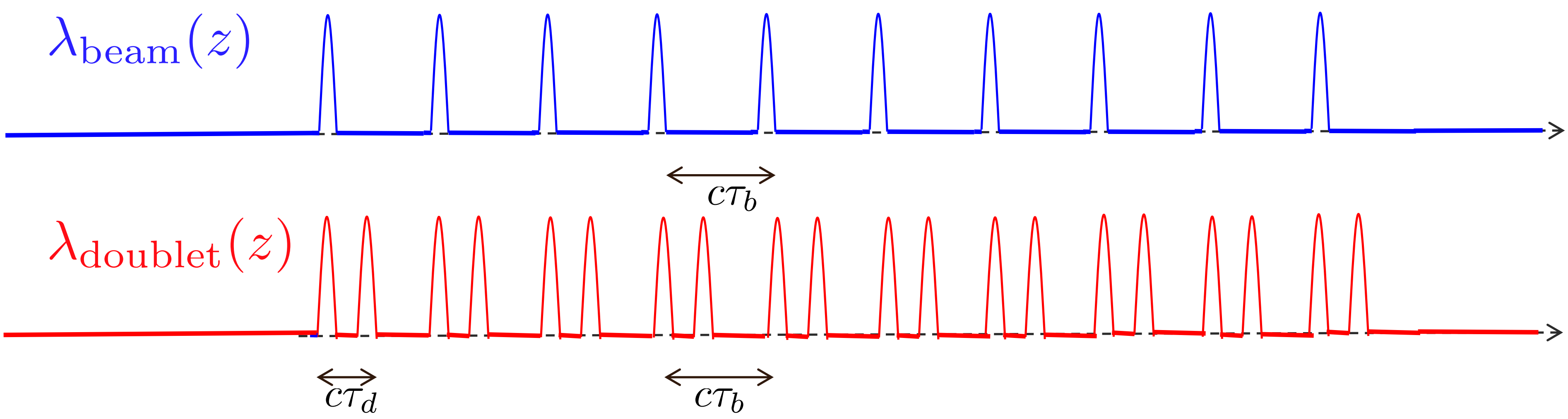
★ Estimation (impedance + synchrotron rad.)

—— Heat load measurement from cryogenics

$$\Delta E_{\text{beam}} = \frac{e^2\omega_0}{2\pi}\sum_{p=-\infty}^{\infty}|\tilde{\lambda}(p\omega_0)|^2\text{Re}\left[Z_{||}(p\omega_0)\right]\cdot\left[\frac{1-\cos(\frac{2\pi Mp}{h})}{1-\cos(\frac{2\pi p}{h})}\right]$$

- The heat dissipated on the beam screen **can be calculated for a beam made of bunches spaced by 50 ns** and compared to the measurement from cryogenics

Beam energy loss: a doublet beam



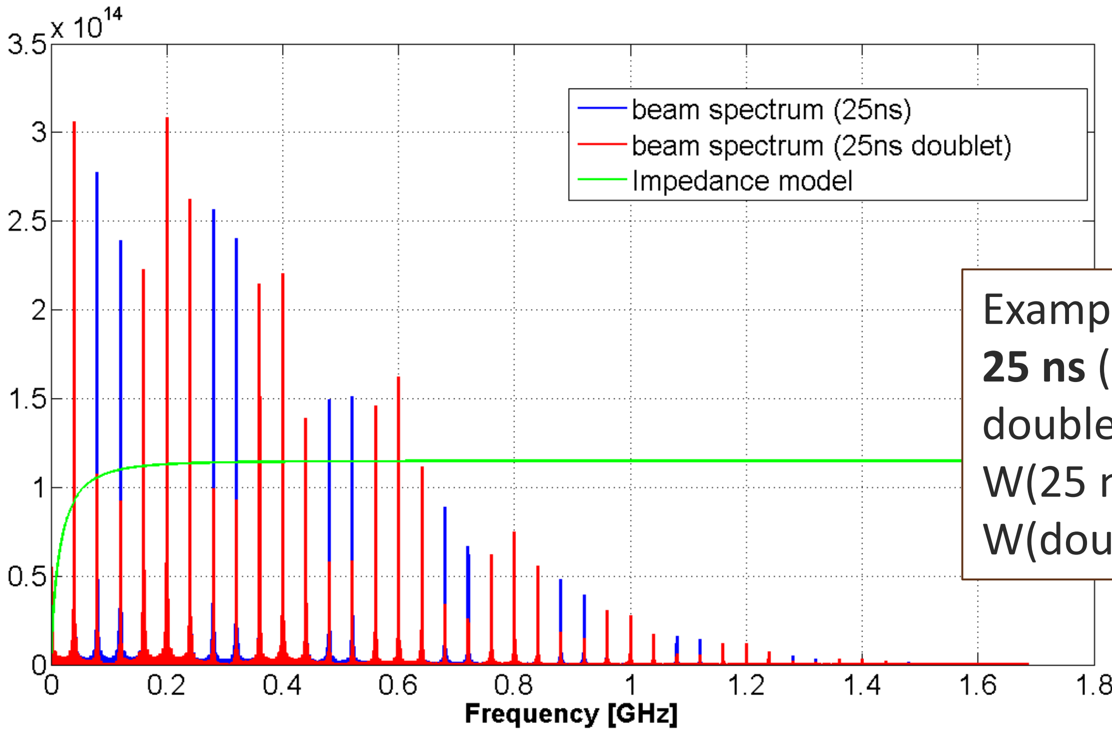
$$\Lambda_{\mathrm{beam}}(\omega) \rightarrow \Lambda_{\mathrm{doublet}}(\omega) = \Lambda_{\mathrm{beam}}(\omega)\left[1 + \exp(-i\omega\tau_d)\right]$$

$$\Delta E_{\mathrm{doublet}} = \frac{2e^2\omega_0}{\pi} \sum_{p=-\infty}^{\infty} \left|\Lambda_{\mathrm{beam}}(p\omega_0)\right|^2 \cos^2\left(\frac{p\omega_0\tau_d}{2}\right) \mathrm{Re}\left[Z_{||}(p\omega_0)\right]$$

