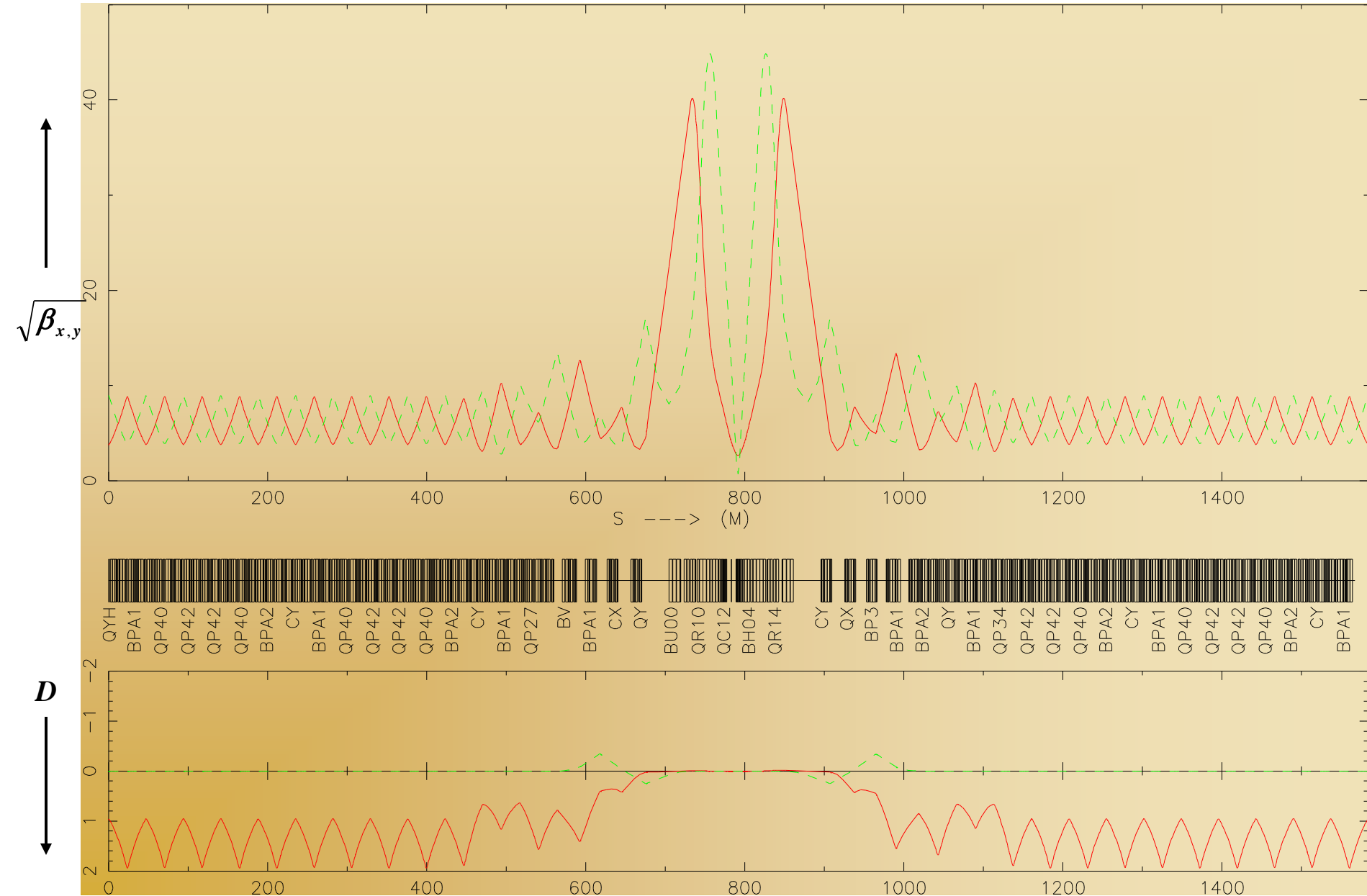


Lattice Design II: Insertions

Bernhard Holzer, CERN



I.) Reminder:

equation of motion

$$x'' + K(s) * x = 0$$

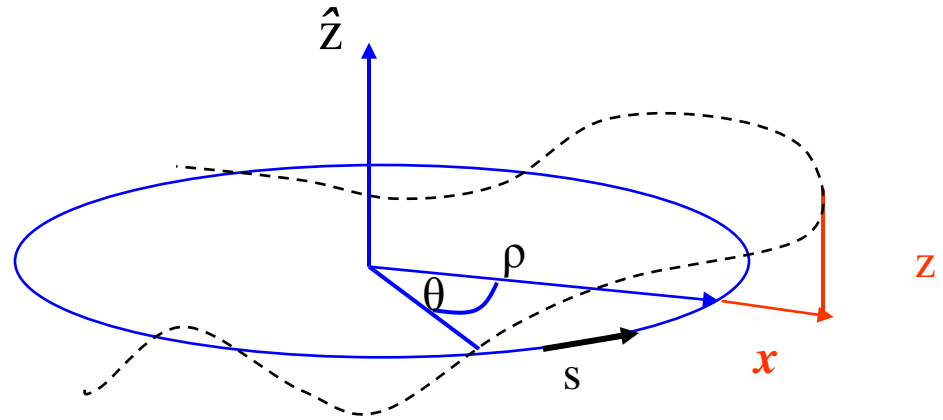
$$K = -k + \frac{1}{\rho^2}$$

single particle trajectory

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M * \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

e.g. matrix for a quadrupole lens:

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s) \\ -\sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$



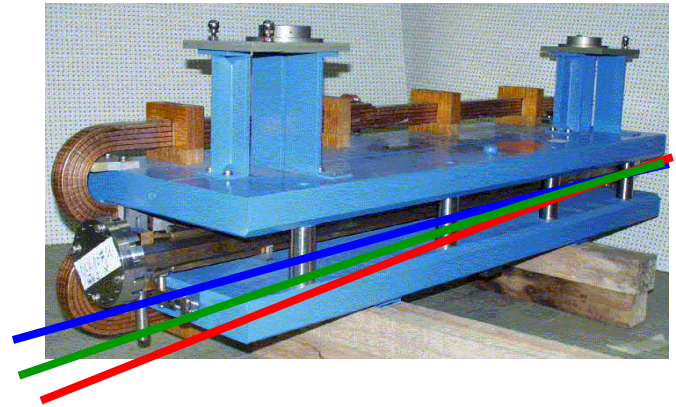
APS Light Source

II.) Dispersion

momentum error:

$$\frac{\Delta p}{p} \neq 0$$

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$



general solution:

$$\left. \begin{aligned} x(s) &= x_h(s) + x_i(s) \\ D(s) &= \frac{x_i(s)}{\frac{\Delta p}{p}} \end{aligned} \right\}$$

$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p} \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p} \end{pmatrix}_0$$

Dispersion:

the dispersion function $D(s)$ is (...obviously) defined by the focusing properties of the lattice and is given by:

$$D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

! weak dipoles \rightarrow large bending radius \rightarrow small dispersion

Example: Drift

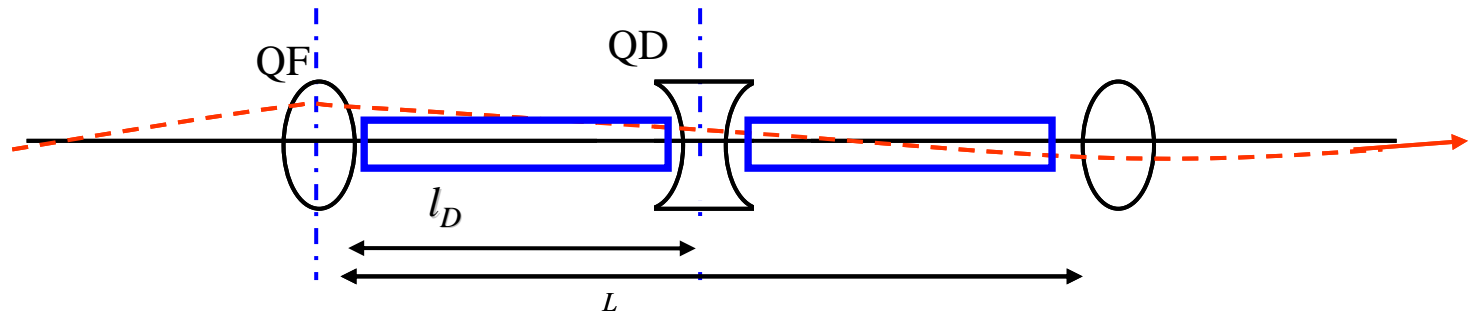
$$M_D = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

$$D(s) = S(s) * \underbrace{\int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s}}_{=0} - C(s) * \underbrace{\int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}}_{=0}$$

$$\rightarrow M_D = \begin{pmatrix} 1 & \ell & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

*...in similar way for quadrupole matrices,
!!! in a quite different way for dipole matrix (see appendix)*

Dispersion in a FoDo Cell:



!! we have now introduced dipole magnets in the FoDo:

- we **still neglect** the **weak focusing** contribution $1/\rho^2$
- but **take into account** $1/\rho$ for the **dispersion effect**
- assume: length of the dipole = l_D

