Introduction to Transverse Beam Dynamics

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The „ not so ideal world "

III.) Acceleration and Momentum Spread

The „ not so ideal world “

15.) The „ $\Delta p / p \neq 0$" Problem

ideal accelerator: *all particles will see the same accelerating voltage.*

→ $\Delta p / p = 0$

„nearly ideal“ accelerator: *Cockroft Walton or van de Graaf*

$\Delta p / p \approx 10^{-5}$

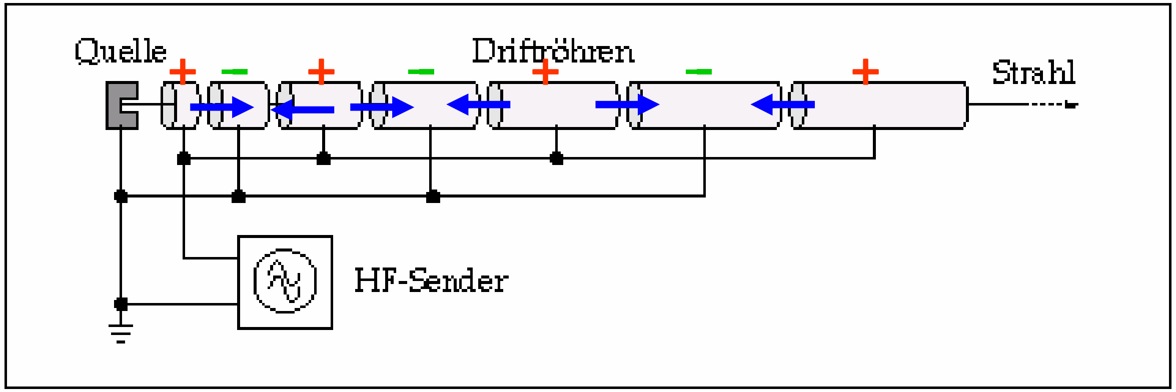
Vivitron, Straßbourg, inner structure of the acc. section

MP Tandem van de Graaf Accelerator at MPI for Nucl. Phys. Heidelberg

Linear Accelerator

1928, Wideroe

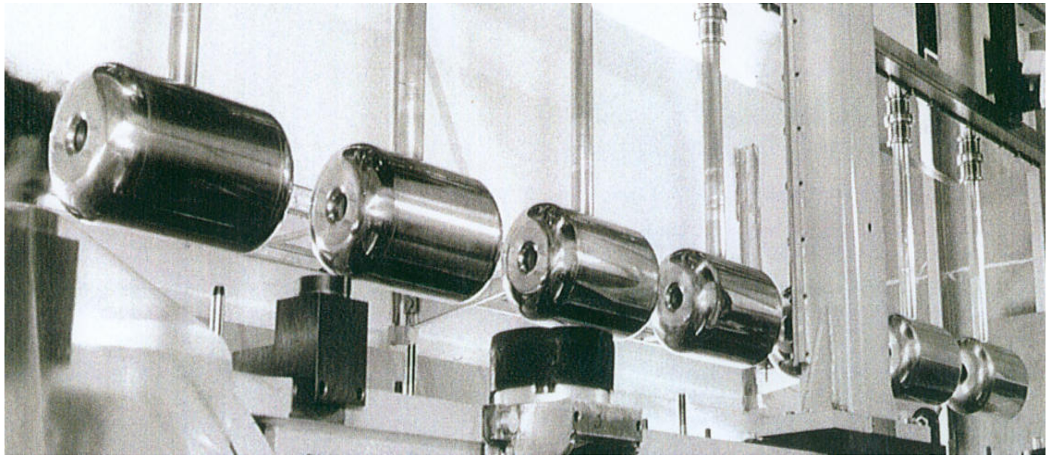
schematic Layout:



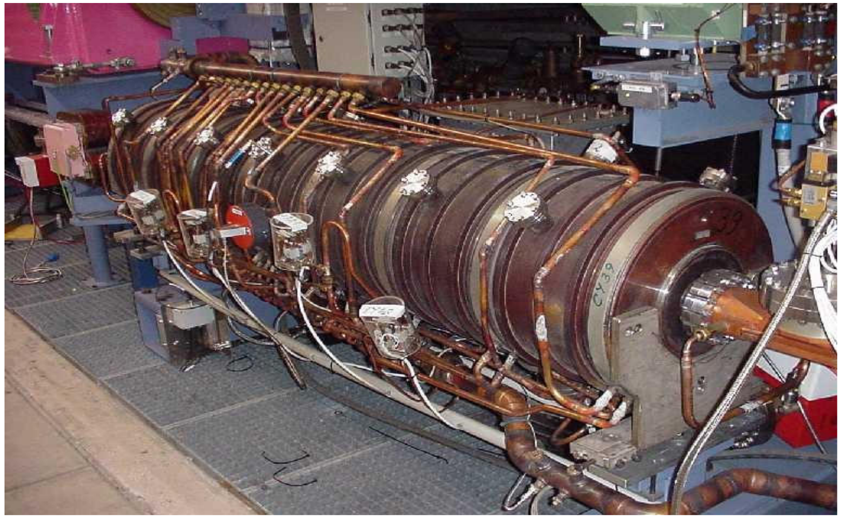
Energy Gain per „Gap“:

$$W = q\,U_0 \sin \omega_{RF}\,t$$

drift tube structure at a proton linac



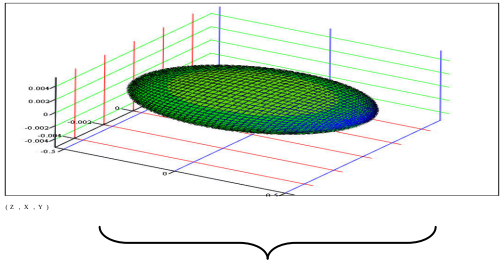
500 MHz cavities in an electron storage ring



* RF Acceleration: multiple application of the same acceleration voltage; brillant idea to gain higher energies

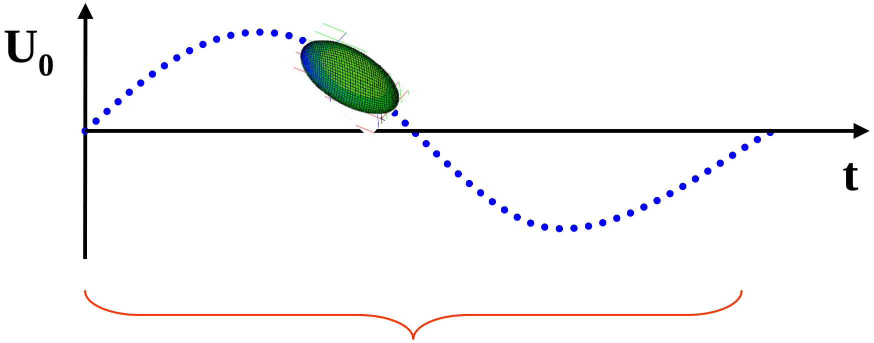
Problem: panta rhei !!!

(Heraklit: 540-480 v. Chr.)



Bunch length of Electrons ≈ 1cm

Example: HERA RF:



$$\lambda = 60\ cm$$

$$\left.\begin{array}{l} \nu = 500\ \ \boldsymbol{MHz} \\ \boldsymbol{c} = \lambda\ \nu \end{array}\right\} \qquad \lambda = 60\ cm$$

$$\sin(90^o) = 1$$

$$\sin(84^o) = 0.994$$

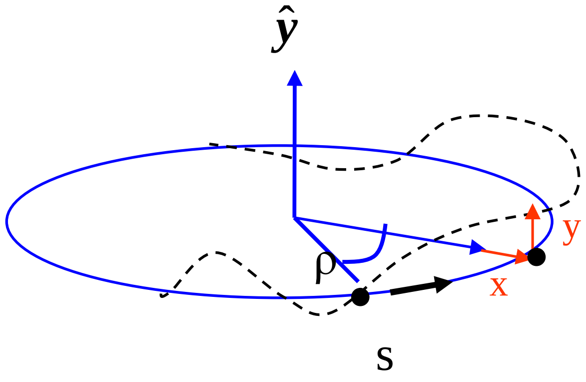
$$\frac{\Delta \boldsymbol{U}}{\boldsymbol{U}} = 6.0\ \ 10^{-3}$$

typical momentum spread of an electron bunch:

$$\frac{\Delta \boldsymbol{p}}{\boldsymbol{p}} \approx 1.0\ \ 10^{-3}$$

16.) Dispersion: trajectories for $\Delta p / p \neq 0$

Question: do you remember last session, page 12 ? … sure you do



Force acting on the particle

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} \;=\; e\,B_y\,v$$

remember: x ≈ mm , ρ ≈ m … → develop for small x

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}(1-\frac{x}{\rho}) = eB_y v$$

consider only linear fields, and change independent variable: $t \to s$

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

$$x'' - \frac{1}{\rho}(1-\frac{x}{\rho}) = \frac{e\;\;B_0}{\boxed{mv}} + \frac{e\;\;x\;\;g}{\boxed{mv}}$$

$p=p_0+\Delta p$

… but now take a small momentum error into account !!!

Dispersion:

develop for small momentum error $\Delta p \ll p_0 \Rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{e\;B_0}{p_0}}_{-\frac{1}{\rho}} - \frac{\Delta p}{p_0^2} eB_0 + \underbrace{\frac{xeg}{p_0}}_{k*x} - \underbrace{xeg\,\frac{\Delta p}{p_0^2}}_{\approx 0}$$

$$x'' + \frac{x}{\rho^2} \approx \frac{\Delta p}{p_0} * \underbrace{\frac{(-eB_0)}{p_0}}_{\frac{1}{\rho}} + k * x = \frac{\Delta p}{p_0} * \frac{1}{\rho} + k * x$$

$$x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0}\frac{1}{\rho} \quad \longrightarrow \quad \boxed{x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0}\frac{1}{\rho}}$$

Momentum spread of the beam adds a term on the r.h.s. of the equation of motion.
→ inhomogeneous differential equation.

Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\left\{\begin{aligned} & x_h''(s) + K(s) \cdot x_h(s) = 0 \\ & x_i''(s) + K(s) \cdot x_i(s) = \frac{1}{\rho} \cdot \frac{\Delta p}{p} \end{aligned}\right.$$

Normalise with respect to Δp/p:

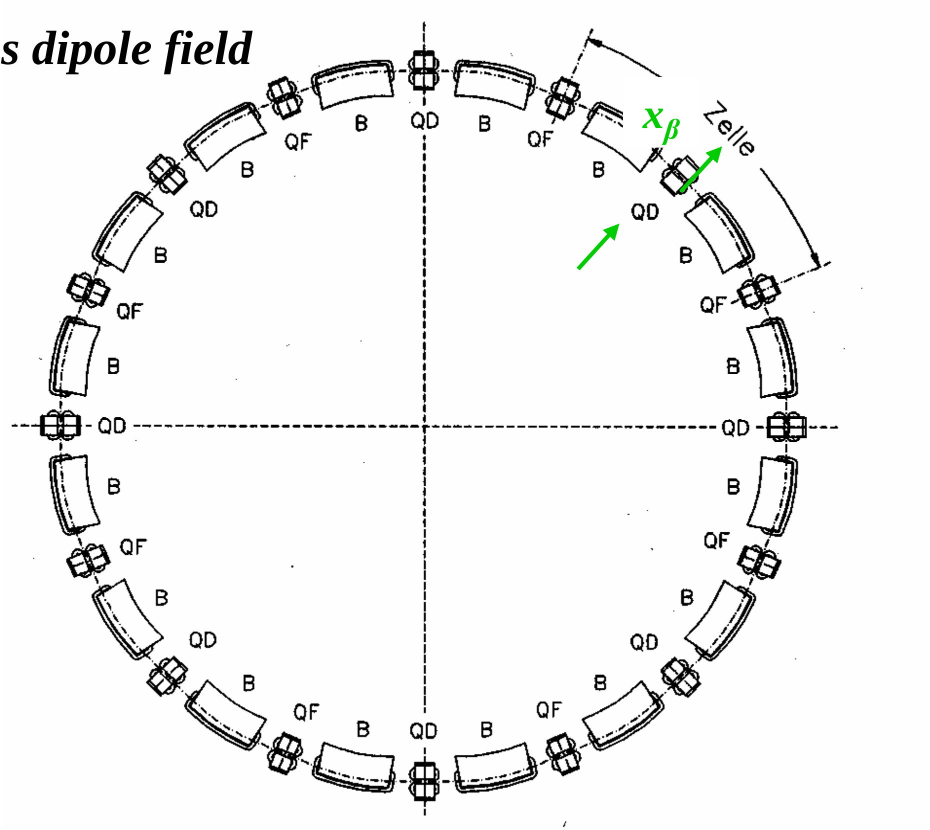
$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