N.B. in this example the doublet has double total intensity than single beam

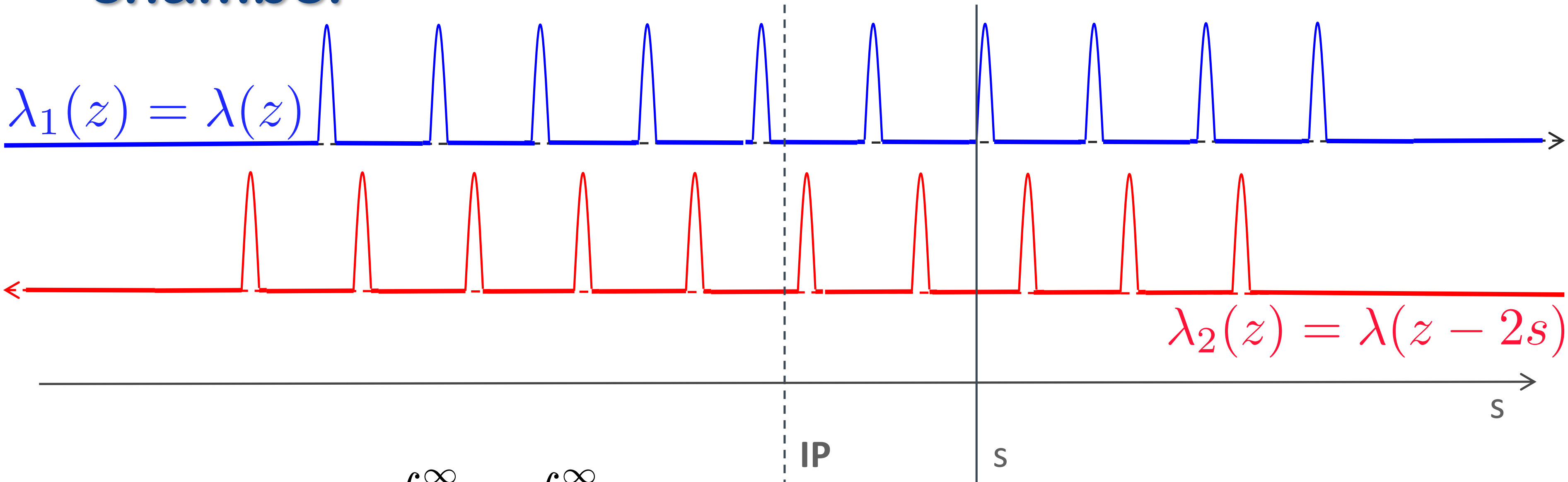
Beam energy loss: a doublet beam

- No additional impedance energy loss is expected with the doublet beam with respect to nominal beam for same total intensity
  - Beam power spectrum is modulated with $\cos^2$ function and lines are weakened by this modulation
  - For higher doublet intensity, global effect depends on the impedance spectrum
  - Example → LHC injection beam stopper (TDI)

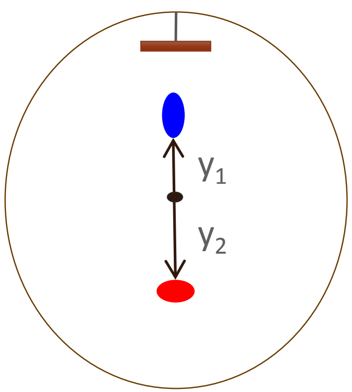


Example for LHC beam in TDI:
25 ns (1.2 x 10^{11} p/b) vs. **20+5 ns** doublet (1.5 x 10^{11} p/doublet)
W(25 ns) = 456 W
W(doublet) = 338 W

Beam energy loss: collider's common chamber



$$\Delta E_{\mathrm{beam1}}(s) = e^2 \int_{-\infty}^{\infty} \lambda(z) \int_{-\infty}^{\infty} \left[\lambda(z')W_{||b1}(z-z') - \lambda(z'-2s)W_{||b2}(z-z')\right] dz'dz$$