Calculate the matrix of the FoDo half cell in thin lens approximation:

in analogy to the derivations of $\hat{\beta}$, $\check{\beta}$

* *thin lens approximation:*

$$f = \frac{1}{k\ell_Q} \gg \ell_Q$$

* *length of quad negligible*

$$\ell_Q \approx 0, \rightarrow \ell_D = \frac{1}{2} L$$

* *start at half quadrupole*

$$\frac{1}{\tilde{f}} = \frac{1}{2f}$$

Matrix of the half cell

$$M_{HalfCell} = M_{\frac{QD}{2}} * M_B * M_{\frac{QF}{2}}$$

$$M_{Half\ Cell} = \begin{pmatrix} 1 & 0 \\ \frac{1}{\tilde{f}} & 1 \end{pmatrix} * \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ -\frac{1}{\tilde{f}} & 1 \end{pmatrix}$$

$$M_{Half\ Cell} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 - \frac{\ell}{\tilde{f}} & \ell \\ -\frac{\ell}{\tilde{f}^2} & 1 + \frac{\ell}{\tilde{f}} \end{pmatrix}$$

calculate the dispersion terms D, D' from the matrix elements

$$D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

$$D(\ell) = \underbrace{\ell}_{\mathbf{S(s)}} * \underbrace{\frac{1}{\rho}}_{\mathbf{C(s)}} * \underbrace{\int_0^\ell \left(1 - \frac{s}{\tilde{f}}\right) ds}_{\mathbf{C(s)}} - \underbrace{\left(1 - \frac{\ell}{\tilde{f}}\right)}_{\mathbf{C(s)}} * \underbrace{\frac{1}{\rho}}_{\mathbf{C(s)}} * \underbrace{\int_0^\ell s ds}_{\mathbf{S(s)}}$$

$$D(\ell) = \frac{\ell}{\rho} \left(\ell - \frac{\ell^2}{2\tilde{f}} \right) - \left(1 - \frac{\ell}{\tilde{f}} \right) * \frac{1}{\rho} * \frac{\ell^2}{2} = \frac{\ell^2}{\rho} - \frac{\ell^3}{2\tilde{f}\rho} - \frac{\ell^2}{2\rho} + \frac{\ell^3}{2\tilde{f}\rho}$$

$$D(\ell) = \frac{\ell^2}{2\rho}$$

in full analogy on derives for D' :

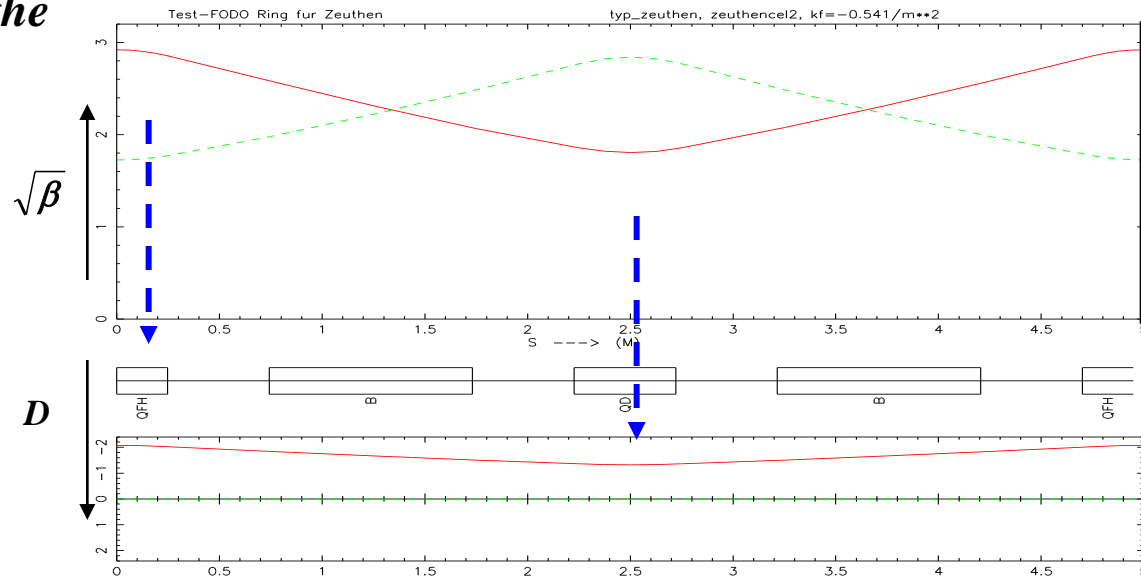
$$D'(s) = \frac{\ell}{\rho} \left(1 + \frac{\ell}{2\tilde{f}} \right)$$

and we get the complete matrix including the dispersion terms D, D'

$$M_{halfCell} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{\ell}{\tilde{f}} & \ell & \frac{\ell^2}{2\rho} \\ -\frac{\ell}{\tilde{f}^2} & 1 + \frac{\ell}{\tilde{f}} & \frac{\ell}{\rho} \left(1 + \frac{\ell}{2\tilde{f}}\right) \\ 0 & 0 & 1 \end{pmatrix}$$

boundary conditions for the transfer from the center of the foc. to the center of the defoc. quadrupole

$$\begin{pmatrix} \check{D} \\ 0 \\ 1 \end{pmatrix} = M_{1/2} * \begin{pmatrix} \hat{D} \\ 0 \\ 1 \end{pmatrix}$$



Dispersion in a FoDo Cell

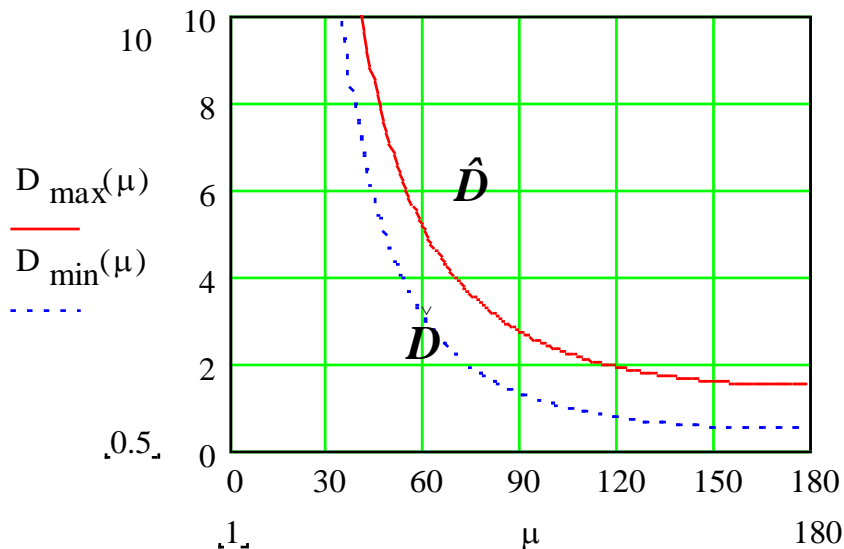
$$\hat{D} = \frac{\ell^2}{\rho} * \frac{(1 + \frac{1}{2} \sin \frac{\mu}{2})}{\sin^2 \frac{\mu}{2}}$$

$$\check{D} = \frac{\ell^2}{\rho} * \frac{(1 - \frac{1}{2} \sin \frac{\mu}{2})}{\sin^2 \frac{\mu}{2}}$$

$$\rightarrow \check{D} = \hat{D}(1 - \frac{\ell}{\tilde{f}}) + \frac{\ell^2}{2\rho}$$

$$\rightarrow 0 = -\frac{\ell}{\tilde{f}^2} * \hat{D} + \frac{\ell}{\rho}(1 + \frac{\ell}{2\tilde{f}})$$

where μ denotes the phase advance of the full cell
and $\ell/\tilde{f} = \sin(\mu/2)$



Nota bene:

! small dispersion needs strong focusing
 \rightarrow large phase advance

!! \leftrightarrow there is an optimum phase for small β

!!! ...do you remember the stability criterion?
 $\frac{1}{2} \text{ trace} = \cos \mu \leftrightarrow \mu < 180$

!!!! ... life is not easy

III.) Lattice Design: Insertions

*1.) ... the most complicated one: **the drift space***

Question to the auditorium: what will happen to the beam parameters α , β , γ if we
stop focusing for a while ...?

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

transfer matrix for a drift:

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

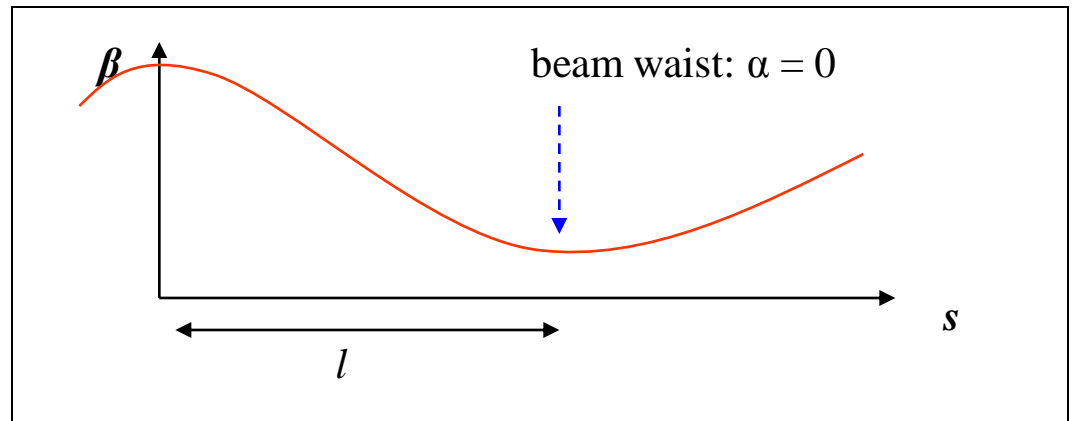
$$\alpha(s) = \alpha_0 - \gamma_0 s$$

$$\gamma(s) = \gamma_0$$

*„0“ refers to the position of the last
lattice element*

„s“ refers to the position in the drift

location of the waist:



given the initial conditions $\alpha_0, \beta_0, \gamma_0$: where is the point of smallest beam dimension in the drift ... or at which location occurs the beam waist ?

beam waist:

$$\alpha(s) = 0 \quad \rightarrow \quad \alpha_0 = \gamma_0 * s$$

$$\ell = \frac{\alpha_0}{\gamma_0}$$

beam size at that position:

$$\left. \begin{array}{l} \gamma(\ell) = \gamma_0 \\ \alpha(\ell) = 0 \end{array} \right\} \rightarrow \gamma(\ell) = \frac{1 + \alpha^2(\ell)}{\beta(\ell)} = \frac{1}{\beta(\ell)}$$

$$\beta(\ell) = \frac{1}{\gamma_0}$$

β -Function in a Drift:

let's assume we are at a **symmetry point** in the center **of a drift**.