- ***is that special orbit, an ideal particle would have for Δp/p = 1***
- ***the orbit of any particle is the sum of the well known x_β and the dispersion***
- ***as D(s) is just another orbit it will be subject to the focusing properties of the lattice***

Dispersion

Example: homogeneous dipole field



it for $\Delta p/p > 0$

$$D(s)\cdot\frac{\Delta p}{p}$$

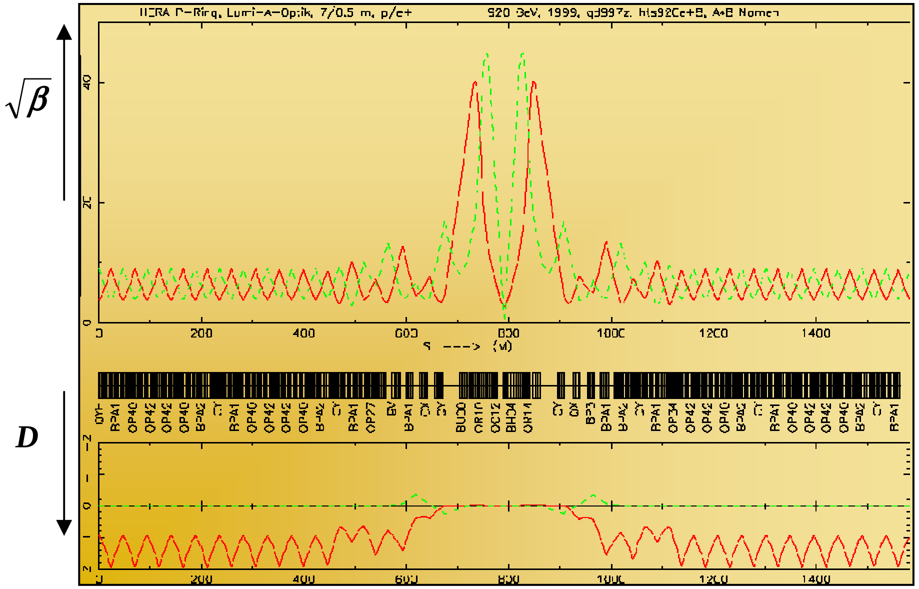
Matrix formalism:

$$\left.\begin{aligned} x(s) &= x_\beta(s) + D(s)\cdot\frac{\Delta p}{p} \\ x(s) &= C(s)\cdot x_0 + S(s)\cdot x_0' + D(s)\cdot\frac{\Delta p}{p} \end{aligned}\right\}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p}\begin{pmatrix} D \\ D' \end{pmatrix}$$

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$



Example HERA

$$x_\beta = 1 \ldots 2\ mm$$
$$D(s) \approx 1 \ldots 2\ m$$
$$\Delta p/p \approx 1 \cdot 10^{-3}$$

Amplitude of Orbit oscillation

contribution due to Dispersion ≈ beam size

→ Dispersion must vanish at the collision point

!

Calculate D, D´

$$D(s) = S(s)\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})\,d\tilde{s} - C(s)\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})\,d\tilde{s}$$

(proof: see appendix)

Example: Drift

$$M_{Drift}=\begin{pmatrix}1 & l\\ 0 & 1\end{pmatrix}$$

$$D(s)=S(s)\underbrace{\int_{s0}^{s1}\frac{1}{\rho}C(\tilde{s})d\tilde{s}}_{=0}-C(s)\underbrace{\int_{s0}^{s1}\frac{1}{\rho}S(\tilde{s})d\tilde{s}}_{=0}$$

Example: Dipole

$$M_{foc}=\begin{pmatrix}\cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}s\\ -\sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s)\end{pmatrix}_0$$

$$\boldsymbol{K}=\frac{1}{\rho^2}\cancel{-k}$$

$$\boldsymbol{s}=\boldsymbol{l_B}$$

$$M_{Dipole}=\begin{pmatrix}\cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho}\\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho}\end{pmatrix}\quad\rightarrow$$

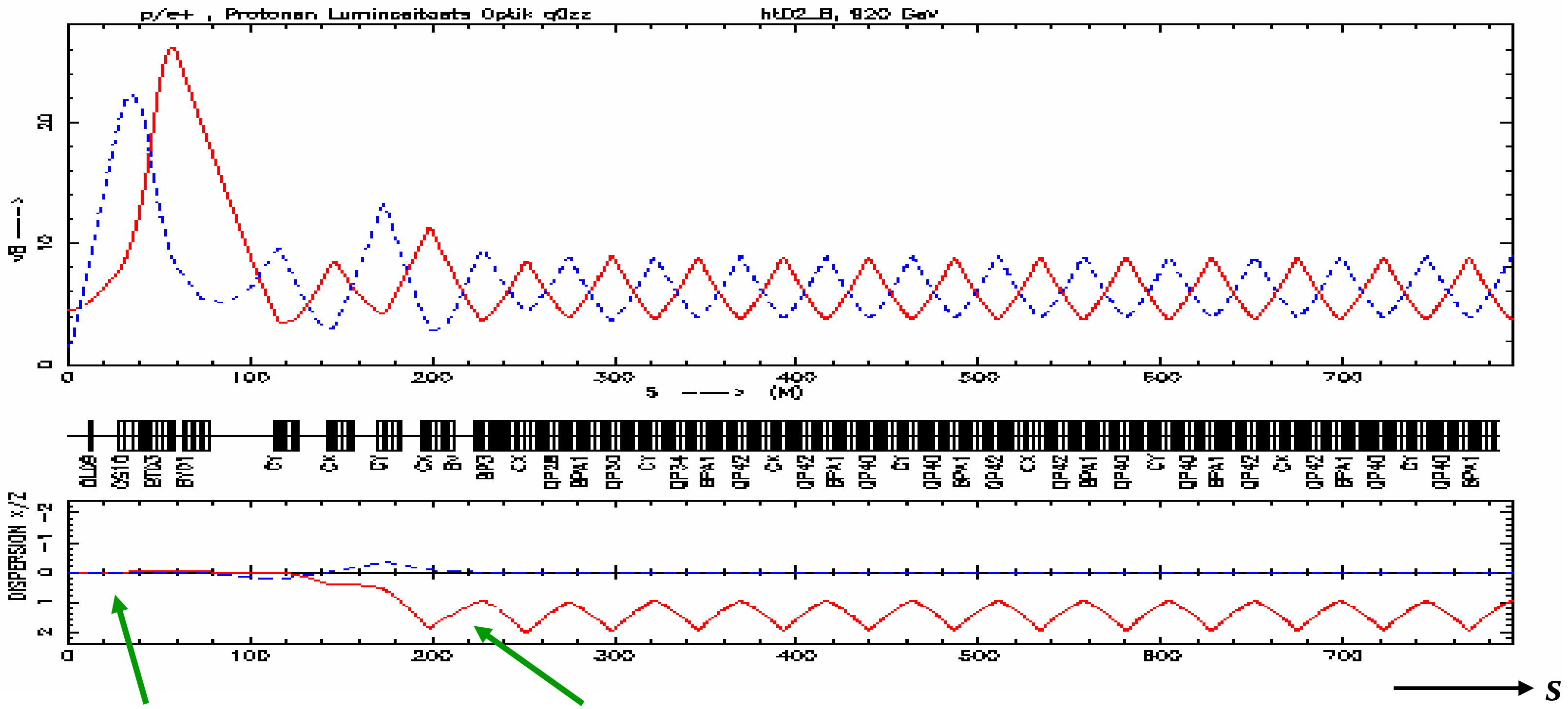
$$D(s)=\rho\cdot(1-\cos\frac{l}{\rho})$$

$$D'(s)=\sin\frac{l}{\rho}$$

Example: Dispersion, calculated by an optics code for a real machine

$$x_D = D(s)\,\frac{\Delta p}{p}$$

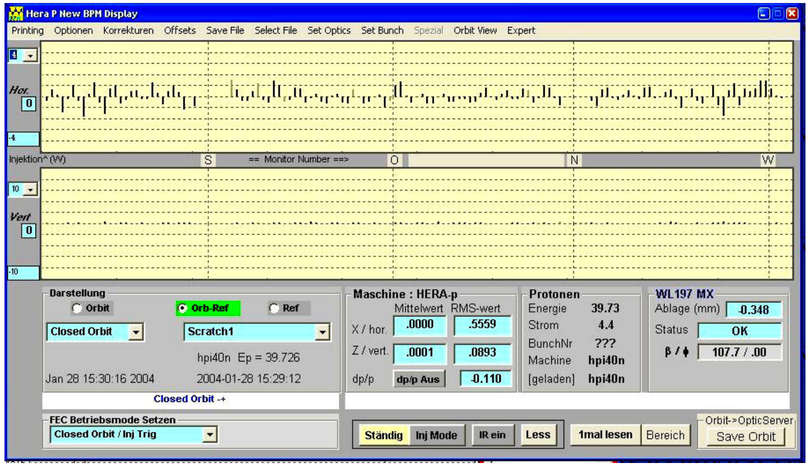
- ** D(s) is created by the dipole magnets*
  … and afterwards focused by the quadrupole fields



Mini Beta Section,
→ no dipoles !!!

D(s) ≈ 1 … 2 m

Dispersion is visible



Darstellung: ○ Orbit ● Orb-Ref ○ Ref
Closed Orbit | Scratch1
hpi40n Ep = 39.726
Jan 28 15:30:16 2004 | 2004-01-28 15:29:12
Closed Orbit -+
FEC Betriebsmode Setzen: Closed Orbit / Inj Trig

Maschine : HERA-p

	Mittelwert	RMS-wert
X / hor.	.0000	.5559
Z / vert.	.0001	.0893
dp/p	dp/p Aus	-0.110

Protonen
Energie 39.73
Strom 4.4
BunchNr ???
Machine hpi40n
[geladen] hpi40n

WL197 MX
Ablage (mm) -0.348
Status OK
β / ϕ 107.7 / .00

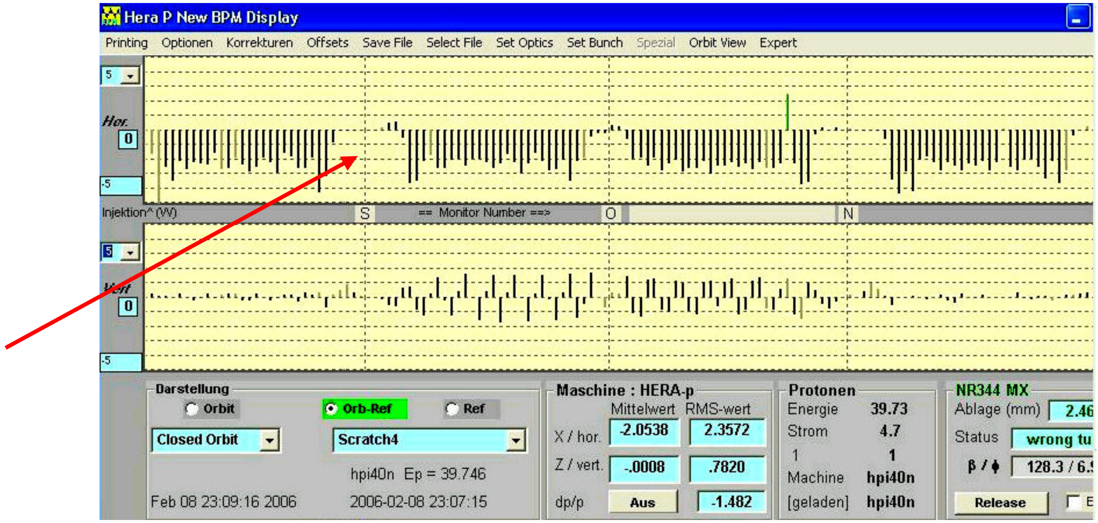
HERA Standard Orbit

dedicated energy change of the stored beam
→ closed orbit is moved to a
dispersions trajectory

$$x_D = D(s) * \frac{\Delta p}{p}$$

Attention: at the Interaction Points
we require D=D´= 0

HERA Dispersion Orbit



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Closed Orbit | Scratch4
hpi40n Ep = 39.746
Feb 08 23:09:16 2006 | 2006-02-08 23:07:15

Maschine : HERA-p

	Mittelwert	RMS-wert
X / hor.	-2.0538	2.3572
Z / vert.	-.0008	.7820
dp/p	Aus	-1.482

Protonen
Energie 39.73
Strom 4.7
1 1
Machine hpi40n
[geladen] hpi40n

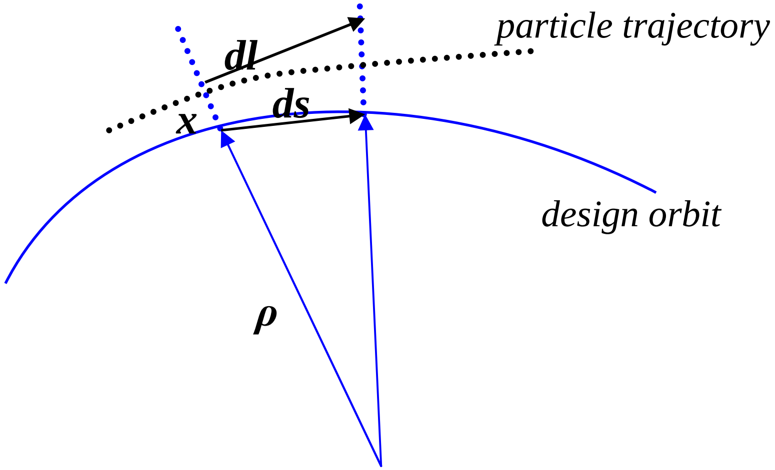
NR344 MX
Ablage (mm) 2.46
Status wrong tu
β / ϕ 128.3 / 6.