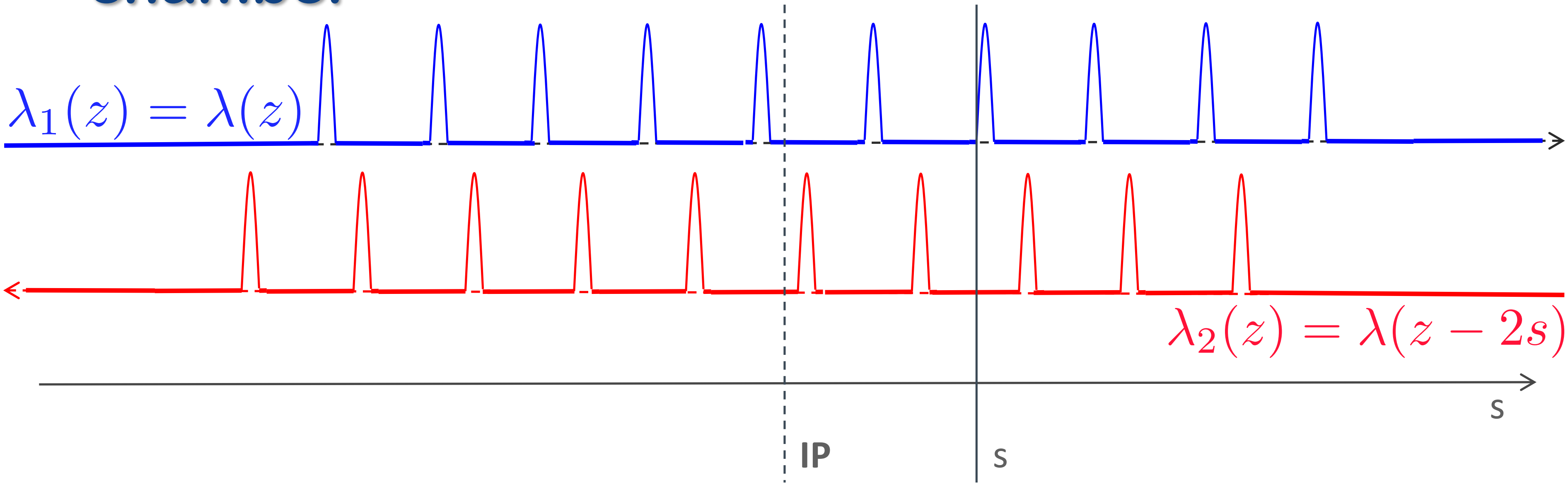


$$W_{||b1}(z) = W_{||}^{(0)}(z) + W_{||}^{(1d)}(z)y_1 + W_{||}^{(1q)}(z)y_1$$

$$W_{||b2}(z) = W_{||}^{(0)}(z) + W_{||}^{(1d)}(z)y_2 + W_{||}^{(1q)}(z)y_1$$

$$\text{with} \quad W_{||}^{1d}(z) = W_{||}^{1q}(z)$$

Beam energy loss: collider's common chamber



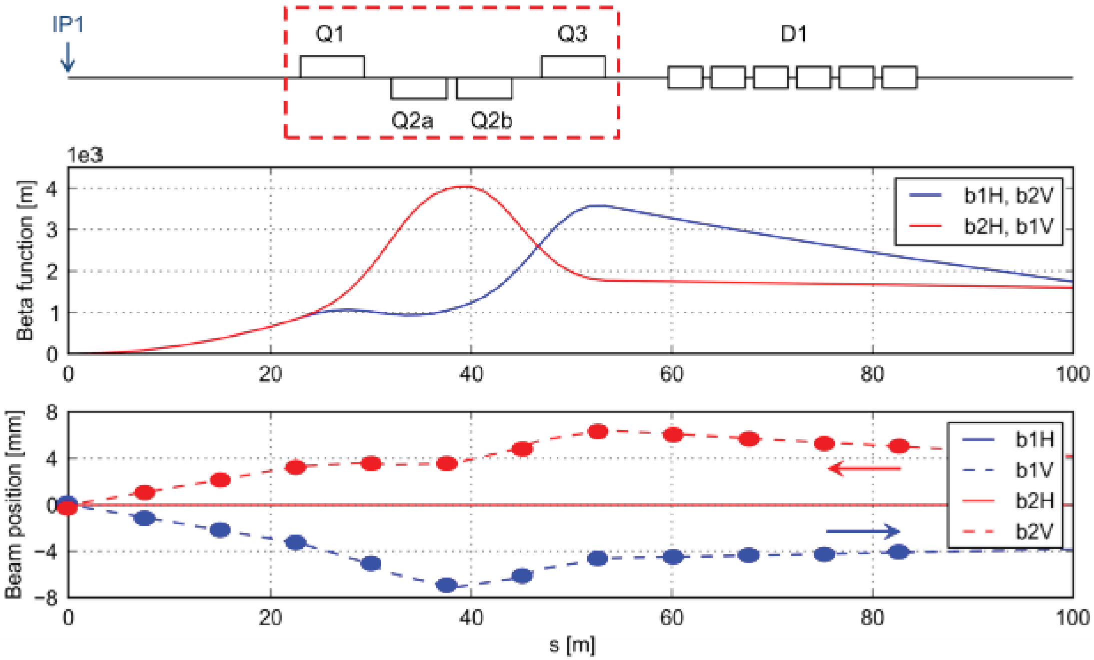
$$W_{||}^{1d}(z),\, W_{||}^{1q}(z) \overset{\mathcal{F}}{\Longleftrightarrow} Z_{||}^{1}(\omega) \qquad\qquad W_{||}^{0}(z) \overset{\mathcal{F}}{\Longleftrightarrow} Z_{||}^{0}(\omega)$$

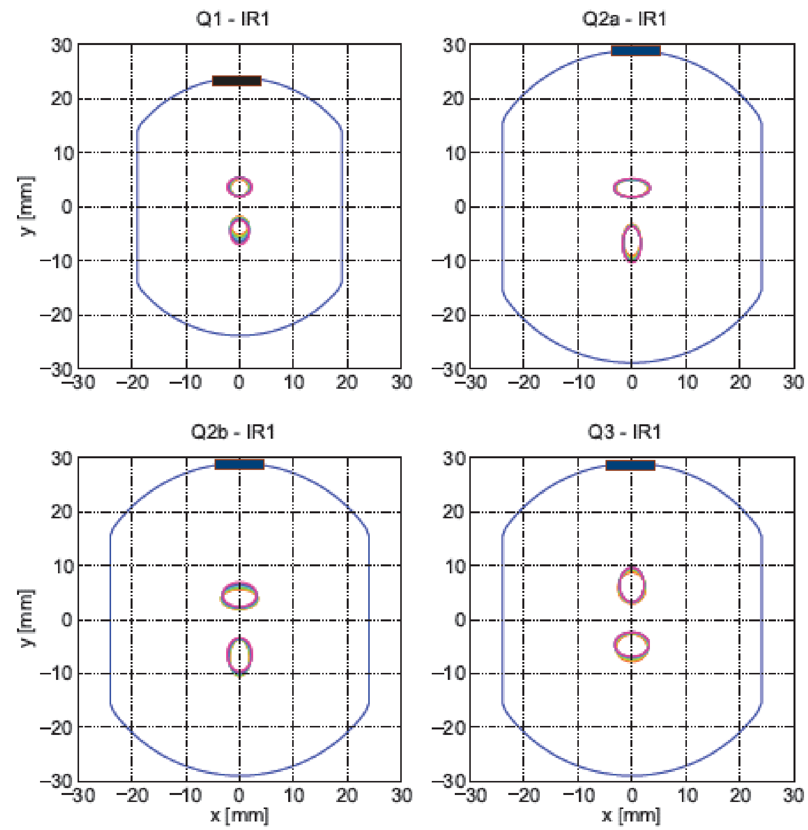
$$\Delta E_{\text{beam1}}(s) + \Delta E_{\text{beam2}}(s) =$$