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

$$\text{as } \alpha_0 = 0, \rightarrow \gamma_0 = \frac{1 + \alpha_0^2}{\beta_0} = \frac{1}{\beta_0}$$

and we get for the β function in the neighborhood of the symmetry point

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0} \quad !!!$$

Nota bene:

- 1.) this is very bad !!!*
- 2.) this is a direct consequence of the conservation of phase space density (... in our words: $\varepsilon = \text{const}$) ... and there is no way out.*
- 3.) Thank you, Mr. Liouville !!!*



***Joseph Liouville,
1809-1882***

β -Function in a Drift:

If we cannot fight against Liouville theorem ... at least we can optimise

Optimisation of the beam dimension:

$$\beta(\ell) = \beta_0 + \frac{\ell^2}{\beta_0}$$

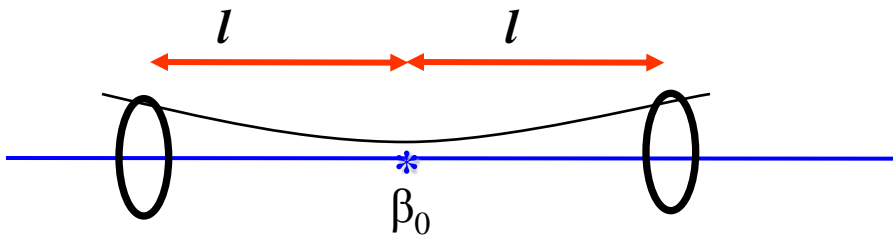
Find the β at the center of the drift that leads to the lowest maximum β at the end:

!

$$\frac{d\hat{\beta}}{d\beta_0} = 1 - \frac{\ell^2}{\beta_0^2} = 0$$

$$\rightarrow \beta_0 = \ell$$

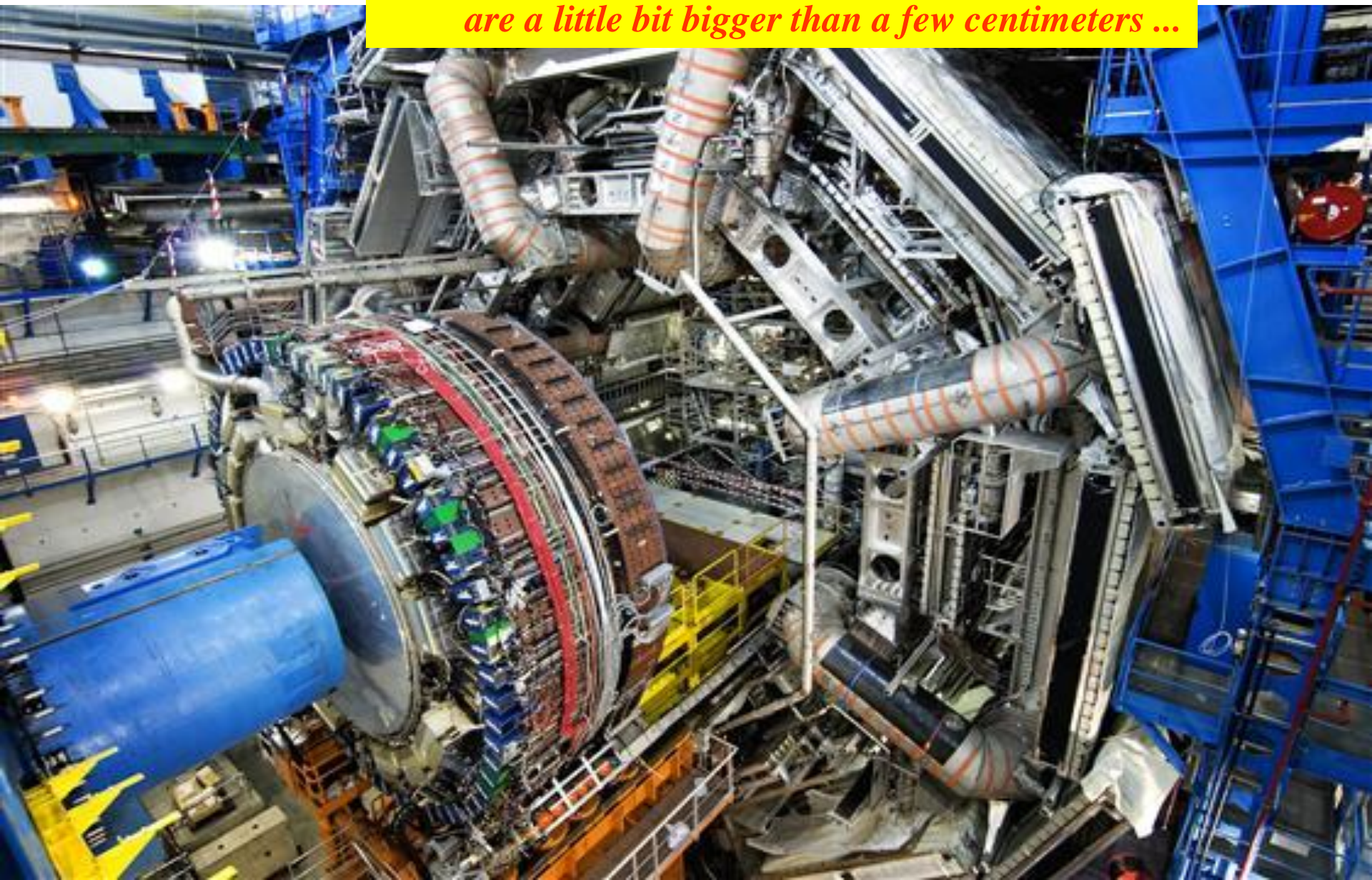
$$\rightarrow \hat{\beta} = 2\beta_0$$



*If we **choose** $\beta_0 = \ell$ we get the smallest β at the end of the drift and the maximum β is just twice the distance ℓ*

... clearly there is an

*But: ... unfortunately ... in general
high energy detectors that are
installed in that drift spaces
are a little bit bigger than a few centimeters ...*

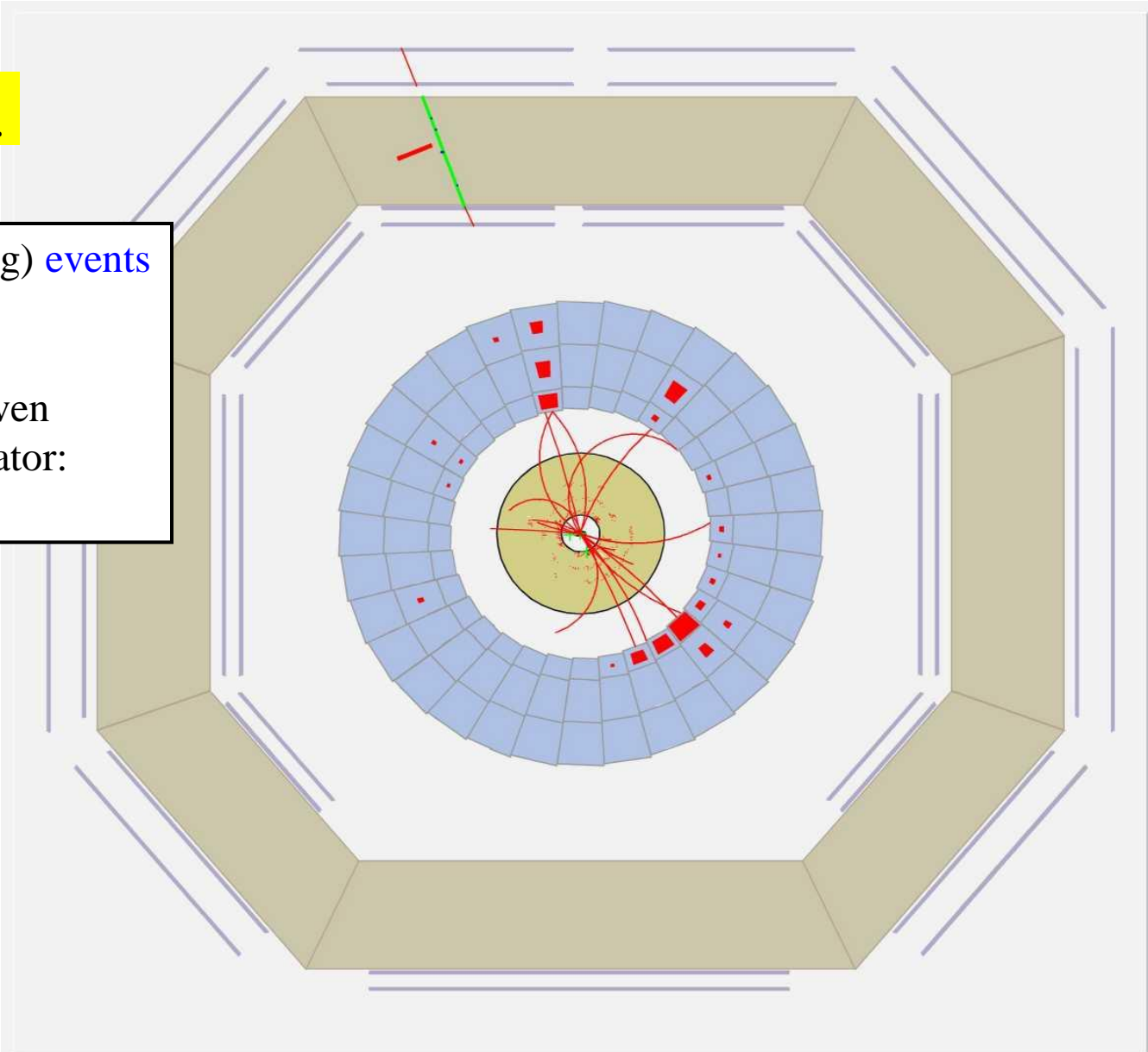


2.) The Mini- β Insertion:

$$R = L * S_{react.}$$

production rate of (scattering) events
is determined by the
cross section Σ_{react}
and a parameter L that is given
by the design of the accelerator:
... the luminosity