17.) Momentum Compaction Factor: α_p

particle with a displacement x to the design orbit
→ path length dl ...

$$\frac{dl}{ds} = \frac{\rho + x}{\rho}$$

$$\rightarrow \quad dl = \left(1 + \frac{x}{\rho(s)}\right) ds$$



circumference of an off-energy closed orbit

$$l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}}{\rho(s)}\right) ds$$

remember:

$$x_{\Delta E}(s) = D(s) \frac{\Delta p}{p}$$

$$\delta l_{\Delta E} = \frac{\Delta p}{p} \oint \left(\frac{D(s)}{\rho(s)}\right) ds$$

**** The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.***

Definition:

$$\frac{\delta l_\varepsilon}{L} = \alpha_p \frac{\Delta p}{p}$$

$$\to \alpha_p = \frac{1}{L}\oint\left(\frac{D(s)}{\rho(s)}\right)ds$$

For first estimates assume:

$$\frac{1}{\rho} = const.$$

$$\int_{dipoles} D(s)\,ds \approx l_{\Sigma(dipoles)} \cdot \langle D \rangle_{dipole}$$

$$\alpha_p = \frac{1}{L}\, l_{\Sigma(dipoles)} \cdot \langle D \rangle \frac{1}{\rho} = \frac{1}{L}\, 2\pi\rho \cdot \langle D \rangle \frac{1}{\rho} \quad \to \quad \alpha_p \approx \frac{2\pi}{L}\langle D \rangle \approx \frac{\langle D \rangle}{R}$$

Assume: $v \approx c$

$$\to \quad \frac{\delta T}{T} = \frac{\delta l_\varepsilon}{L} = \alpha_p \frac{\Delta p}{p}$$

α_p ***combines via the dispersion function the momentum spread with the longitudinal motion of the particle.***

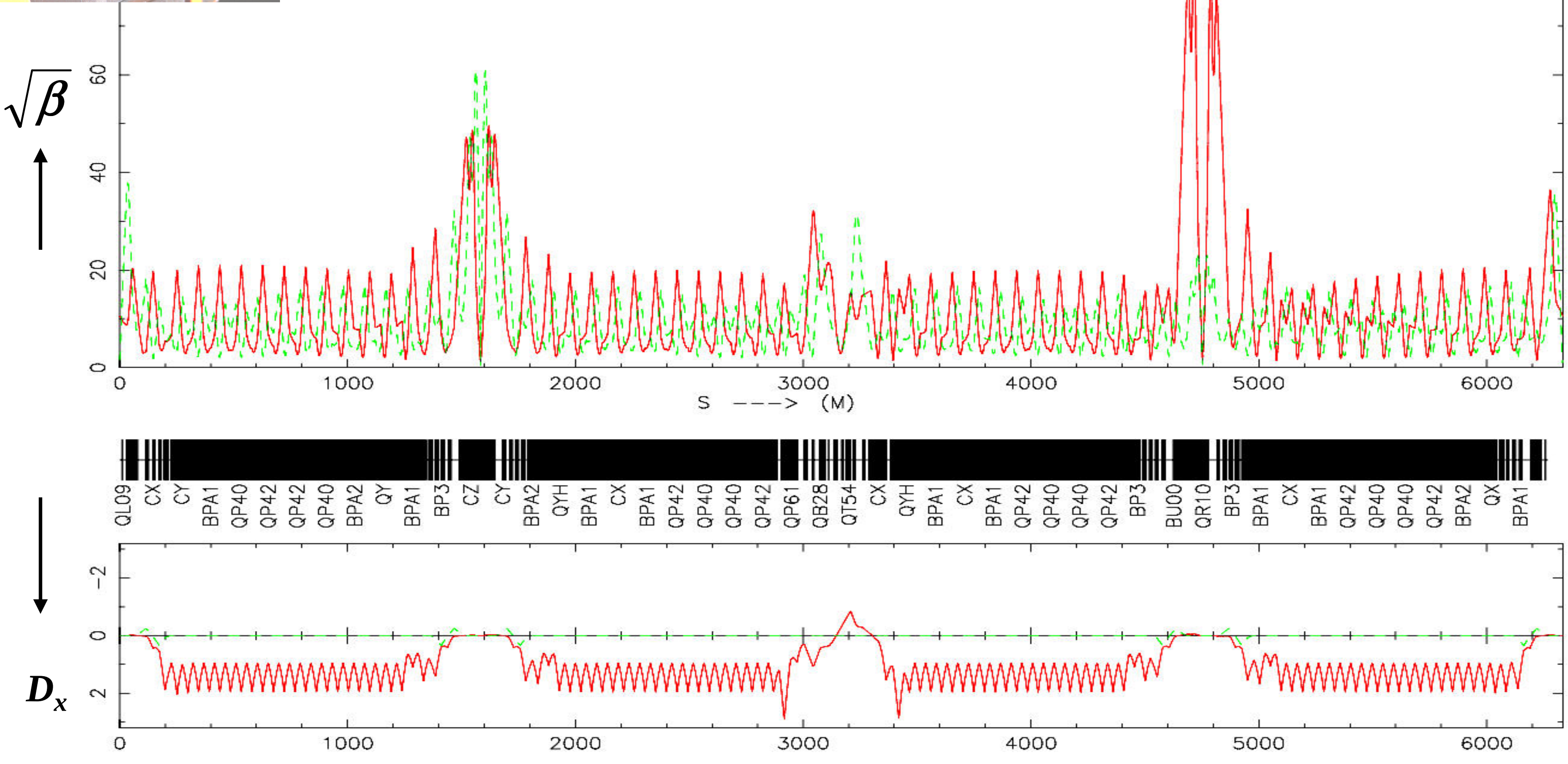
Introduction to Transverse Beam Optics

Bernhard Holzer

IV.) Errors in Field and Gradient

The „ überhaupt nicht ideal world "

18.) Quadrupole Errors



Quadrupole Errors

go back to Lecture I, page 1

single particle trajectory

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M_{QF} * \begin{pmatrix} x \\ x' \end{pmatrix}_1$$

Solution of equation of motion

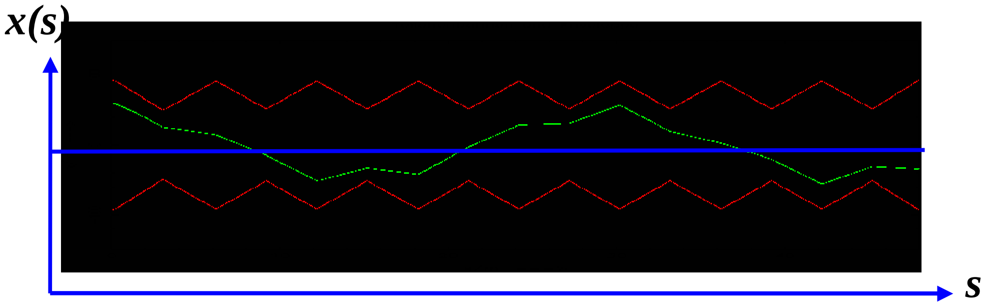
$$x = x_0 \cos(\sqrt{k}\, l_q) + x'_0 \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, l_q)$$

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{k}\, l_q) & \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, l_q) \\ -\sqrt{k} \sin(\sqrt{k}\, l_q) & \cos(\sqrt{k}\, l_q) \end{pmatrix} \quad , \quad M_{thinlens} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

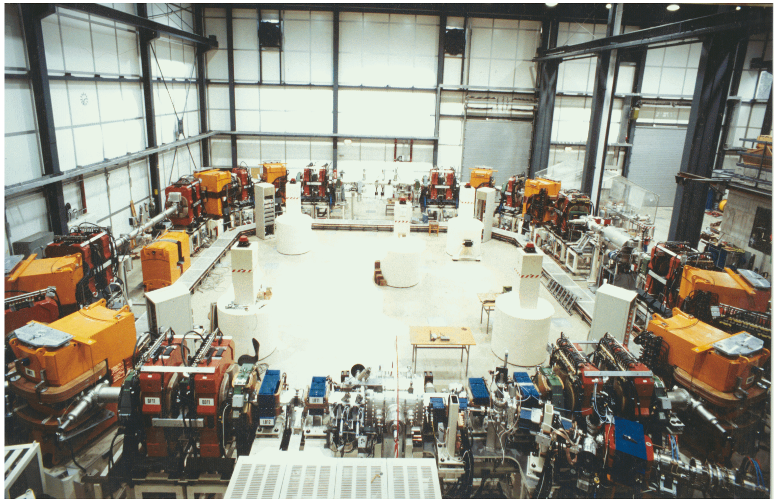
$$M_{turn} = M_{QF} * M_{D1} * M_{QD} * M_{D2} * M_{QF} \ \ldots$$

Definition: ***phase advance of the particle oscillation per revolution in units of 2π is called tune***

$$Q = \frac{\psi_{turn}}{2\pi}$$



Matrix in Twiss Form



Transfer Matrix from point „0“ in the lattice to point „s“:

$$\boldsymbol{M}(\boldsymbol{s})=\begin{pmatrix}\sqrt{\frac{\beta_s}{\beta_0}}(\cos\psi_s+\alpha_0\sin\psi_s) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0-\alpha_s)\cos(\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}(\cos(\psi_s-\alpha_0\sin\psi_s)\end{pmatrix}$$

For one complete turn the Twiss parameters have to obey periodic bundary conditions:

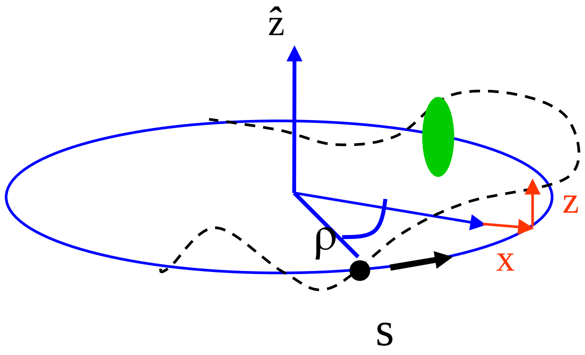
$$\beta(s+L)=\beta(s)$$
$$\alpha(s+L)=\alpha(s)$$
$$\gamma(s+L)=\gamma(s)$$

$$\boldsymbol{M}(\boldsymbol{s})=\begin{pmatrix}\cos\psi_{turn}+\alpha_s\sin\psi_{turn} & \beta_s\sin\psi_{turn} \\ -\gamma_s\sin\psi_s & \cos\psi_{turn}-\alpha_s\sin\psi_{turn}\end{pmatrix}$$

Quadrupole Error in the Lattice

optic perturbation described by thin lens quadrupole

$$\boldsymbol{M}_{dist} = \boldsymbol{M}_{\Delta k} \cdot \boldsymbol{M}_0 = \underbrace{\begin{pmatrix} 1 & 0 \\ \Delta \boldsymbol{kds} & 1 \end{pmatrix}}_{\textit{quad error}} \cdot \underbrace{\begin{pmatrix} \cos\psi_{turn} + \alpha \sin\psi_{turn} & \beta \sin\psi_{turn} \\ -\gamma \sin\psi_{turn} & \cos\psi_{turn} - \alpha \sin\psi_{turn} \end{pmatrix}}_{\textit{ideal storage ring}}$$



$$\boldsymbol{M}_{dist} = \begin{pmatrix} \cos\psi_0 + \alpha \sin\psi_0 & \beta \sin\psi_0 \\ \Delta \boldsymbol{kds}(\cos\psi_0 + \alpha \sin\psi_0) - \gamma \sin\psi_0 & \Delta \boldsymbol{kds}\beta \sin\psi_0 + \cos\psi_0 - \alpha \sin\psi_0 \end{pmatrix}$$

rule for getting the tune

$$\boldsymbol{Trace}(\boldsymbol{M}) = 2\cos\psi = 2\cos\psi_0 + \Delta \boldsymbol{kds}\beta \sin\psi_0$$

Quadrupole error → Tune Shift

$$\boxed{\psi = \psi_0 + \Delta\psi} \longrightarrow \cos(\psi_0 + \Delta\psi) = \cos\psi_0 + \frac{\Delta kds\,\beta\sin\psi_0}{2}$$

remember the old fashioned trigonometric stuff and assume that the error is small !!!