$$\frac{4e^2\omega_0}{\pi}\sum_{p=0}^{\infty}|\Lambda(p\omega_0)|^2\left\{\mathrm{Re}\left[Z_{||}^{0}(p\omega_0)\right] + [y_1(s)+y_2(s)]\,\mathrm{Re}\left[Z_{||}^{1}(p\omega_0)\right]\right\}\cdot\sin^2\left(\frac{p\omega_0 s}{c}\right)$$

$$\Delta W_{CC} = \frac{\omega_0}{2\pi}\int_{-s_0}^{s_0}\left[\Delta E_{\text{beam1}}(s) + \Delta E_{\text{beam2}}(s)\right] ds$$

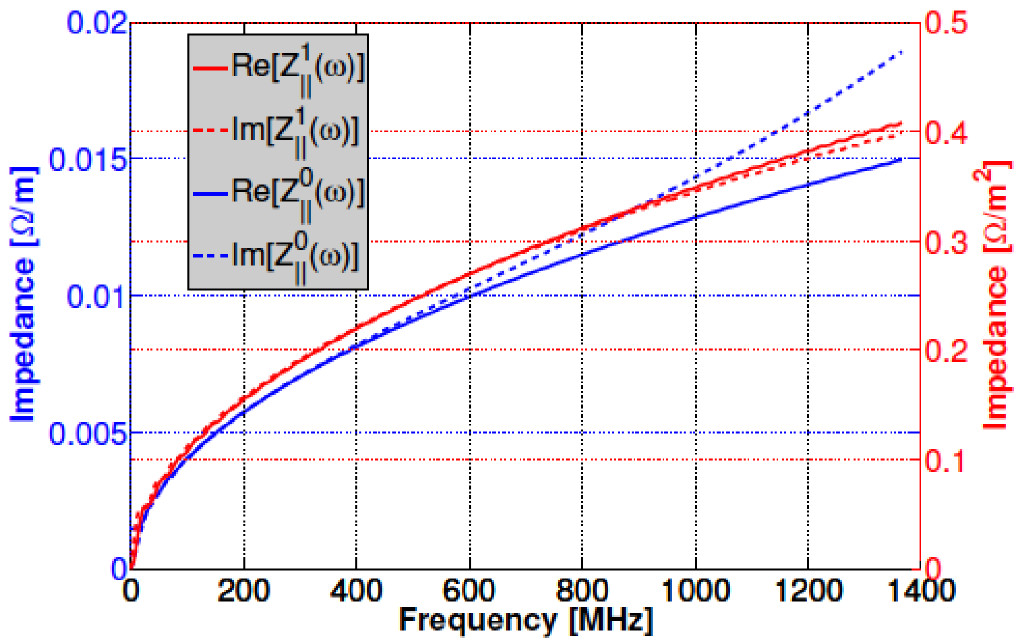
Beam energy loss in the LHC triplets





- Application to the LHC inner triplets
  - Beams are separated vertically (IP1) or horizontally (IP5)
  - Strongly off-axis for ~30m, all relative delays between beams swept
  - Asymmetric chamber in the direction of separation because of the weld

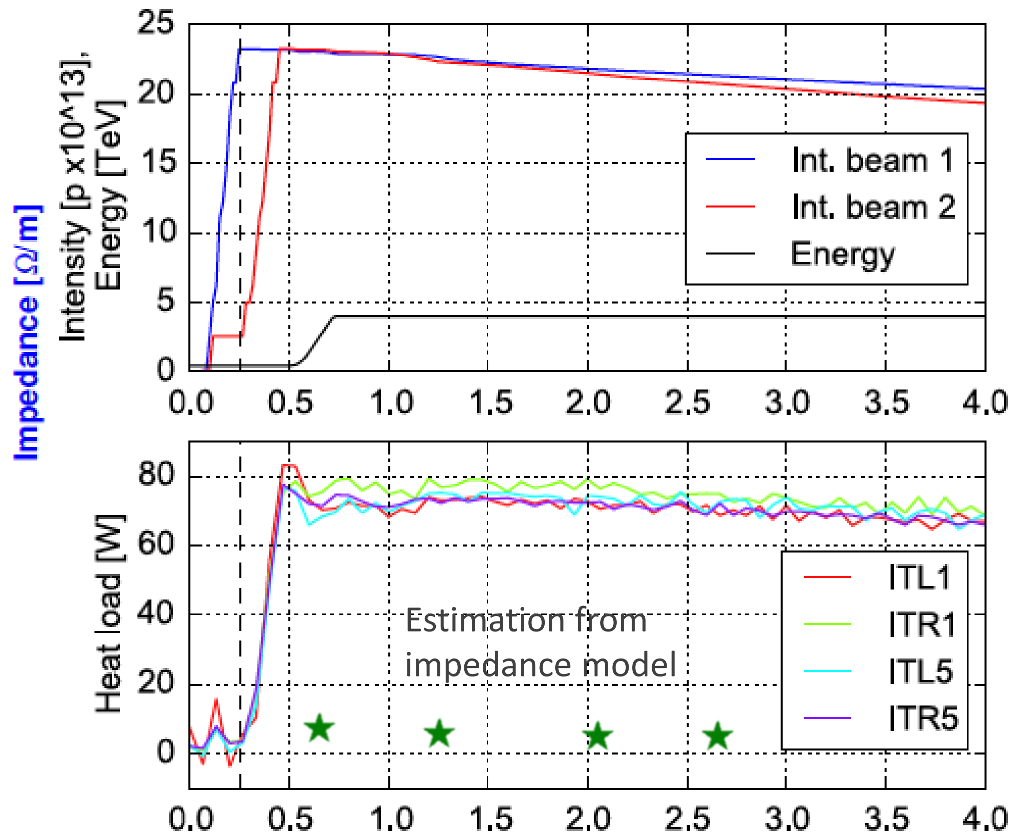
Beam energy loss in the LHC triplets



$$\Delta W_{IT} = 4\,\mathrm{W}$$

for a typical 50 ns fill of the LHC

Beam energy loss in the LHC triplets



$$\Delta W_{IT} = 4\,W$$

for a typical 50 ns fill of the LHC

- Comparison with measured data (L. Tavian)
  - Estimated heat load more than a factor 10 below measurement
  - Indication of a dominant contribution from electron cloud, also enhanced by the two-beam effect

