RF Engineering Basic Concepts: The Smith Chart

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Contents

Motivation

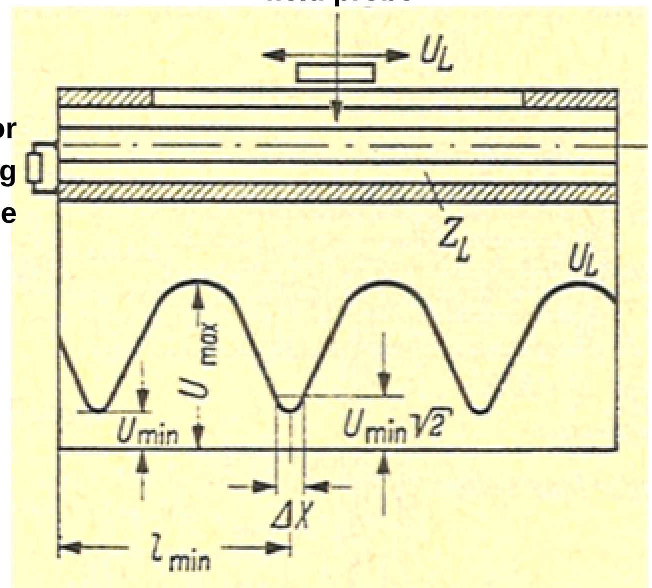
- The Smith Chart was invented by Phillip Smith in 1939 in order to provide an easily usable graphical representation of the complex reflection coefficient Γ and reading of the associated complex terminating impedance
- Γ is defined as the ratio of electrical field strength of the reflected versus forward travelling wave
- Why not the magnetic field strength? – Simply, since the electric field is easier measurable as compared to the magnetic field

Motivation

- In the old days when no network analyzers were available, the reflection coefficient was measured using a coaxial measurement line with a slit in axial direction:

- There was a little electric field probe protruding into the field region of this coaxial line near the outer conductor and the signal picked up was rectified in a microwave diode and displayed on a micro volt meter

- Going along this RF measurement line, one could find minima and maxima and determine their position, spacing and the ratio of maximum to minimum voltage reading. This is the origin of the VSWR (voltage standing wave ratio) which we will discuss later again in more detail

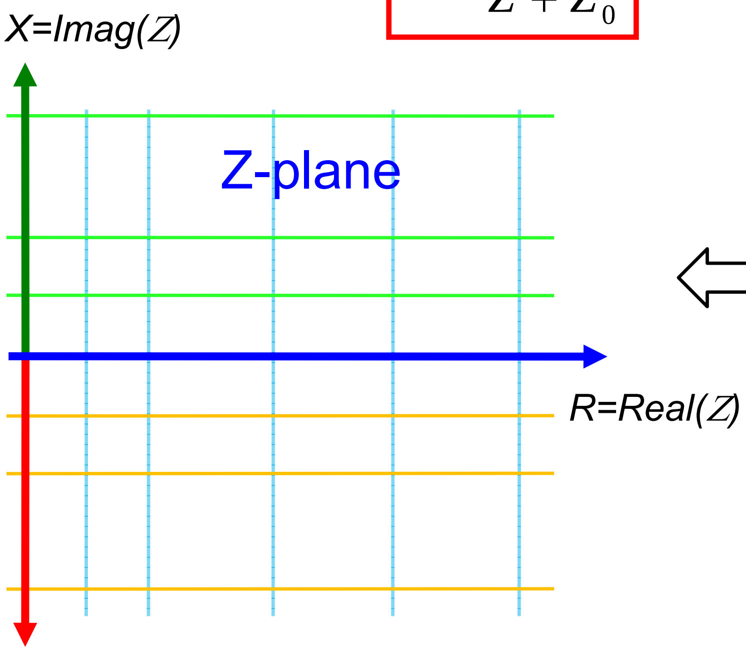


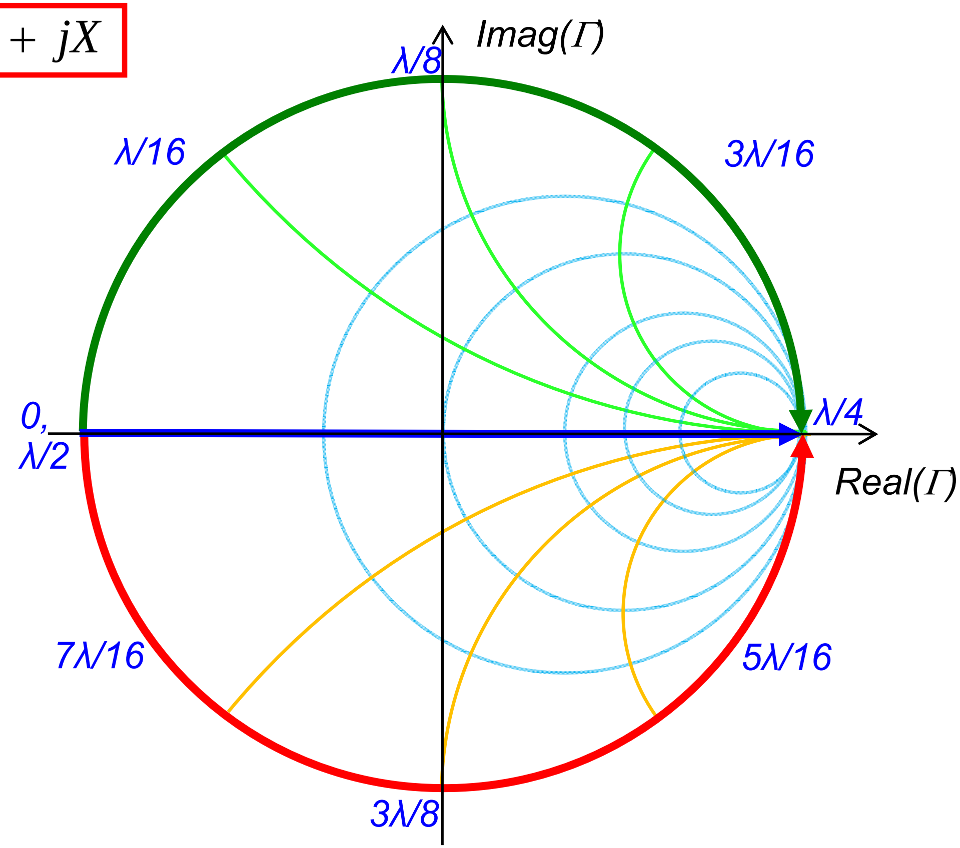
- RF measurements like this are now obsolete, but the Smith Chart, the VSWR and the reflection coefficient Γ are still very important and used in the everyday life of the microwave engineer both for simulations and measurements

The Smith Chart (1)

The Smith Chart represents the complex Γ-plane within the unit circle. It is a conformal mapping (=bilinear) of the complex Z-plane onto the complex Γ plane using the transformation

$$\boxed{\Gamma = \frac{Z - Z_0}{Z + Z_0}} \quad \text{with} \quad \boxed{Z = R + jX}$$





→The half plane with positive real part of Z is thus transformed into the interior of the unit circle of the Γ plane and vice versa!

The Smith Chart (2)

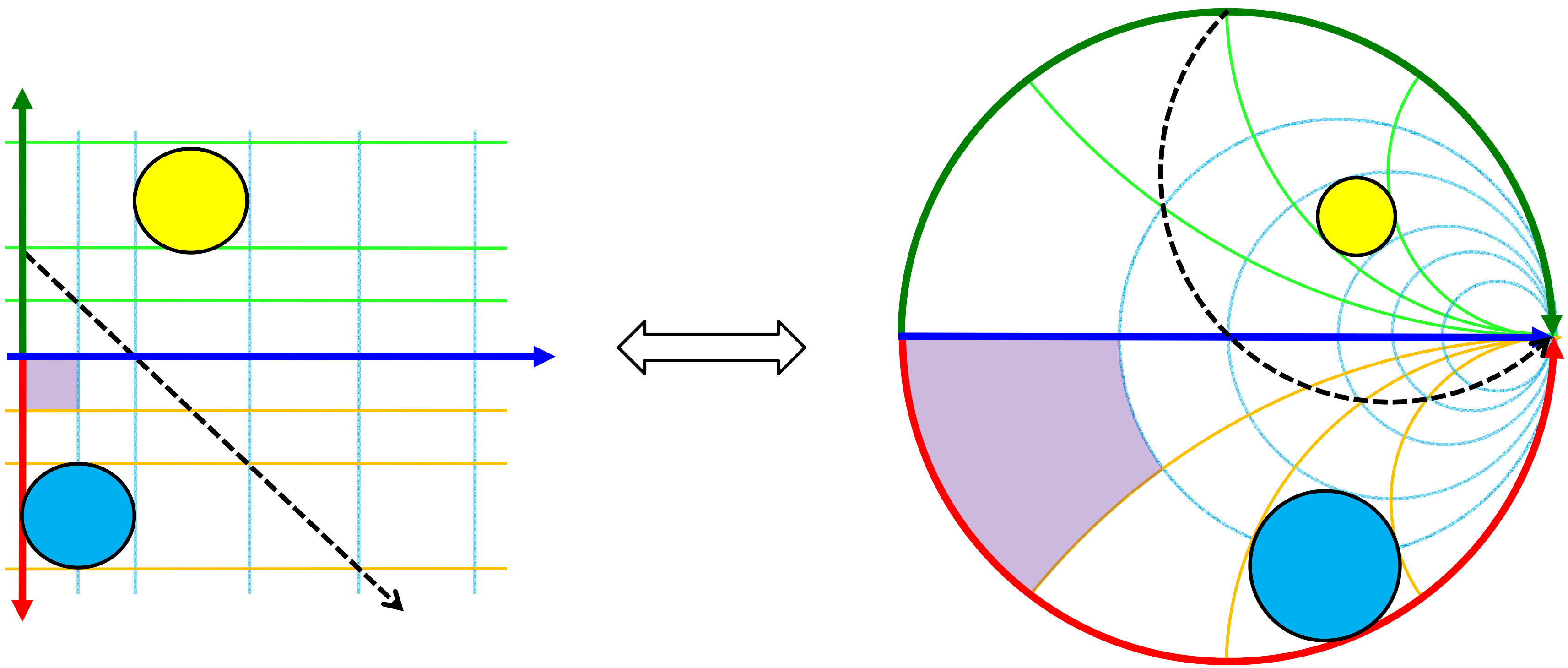
This bilinear transformation has the following properties:

- *generalized circles are transformed into generalized circles*
  - *circle → circle*
  - *straight line → circle*
  - *circle → straight line*
  - *straight line → straight line*
- *angles are preserved locally*

a straight line is nothing else than a circle with infinite radius

a circle is defined by 3 points

a straight line is defined by 2 points



The Smith Chart (3)

Impedances Z are usually first normalized by

$$z = \frac{Z}{Z_0}$$

where Z_0 is some reference impedance which is often the characteristic impedance of the connecting coaxial cables (e.g. 50 Ohm). The general form of the transformation can then be written as

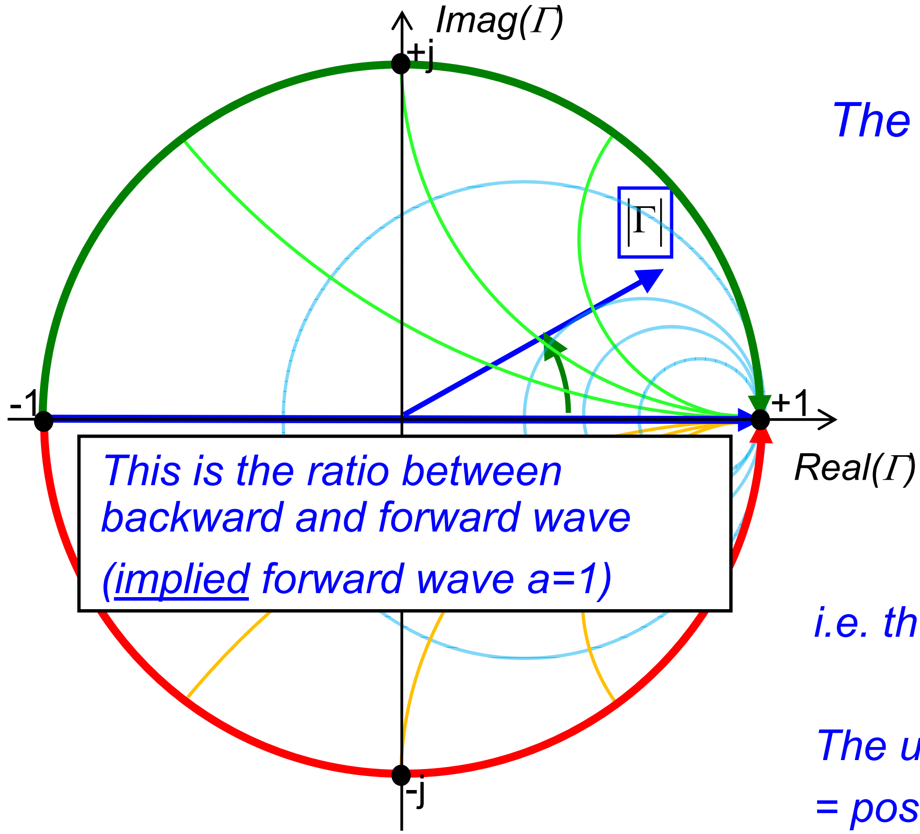
$$\Gamma = \frac{z-1}{z+1} \quad resp. \quad z = \frac{1+|\Gamma|}{1-|\Gamma|}$$

This mapping offers several practical advantages:

1. *The diagram includes all "passive" impedances, i.e. those with positive real part, from zero to infinity in a handy format. Impedances with negative real part ("active device", e.g. reflection amplifiers) would show up outside the (normal) Smith chart.*

2. *The mapping converts impedances (z) or admittances (y) into reflection factors and vice-versa. This is particularly interesting for studies in the radiofrequency and microwave domain where electrical quantities are usually expressed in terms of "direct" or "forward" waves and "reflected" or "backward" waves. This replaces the notation in terms of currents and voltages used at lower frequencies. Also the reference plane can be moved very easily using the Smith chart.*

The Smith Chart (4)



The Smith Chart ("Abaque Smith" in French) is the linear representation of the complex reflection factor

$$\Gamma = \frac{b}{a}$$

i.e. the ratio of backward/forward wave.