*ZEUS detector: inelastic
scattering event of e^+/p*

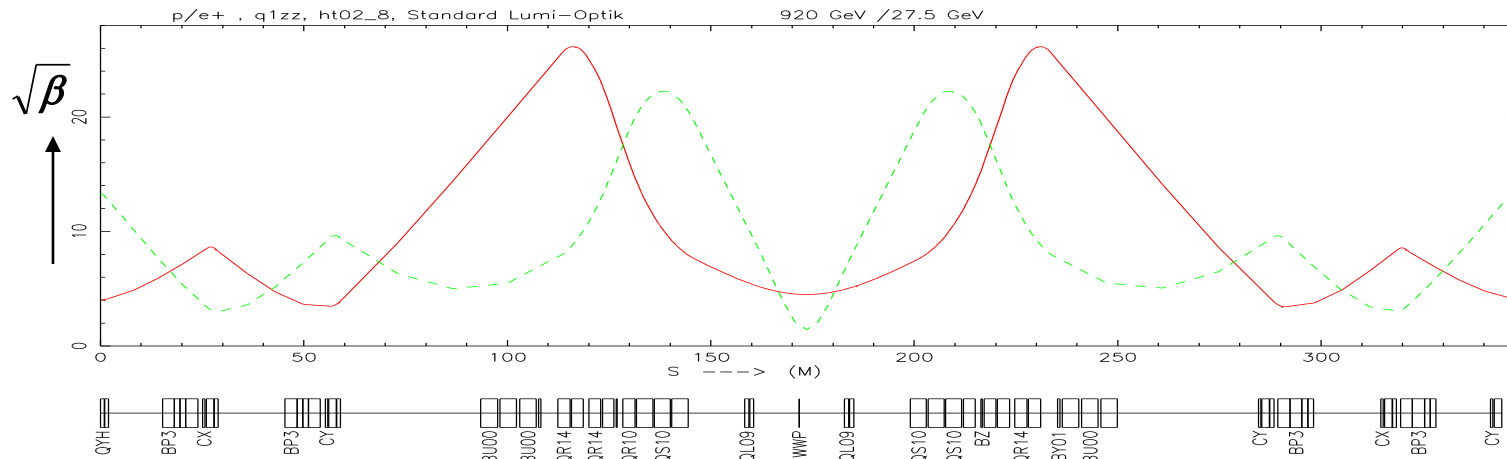


The Mini- β Insertion:

Event rate of a collider ring: $R = \sigma_R * L$

Luminosity: given by the total stored **beam currents** and the **beam size** at the collision point (IP)

$$L = \frac{1}{4\pi e^2 f_0 b} * \frac{I_1 * I_2}{\sigma_x^* * \sigma_y^*}$$



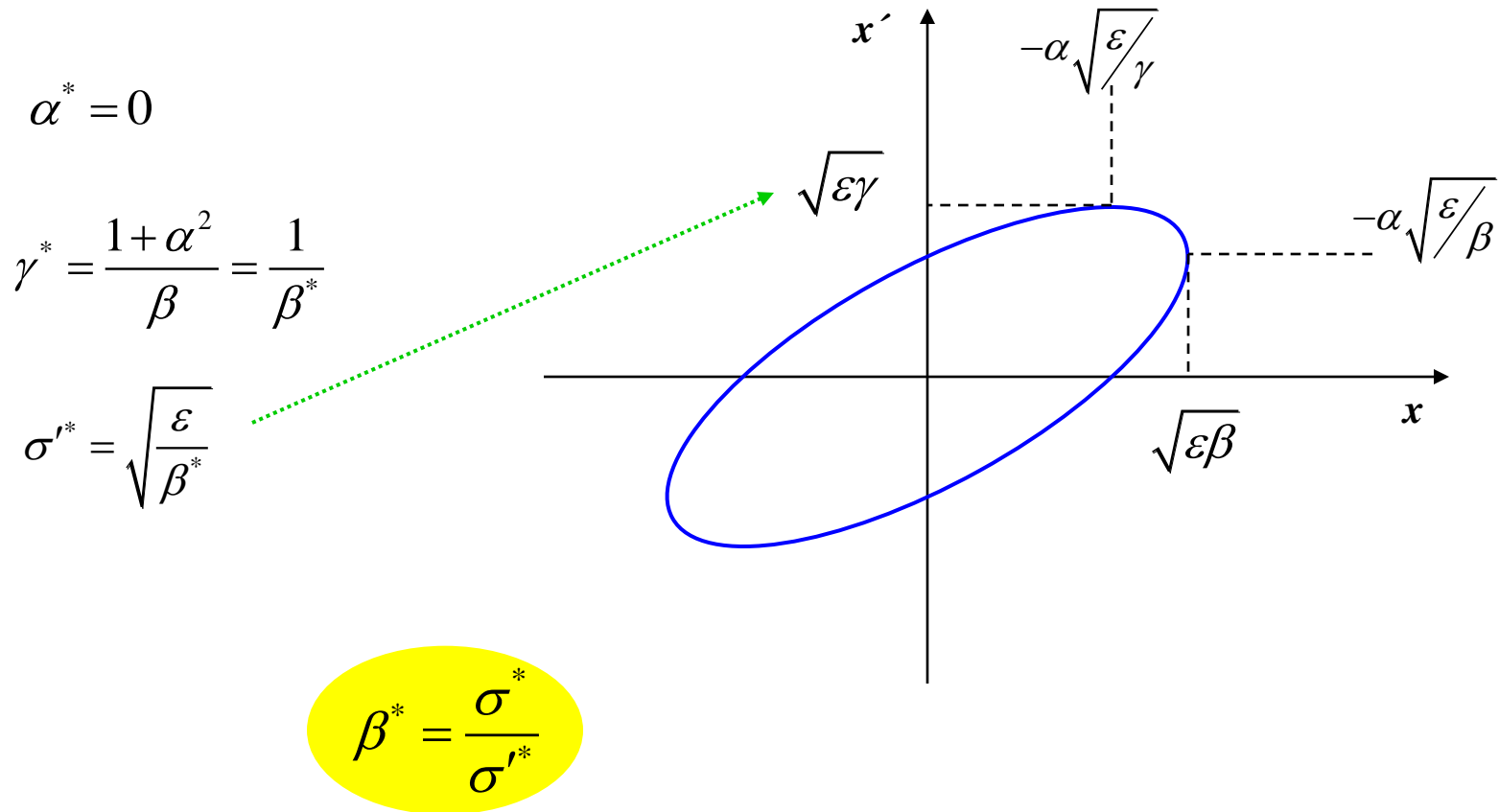
How to create a mini β insertion:

- * **symmetric drift space** (length adequate for the experiment)
- * **quadrupole doublet** on each side (as close as possible)
- * **additional quadrupole lenses to match twiss parameters** to the periodic cell in the arc

Mini- β Insertions: Betafunctions

A mini- β insertion is always a kind of **special symmetric drift space**.

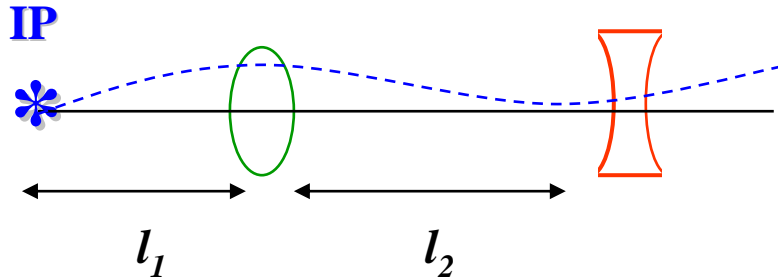
→ greetings from Liouville



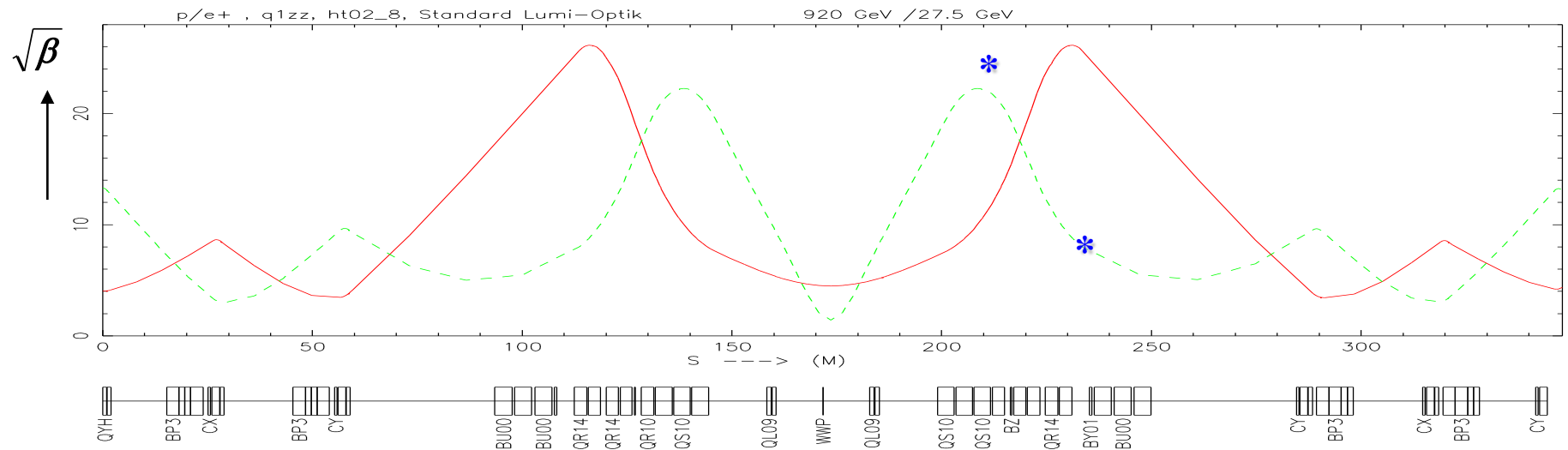
at a symmetry point β is just the ratio of beam dimension and beam divergence.