$$\cos\psi_0\underbrace{\cos\Delta\psi}_{\approx 1} - \sin\psi_0\underbrace{\sin\Delta\psi}_{\approx\Delta\psi} = \cos\psi_0 + \frac{kds\,\beta\sin\psi_0}{2}$$

$$\Delta\psi = \frac{kds\,\beta}{2}$$

and referring to Q instead of ψ:

$$\psi = 2\pi Q$$

$$\boxed{\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k(s)\beta(s)ds}{4\pi}}$$

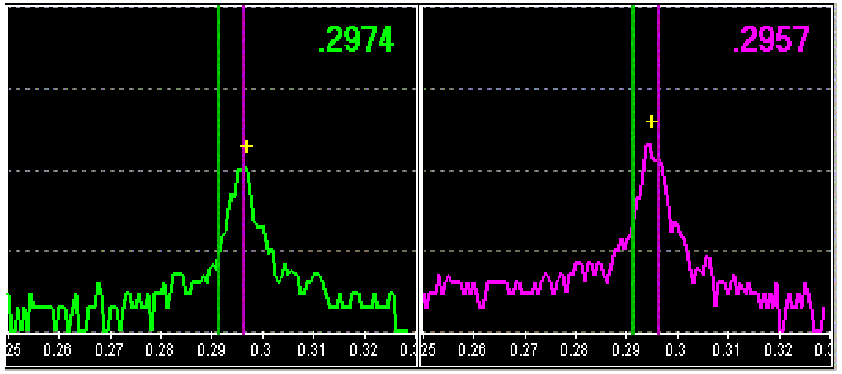
! *the tune shift is proportional to the β-function at the quadrupole*

!! *field quality, power supply tolerances etc are much tighter at places where β is large*

!!! *mini beta quads: β ≈ 1900 m*
arc quads: β ≈ 80 m

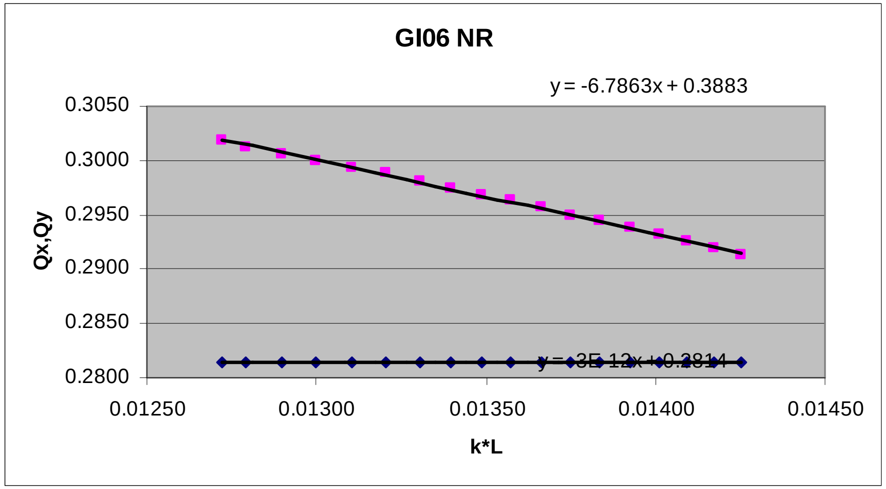
!!!! *β is a measure for the sensitivity of the beam*

a quadrupol error leads to a shift of the tune:



$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k\,\beta(s)}{4\pi}\,ds \approx \frac{\Delta k l_{quad}\,\overline{\beta}}{4\pi}$$

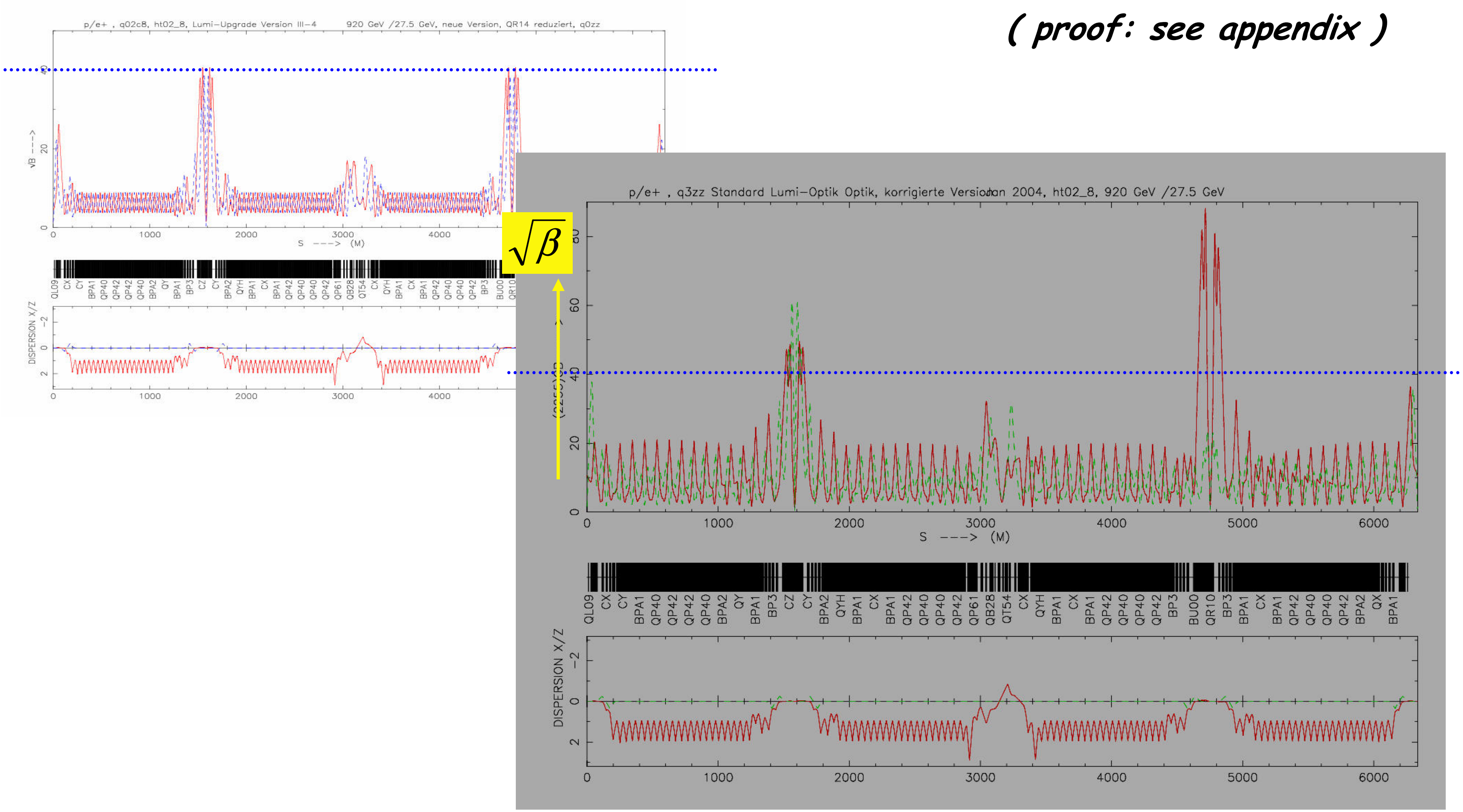
Example: measurement of β in a storage ring:
tune spectrum



Quadrupole error: Beta Beat

$$\Delta\beta(s_0) = \frac{\beta_0}{2\sin 2\pi Q}\int_{s1}^{s1+l}\beta(s_1)\Delta K\cos\left(2\left|\psi_{s1}-\psi_{s0}\right|-2\pi Q\right)ds$$

(proof: see appendix)

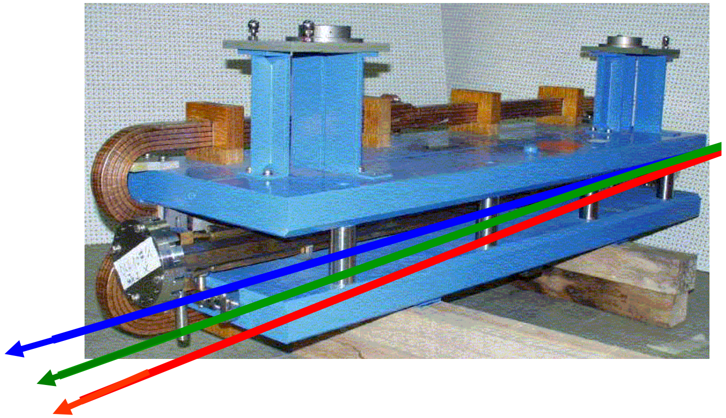


$\sqrt{\beta}$

19.) Chromaticity:
A Quadrupole Error for $\Delta p/p \neq 0$

Influence of external fields on the beam: *prop. to magn. field & prop. zu 1/p*

dipole magnet $\alpha = \frac{\int B\,dl}{p/e}$



$$x_D(s) = D(s)\,\frac{\Delta p}{p}$$

focusing lens $k = \frac{g}{p/e}$

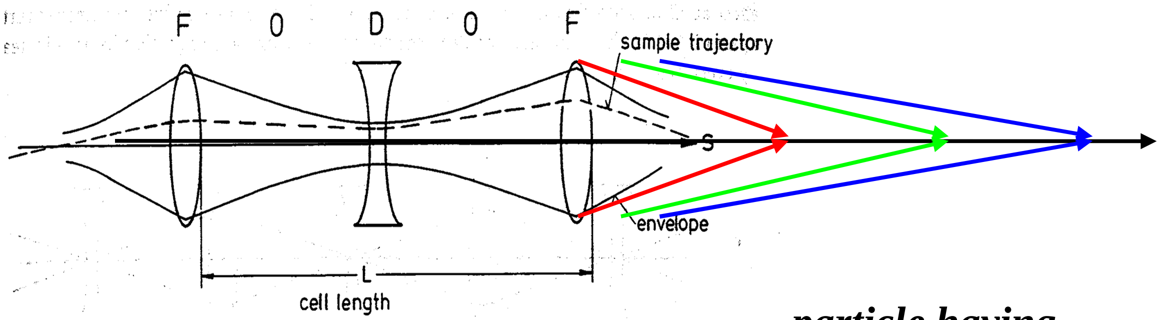


Figure 29: FODO cell

particle having ...
to high energy
to low energy
ideal energy

Chromaticity: Q'

$$k = \frac{g}{p/e} \qquad\qquad p = p_0 + \Delta p$$

in case of a momentum spread:

$$\boldsymbol{k = \frac{eg}{p_0 + \Delta p} \approx \frac{e}{p_0}\left(1 - \frac{\Delta p}{p_0}\right) g = k_0 + \Delta k}$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

… which acts like a quadrupole error in the machine and leads to a tune spread:

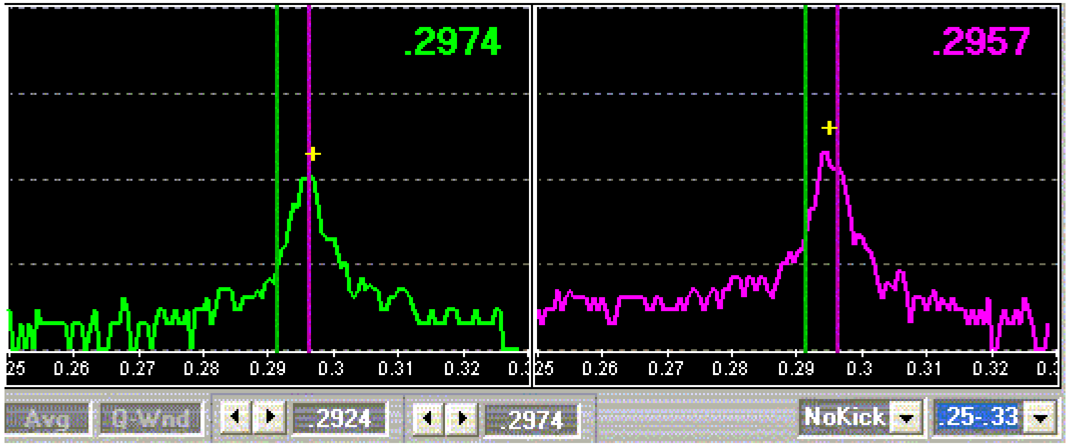
$$\Delta Q = -\frac{1}{4\pi} \frac{\Delta p}{p_0} k_0 \beta(s) ds$$