The upper half of the Smith-Chart is "inductive" = positive imaginary part of impedance, the lower half is "capacitive" = negative imaginary part.

Important points

Important Points:

- *Short Circuit*

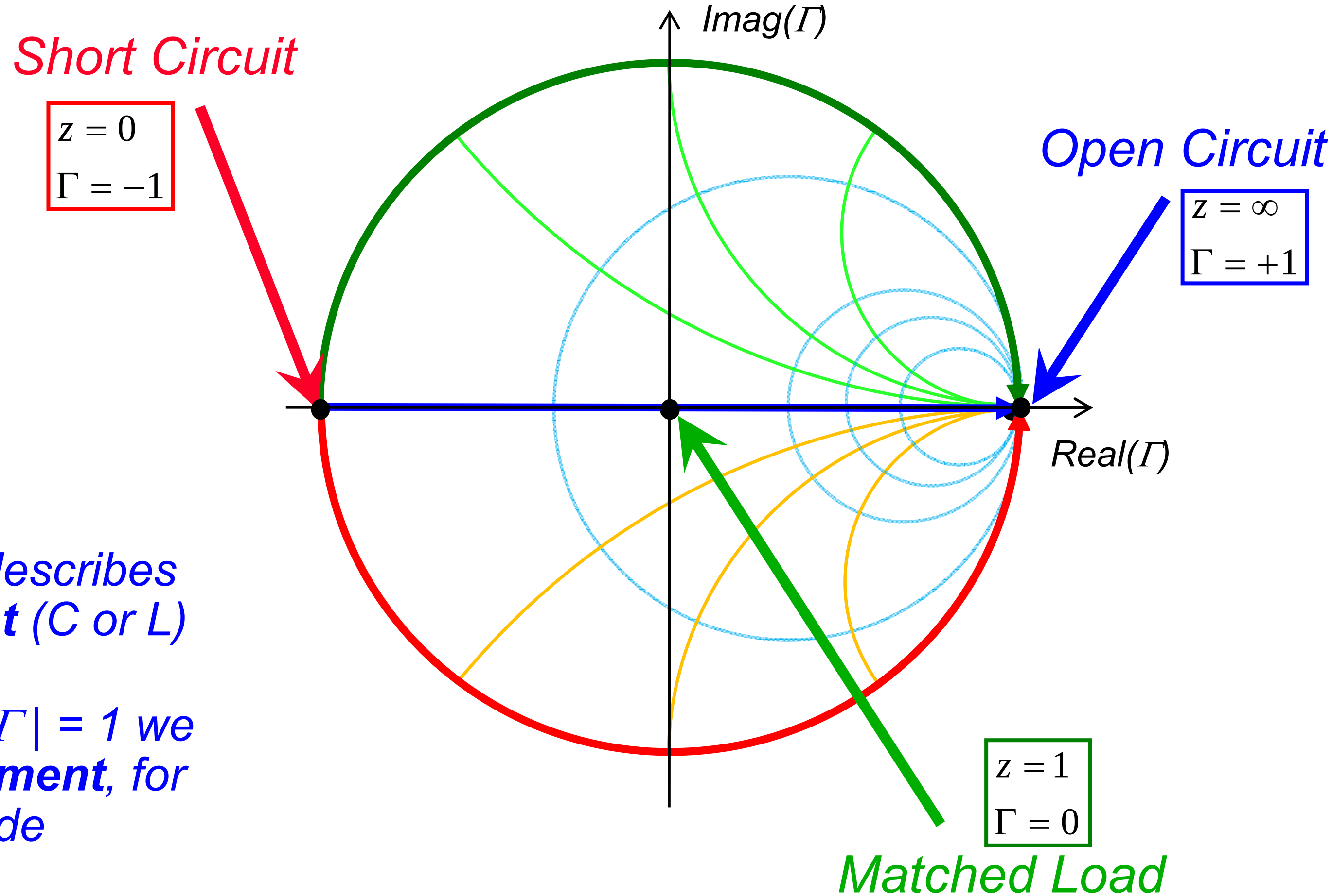
  $\Gamma = -1$, $z = 0$
- *Open Circuit*

  $\Gamma = 1$, $z \rightarrow \infty$
- *Matched Load*

  $\Gamma = 0$, $z = 1$

- *The circle $|\Gamma| = 1$ describes a* ***lossless element*** *(C or L)*

- *Outside the circle $|\Gamma| = 1$ we have an* ***active element****, for instance tunnel diode reflection amplifier*

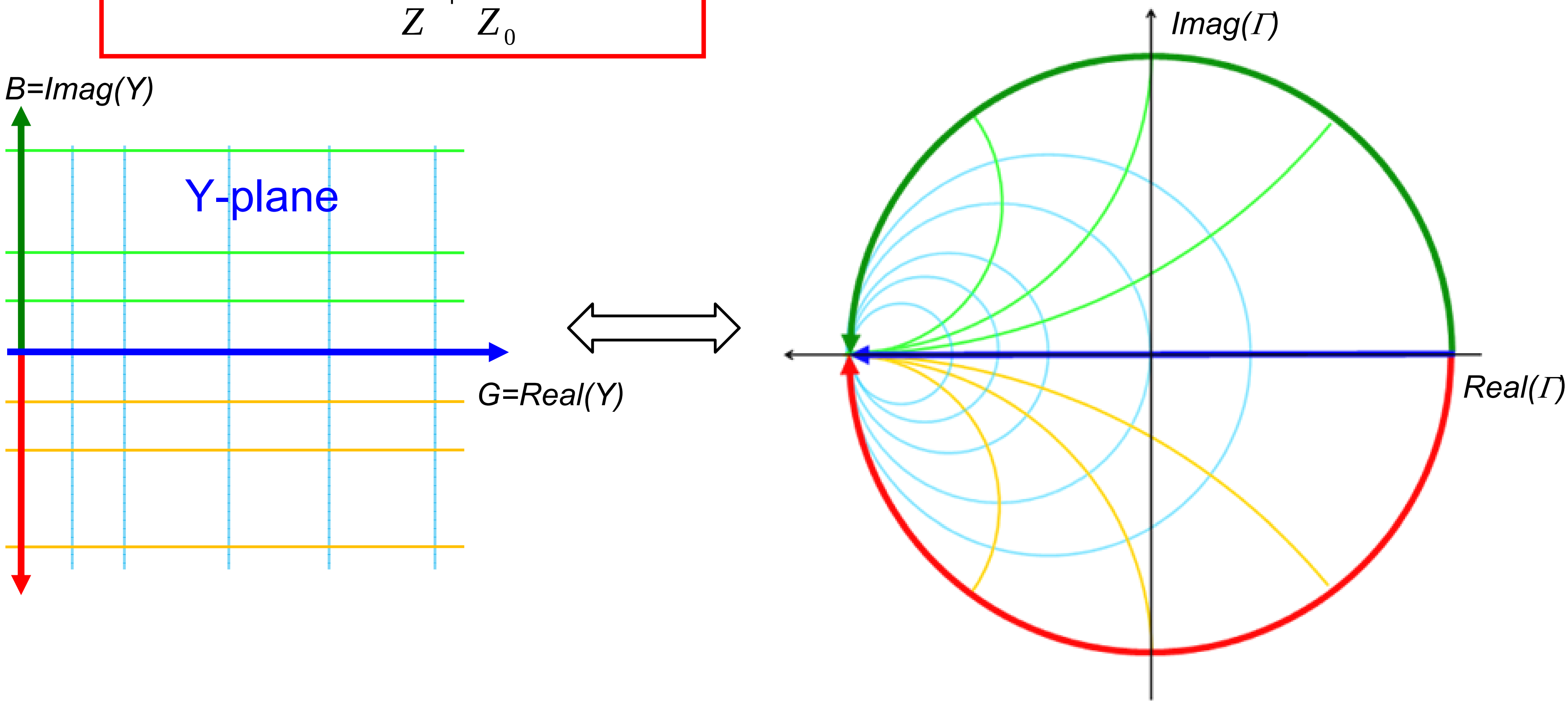


The Smith Chart (5)

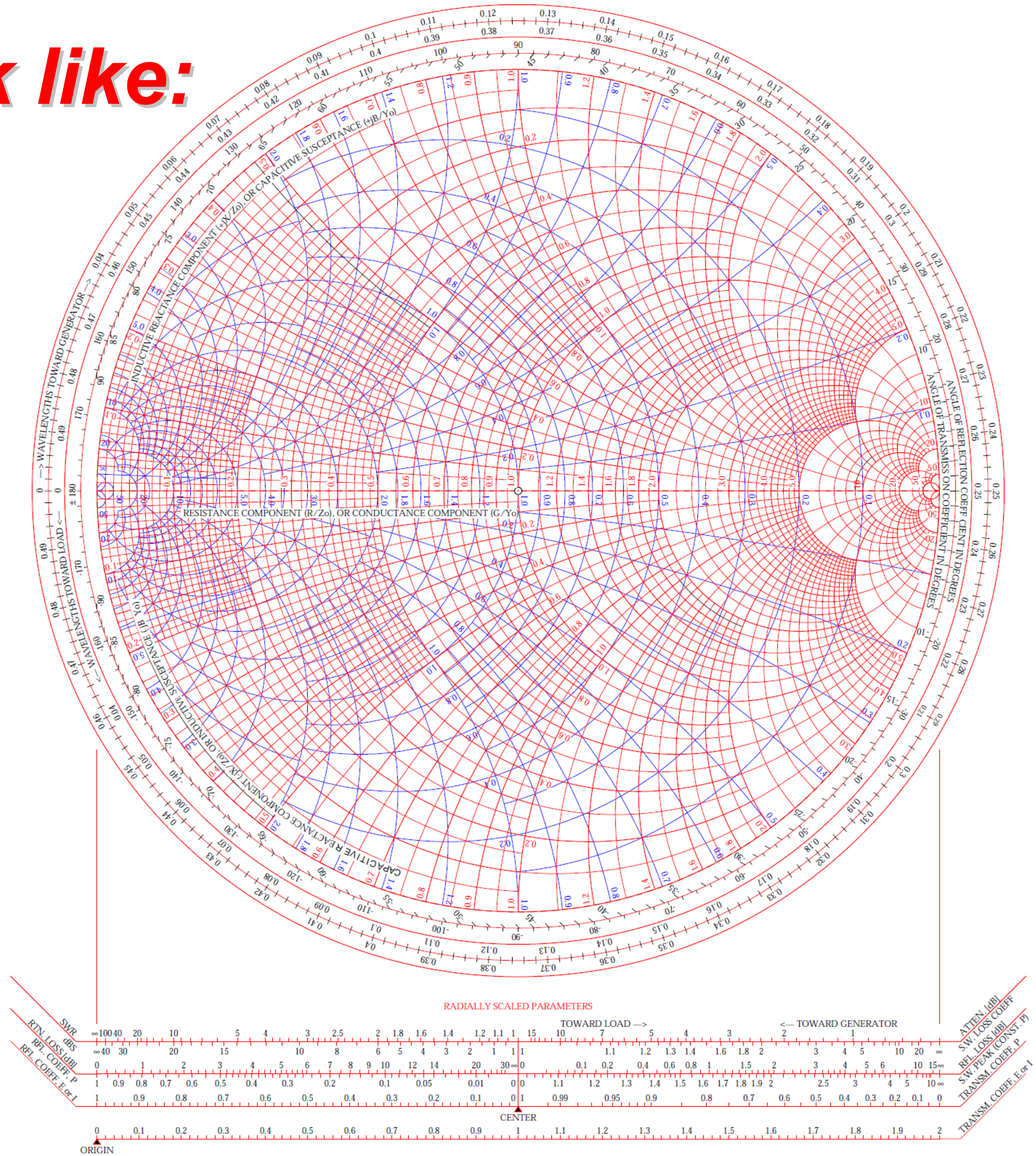
The Smith Chart can also represent the complex Γ-plane within the unit circle using a conformal mapping of the complex Y-plane onto itself using the transformation

$$\boxed{\Gamma = -\frac{Y - Y_0}{Y + Y_0} = -\frac{\frac{1}{Z} - \frac{1}{Z_0}}{\frac{1}{Z} + \frac{1}{Z_0}} = \frac{Z - Z_0}{Z + Z_0}}$$

with $\boxed{Y = G + jB}$



How does it look like:



Answer:

VERY CONFUSING!