size of β at the second quadrupole lens (in thin lens approx):

... after some transformations and a couple of beer ...



$$\beta(s) = \left(1 + \frac{l_2}{f_1}\right)^2 * \beta^* + \frac{1}{\beta^*} \left(l_1 + l_2 + \frac{l_1 l_2}{f_1}\right)^2$$



Mini- β Insertions: Phase advance

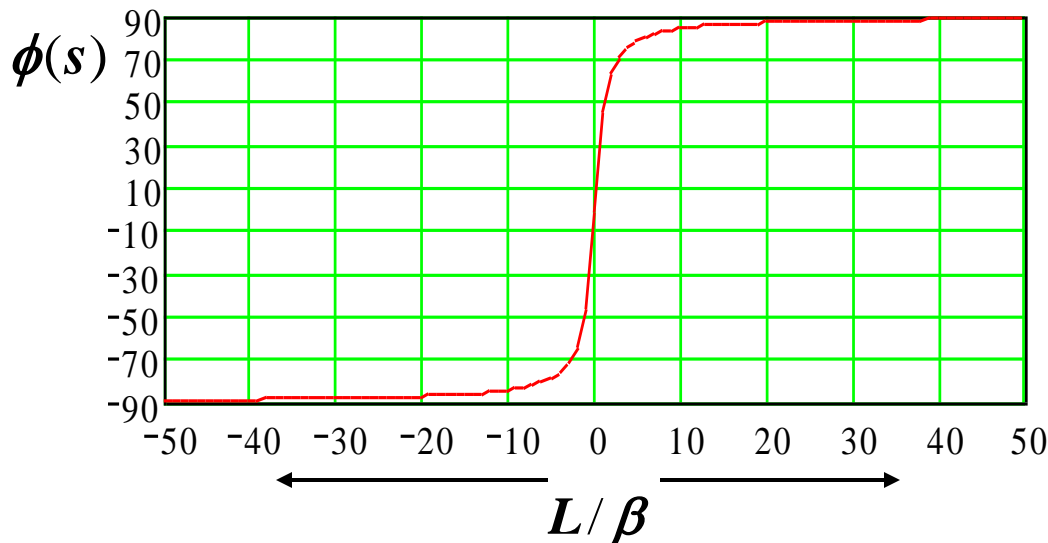
By definition the phase advance is given by:

$$\Phi(s) = \int \frac{1}{\beta(s)} ds$$

Now in a mini β insertion:

$$\beta(s) = \beta_0 \left(1 + \frac{s^2}{\beta_0^2}\right)$$

$$\rightarrow \Phi(s) = \frac{1}{\beta_0} \int_0^L \frac{1}{1 + s^2 / \beta_0^2} ds = \arctan \frac{L}{\beta_0}$$



*Consider the drift spaces on both sides of the IP: the **phase advance** of a mini β insertion is approximately π , in other words: the **tune will increase by half an integer.***

Are there any problems ??

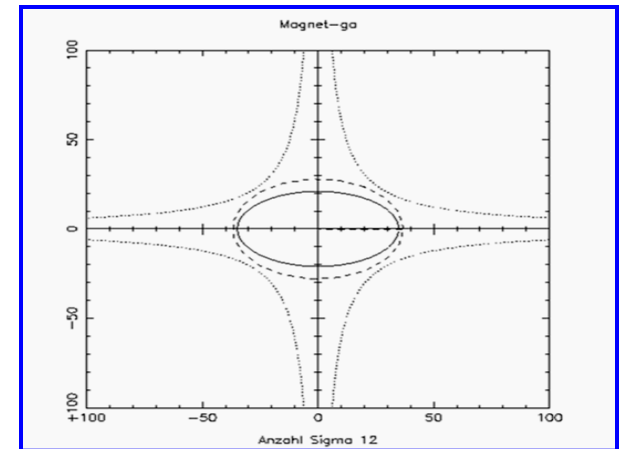
sure there are...

- * *large β values at the doublet quadrupoles \rightarrow large contribution to chromaticity ξ ... and no local correction*

$$\xi = \frac{1}{4\pi} \oint \{ K(s) \beta(s) \} ds$$

- * *aperture of mini β quadrupoles limit the luminosity*

beam envelope at the first
mini β quadrupole lens in
the HERA proton storage ring



- * *field quality and magnet stability most critical at the high β sections*
effect of a quad error:

$$\Delta Q = \int_{s_0}^{s_0+l} \frac{\Delta K(s) \beta(s) ds}{4\pi}$$

\rightarrow keep distance „s“ to the first mini β quadrupole as small as possible

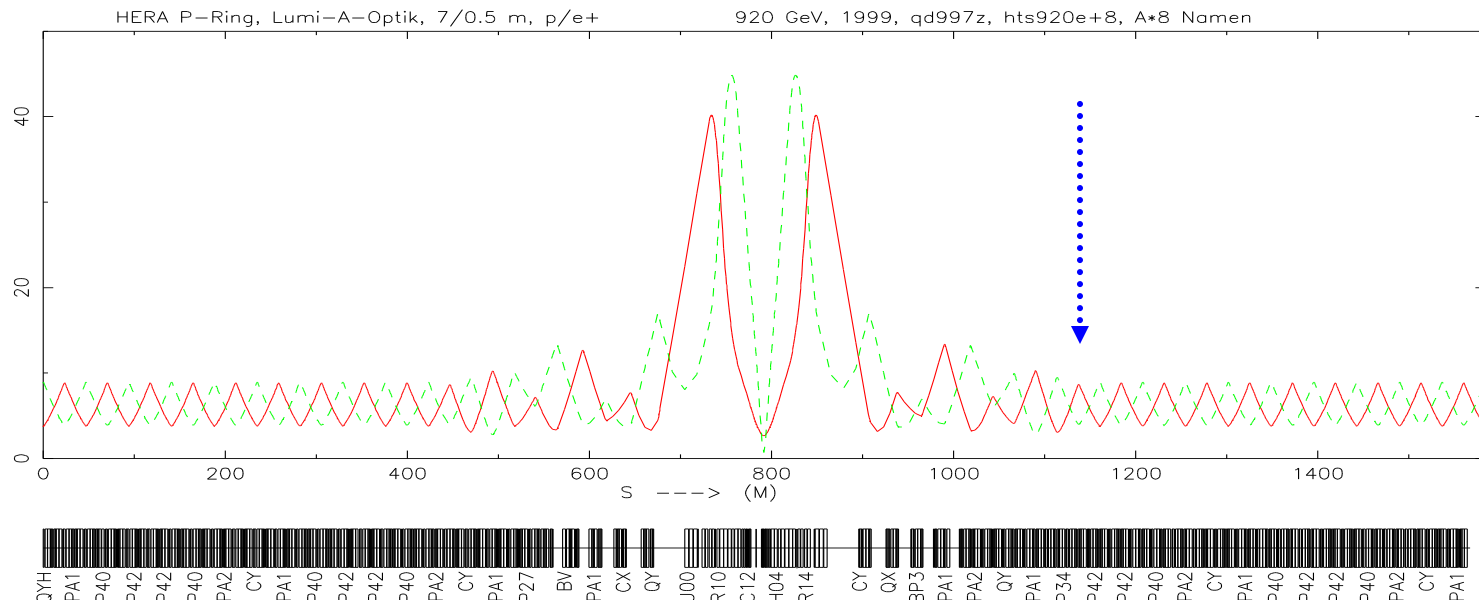
Mini- β Insertions: some guide lines

- * calculate the **periodic solution in the arc**
- * **introduce the drift space** needed for the insertion device (detector ...)
- * put a **quadrupole doublet** (triplet ?) **as close as possible**
- * introduce **additional quadrupole lenses** to match the beam parameters to the values at the beginning of the arc structure

parameters to be optimised & matched to the periodic solution:

$$\begin{array}{ll} \alpha_x, \beta_x & D_x, D_x' \\ \alpha_y, \beta_y & Q_x, Q_y \end{array}$$

**8 individually
powered quad
magnets are
needed to match
the insertion
(... at least)**



3.) Dispersion Suppressors

There are *two rules* of paramount importance *about dispersion*:

! it is nasty

!! it is not easy to get rid of it.

remember: oscillation amplitude for a particle
with momentum deviation

$$x(s) = x_{\beta}(s) + D(s) * \frac{\Delta p}{p}$$

beam size at the IP

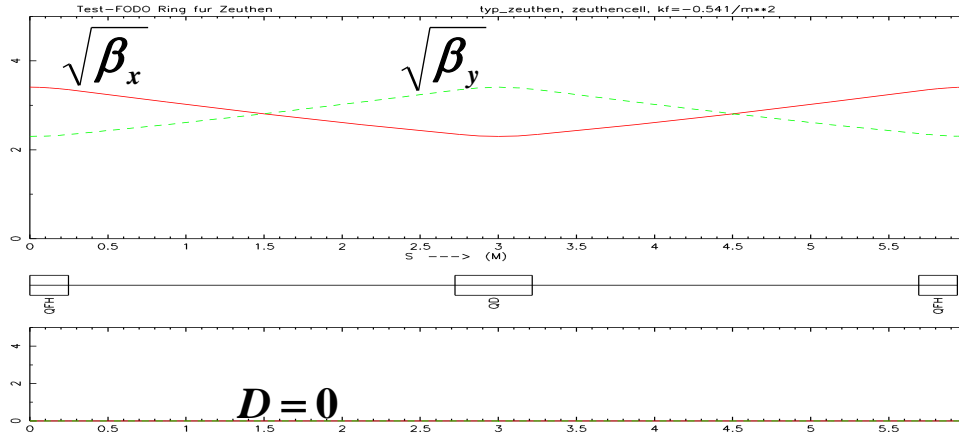
$$\sigma_x^* = 118 \text{ } \mu m, \sigma_y^* = 32 \text{ } \mu m$$

dispersion trajectory

$$\left. \begin{array}{l} \bar{D}(s) = 1.5 \text{ } m \\ \frac{\Delta p}{p} \approx 5 * 10^{-4} \end{array} \right\} \rightarrow x_D \approx 0.75 \text{ } mm$$