Panofsky-Wenzel Theorem

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Source terms (displaced point charge traveling along s with speed v) in Cartesian coordinates:

$$\rho(x, y, s, t) = q_1 \delta(x - x_1)\delta(y - y_1)\delta(s - vt)$$

$$\vec{j}(x, y, s, t) = \rho(x, y, s, t)\vec{v}$$

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Source terms (displaced point charge traveling along s with speed v) in cylindrical coordinates:

$$\rho(r, \theta, s, t) = \frac{q_1}{r_1} \delta(r - r_1) \delta_P(\theta) \delta(s - vt) =$$

$$= \frac{q_1}{r_1} \delta(r - r_1) \delta(s - vt) \sum_{m=0}^{\infty} \frac{\cos m\theta}{\pi(1 + \delta_{m0})}$$

$$\vec{j}(r, \theta, s, t) = \rho(r, \theta, s, t) \vec{v}$$

$$v = \beta c \;\; \text{with} \;\; \beta \approx 1$$

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_s}{\partial s} = \frac{\rho}{\epsilon_0}$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_s}{\partial s} = 0$$

$$\frac{\partial B_s}{\partial y} - \frac{\partial B_y}{\partial s} - \frac{1}{c^2}\frac{\partial E_x}{\partial t} = 0$$

$$\frac{\partial B_x}{\partial s} - \frac{\partial B_s}{\partial x} - \frac{1}{c^2}\frac{\partial E_y}{\partial t} = 0$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} - \frac{1}{c^2}\frac{\partial E_s}{\partial t} = \mu_0 \rho c$$

$$\frac{\partial E_s}{\partial y} - \frac{\partial E_y}{\partial s} + \frac{\partial B_x}{\partial t} = 0$$

$$\frac{\partial E_x}{\partial s} - \frac{\partial E_s}{\partial x} + \frac{\partial B_y}{\partial t} = 0$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial B_s}{\partial t} = 0$$

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

We want to find relations between the forces on the witness charge:

$$\vec{F}_\perp = q_2[(E_x - cB_y)\hat{x} + (E_y + cB_x)\hat{y}]$$

$$F_s = q_2 E_s$$

with

$$s - ct = z$$

$$\Downarrow$$

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial z} = -\frac{1}{c}\frac{\partial}{\partial t}$$

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_s}{\partial s} = \frac{\rho}{\epsilon_0}$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_s}{\partial s} = 0$$

We first use this set of equations

$$\frac{\partial B_s}{\partial y} - \frac{\partial B_y}{\partial s} - \frac{1}{c^2}\frac{\partial E_x}{\partial t} = 0$$

$$\frac{\partial B_x}{\partial s} - \frac{\partial B_s}{\partial x} - \frac{1}{c^2}\frac{\partial E_y}{\partial t} = 0$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} - \frac{1}{c^2}\frac{\partial E_s}{\partial t} = \mu_0 \rho c$$

$$\boxed{\begin{aligned} \frac{\partial E_s}{\partial y} - \frac{\partial E_y}{\partial s} + \frac{\partial B_x}{\partial t} &= 0 \\ \frac{\partial E_x}{\partial s} - \frac{\partial E_s}{\partial x} + \frac{\partial B_y}{\partial t} &= 0 \end{aligned}}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial B_s}{\partial t} = 0$$

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

$$\frac{\partial F_x}{\partial z} - \frac{\partial F_s}{\partial x} = 0$$

$$\frac{\partial F_s}{\partial y} - \frac{\partial F_y}{\partial z} = 0$$

$$\frac{\partial \vec{F}_\perp}{\partial z} = \nabla_\perp F_s$$

$$\boxed{\frac{\partial \int_0^L \vec{F}_\perp ds}{\partial z} = \nabla_\perp \int_0^L F_s ds}$$

Result known as Panofsky-Wenzel theorem

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

$$\boxed{\frac{\partial \int_0^L \vec{F}_\perp ds}{\partial z} = \nabla_\perp \int_0^L F_s ds}$$

$$\int_0^L F_x ds = W_x(z)\Delta x_1 + W_{Qx}(z)x$$

$$\Downarrow$$

$$W_x'(z) = W_{||}^{(dq)}(z) \quad \stackrel{\mathcal{F}}{\Longleftrightarrow} \quad \frac{\omega}{c}Z_x(\omega) = Z_{||}^{(dq)}(\omega)$$

$$W_{Qx}'(z) = 2W_{||}^{(2q)}(z) \quad \stackrel{\mathcal{F}}{\Longleftrightarrow} \quad \frac{\omega}{c}Z_{Qx}(\omega) = 2Z_{||}^{(2q)}(\omega)$$

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

The CERN Accelerator School

$$\boxed{\frac{\partial \int_0^L \vec{F}_\perp ds}{\partial z} = \nabla_\perp \int_0^L F_s ds}$$

$$\int_0^L F_x ds = W_x(z)\Delta x_1 + W_{Qx}(z)x$$

$$W'_x(z) = W \;\ldots\; {}^{(dq)}_{||}(\omega)$$

The longitudinal and transverse wake functions are not independent, although in general no relation can be established between $W_{||}(z)$ and $W_{x,y}(z)$, which are the main wakes in the longitudinal and transverse planes, respectively.