The Smith Chart (6)

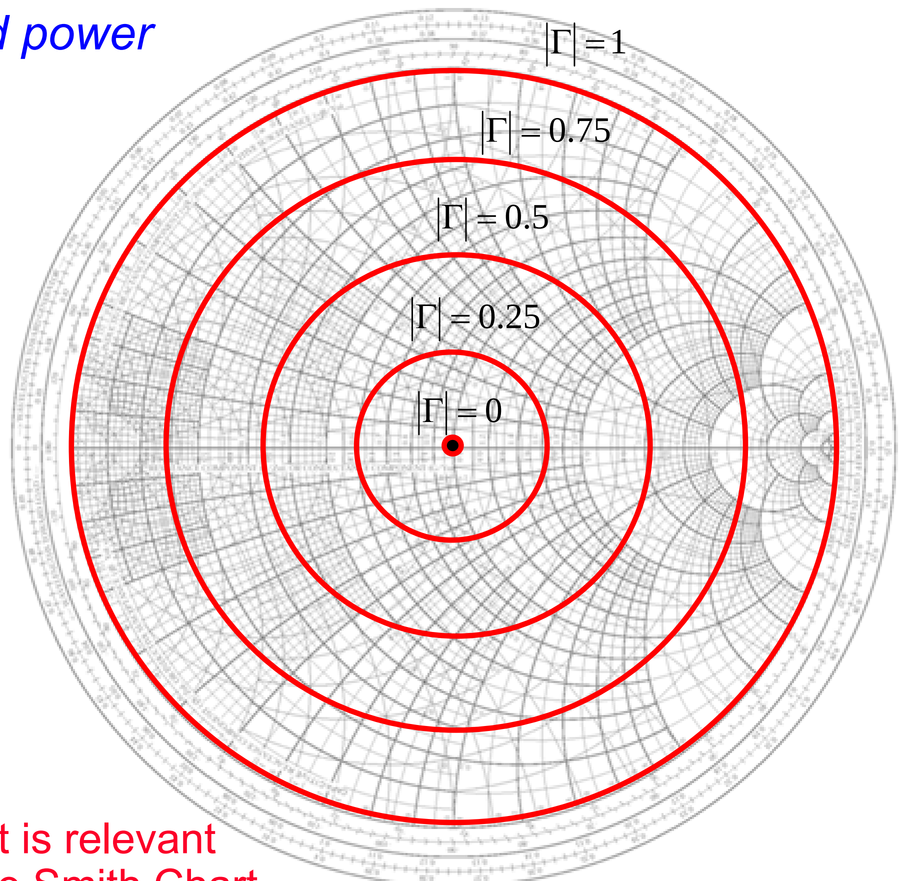
3. *The distance from the center of the diagram is directly proportional to the magnitude of the reflection factor. In particular, the perimeter of the diagram represents full reflection, $|\Gamma|=1$. Problems of matching are clearly visualize.*

Power into the load = forward power – reflected power

$$P = |a|^2 - |b|^2$$
$$= |a|^2 \left(1 - |\Gamma|^2\right)$$

available power from the source

"(mismatch)" loss



Here the US notion is used, where power = $|a|^2$.
European notation (often): power = $|a|^2/2$
These conventions have no impact on S parameters, but is relevant for **absolute** power calculation which is rarely used in the Smith Chart

The Smith Chart (7)

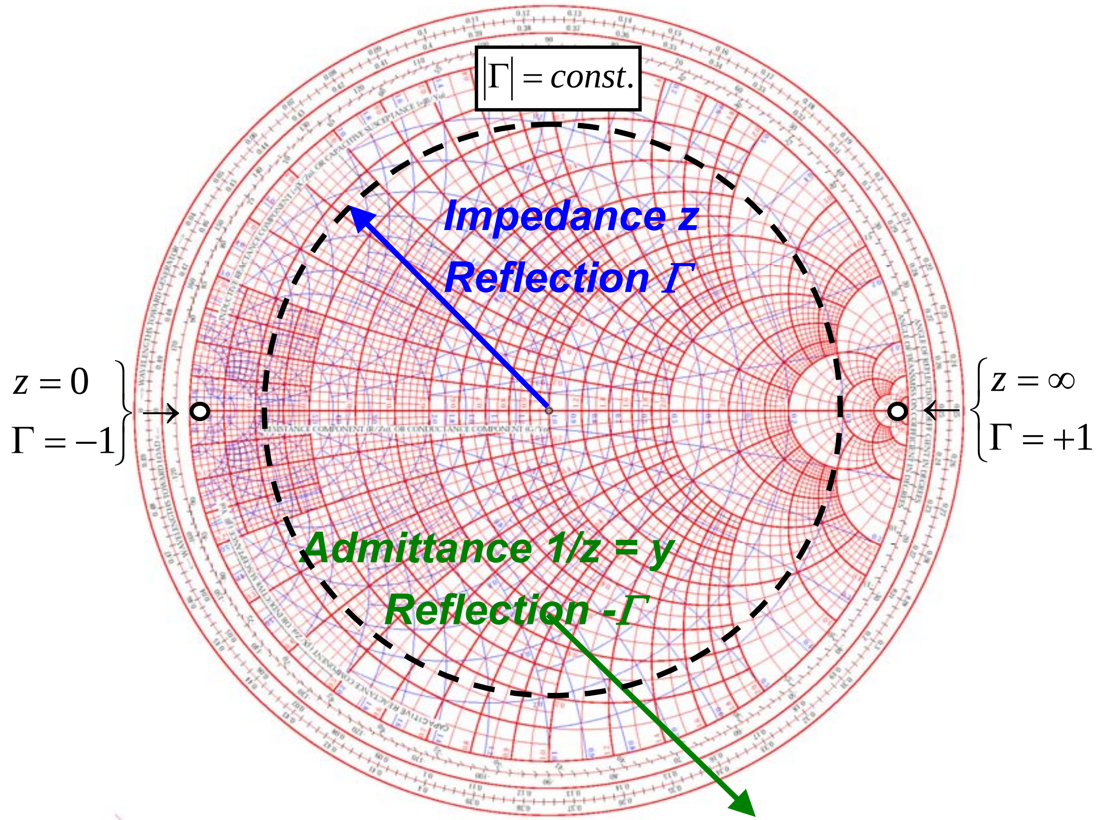
4. *The transition*

impedance ⇔ admittance

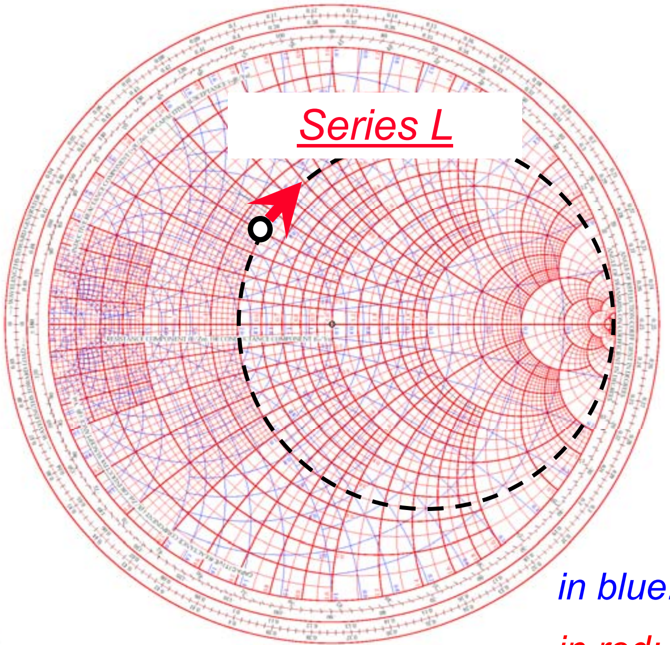
and vice-versa is particularly easy.

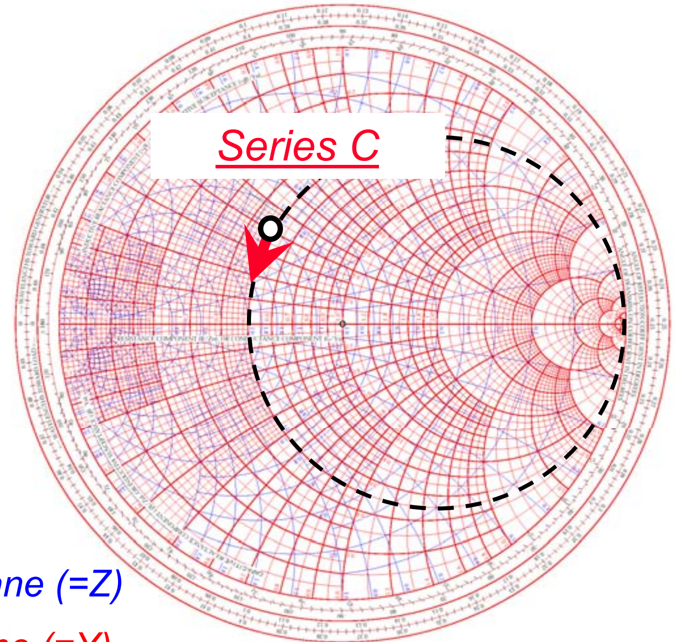
$$\Gamma(1/z) = \frac{1/z - 1}{1/z + 1} = \frac{1 - z}{1 + z} = -\left(\frac{z - 1}{z + 1}\right)$$

$$\underline{\Gamma(1/z) = -\Gamma(z)}$$



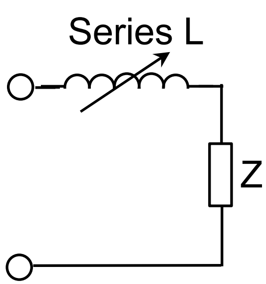
Navigation in the Smith Chart (1)

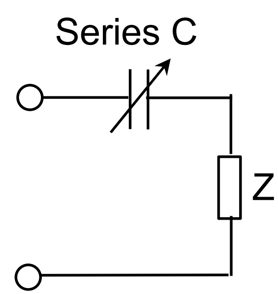




in blue: Impedance plane (=Z)

in red: Admittance plane (=Y)





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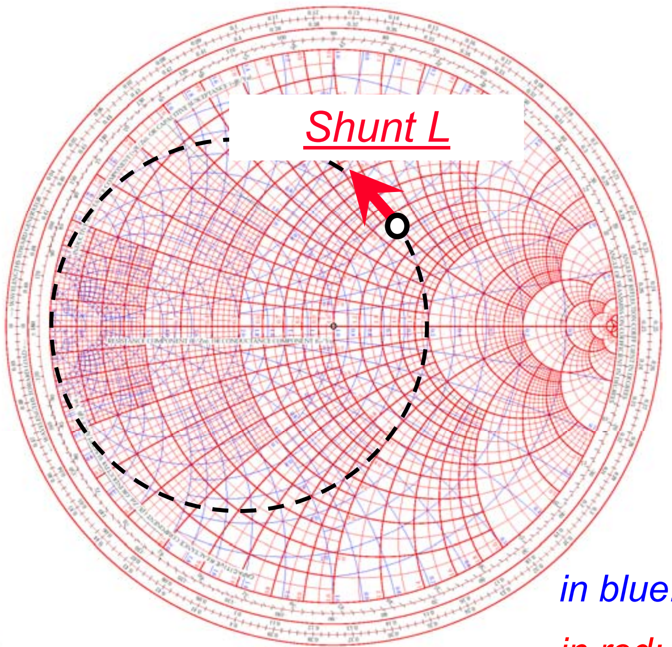
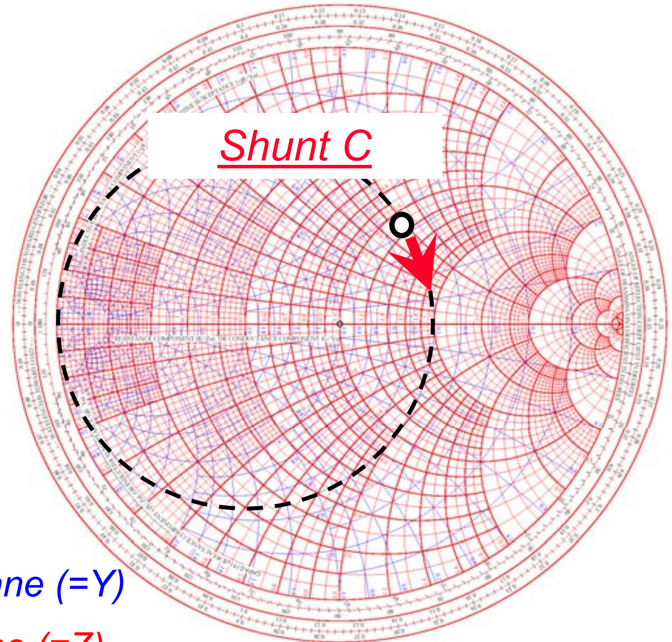
Navigation in the Smith Chart (2)

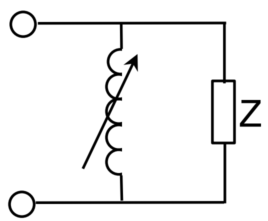




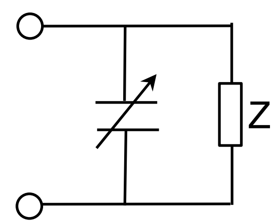
in blue: Admittance plane (=Y)

in red: Impedance plane (=Z)

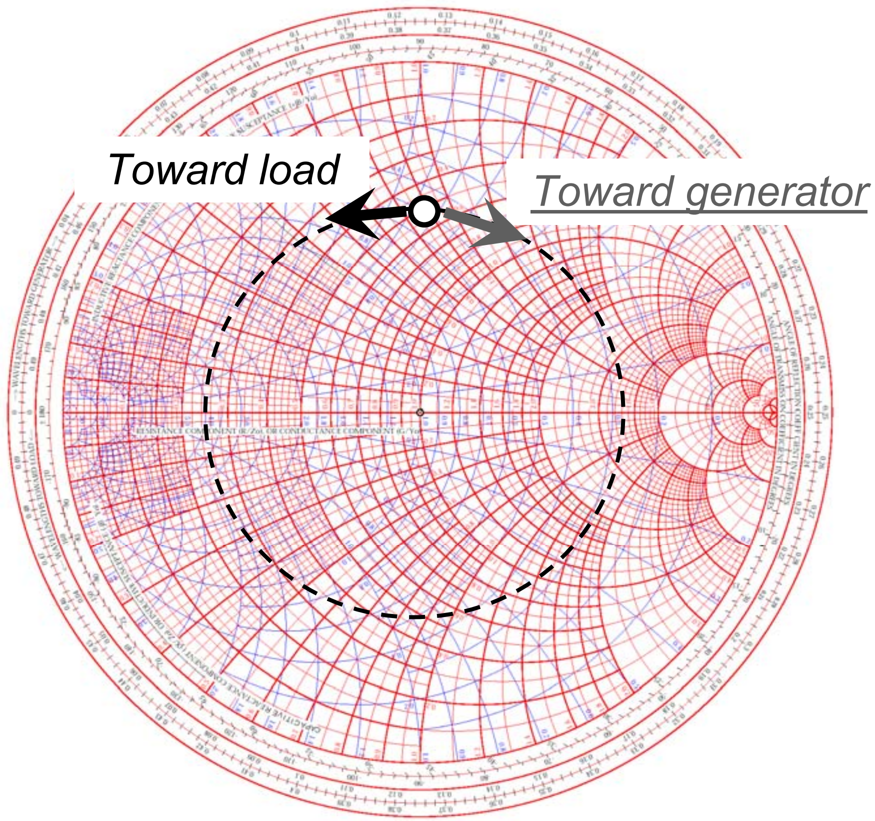
Shunt L



Shunt C

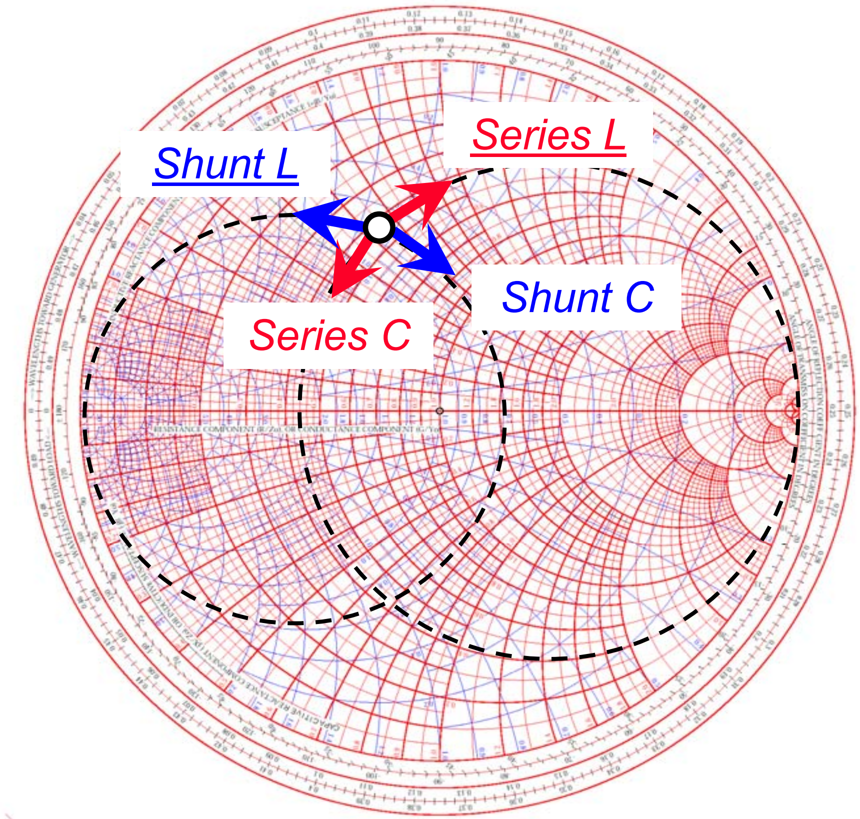


Navigation in the Smith Chart (3)