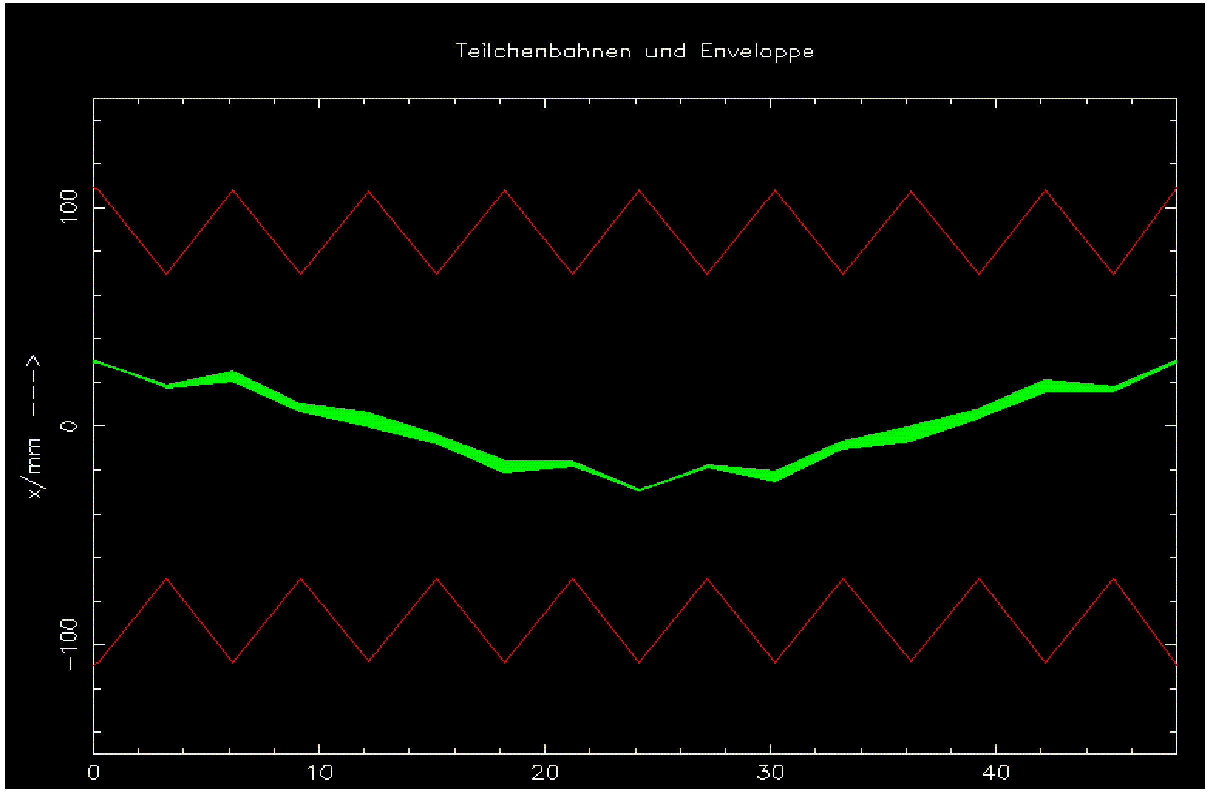
definition of chromaticity:

$$\Delta Q = Q' \frac{\Delta p}{p} \quad ; \qquad Q' = -\frac{1}{4\pi} \oint k(s) \beta(s) ds$$

Where is the Problem ?

Tunes and Resonances





avoid resonance conditions:

$$m\,Q_x + n\,Q_y + l\,Q_s = \textit{integer}$$

… for example: $1\,Q_x = 1$

… and now again about Chromaticity:

Problem: chromaticity is generated by the lattice itself !!

Q' is a number indicating the size of the tune spot in the working diagram,
Q' is always created if the beam is focussed
→ it is determined by the focusing strength k of all quadrupoles

$$\boxed{Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds}$$

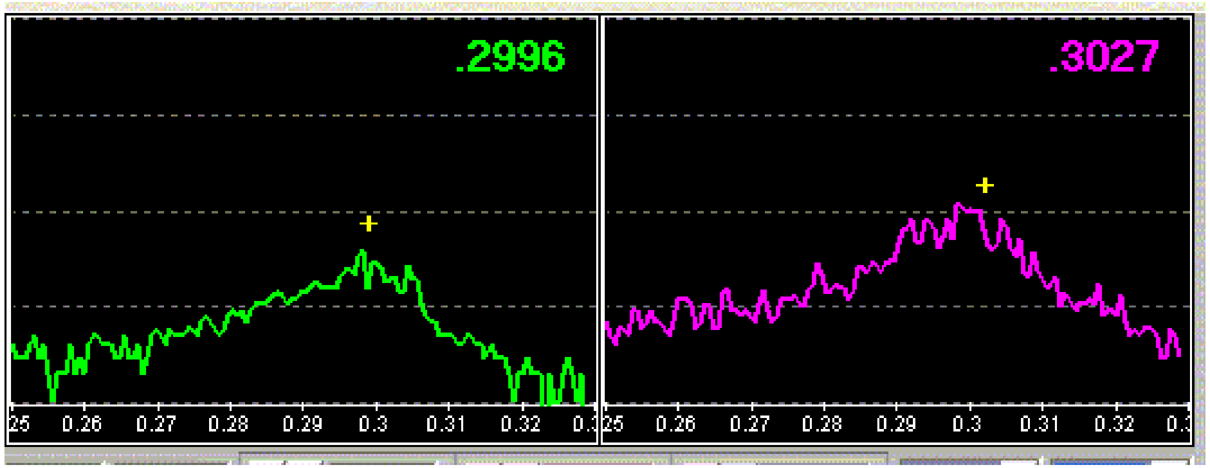
k = quadrupole strength
β = betafunction indicates the beam size … and even more the sensitivity of
the beam to external fields

Example: HERA

HERA-p: ***Q' = -70 … -80***
$\Delta p/p = 0.5 * 10^{-3}$
$\Delta Q = 0.257 \dots 0.337$

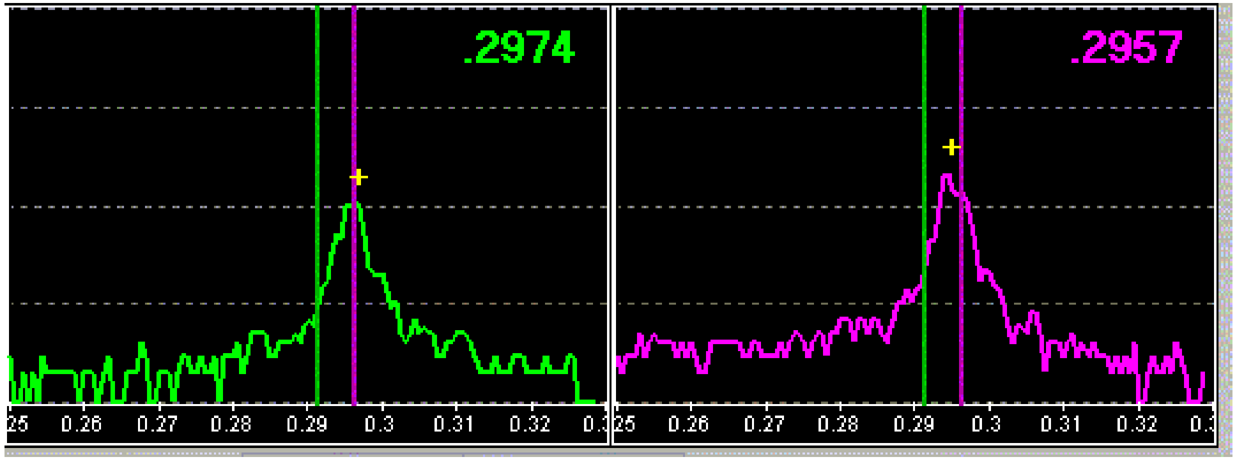
→Some particles get very close to resonances and are lost

in other words: the tune is not a point
it is a pancake



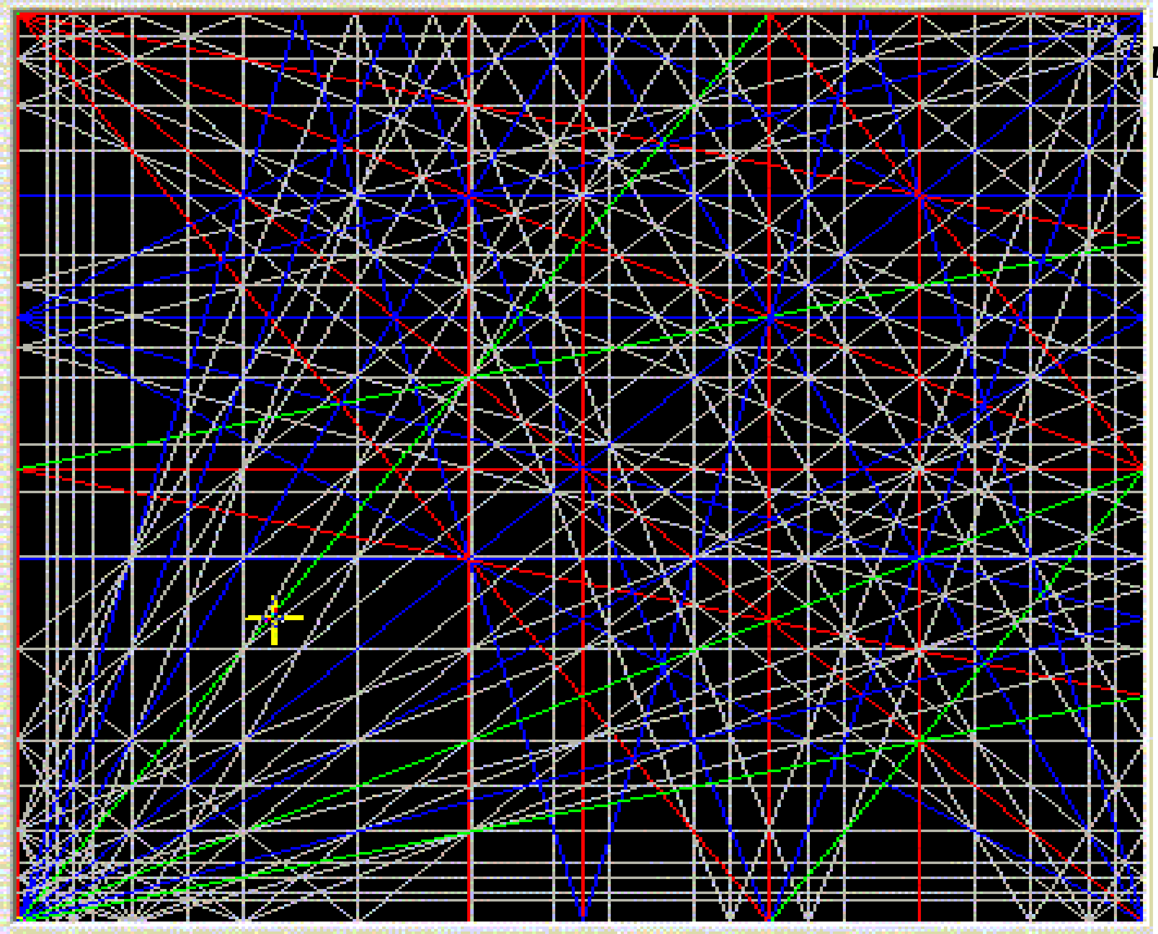
Tune signal for a nearly
uncompensated cromaticity
(Q' ≈ 20)

Ideal situation: cromaticity well corrected,
(Q' ≈ 1)



Tune and Resonances

$$m*Q_x+n*Q_y+l*Q_s = \text{integer}$$



RA e Tune diagram up to 3rd order

… and up to 7th order

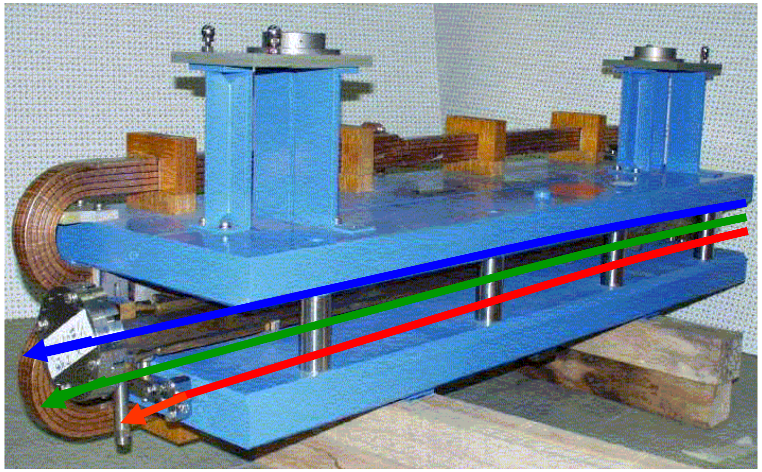
Homework for the operateurs:
find a nice place for the tune
where against all probability
the beam will survive

Correction of Q':