$$W'_{Qx}(z) = 2W_{||}^{(2q)}(z) \quad \stackrel{\mathcal{F}}{\Longleftrightarrow} \quad \frac{\omega}{c} Z_{Qx}(\omega) = 2Z_{||}^{(2q)}(\omega)$$

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

$$\boxed{\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_s}{\partial s} = \frac{\rho}{\epsilon_0}}$$

$$\boxed{\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_s}{\partial s} = 0}$$

$$\frac{\partial B_s}{\partial y} - \frac{\partial B_y}{\partial s} - \frac{1}{c^2}\frac{\partial E_x}{\partial t} = 0$$

$$\frac{\partial B_x}{\partial s} - \frac{\partial B_s}{\partial x} - \frac{1}{c^2}\frac{\partial E_y}{\partial t} = 0$$

$$\frac{\partial E_s}{\partial y} - \frac{\partial E_y}{\partial s} + \frac{\partial B_x}{\partial t} = 0$$

$$\frac{\partial E_x}{\partial s} - \frac{\partial E_s}{\partial x} + \frac{\partial B_y}{\partial t} = 0$$

We can now use also these two sets of equations to find additional properties of the wakes

$$\boxed{\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} - \frac{1}{c^2}\frac{\partial E_s}{\partial t} = \mu_0 \rho c}$$

$$\boxed{\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial B_s}{\partial t} = 0}$$

Relation between transverse and longitudinal wakes (Panofsky-Wenzel theorem)

$$\frac{\partial F_x}{\partial x} = -\frac{\partial F_y}{\partial y}$$

$$\frac{\partial \int_0^L F_x ds}{\partial x} = -\frac{\partial \int_0^L F_y ds}{\partial y}$$

$$\boxed{W_{Qx}(z) = -W_{Qy}(z)}$$

This is an interesting result!
The quadrupolar wakes in x and y must be equal with opposite signs

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

$$\frac{\partial \int_0^L F_x ds}{\partial y} = \frac{\partial \int_0^L F_y ds}{\partial x}$$

This relation means that the cross-wakes between x and y must be equal.
We have so far ignored these terms in our derivations.

Instabilities

Synchrotron tune shift

- The **equilibrium distribution** in the presence of a longitudinal wake field can be found analytically. The (linearized) **longitudinal Hamiltonian** with longitudinal wake fields is given as:

$$H = -\frac{1}{2}\eta\,\delta^2 - \frac{1}{2\eta}\left(\frac{\omega_s}{\beta c}\right)^2 z^2 + \frac{e^2}{\beta^2 EC}\sum_k \int dz'' \int dz'\, \lambda(z'+kC) W_0'(z''-z'-kC)$$

- Remember the example of the harmonic oscillator:

$$H = \frac{1}{2}p^2 \qquad + \frac{1}{2}\boxed{W}\,q^2$$

Coefficient determines frequency/tune

Synchrotron tune shift

- The **equilibrium distribution** in the presence of a longitudinal wake field can be found analytically. The (linearized) **longitudinal Hamiltonian** with longitudinal wake fields is given as:

$$H = -\frac{1}{2}\eta\,\delta^2 - \frac{1}{2\eta}\left(\frac{\omega_s}{\beta c}\right)^2 z^2 + \underbrace{\frac{e^2}{\beta^2 EC}\sum_k \int dz'' \int dz'\,\lambda(z'+kC)W_0'(z''-z'-kC)}_{\text{we make an expansion in } z \text{ -- factor out } \frac{1}{2\eta\beta^2c^2}}$$

- Remember the example of the harmonic oscillator:

$$H = \frac{1}{2}p^2 \boxed{+\,V\,q} + \frac{1}{2}\boxed{W}\,q^2$$

Coefficient determines frequency/tune

Term determines center position/orbit

Synchrotron tune shift

- The **equilibrium distribution** in the presence of a longitudinal wake field can be found analytically. The (linearized) **longitudinal Hamiltonian** with longitudinal wake fields is given as:

$$H = -\frac{1}{2}\eta\,\delta^2 - \frac{1}{2\eta}\left(\frac{\omega_s}{\beta c}\right)^2 z^2 + \underbrace{\frac{e^2}{\beta^2 EC}\sum_k \int dz'' \int dz'\,\lambda(z'+kC)W_0'(z''-z'-kC)}_{\text{expansion in } z \text{ - factor } \frac{1}{2\eta\beta^2c^2}}$$

- It follows then quite easily that:

$$\begin{aligned}\Delta\omega_s &\approx -\frac{1}{2\omega_s}\frac{e^2\,\eta c^2}{EC}\int dz'\,\lambda(z')W''0(z-z') \\ &= -\frac{i}{4\pi}\frac{e^2\,\eta c^2}{\omega_s\,EC}\int d\omega\,\hat{\lambda}(\omega)Z_0(\omega)\frac{\omega}{c}\end{aligned}$$

Remember, we make use of:
$\Omega^2 - \omega_s^2 \approx 2\omega_s\,\Delta\omega_s$

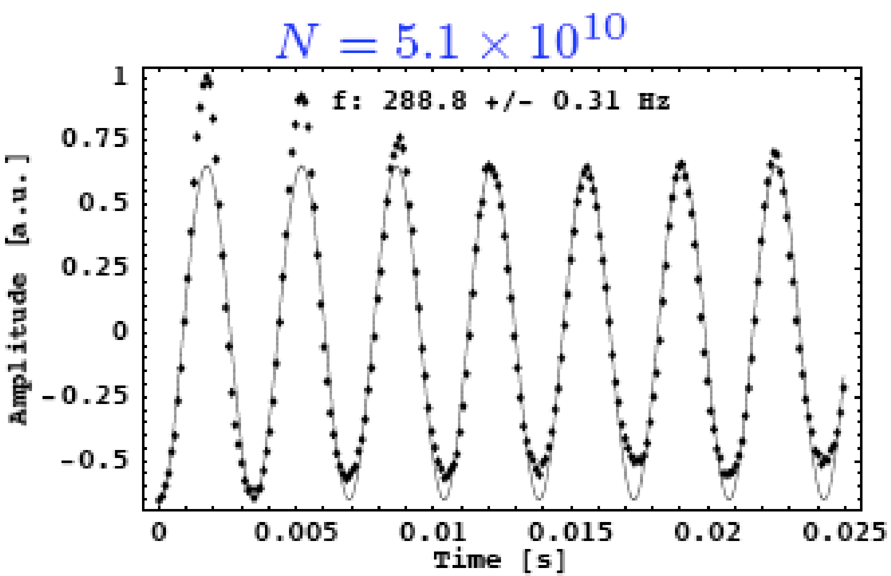
- The synchrotron tune shift from an impedance is, hence, given as:

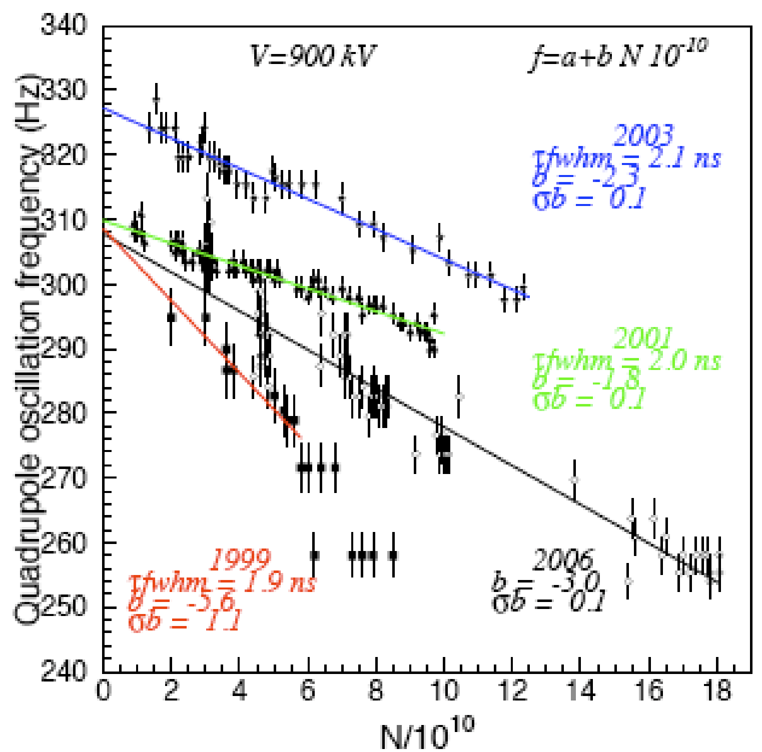
$$\Delta Q_s = -\frac{1}{4\omega_s}\frac{e^2\,\eta}{(2\pi^2)E}\int d\omega\,\omega\hat{\lambda}(\omega)\,\mathrm{Im}[Z_0(\omega)]$$

Measurements of synchrotron tune shift at SPS

The CERN Accelerator School

- The slope of the **incoherent synchrotron tune shift with intensity**, measured in reproducible conditions over the years, shows the evolution of the **imaginary part of the machine impedance** (E. Shaposhnikova, T. Bohl, J. Tuckmantel)
  - The technique uses the quadrupole oscillations of a bunch injected with a mismatch
  - Qs can be extrapolated from bunch length or peak amplitude measurements