Red arcs	**Resistance R**
Blue arcs	**Conductance G**
Con-centric circle	**Transmission line going Toward load** **Toward generator**

Navigation in the Smith Chart - Summary

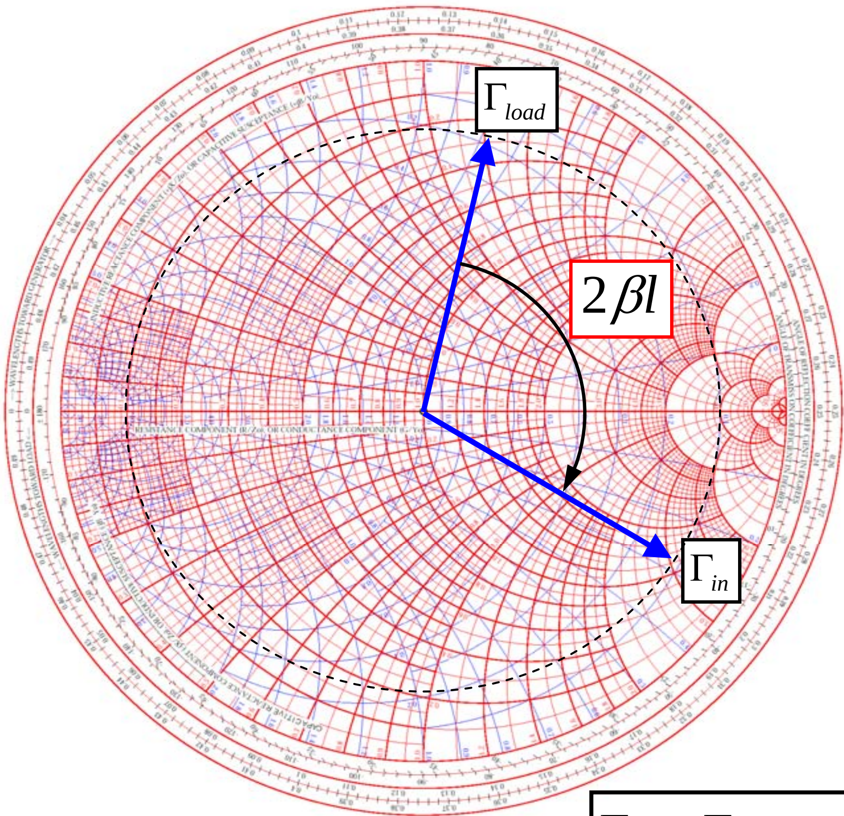


in blue: Admittance plane (=Y)

in red: Impedance plane (=Z)

	Up	Down
Red circles	**Series L**	**Series C**
Blue circles	**Shunt L**	**Shunt C**

Impedance transformation by transmission lines



$$\Gamma_{in} = \Gamma_{load}\, e^{-j2\beta l}$$

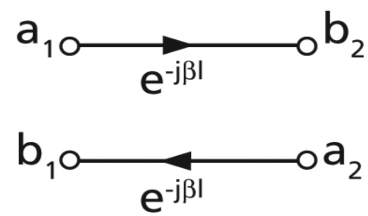
How to remember that when adding a section of line we have to turn clockwise: assume we are at Γ=-1 (short circuit) and add a very short piece of coaxial cable. Then we have made an inductance thus we are in the upper half of the Smith-Chart.

The S-matrix for an ideal, lossless transmission line of length l is given by

$$\mathbf{S} = \begin{bmatrix} 0 & e^{-j\beta l} \\ e^{-j\beta l} & 0 \end{bmatrix}$$

where $\beta = 2\pi / \lambda$

is the propagation coefficient with the wavelength λ (this refers to the wavelength on the line containing some dielectric). For ε_r=1 we denote λ=λ_0.

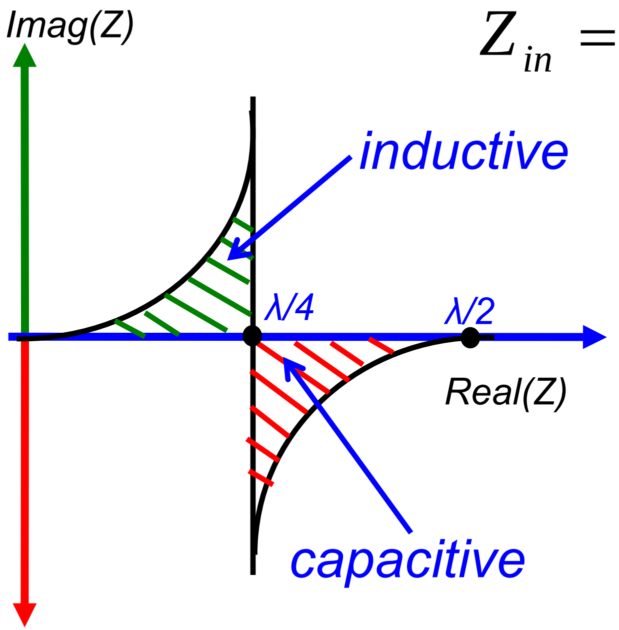


N.B.: It is supposed that the reflection factors are evaluated with respect to the characteristic impedance Z_0 of the line segment.

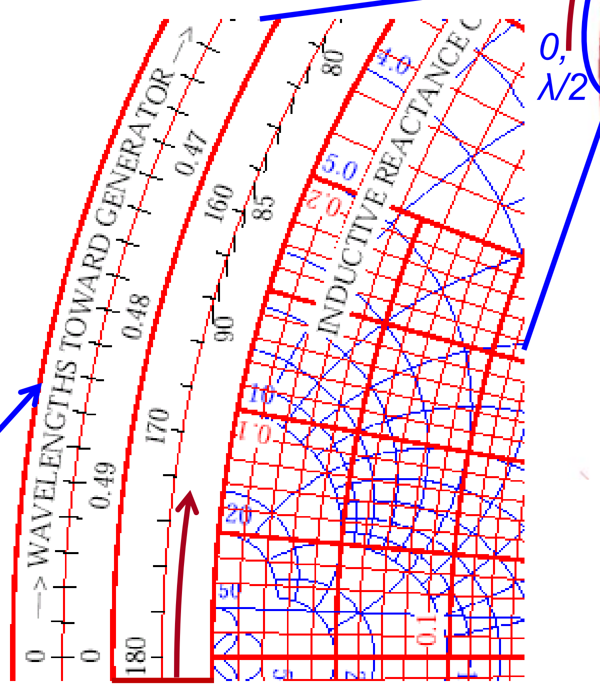
Transformation over a lossless transmission line

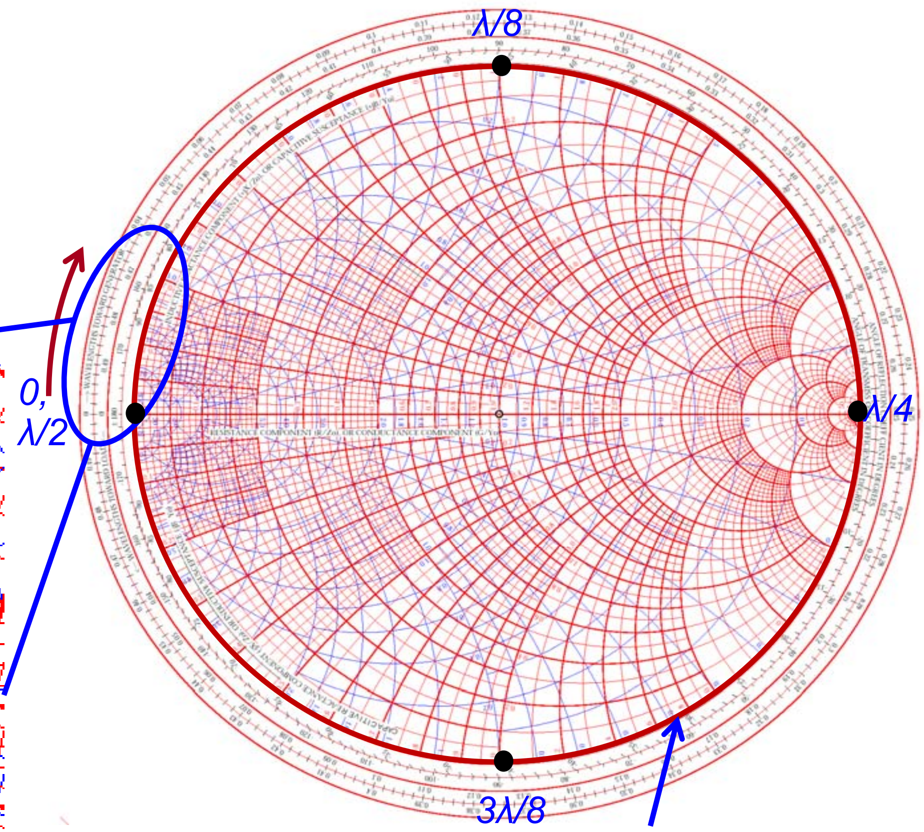
Impedance of a shortened coaxial line is given by:

$$Z_{in} = jZ_0 \tan(\beta l)$$



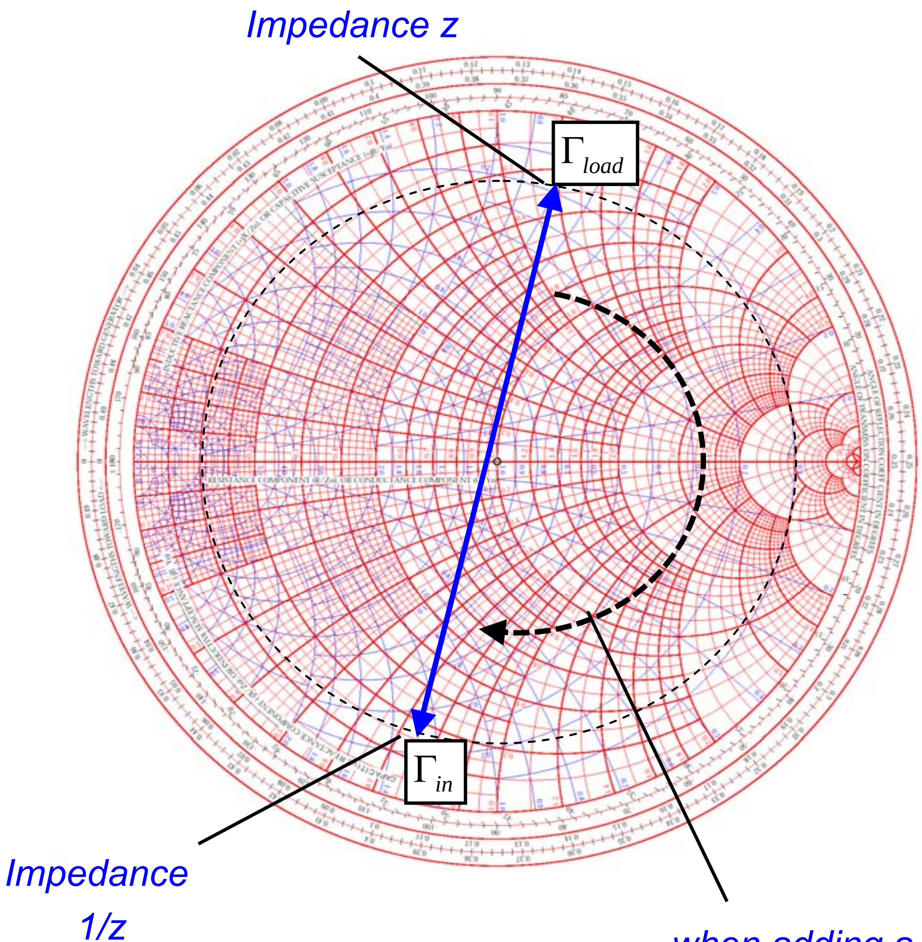
From here, we read l/λ which is the parameterization of this outermost circle





We go along the red circle when adding a coaxial line (Imagine we have a coaxial trombone terminated by a short and we vary its length)

λ/4 - Line transformations



when adding a transmission line
to some terminating impedance we move
clockwise through the Smith-Chart.