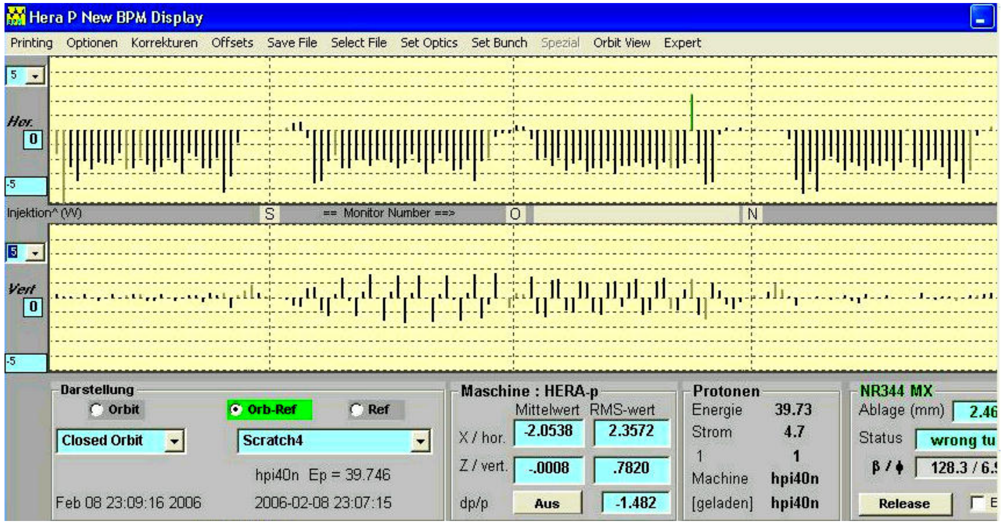
Need: additional quadrupole strength for each momentum deviation Δp/p

1.) sort the particles according to their momentum

$$x_D(s) = D(s)\frac{\Delta p}{p}$$



… using the dispersion function



Darstellung
○ Orbit ● Orb-Ref ○ Ref
Closed Orbit
Scratch4
hpi40n Ep = 39.746
Feb 08 23:09:16 2006
2006-02-08 23:07:15

Maschine : HERA-p

	Mittelwert	RMS-wert
X / hor.	-2.0538	2.3572
Z / vert.	-.0008	.7820
dp/p	Aus	-1.482

Protonen
Energie 39.73
Strom 4.7
1 1
Machine hpi40n
[geladen] hpi40n

NR344 MX
Ablage (mm) 2.46
Status wrong tu
β / ϕ 128.3 / 6.
☐ E

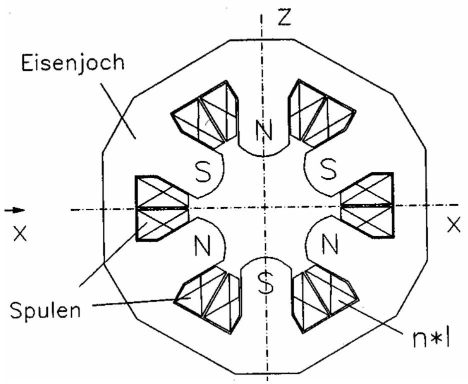
Correction of Q':

2.) apply a magnetic field that rises quadratically with x (sextupole field)

$$\left.\begin{aligned} B_x &= \tilde{g}xy \\ B_y &= \frac{1}{2}\tilde{g}(x^2 - y^2) \end{aligned}\right\} \quad \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} = \tilde{g}x$$

linear rising „gradient“:

Sextupole Magnet:



normalised quadrupole strength:

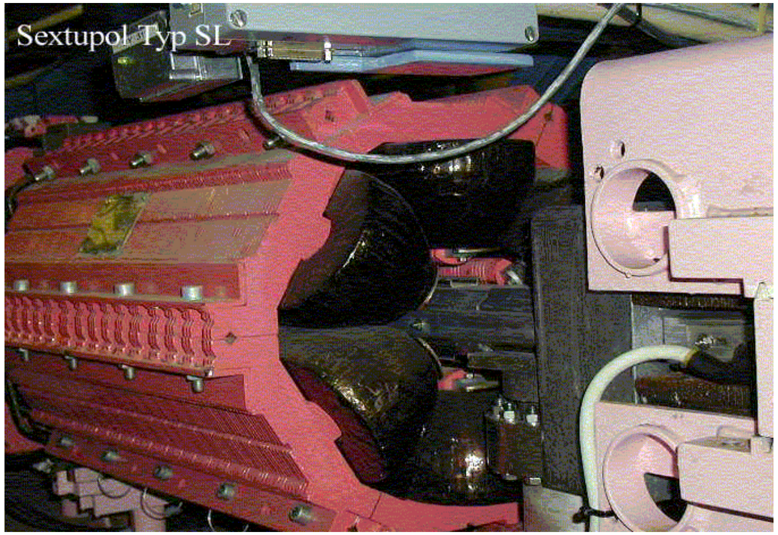
$$k_{sext} = \frac{\tilde{g}x}{p/e} = m_{sext.}x$$

$$k_{sext} = m_{sext.}D\frac{\Delta p}{p}$$

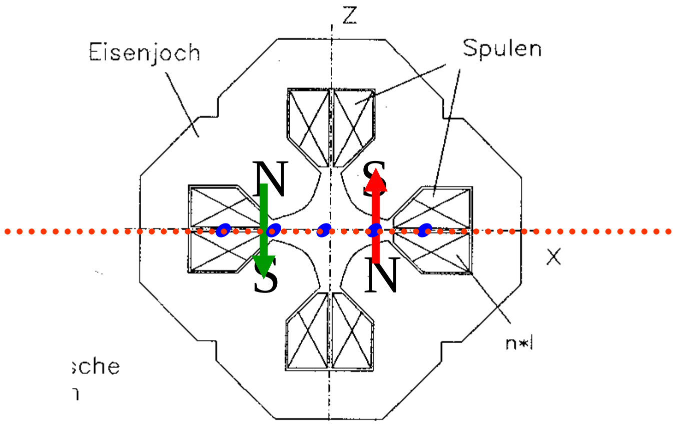
corrected chromaticity:

$$\boxed{Q' = -\frac{1}{4\pi}\oint \{k(s) - mD(s)\}\beta(s)ds}$$

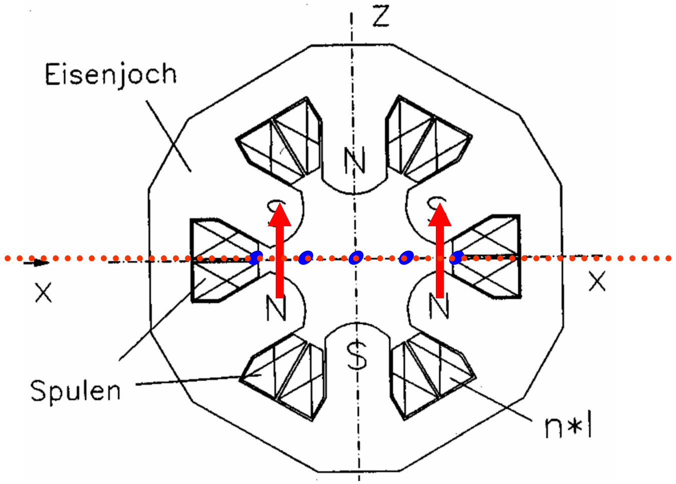
sextupole magnet in a storage ring
… placed close to the quadrupole lens



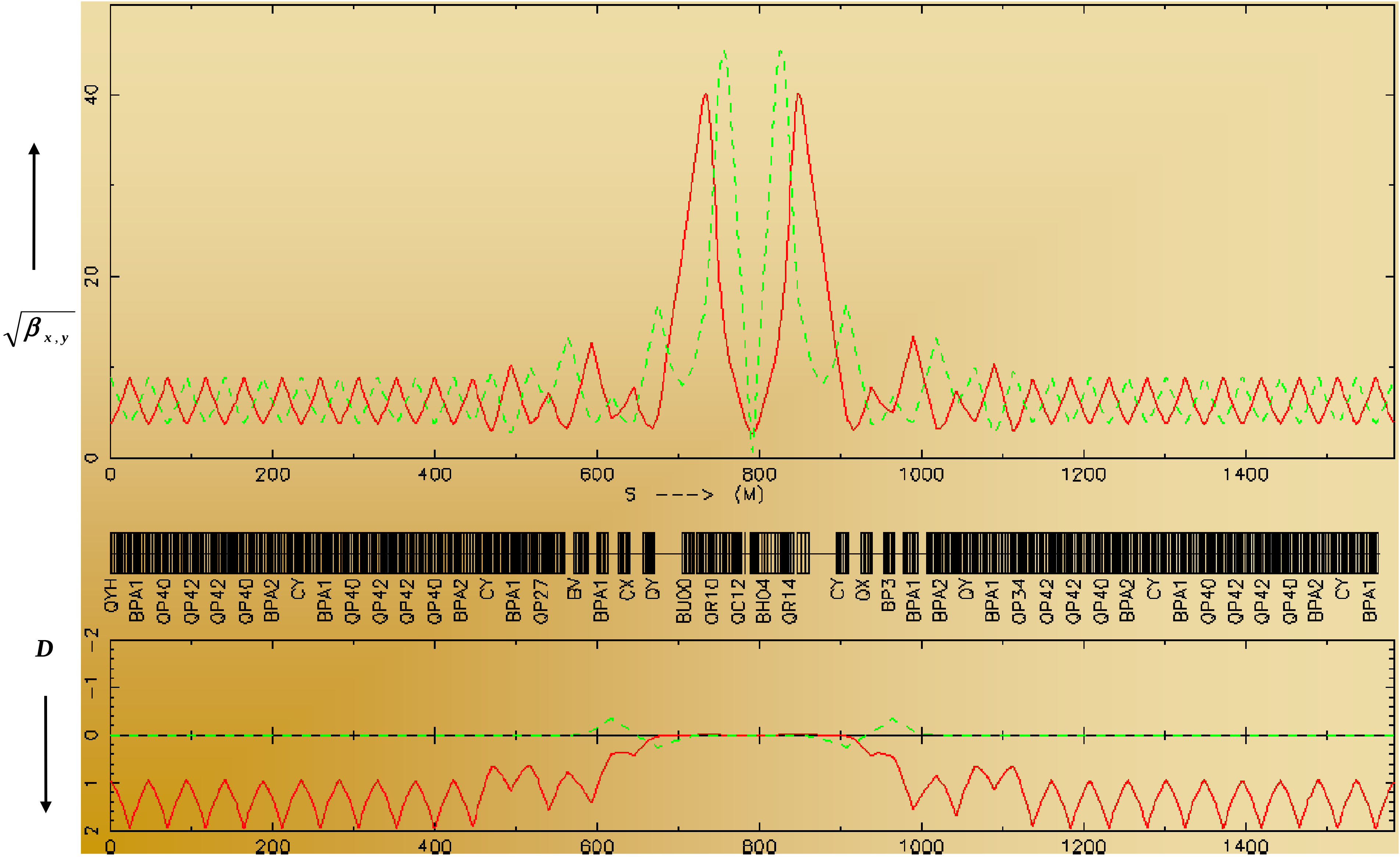
quadrupole magnet



sextupole magnet



20.) Insertions



Insertions

... the most complicated one: the drift space

Question to the audience: what will happen to the beam parameters α, β, γ if we stop focusing for a while ...?

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_S = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

transfer matrix for a drift:

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \quad \longrightarrow$$

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

$$\alpha(s) = \alpha_0 - \gamma_0 s$$

$$\gamma(s) = \gamma_0$$

β-Function in a Drift:

let‘s assume we are at a symmetry point in the center of a drift.

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

as $\alpha_0 = 0, \quad \rightarrow \quad \gamma_0 = \frac{1+\alpha_0^2}{\beta_0} = \frac{1}{\beta_0}$

and we get for the β function in the neighborhood of the symmetry point

$$\boxed{\beta(s) = \beta_0 + \frac{s^2}{\beta_0}}$$

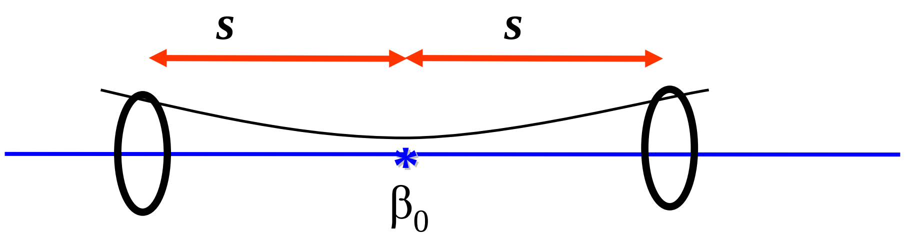
! ! !

Nota bene:

1.) this is very bad !!!