Dispersion Suppressors

$$D(s) = S(s) \int_{s_0}^{s_1} \frac{1}{\rho} C(\tilde{s}) d\tilde{s} - C(s) \int_{s_0}^{s_1} \frac{1}{\rho} S(\tilde{s}) d\tilde{s}$$



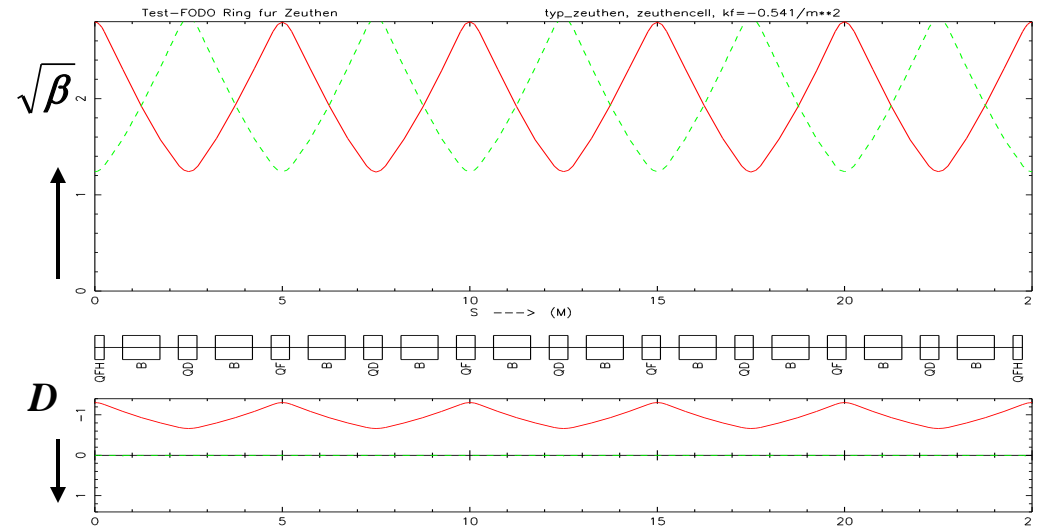
optical functions of a FoDo cell **without** dipoles: **D=0**

Remember: Dispersion in a FoDo cell **including** dipoles

$$\hat{D} = \frac{\ell^2}{\rho} * \frac{(1 + \frac{1}{2} \sin \frac{\mu}{2})}{\sin^2 \frac{\mu}{2}}$$

$$\check{D} = \frac{\ell^2}{\rho} * \frac{(1 - \frac{1}{2} \sin \frac{\mu}{2})}{\sin^2 \frac{\mu}{2}}$$

FoDo cell including the effect of the bending magnets



4.) Dispersion Suppressor Schemes

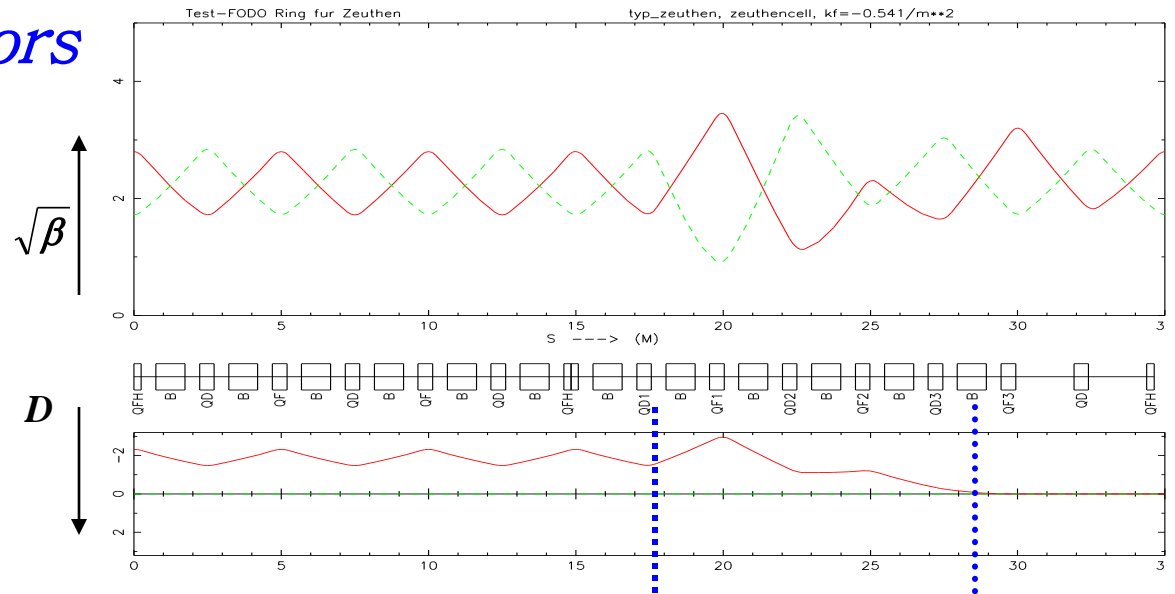
a.) The *straight forward* one: use *additional quadrupole lenses* to match the optical parameters ... including the $D(s)$, $D'(s)$ terms

* Dispersion suppressed by 2 quadrupole lenses,

* β and α restored to the values of the periodic solution by 4 additional quadrupoles

$$\left. \begin{array}{l} D(s), \quad D'(s) \\ \beta_x(s), \alpha_x(s) \\ \beta_y(s), \alpha_y(s) \end{array} \right\} \rightarrow \text{6 additional quadrupole lenses required}$$

Dispersion Suppressors Quadrupole scheme



*periodic FoDo
structure*

*matching section
including 6 additional
quadrupoles*

*dispersion free
section, regular
FoDo without dipoles*

Advantage:

- ! *easy,*
- ! *flexible: it works for any phase advance per cell*
- ! *does not change the geometry of the storage ring,*
- ! *can be used to match between different lattice structures (i.e. phase advances)*

Disadvantage:

- ! *additional power supplies needed (→ expensive)*
- ! *requires stronger quadrupoles*
- ! *due to higher β values: more aperture required*

b.) The Missing Bend Dispersion Suppressor

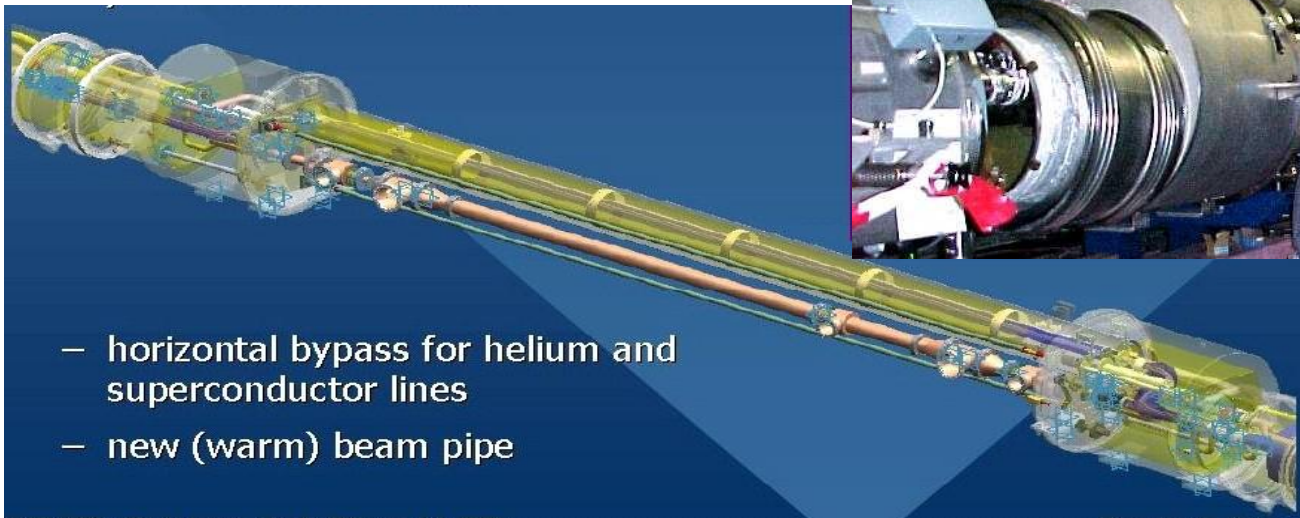
... turn it the other way round:

Start with $D(s) = \hat{D}, \quad D'(s) = 0$

and create dispersion – using dipoles - in such a way, that it fits exactly the conditions at the centre of the first regular quadrupoles:

$$D(s) = S(s) \int_{s_0}^{s_1} \frac{1}{\rho} C(\tilde{s}) d\tilde{s} - C(s) \int_{s_0}^{s_1} \frac{1}{\rho} S(\tilde{s}) d\tilde{s}$$

at the end of the arc: add m cells without dipoles followed by n regular arc cells.