A transmission line of length

$$l = \lambda / 4$$

transforms a load reflection Γ_{load} to its input as

$$\Gamma_{in} = \Gamma_{load}\, e^{-j2\beta l} = \Gamma_{load}\, e^{-j\pi} = \underline{-\Gamma_{load}}$$

Thus a normalized load impedance z is transformed into 1/z.

In particular, a short circuit at one end is transformed into an open circuit at the other. This is a particular property of the λ/4 transformers.

Looking through a 2-port

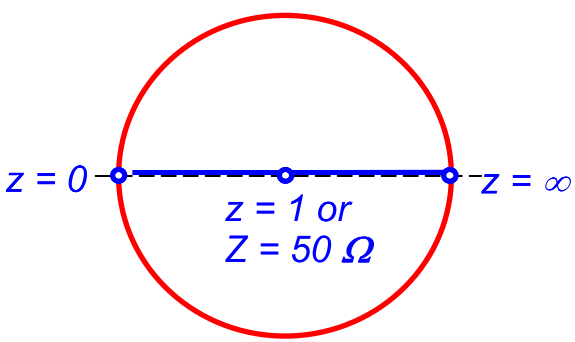
In general:

$$\Gamma_{in} = S_{11} + \frac{S_{12} S_{21} \Gamma_L}{1 - S_{22} \Gamma_L}$$

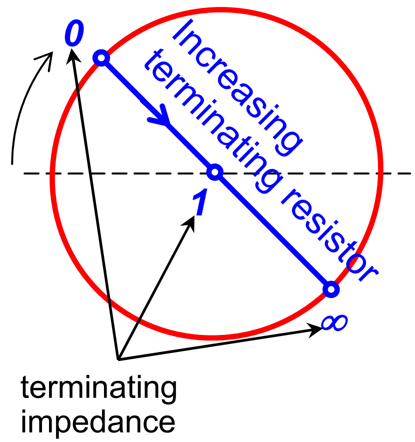
were Γ_{in} is the reflection coefficient when looking through the 2-port and Γ_{load} is the load reflection coefficient.

The outer circle and the real axis in the simplified Smith diagram below are mapped to other circles and lines, as can be seen on the right.

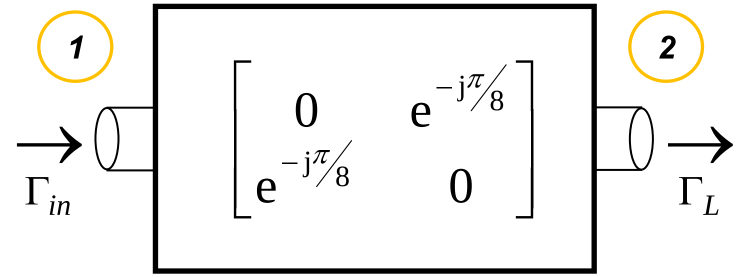
Variable load resistor ($0<z<\infty$):



Line $\lambda/16$ (= $\pi/8$):



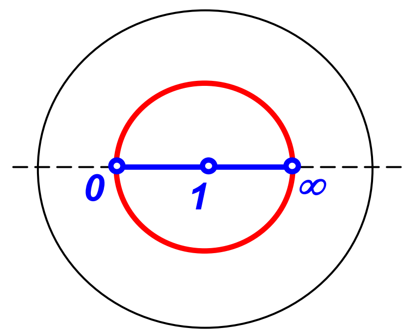
Transformer with $e^{-2\beta l}$

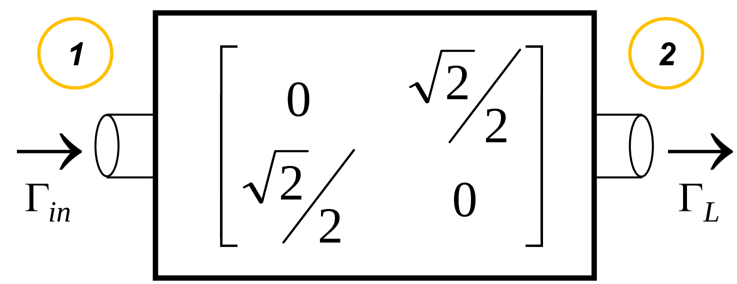


$$\begin{bmatrix} 0 & e^{-j\pi/8} \\ e^{-j\pi/8} & 0 \end{bmatrix}$$

$$\Rightarrow \Gamma_{in} = \Gamma_L \, e^{-j\pi/4}$$

Attenuator 3dB:



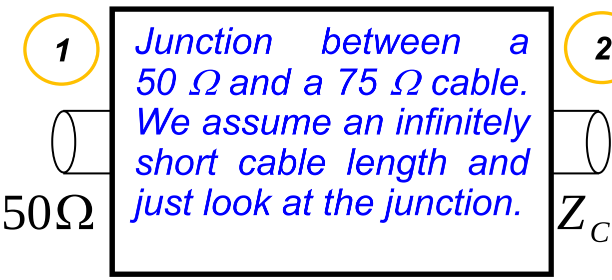


$$\begin{bmatrix} 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 \end{bmatrix}$$

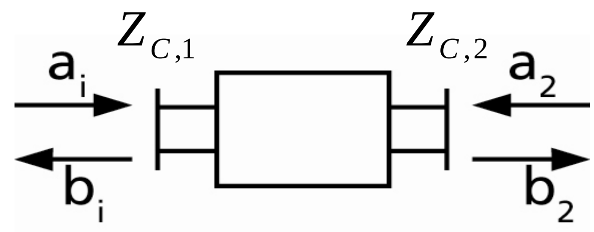
$$\Rightarrow \Gamma_{in} = \Gamma_L / 2$$

Example: a Step in Characteristic Impedance (1)

Consider a junction of two coaxial cables, one with $Z_{C,1}$ = 50 Ω characteristic impedance, the other with $Z_{C,2}$ = 75 Ω characteristic impedance.



$$Z_{C,1} = 50\Omega \qquad Z_{C,2} = 75\Omega$$



Step 1: *Calculate the reflection coefficient and keep in mind: all ports have to be terminated with their respective characteristic impedance, i.e. 75 Ω for port 2.*

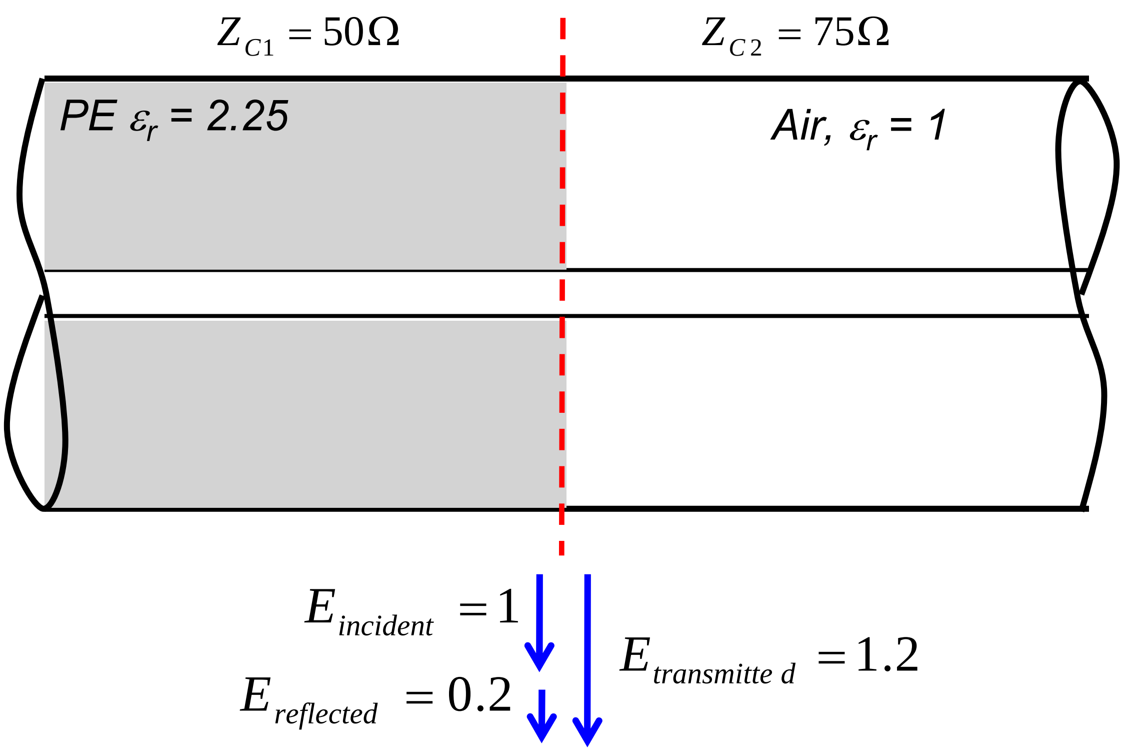
$$\Gamma_1 = \frac{Z - Z_{C,1}}{Z + Z_{C,1}} = \frac{75 - 50}{75 + 50} = 0.2$$

Thus, the voltage of the reflected wave at port 1 is 20% of the incident wave and the reflected power at port 1 (proportional Γ^2) is 0.2^2 = 4%. As this junction is lossless, the transmitted power must be 96% (conservation of energy). From this we can deduce b_2^2 = 0.96. But: how do we get the voltage of this outgoing wave?

Example: a Step in Characteristic Impedance (2)

***Step 2**: Remember, a and b are power-waves and defined as voltage of the forward- or backward travelling wave normalized to $\sqrt{Z_C}$.*

The tangential electric field in the dielectric in the 50 Ω and the 75 Ω line, respectively, must be continuous.



t = voltage transmission coefficient in this case. $t = 1 + \Gamma$

This may appear counterintuitive, as one might expect 1-Γ for the transmitted wave. ***Note that the voltage of the transmitted wave is higher than the voltage of the incident wave.*** *But we have to normalize to get the corresponding S-parameter. $S_{12} = S_{21}$ via reciprocity! But $S_{11} \neq S_{22}$, i.e. the structure is NOT symmetric.*

Example: a Step in Characteristic Impedance (3)

Once we have determined the voltage transmission coefficient, we have to normalize to the ratio of the characteristic impedances, respectively. Thus we get for

$$S_{21} = 1.2\sqrt{\frac{50}{75}} = 1.2 \cdot 0.816 = 0.9798$$

We know from the previous calculation that the reflected <u>power</u> (proportional Γ^2) is 4% of the incident power. Thus 96% of the power are transmitted.

Check done

$$S_{21}^{\ 2} = 1.44\,\frac{1}{1.5} = 0.96 = (0.9798)^2$$

$$S_{22} = \frac{50-75}{50+75} = -0.2$$

To be compared with S_{11} = +0.2!

Example: a Step in Characteristic Impedance (4)

Visualization in the Smith chart

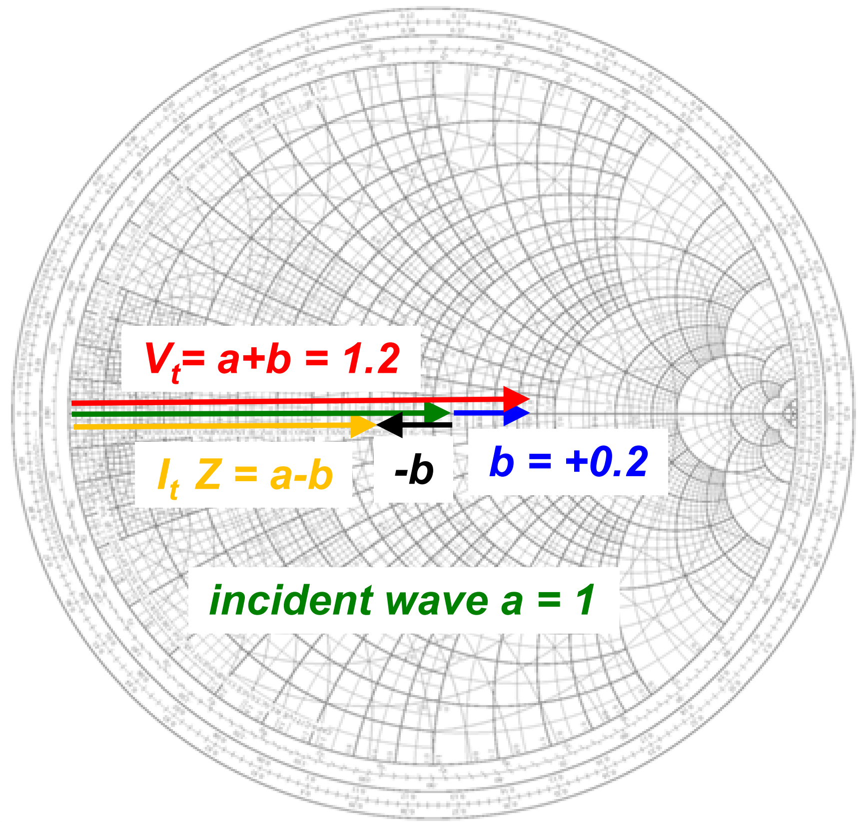
As shown in the previous slides the transmitted voltage is

$t = 1 + \Gamma$

$V_t = a + b$ and subsequently the current is

$I_t Z = a - b$.

Remember: the reflection coefficient Γ is defined with respect to voltages. For currents the sign inverts. Thus a positive reflection coefficient in the normal (=voltage) definition leads to a subtraction of currents or is negative with respect to current.