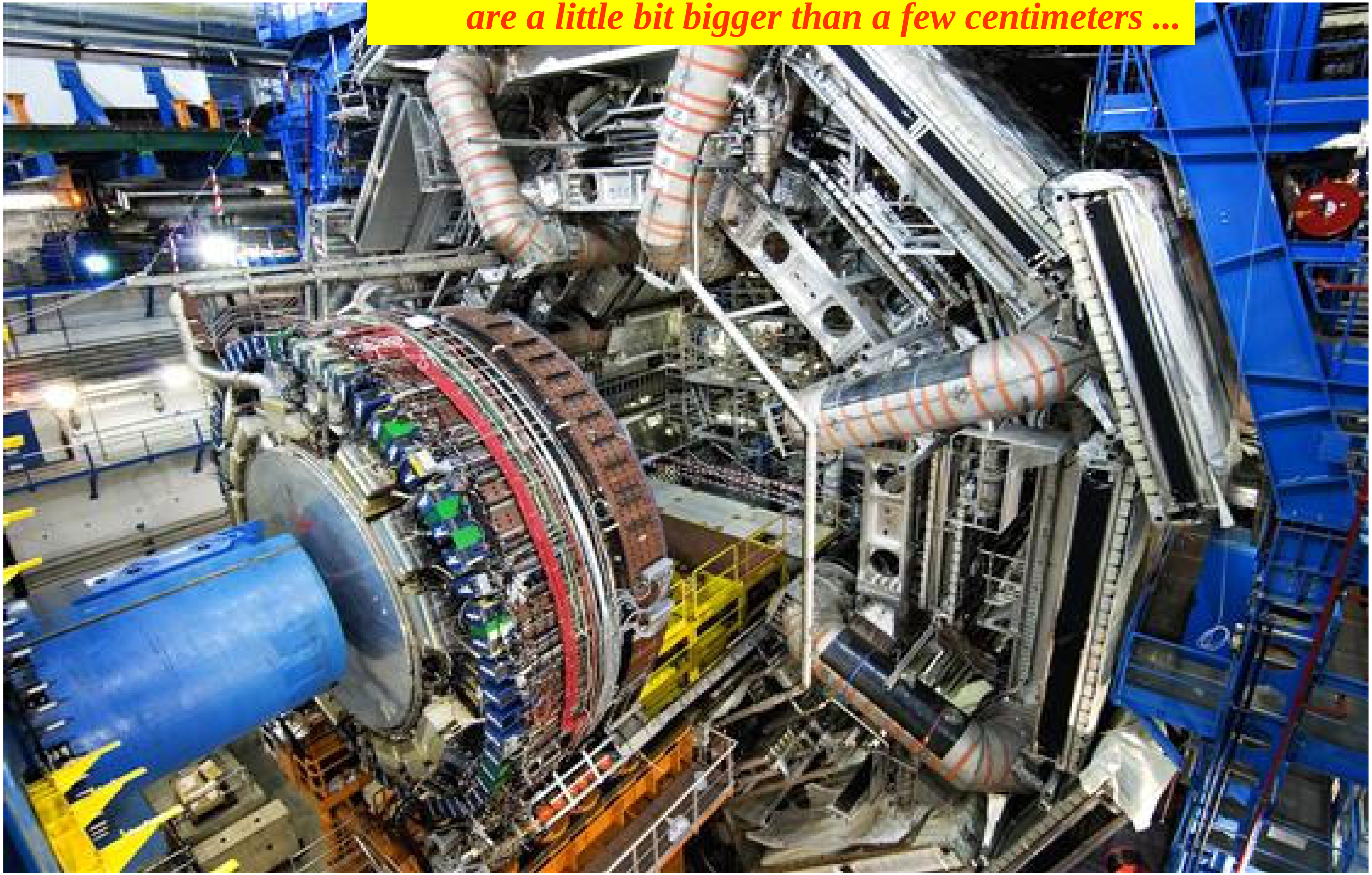
2.) this is a direct consequence of the conservation of phase space density (... in our words: ε = const) … and there is no way out.

3.) Thank you, Mr. Liouville !!!



... clearly there is an

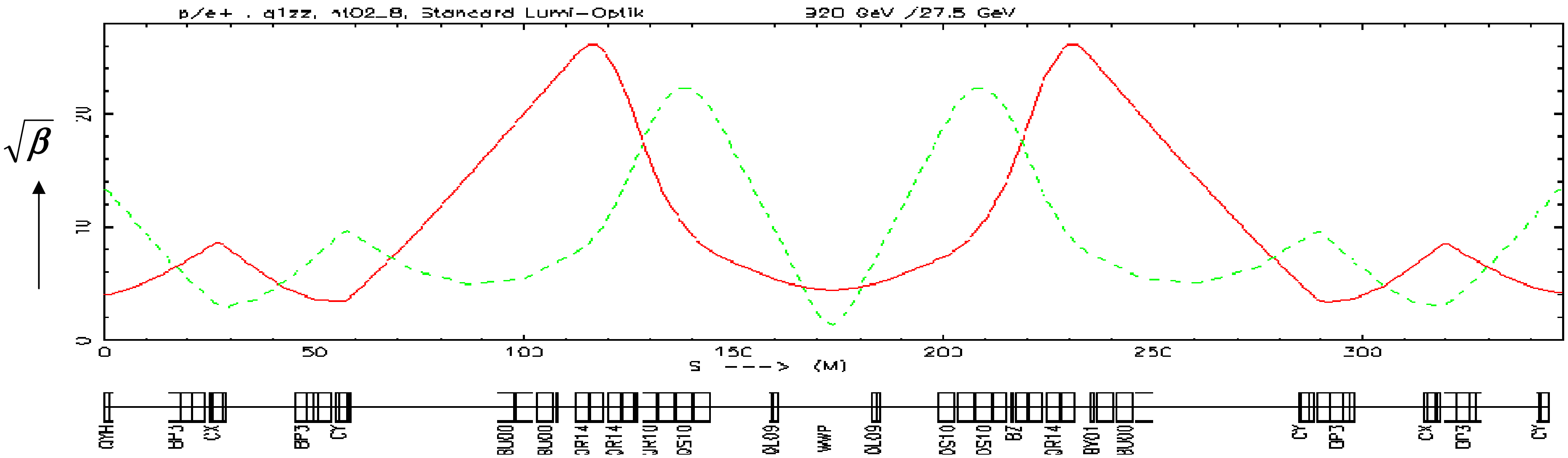
But: ... unfortunately ... in general
high energy detectors that are
installed in that drift spaces
are a little bit bigger than a few centimeters ...



Example of a long Drift: The Mini-β Insertion:

Luminosity: given by the total stored beam currents and the beam size at the collision point (IP)

$$L = \frac{1}{4\pi e^2 f_{rev} n_b} \frac{I_1 I_2}{\sigma_x^* \sigma_y^*}$$



How to create a mini β insertion:

* symmetric drift space (length adequate for the experiment)

* make the beat values as small as possible $\sigma = \sqrt{\varepsilon\beta}$

* ... where is the limit ???

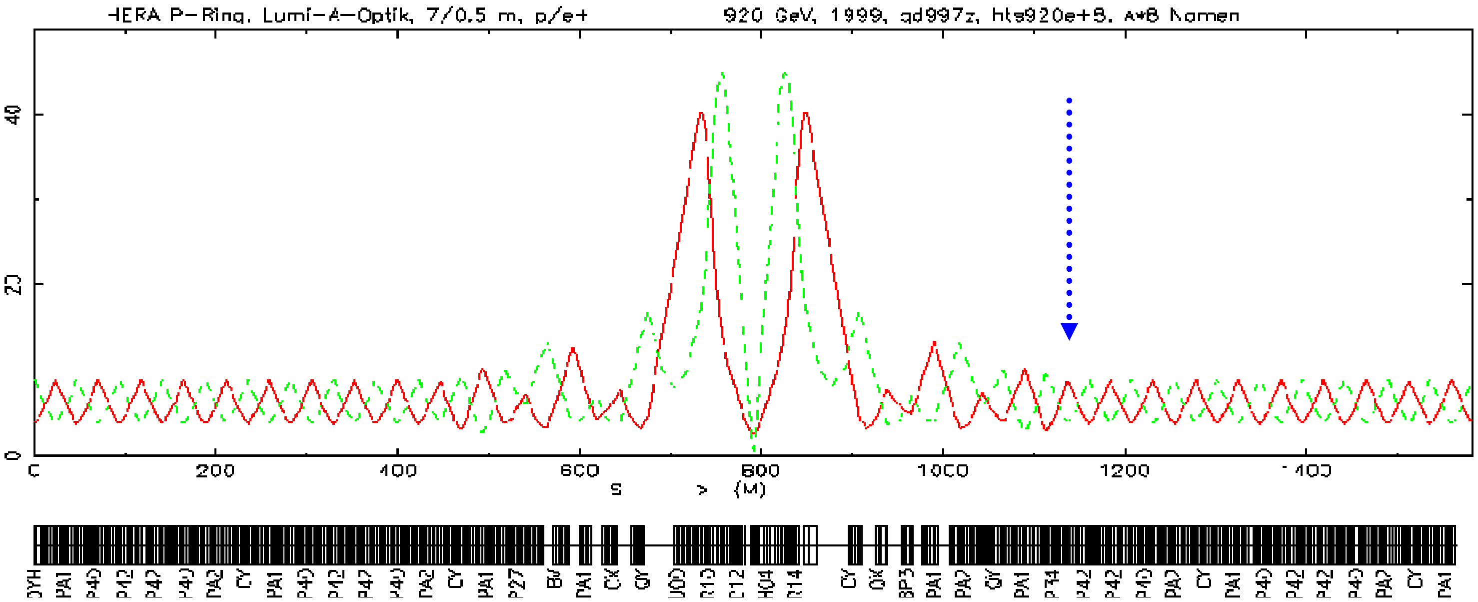
Mini-β Insertions: some guide lines

- *calculate the periodic solution in the arc*
- *introduce the drift space needed for the insertion device (detector ...)*
- *put a quadrupole doublet (triplet ?) as close as possible*
- *introduce additional quadrupole lenses to match the beam parameters to the values at the beginning of the arc structure*

parameters to be optimised & matched to the periodic solution:

$$\alpha_x,\ \beta_x \qquad D_x,\ D_x'$$
$$\alpha_y,\ \beta_y \qquad Q_x,\ Q_y$$

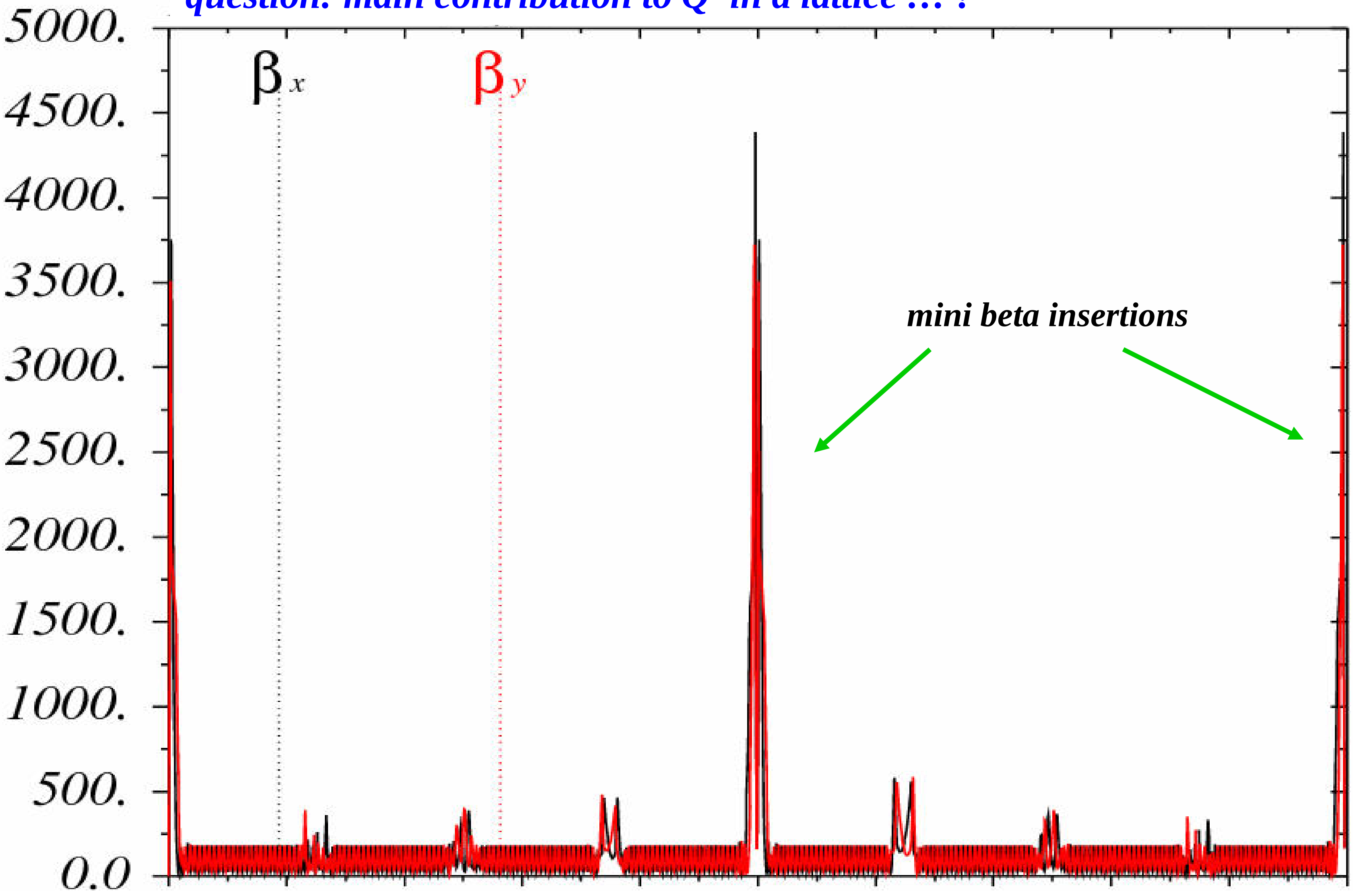
8 individually powered quad magnets are needed to match the insertion
(... at least)



… and now back to the Chromaticity

$$\boxed{Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)\,ds}$$

question: main contribution to Q' in a lattice … ?