The Missing Bend Dispersion Suppressor

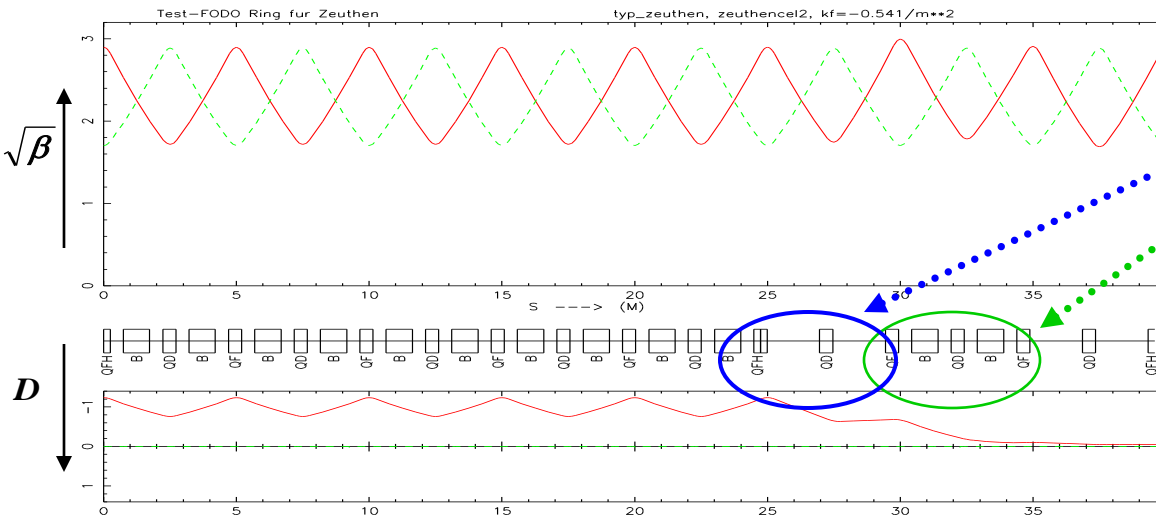
conditions for the (missing) dipole fields:

$$\frac{2m+n}{2} \Phi_C = (2k+1) \frac{\pi}{2}$$

$$\sin \frac{n\Phi_C}{2} = \frac{1}{2}, \quad k = 0, 2 \dots \text{ or}$$

$$\sin \frac{n\Phi_C}{2} = \frac{-1}{2}, \quad k = 1, 3 \dots$$

*m = number of cells without dipoles
followed by n regular arc cells.*



Example:

phase advance in the arc $\Phi_C = 60^\circ$

number of suppr. cells $m = 1$

number of regular cells $n = 1$

c.) *The Half Bend Dispersion Suppressor*

condition for vanishing dispersion: $2 * \delta_{\text{supr}} * \sin^2\left(\frac{n\Phi_c}{2}\right) = \delta_{\text{arc}}$

so if we require

$$\delta_{\text{supr}} = \frac{1}{2} * \delta_{\text{arc}}$$

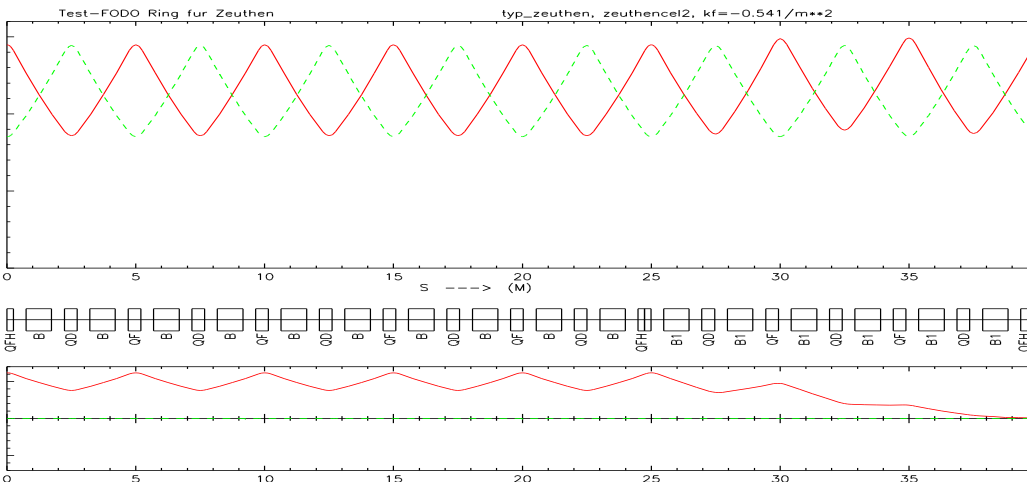
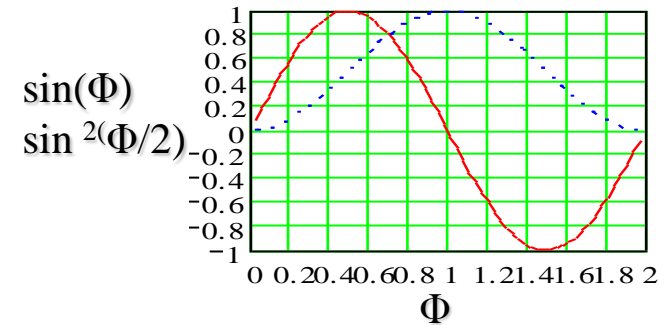
we get

$$\sin^2\left(\frac{n\Phi_c}{2}\right) = 1$$

or, which is equivalent

$$\sin(n\Phi_c) = 0$$

$$n\Phi_c = k * \pi, \quad k = 1, 3, \dots$$



in the n suppressor cells the phase advance has to accumulate to a odd multiple of π

strength of suppressor dipoles is half as strong as that of arc dipoles, $\delta_{\text{suppr}} = 1/2 \delta_{\text{arc}}$

Example: *phase advance in the arc $\Phi_c = 60^\circ$
number of suppr. cells $n = 3$*

Resume'

1.) Dispersion in a FoDo cell:

small dispersion \leftrightarrow large bending radius
short cells
strong focusing

$$\hat{D} = \frac{l^2}{\rho} * \frac{1 + \frac{1}{2} \sin \frac{\mu}{2}}{\sin^2 \frac{\mu}{2}}$$

2.) Chromaticity of a cell:

small $\xi \leftrightarrow$ weak focusing
small β

$$\xi_{total} = -\frac{1}{4\pi} \oint \{K(s) - mD(s)\} \beta(s) ds$$

3.) Position of a waist at the cell end:

α_0, β_0 = values at the end of the cell

$$l = \frac{\alpha_0}{\gamma_0} \quad , \quad \beta(s) = \frac{1}{\gamma_0}$$

4.) β function in a drift

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

5.) Mini β insertion

small $\beta \leftrightarrow$ short drift space required
phase advance ≈ 180

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

Appendix I: Dispersion:

Solution of the Inhomogenous Equation of Motion

the dispersion function is given by

$$D(s) = S(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

proof: $D'(s) = S'(s) * \int \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} + S(s) * \frac{C(\tilde{s})}{\rho(\tilde{s})} - C'(s) * \int \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s} - C(s) \frac{S(\tilde{s})}{\rho(\tilde{s})}$

$$D'(s) = S'(s) * \int \frac{C}{\rho} d\tilde{s} - C'(s) * \int \frac{S}{\rho} d\tilde{s}$$

$$D''(s) = S''(s) * \int \frac{C}{\rho} d\tilde{s} + S' \frac{C}{\rho} - C''(s) * \int \frac{S}{\rho} d\tilde{s} - C' \frac{S}{\rho}$$

$$D''(s) = S''(s) * \int \frac{C}{\rho} d\tilde{s} - C''(s) * \int \frac{S}{\rho} d\tilde{s} + \underbrace{\frac{1}{\rho} (CS' - SC')}_{= \det(M) = 1}$$

$$D''(s) = S''(s) * \int \frac{C}{\rho} d\tilde{s} - C''(s) * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

now the principal trajectories S and C fulfill the homogeneous equation

$$S''(s) = -K * S, \quad C''(s) = -K * C$$

and so we get:

$$D''(s) = -K * S(s) * \int \frac{C}{\rho} d\tilde{s} + K * C(s) * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

$$D''(s) = -K * D(s) + \frac{1}{\rho}$$

$$D''(s) + K * D(s) = \frac{1}{\rho}$$

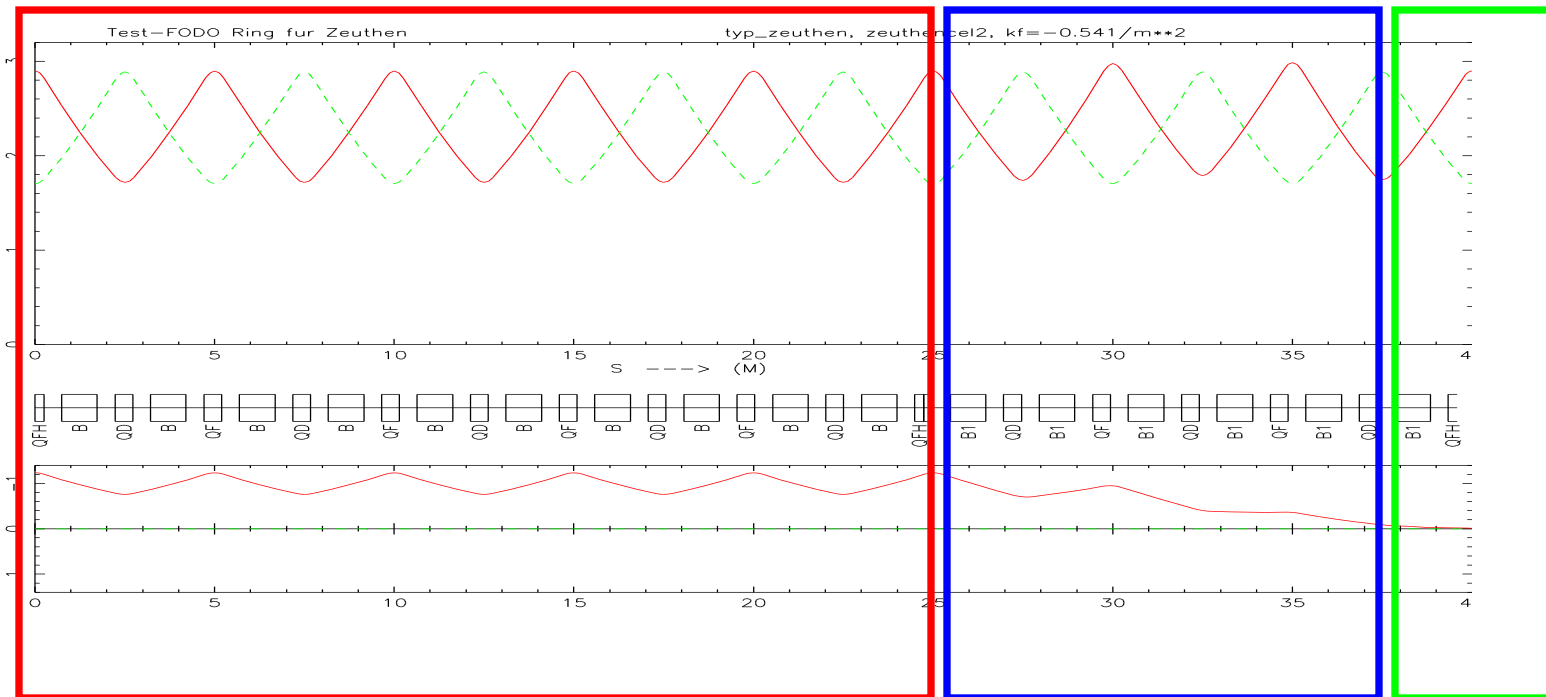
qed.