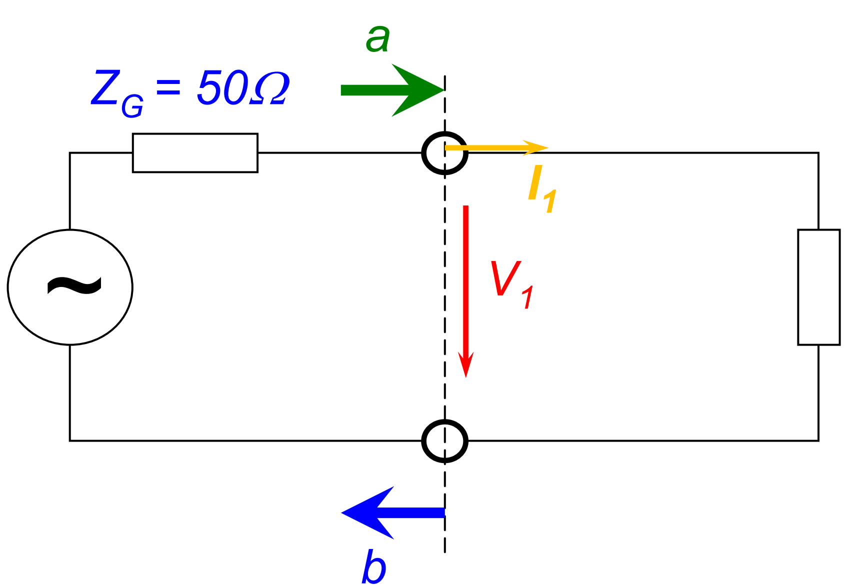


Note: here Z_{load} is real

Example: a Step in Characteristic Impedance (5)

General case

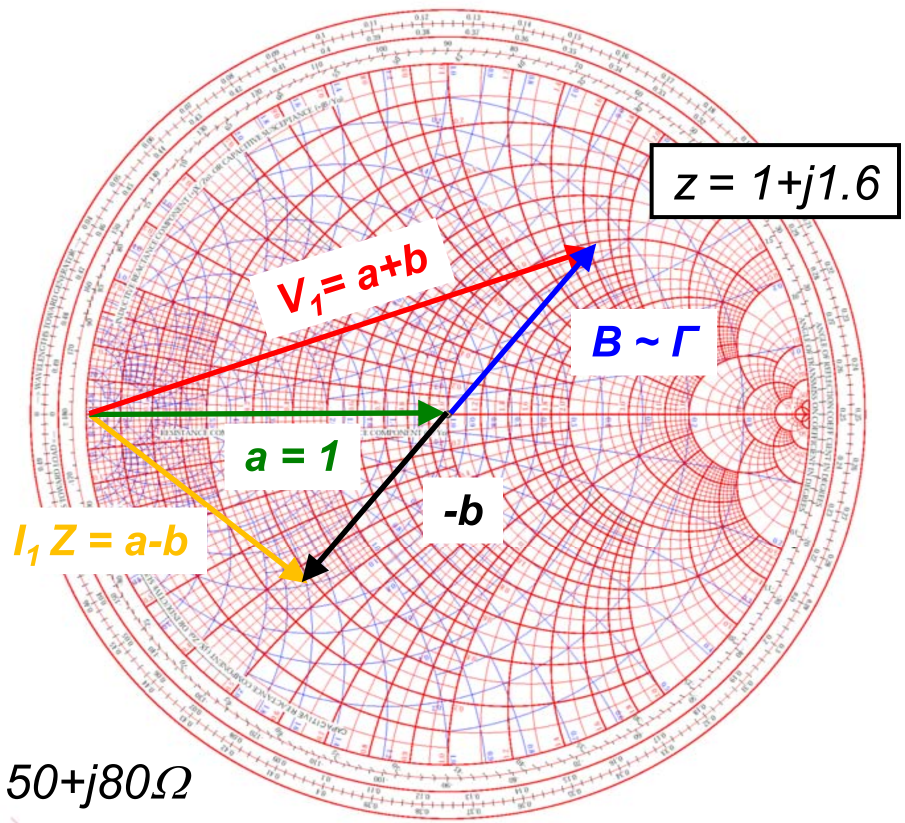
Thus we can read from the Smith chart immediately the amplitude and phase of voltage and current on the load (of course we can calculate it when using the complex voltage divider).



$Z = 50+j80\Omega$

(load impedance)

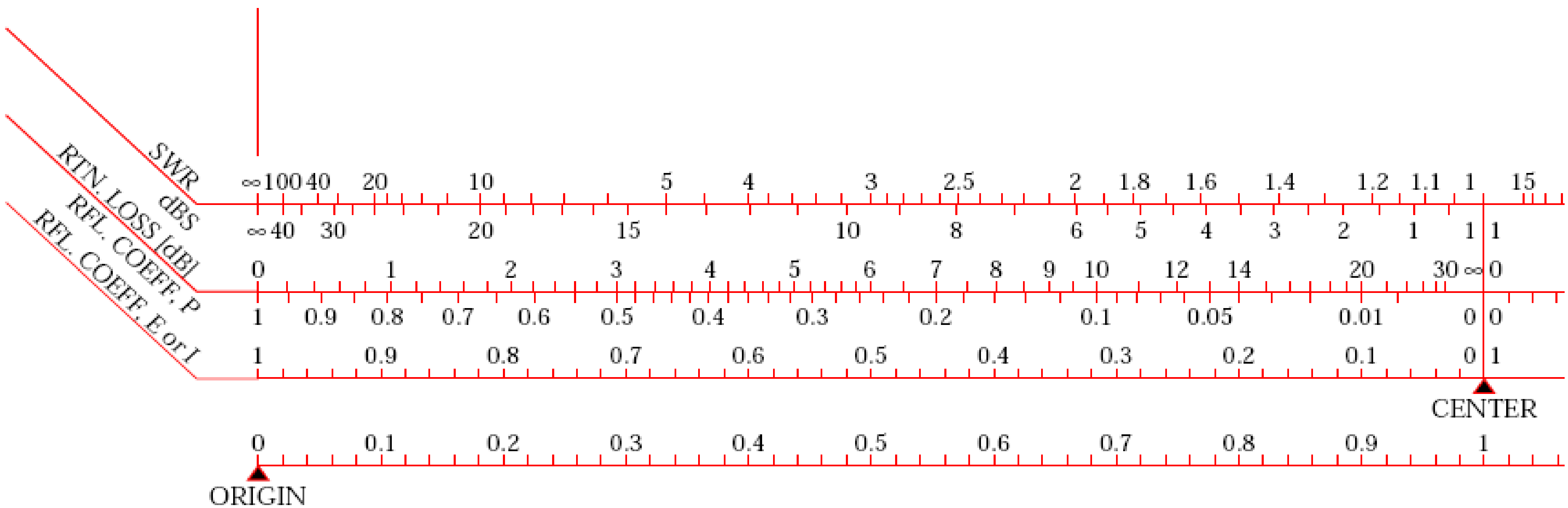
$\Rightarrow$ $z = 1+j1.6$



What about all these rulers below the Smith chart (1)

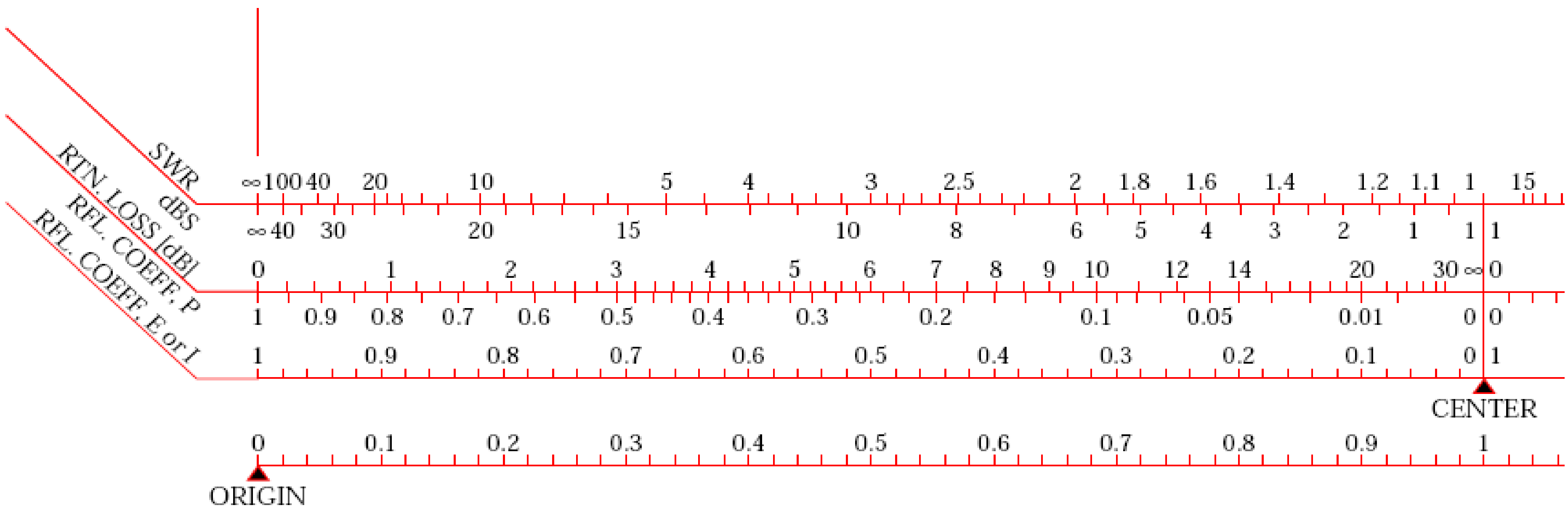
How to use these rulers:

You take the modulus of the reflection coefficient of an impedance to be examined by some means, either with a conventional ruler or better take it into the compass. Then refer to the coordinate denoted to CENTER and go to the left or for the other part of the rulers (not shown here in the magnification) to the right except for the lowest line which is marked ORIGIN at the left.



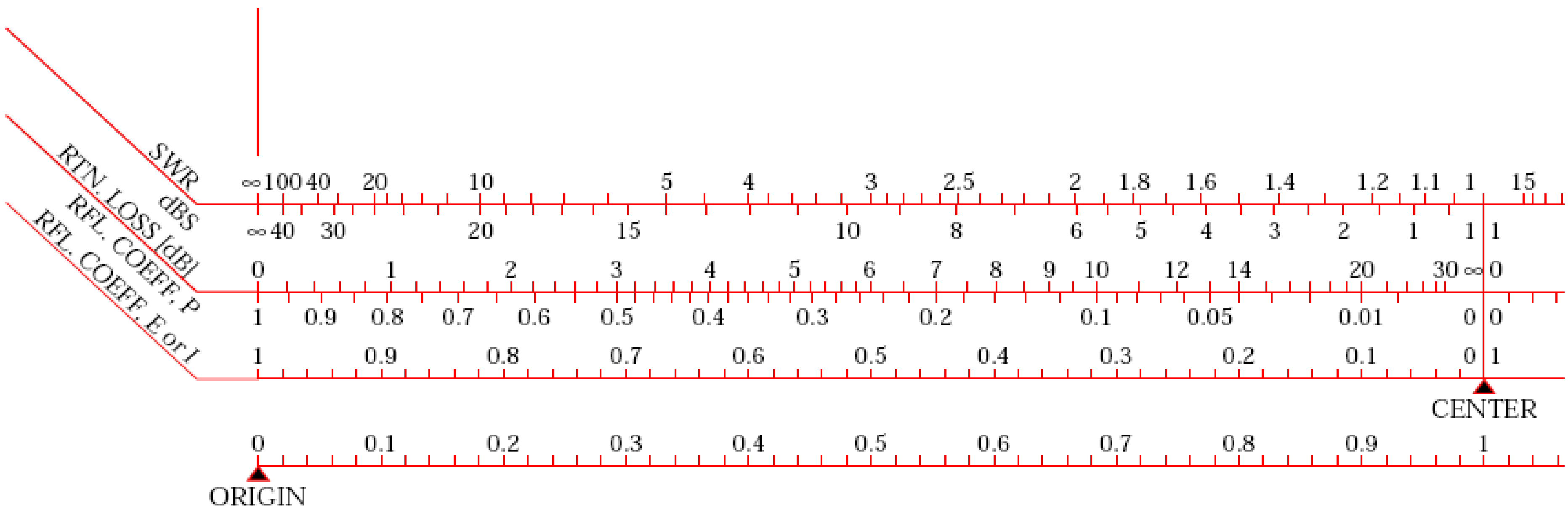
What about all these rulers below the Smith chart (2)

First ruler / left / upper part, marked SWR. This means VSWR, i.e. Voltage Standing Wave Ratio, the range of value is between one and infinity. One is for the matched case (center of the Smith chart), infinity is for total reflection (boundary of the SC). The upper part is in linear scale, the lower part of this ruler is in dB, noted as dBS (dB referred to Standing Wave Ratio). Example: SWR = 10 corresponds to 20 dBS, SWR = 100 corresponds to 40 dBS [voltage ratios, not power ratios].



What about all these rulers below the Smith chart (3)

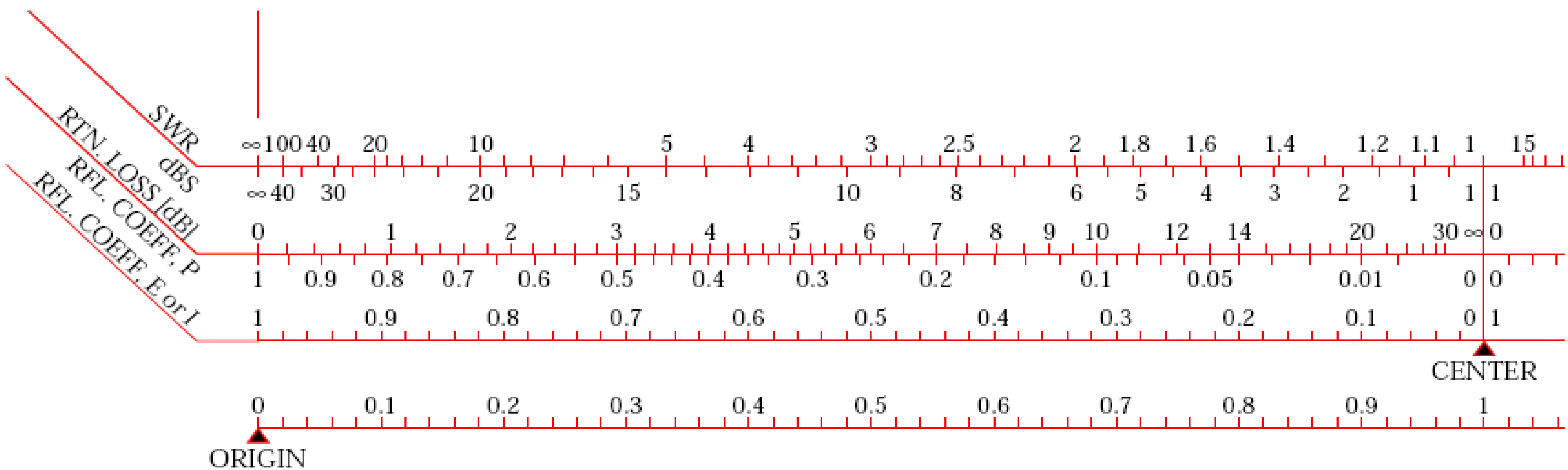
Second ruler / left / upper part, marked as RTN.LOSS = return loss in dB. This indicates the amount of reflected wave expressed in dB. Thus, in the center of SC nothing is reflected and the return loss is infinite. At the boundary we have full reflection, thus return loss 0 dB. The lower part of the scale denoted as RFL.COEFF. P = reflection coefficient in terms of POWER (proportional $|\Gamma|^2$). No reflected power for the matched case = center of the SC, (normalized) reflected power = 1 at the boundary.