Resume´:

quadrupole error: *tune shift*

$$\Delta Q \approx \int_{s0}^{s0+l} \frac{\Delta k(s)\,\beta(s)}{4\pi}\,ds \approx \frac{\Delta k(s)\;\, l_{quad}\;\, \overline{\beta}}{4\pi}$$

beta beat

$$\Delta\beta(s_0) = \frac{\beta_0}{2\sin 2\pi Q}\int_{s1}^{s1+l} \beta(s_1)\Delta k\cos\left(2(\psi_{s1}-\psi_{s0})-2\pi Q\right)ds$$

chromaticity

$$\Delta Q = Q'\,\frac{\Delta p}{p}$$

$$Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds$$

momentum compaction

$$\frac{\delta\, l_{\varepsilon}}{L} = \alpha_p\,\frac{\Delta p}{p}$$

$$\alpha_p \approx \frac{2\pi}{L}\left\langle D\right\rangle \approx \frac{\left\langle D\right\rangle}{R}$$

Appendix I:

Dispersion: Solution of the inhomogenious equation of motion

Ansatz:

$$D(s) = S(s)\int_{s0}^{s1}\frac{1}{\rho}C(\tilde{s})d\tilde{s} - C(s)\int_{s0}^{s1}\frac{1}{\rho}S(\tilde{s})d\tilde{s}$$

$$D'(s) = S'*\int\frac{1}{\rho}C\ dt + \cancel{S\frac{1}{\rho}C} - C'*\int\frac{1}{\rho}S\ dt - \cancel{C\frac{1}{\rho}S}$$

$$D'(s) = S'*\int\frac{C}{\rho}dt\ -\ C'*\int\frac{S}{\rho}dt$$

$$D''(s) = S''*\int\frac{C}{\rho}d\tilde{s} + S'\frac{C}{\rho}\ -\ C''*\int\frac{S}{\rho}d\tilde{s}\ -\ C'\frac{S}{\rho}$$

$$= S''*\int\frac{C}{\rho}d\tilde{s} -\ C''*\int\frac{S}{\rho}d\tilde{s} + \frac{1}{\rho}\underbrace{(CS' - S\,C')}_{=\ \det M\ =\ 1}$$

---

remember: for Cs) and S(s) to be independent solutions the Wronski determinant has to meet the condition

$$W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} \neq 0$$

and as it is independent of the variable „s“

$$\frac{dW}{ds} = \frac{d}{ds}(CS' - SC') = CS'' - SC'' = -K(CS - SC) = 0$$

we get for the initital conditions that we had chosen …

$$\left.\begin{matrix} C_0 = 1, & C_0' = 0 \\ S_0 = 0, & S_0' = 1 \end{matrix}\right\} \quad W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} = 1$$

$$D'' = S'' * \int \frac{C}{\rho} d\tilde{s} - C'' * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

remember: S & C are solutions of the homog. equation of motion:

$$S'' + K * S = 0$$

$$C'' + K * C = 0$$

$$D'' = -K * S * \int \frac{C}{\rho} d\tilde{s} + K * C * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

$$D'' = -K * \underbrace{\left\{ S \int \frac{C}{\rho} d\tilde{s} + C \int \frac{S}{\rho} d\tilde{s} \right\}}_{=D(s)} + \frac{1}{\rho}$$

$$D'' = -K * D + \frac{1}{\rho}$$

… or

$$\underline{\underline{D'' + K * D = \frac{1}{\rho}}}$$

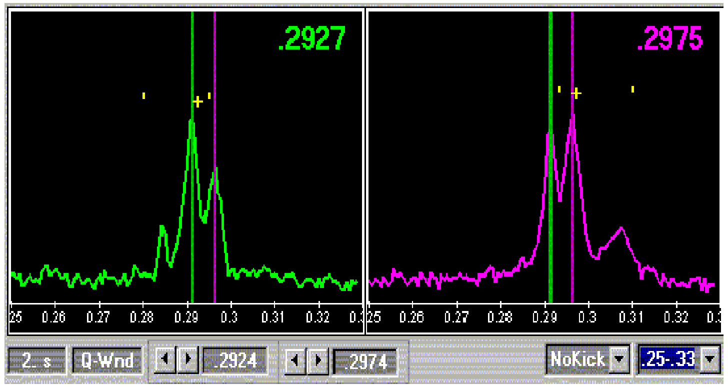
qed

Appendix II:

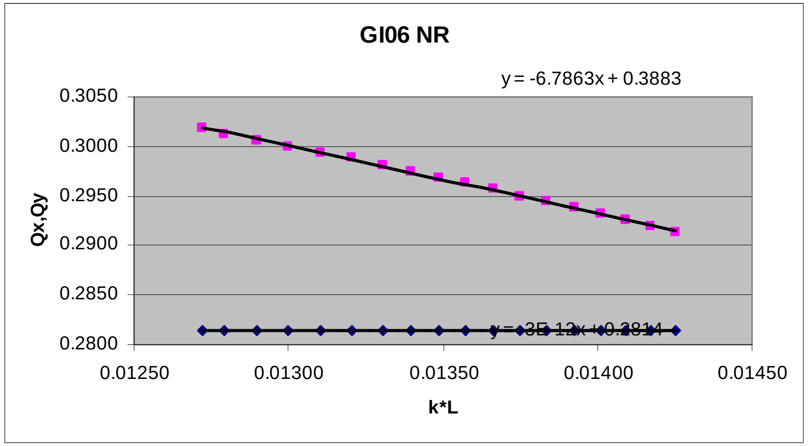
Quadrupole Error and Beta Function

a change of quadrupole strength in a synchrotron leads to tune sift:

$$\Delta Q \approx \int_{s0}^{s0+l} \frac{\Delta k(s)\,\beta(s)}{4\pi}\,ds \approx \frac{\Delta k(s) * l_{quad} * \overline{\beta}}{4\pi}$$



tune spectrum ...



tune shift as a function of a gradient change

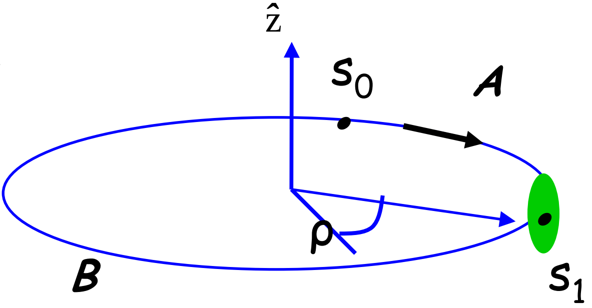
But we should expect an error in the β-function as well …

… shouldn´t we ???

Quadrupole Errors and Beta Function

a quadrupole error will not only influence the oscillation frequency … „tune"
… but also the amplitude … „beta function"

split the ring into 2 parts, described by two matrices A and B



$$M_{turn} = B * A$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

matrix of a quad error between A and B

$$M_{dist} = \begin{pmatrix} m^*_{11} & m^*_{12} \\ m^*_{21} & m^*_{22} \end{pmatrix} = B \begin{pmatrix} 1 & 0 \\ -\Delta kds & 1 \end{pmatrix} A$$

$$M_{dist} = B \begin{pmatrix} a_{11} & a_{12} \\ -\Delta kdsa_{11} + a_{12} & -\Delta kdsa_{12} + a_{22} \end{pmatrix}$$

$$M_{dist} = \begin{pmatrix} \sim & b_{11}a_{12} + b_{12}(-\Delta kdsa_{12} + a_{22}) \\ \sim & \sim \end{pmatrix}$$

the beta function is usually obtained via the matrix element „m12", which is in Twiss form for the undistorted case

$$m_{12} = \beta_0 \sin 2\pi Q$$

and including the error:

$$m^*_{12} = \underbrace{b_{11}a_{12} + b_{12}a_{22}}_{m_{12} = \beta_0 \sin 2\pi Q} - b_{12}a_{12}\Delta k ds$$

$$(1) \quad m^*_{12} = \beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds$$

As M is still a matrix for one complete turn we still can express the element m_{12} in twiss form:*

$$(2) \quad m^*_{12} = (\beta_0 + d\beta) * \sin 2\pi (Q + dQ)$$

Equalising (1) and (2) and assuming a small error

$$\beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds = (\beta_0 + d\beta) * \sin 2\pi (Q + dQ)$$

$$\beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds = (\beta_0 + d\beta) * \sin 2\pi Q \underbrace{\cos 2\pi dQ}_{\approx 1} + \cos 2\pi Q \underbrace{\sin 2\pi dQ}_{\approx 2\pi dQ}$$

$$\cancel{\beta_0 \sin 2\pi Q} - a_{12} b_{12} \Delta k ds = \cancel{\beta_0 \sin 2\pi Q} + \beta_0 2\pi dQ \cos 2\pi Q + d\beta_0 \sin 2\pi Q + \cancel{d\beta_0 2\pi dQ} \cos 2\pi Q$$

ignoring second order terms

$$-a_{12} b_{12} \Delta k ds = \beta_0 2\pi dQ \cos 2\pi Q + d\beta_0 \sin 2\pi Q$$

remember: tune shift dQ due to quadrupole error: $dQ = \dfrac{\Delta k \beta_1 ds}{4\pi}$

(index „1" refers to location of the error)

$$-a_{12} b_{12} \Delta k ds = \frac{\beta_0 \Delta k \beta_1 ds}{2} \cos 2\pi Q + d\beta_0 \sin 2\pi Q$$

solve for dβ

$$d\beta_0 = \frac{-1}{2 \sin 2\pi Q} \left\{ 2 a_{12} b_{12} + \beta_0 \beta_1 \cos 2\pi Q \right\} \Delta k ds$$

express the matrix elements a_{12}, b_{12} in Twiss form

$$M = \begin{pmatrix} \sqrt{\dfrac{\beta_s}{\beta_0}} \left(\cos \psi_s + \alpha_0 \sin \psi_s \right) & \sqrt{\beta_s \beta_0} \sin \psi_s \\ \dfrac{(\alpha_0 - \alpha_s) \cos \psi_s - (1 + \alpha_0 \alpha_s) \sin \psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\dfrac{\beta_0}{\beta s}} \left(\cos \psi_s - \alpha_s \sin \psi_s \right) \end{pmatrix}$$

$$d\beta_0 = \frac{-1}{2\sin 2\pi Q}\left\{2a_{12}b_{12} + \beta_0\beta_1 \cos 2\pi Q\right\}\Delta k ds$$

$$\boldsymbol{a}_{12} = \sqrt{\boldsymbol{\beta}_0\boldsymbol{\beta}_1}\,\sin\Delta\boldsymbol{\psi}_{0\to 1}$$

$$\boldsymbol{b}_{12} = \sqrt{\boldsymbol{\beta}_1\boldsymbol{\beta}_0}\,\sin(2\pi\boldsymbol{Q} - \Delta\boldsymbol{\psi}_{0\to 1})$$

$$d\beta_0 = \frac{-\beta_0\beta_1}{2\sin 2\pi Q}\underbrace{\left\{2\sin\Delta\psi_{12}\sin(2\pi Q - \Delta\psi_{12}) + \cos 2\pi Q\right\}}\Delta k ds$$

… after some TLC transformations … $= \cos(2\Delta\psi_{01} - 2\pi Q)$

$$\Delta\beta(s_0) = \frac{-\beta_0}{\boxed{2\sin 2\pi Q}}\int_{s1}^{s1+l}\beta(s_1)\Delta k\cos\big(2(\psi_{s1} - \psi_{s0}) - 2\pi Q\big)ds$$

Nota bene: ! the beta beat is proportional to the strength of the error Δk

!! and to the β function at the place of the error ,

!!! and to the β function at the observation point,
(… remember orbit distortion !!!)

!!!! there is a resonance denominator