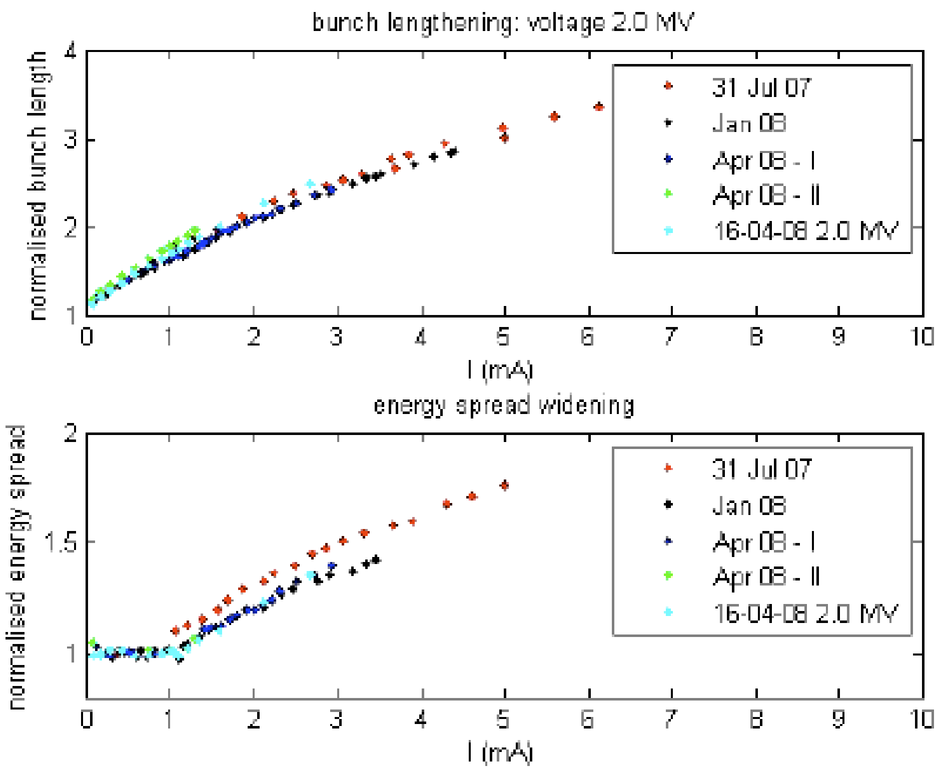
75

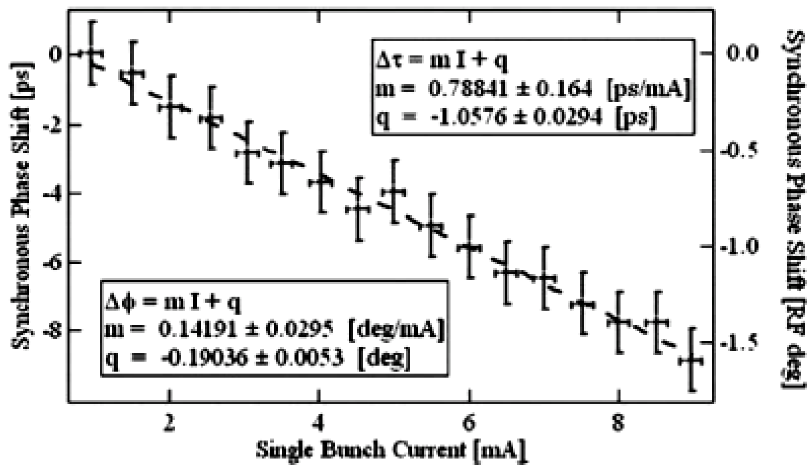
Measurements of potential well distortion
Stable phase and bunch lengthening

Measurements at light sources

⇒ Bunch lengthening @DIAMOND (left, R. Bartolini)

⇒ Energy loss measured through the synchronous phase shift @Australian light source (right, R. Dowd, M. Boland, G. LeBlanc, M. Spencer, Y. Tan, PAC07)





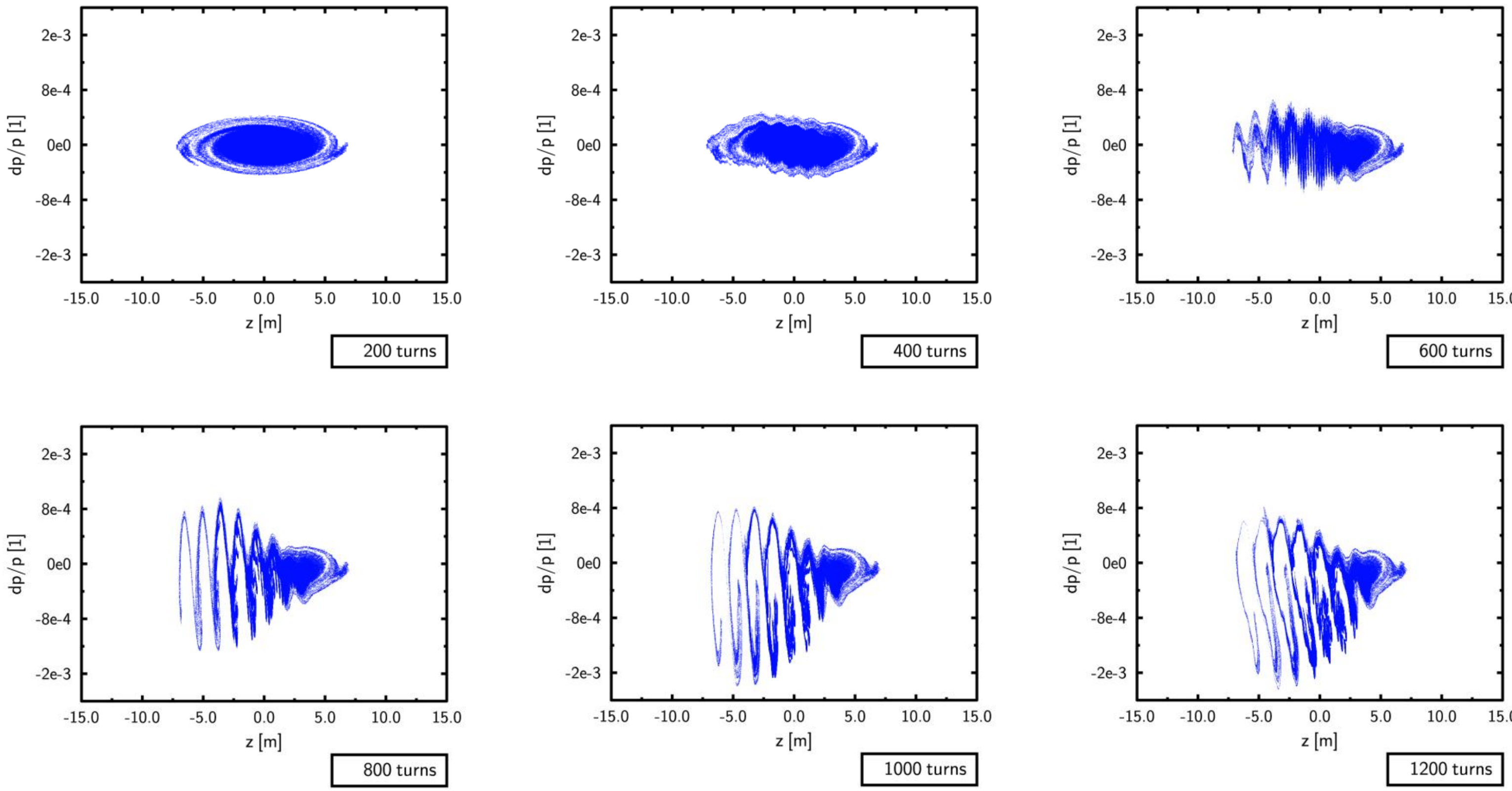
Synchronous phase shift measured with a streak camera in the Australian Synchrotron.

Examples of numerical simulations debunching bunch with SPS impedance model

Microwave instability on a debunching bunch is used at SPS for probing the machine impedance (E. Shaposhnikova, T. Bohl, H. Timkó, et al.)

⇒ Long bunch injected into SPS with RF off slowly debunches due to low momentum spread

⇒ Spectrum of bunch profile reveals important components for the impedance



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⇒ Long bunch injected into SPS with RF off slowly debunches due to low momentum spread

⇒ Spectrum of bunch profile reveals important components for the impedance

⇒ Simulations with impedance model are used to match measured profile