What about all these rulers below the Smith chart (4)

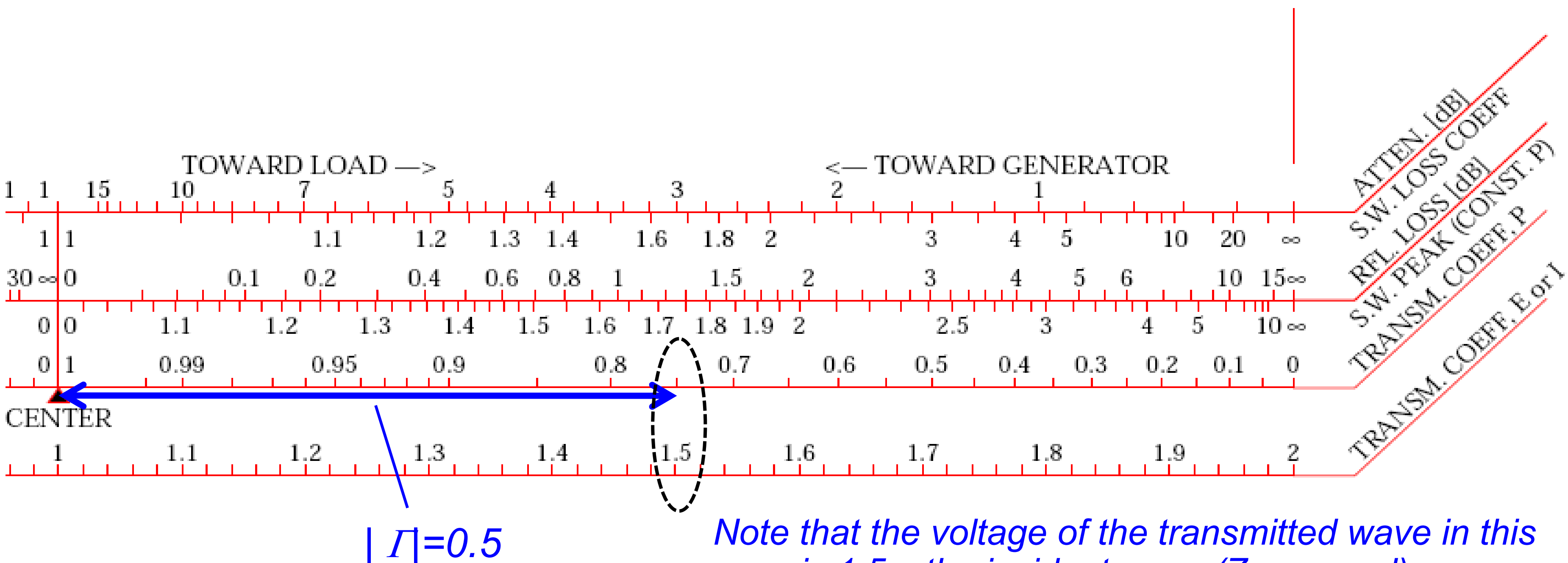
Third ruler / left, marked as RFL.COEFF,E or I = gives us the modulus (= absolute value) of the reflection coefficient in linear scale. Note that since we have the modulus we can refer it both to voltage or current as we have omitted the sign, we just use the modulus. Obviously in the center the reflection coefficient is zero, at the boundary it is one.

The fourth ruler has been discussed in the example of the previous slides: Voltage transmission coefficient. Note that the modulus of the voltage (and current) transmission coefficient has a range from zero, i.e. short circuit, to +2 (open = 1+Γ with Γ=1). This ruler is only valid for Z_{load} = real, i.e. the case of a step in characteristic impedance of the coaxial line.



What about all these rulers below the Smith chart (5)

Third ruler / right, marked as TRANSM.COEFF.P refers to the transmitted power as a function of mismatch and displays essentially the relation $P_t = 1 - |\Gamma|^2$*. Thus, in the center of the SC full match, all the power is transmitted. At the boundary we have total reflection and e.g. for a* Γ *value of 0.5 we see that 75% of the incident power is transmitted.*



Note that the voltage of the transmitted wave in this case is 1.5 x the incident wave (Z_{load} *= real)*

What about all these rulers below the Smith chart (6)

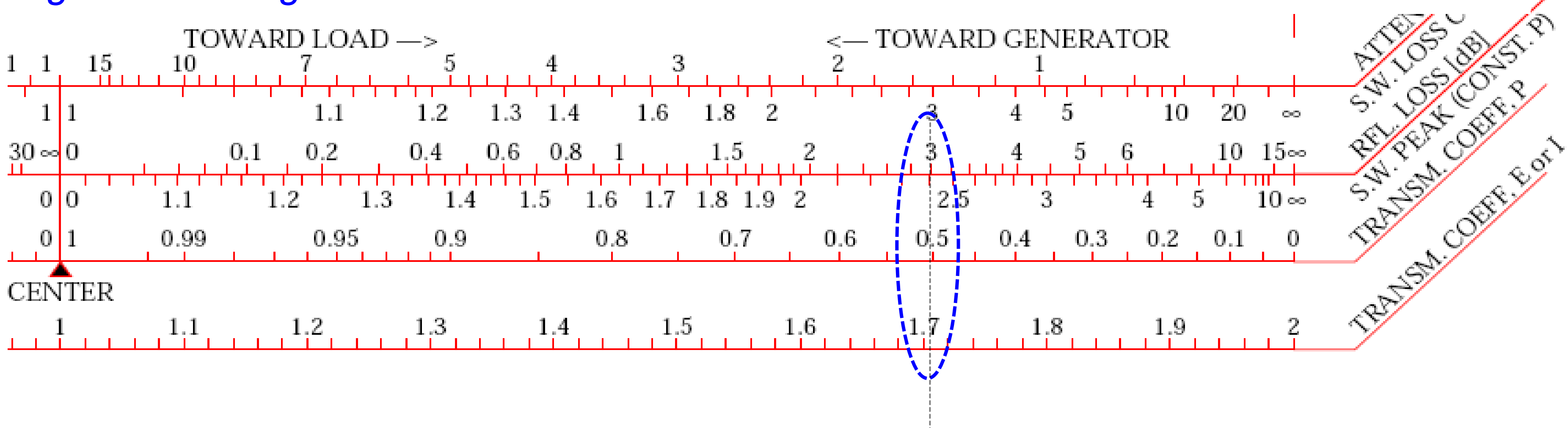
Second ruler / right / upper part, denoted as RFL.LOSS in dB = reflection loss. This ruler refers to the loss in the ***transmitted*** *wave, not to be confounded with the return loss referring to the* ***reflected*** *wave. It displays the relation* $P_t = 1 - |\Gamma|^2$ *in dB.*

Example: $|\Gamma| = 1/\sqrt{2} = 0.707$, *transmitted power = 50% thus loss = 50% = 3dB.*

Note that in the lowest ruler the voltage of the transmitted wave (Z_{load} = real) would be

$$V_t = 1.707 = 1 + 1/\sqrt{2}$$

if referring to the voltage.

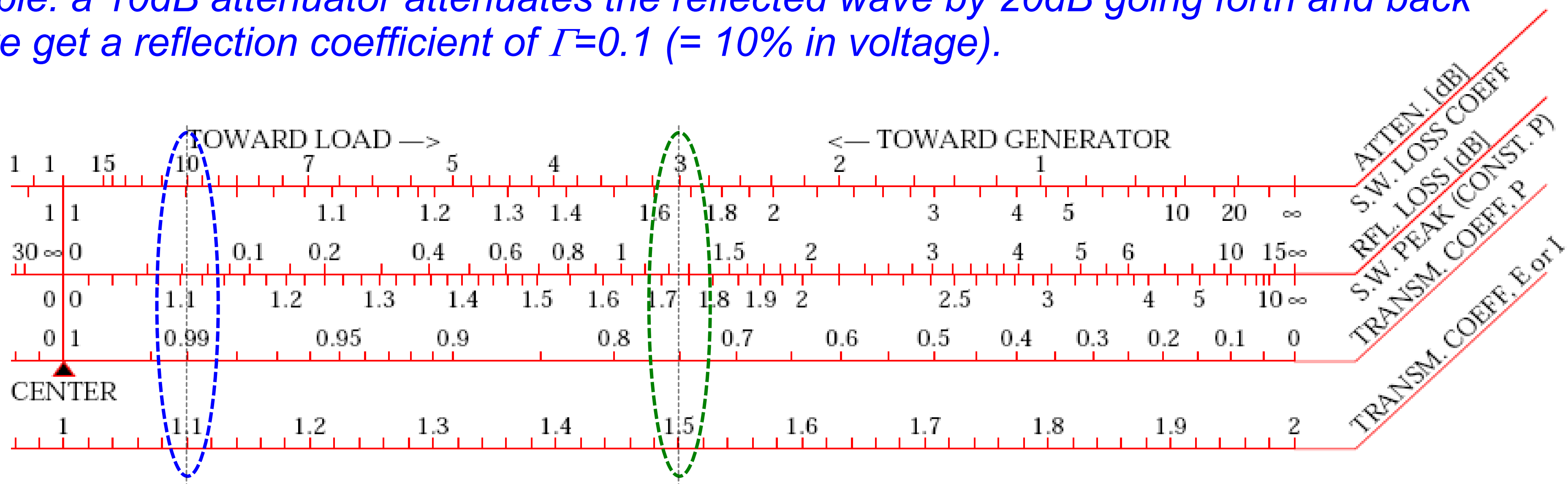


What about all these rulers below the Smith chart (7)

First ruler / right / upper part, denoted as ATTEN. in dB assumes that we are measuring an attenuator (that may be a lossy line) which itself is terminated by an open or short circuit (full reflection). Thus the wave is travelling twice through the attenuator (forward and backward). The value of this attenuator can be between zero and some very high number corresponding to the matched case.

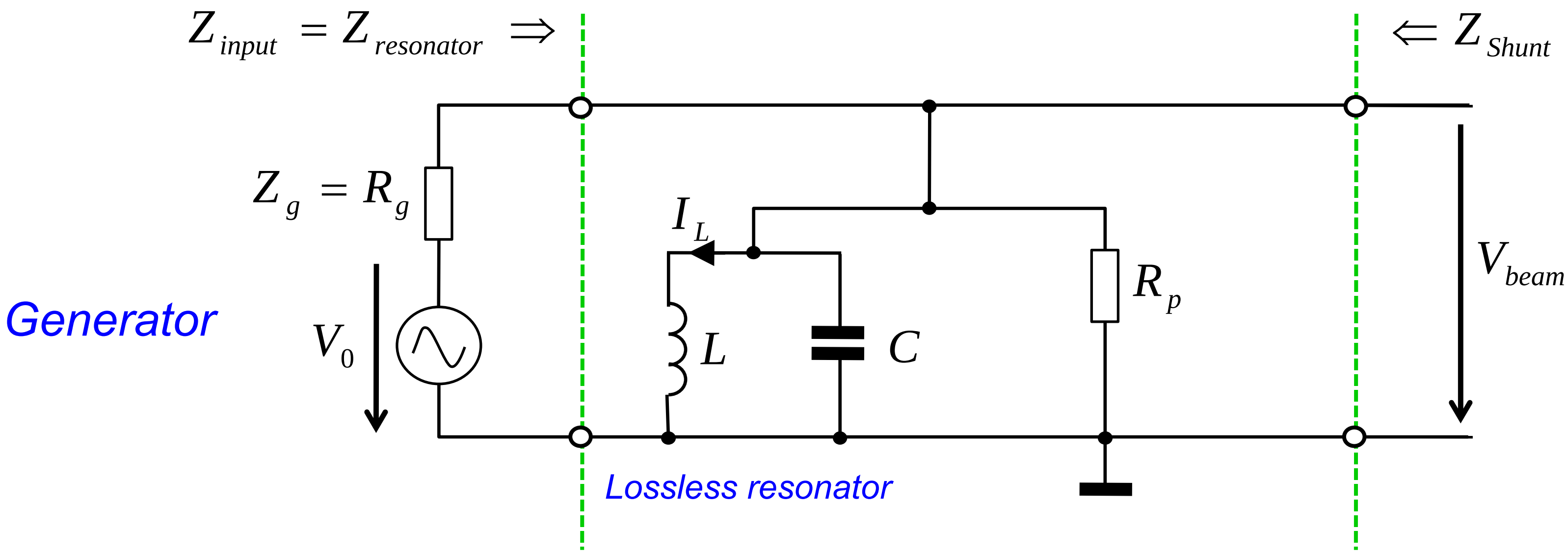
The lower scale of ruler #1 displays the same situation just in terms of VSWR.

Example: a 10dB attenuator attenuates the reflected wave by 20dB going forth and back and we get a reflection coefficient of Γ=0.1 (= 10% in voltage).



Another Example: 3dB attenuator gives forth and back 6dB which is half the voltage.

Quality Factor (1): Equivalent circuit of a cavity



The generator may deliver power to R_p and to the beam, but also the beam can deliver power to R_p and Rg!

The beam is usually considered as a current source with infinite source impedance.

R_p = resistor representing the loss of the parallel RLC equivalent circuit

We have Resonance condition, when $\omega L = \dfrac{1}{\omega C}$

→ Resonance frequency: $\omega_{res} = 2\pi f_{res} = \dfrac{1}{\sqrt{LC}} \Rightarrow \boxed{f_{res} = \dfrac{1}{2\pi}\dfrac{1}{\sqrt{LC}}}$

The Quality Factor (2)

- The quality (Q) factor of a resonant circuit is defined as the ratio of the stored energy *W* over the energy dissipated *P* in one cycle.

$$Q = \frac{\omega W}{P}$$

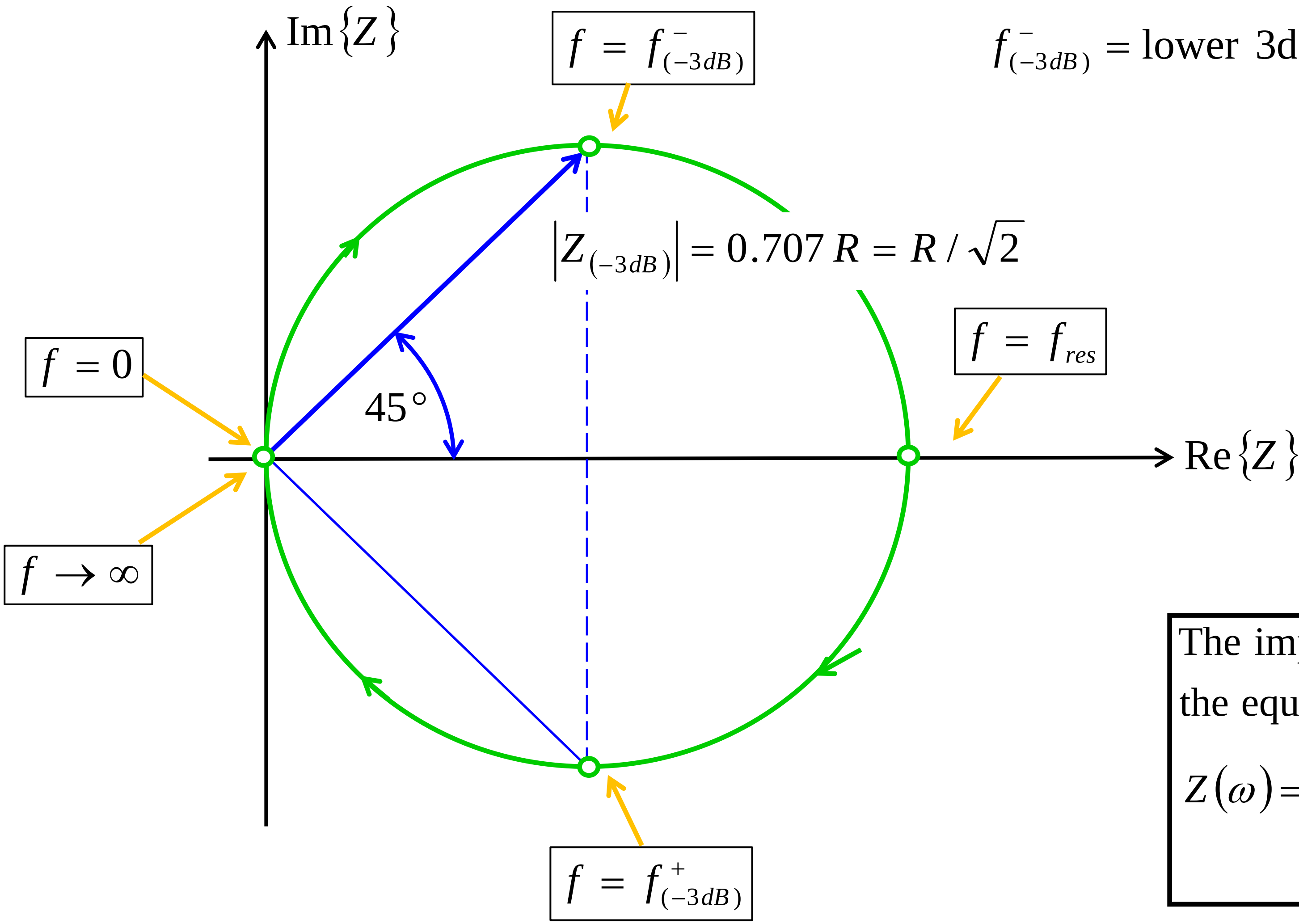
- Q_0: *Unloaded Q factor* of the unperturbed system, e.g. the "stand alone"cavity
- Q_L: *Loaded Q factor*: generator and measurement circuits connected
- Q_{ext}: *External Q factor* describes the degradation of Q_0 due to generator and diagnostic impedances
- These Q factors are related by

$$\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_{ext}}$$

- The Q factor of a resonance can be calculated from the center frequency f_0 and the 3 dB bandwidth Δf as

$$Q = \frac{f_0}{\Delta f}$$

Input Impedance: Z-plane



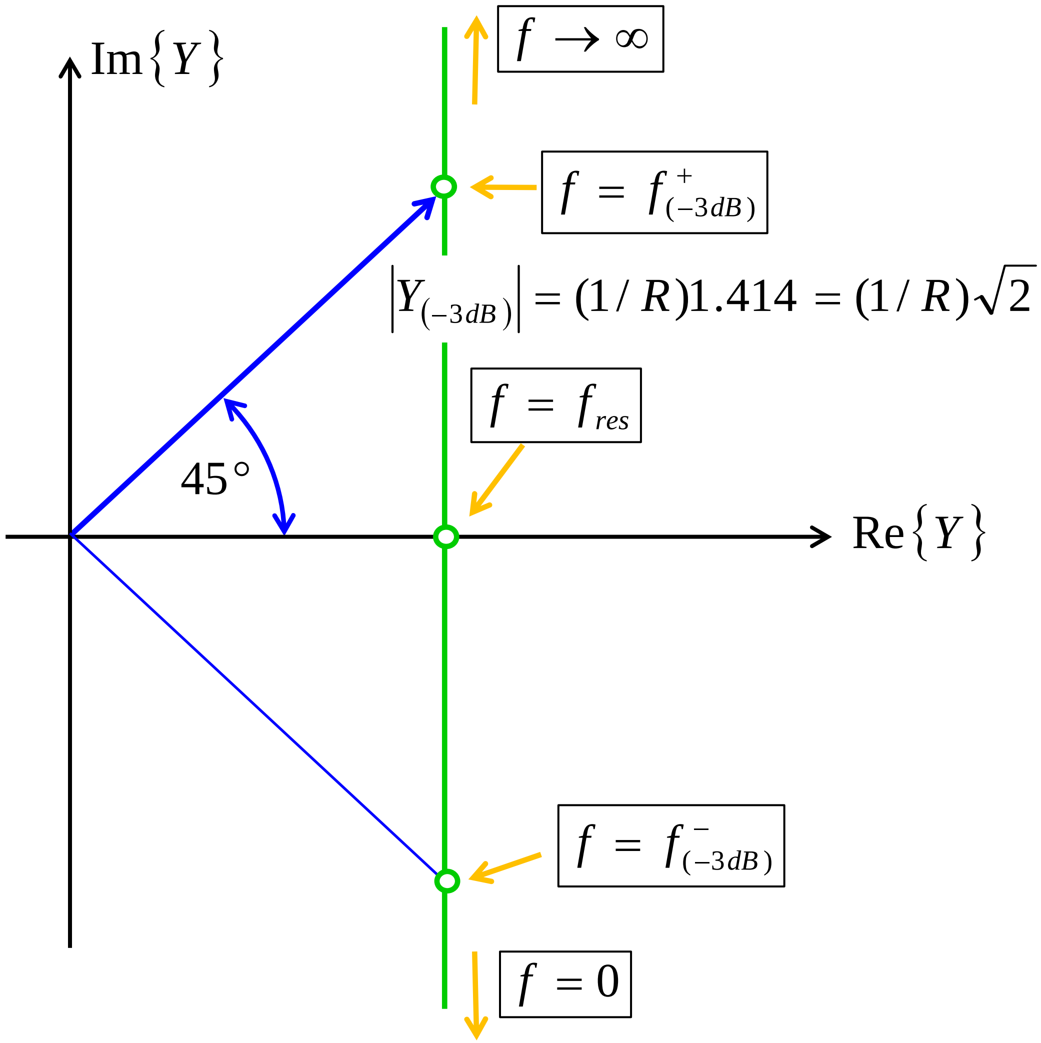
$f^{-}_{(-3dB)}$ = lower 3dB point

$f^{+}_{(-3dB)}$ = upper 3dB point

The impedance Z for the equivalent circuit is:

$$Z(\omega) = \frac{1}{\frac{1}{R} + j\omega C + \frac{1}{j\omega L}}$$

Input Admittance: Y-plane



Evaluating the admittance Y for the equivalent circuit we get

$$Y = \frac{1}{Z} = \frac{1}{R} + j\omega C + \frac{1}{j\omega L}$$

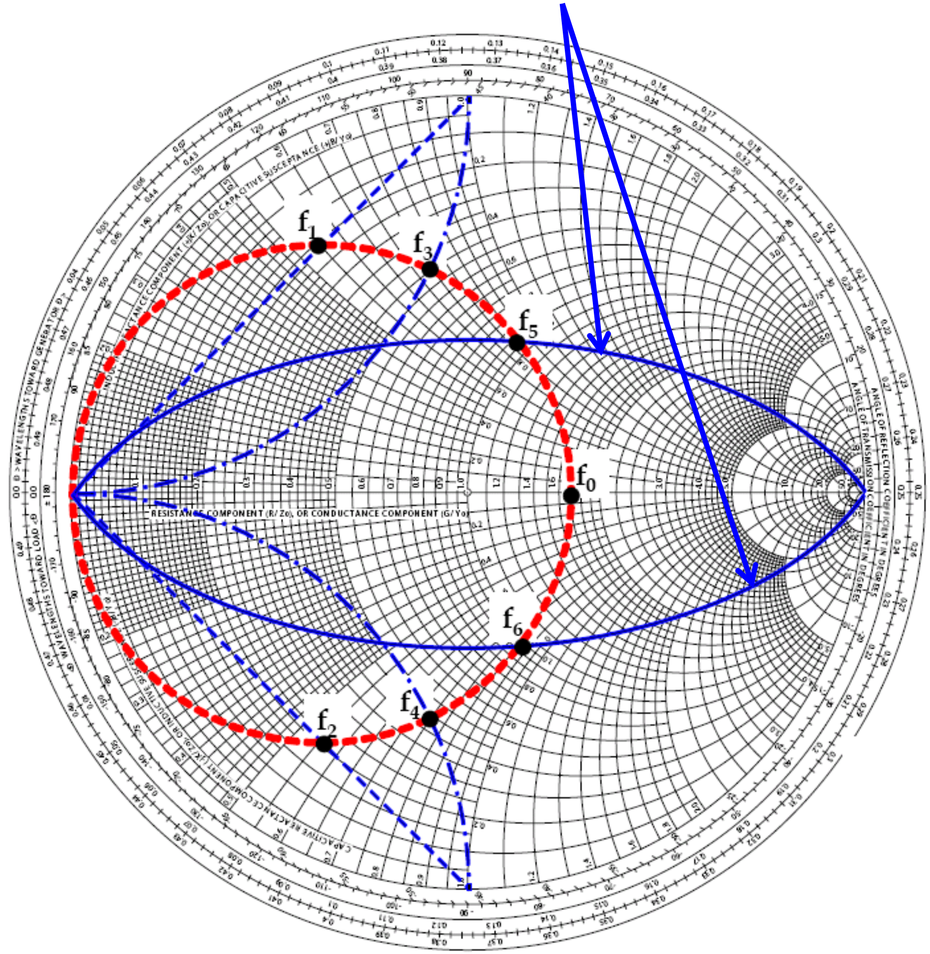
$$= \frac{1}{R} + j(\omega C - \frac{1}{\omega L})$$

$$= \frac{1}{R} + j\frac{1}{R/Q}(\frac{f}{f_{res}} - \frac{f_{res}}{f})$$

Q-Factor Measurement in the Smith Chart

- The typical locus of a resonant circuit in the Smith chart is illustrated as the dashed red circle (shown in the “detuned short” position)
- From the different marked frequency points the 3 dB bandwidth and thus the quality factors $Q_0\,(f_5,f_6)$, $Q_L(f_1,f_2)$ and $Q_{ext}(f_3,f_4)$ can be determined
- The larger the circle, the stronger the coupling
- In practise, the circle may be rotated around the origin due to transmission lines between the resonant circuit and the measurement device

Locus of Re(Z)=Im(Z)



Q-Factor Measurement in the Smith Chart

- The unloaded Q_0 can be determined from f_5 and f_6. Condition: Re{Z} = Im{Z} in detuned short position.
  - Resonator in "detuned short" position
  - Marker format: Z
  - Search for the two points where Re{Z} = Im{Z} $\Rightarrow$ f_5 and f_6
- The loaded Q_L can be calculated from the points f_1 and f_2. Condition: $|\mathrm{Im}\{S_{11}\}| \rightarrow \max$ in detuned short position.
  - Resonator in "detuned short" position
  - Marker format: $\mathrm{Re}\{S_{11}\} + j\mathrm{Im}\{S_{11}\}$
  - Search for the two points where $|\mathrm{Im}\{S_{11}\}| \rightarrow \max \Rightarrow f_1$ and f_2
- The external Q_E can be calculated from f_3 and f_4. Condition: $Z = \pm j$ in detuned open position, which is equivalent to $Y = \pm j$ in detuned short position.
  - Resonator in "detuned open" position
  - Marker format: Z
  - Search for the two points where $Z = \pm j \Rightarrow f_3$ and f_4

Q-Factor Measurement in the Smith Chart

- There are three ranges of the coupling factor β defined by

$$\beta = \frac{Q_0}{Q_{ext}} \qquad \text{or} \quad Q_L = \frac{Q_0}{1+\beta}$$

- This allows us to define:

- Critical Coupling: $\beta = 1$, $Q_L = Q_0/2$.
  The locus of ρ touches the center of the SC. At resonance all the available generator power is coupled to the resonance circuit. The phase swing is 180°.

- Undercritical Coupling: $(0 < \beta < 1)$.
  The locus of ρ in the detuned short position is left of the center of the SC. The phase swing is smaller than 180°.

- Overcritical coupling: $(1 < \beta < \infty)$.
  The center of the SC is inside the locus of ρ. The phase swing is larger than 180°. Example see previous slide