Appendix II: Dispersion Suppressors

... the calculation of the **half bend scheme** in full detail (**for purists only**)

1.) the lattice is split into 3 parts: (*Gallia divisa est in partes tres*)

- | | |
|--|--|
| * periodic solution of the arc | periodic β , periodic dispersion D |
| * section of the dispersion suppressor | periodic β , dispersion vanishes |
| * FoDo cells without dispersion | periodic β , $D = D' = 0$ |



2.) calculate the dispersion **D** in the periodic part of the lattice

transfer matrix of a periodic cell:

$$M_{0 \rightarrow s} = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}} (\cos \phi + \alpha_0 \sin \phi) & \sqrt{\beta_s \beta_0} \sin \phi \\ \frac{(\alpha_0 - \alpha_s) \cos \phi - (1 + \alpha_0 \alpha_s) \sin \phi}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_s}{\beta_0}} (\cos \phi - \alpha_s \sin \phi) \end{pmatrix}$$

for the transformation from one symmetry point to the next (i.e. one cell) we have:

Φ_C = phase advance of the cell, $\alpha = 0$ at a symmetry point. The index “c” refers to the periodic solution of one cell.

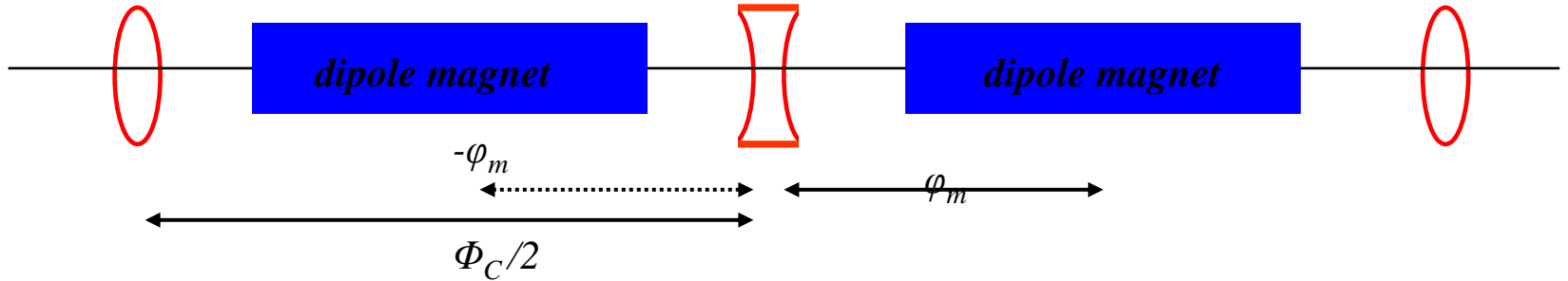
$$M_{Cell} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos \Phi_C & \beta_C \sin \Phi_C & D(l) \\ -\frac{1}{\beta_C} \sin \Phi_C & \cos \Phi_C & D'(l) \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix elements D and D' are given by the C and S elements in the usual way:

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

$$D'(l) = S'(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C'(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

here the values $C(l)$ and $S(l)$ refer to the symmetry point of the cell (middle of the quadrupole) and the integral is to be taken over the dipole magnet where $\rho \neq 0$. For $\rho = \text{const}$ the integral over $C(s)$ and $S(s)$ is approximated by the values in the middle of the dipole magnet.



Transformation of $C(s)$ from the symmetry point to the center of the dipole:

$$C_m = \sqrt{\frac{\beta_m}{\beta_C}} \cos \Delta\Phi = \sqrt{\frac{\beta_m}{\beta_C}} \cos\left(\frac{\Phi_C}{2} \pm \varphi_m\right) \quad S_m = \beta_m \beta_C \sin\left(\frac{\Phi_C}{2} \pm \varphi_m\right)$$

where β_C is the periodic β function at the beginning and end of the cell, β_m its value at the middle of the dipole and φ_m the phase advance from the quadrupole lens to the dipole center.

Now we can solve the integral for D and D' :

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

$$D(l) = \beta_C \sin \Phi_C * \frac{L}{\rho} * \sqrt{\frac{\beta_m}{\beta_C}} * \cos\left(\frac{\Phi_C}{2} \pm \varphi_m\right) - \cos \Phi_C * \frac{L}{\rho} \sqrt{\beta_m \beta_C} * \sin\left(\frac{\Phi_C}{2} \pm \varphi_m\right)$$

$$D(l) = \delta \sqrt{\beta_m \beta_c} \left\{ \sin \Phi_c \left[\cos\left(\frac{\Phi_c}{2} + \varphi_m\right) + \cos\left(\frac{\Phi_c}{2} - \varphi_m\right) \right] - \right. \\ \left. - \cos \Phi_c \left[\sin\left(\frac{\Phi_c}{2} + \varphi_m\right) + \sin\left(\frac{\Phi_c}{2} - \varphi_m\right) \right] \right\}$$

I have put $\delta = L/\rho$ for the strength of the dipole

remember the relations

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} * \cos \frac{x-y}{2}$$

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} * \cos \frac{x-y}{2}$$

$$D(l) = \delta \sqrt{\beta_m \beta_c} \left\{ \sin \Phi_c * 2 \cos \frac{\Phi_c}{2} * \cos \varphi_m - \cos \Phi_c * 2 \sin \frac{\Phi_c}{2} * \cos \varphi_m \right\}$$

$$D(l) = 2\delta \sqrt{\beta_m \beta_c} * \cos \varphi_m \left\{ \sin \Phi_c * \cos \frac{\Phi_c}{2} * -\cos \Phi_c * \sin \frac{\Phi_c}{2} \right\}$$

remember:

$$\sin 2x = 2 \sin x * \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$D(l) = 2\delta \sqrt{\beta_m \beta_c} * \cos \varphi_m \left\{ 2 \sin \frac{\Phi_c}{2} * \cos^2 \frac{\Phi_c}{2} - \left(\cos^2 \frac{\Phi_c}{2} - \sin^2 \frac{\Phi_c}{2} \right) * \sin \frac{\Phi_c}{2} \right\}$$

$$D(l) = 2\delta\sqrt{\beta_m\beta_c} * \cos\varphi_m * \sin\frac{\Phi_c}{2} \left\{ 2\cos^2\frac{\Phi_c}{2} - \cos^2\frac{\Phi_c}{2} + \sin^2\frac{\Phi_c}{2} \right\}$$

$$D(l) = 2\delta\sqrt{\beta_m\beta_c} * \cos\varphi_m * \sin\frac{\Phi_c}{2}$$

in full analogy one derives the expression for D':

$$D'(l) = 2\delta\sqrt{\beta_m/\beta_c} * \cos\varphi_m * \cos\frac{\Phi_c}{2}$$

As we refer the expression for D and D' to a periodic struture, namly a FoDo cell we require periodicity conditons:

$$\begin{pmatrix} D_c \\ D'_c \\ 1 \end{pmatrix} = M_c * \begin{pmatrix} D_c \\ D'_c \\ 1 \end{pmatrix}$$

and by symmetry: $D'_c = 0$

With these boundary conditions the Dispersion in the FoDo is determined:

$$D_c * \cos\Phi_c + \delta\sqrt{\beta_m\beta_c} * \cos\varphi_m * 2\sin\frac{\Phi_c}{2} = D_c$$

(A1)

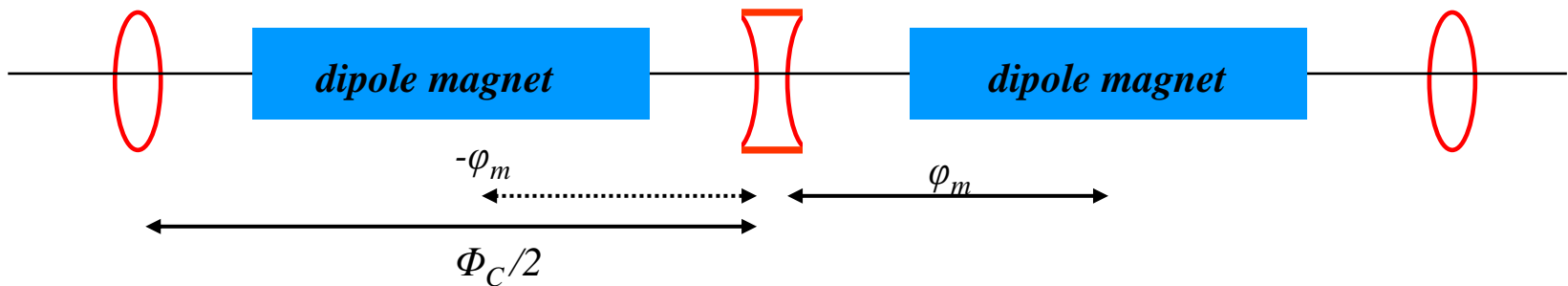
$$D_C = \delta \sqrt{\beta_m \beta_C} * \cos \varphi_m / \sin \frac{\Phi_C}{2}$$

This is the value of the periodic dispersion in the cell evaluated at the position of the dipole magnets.

3.) Calculate the dispersion in the suppressor part:

We will now move to the second part of the dispersion suppressor: The section where ... starting from $D=D'=0$ the dispersion is generated ... or turning it around where the Dispersion of the arc is reduced to zero.

The goal will be to generate the dispersion in this section in a way that the values of the periodic cell that have been calculated above are obtained.



The relation for D, generated in a cell still holds in the same way:

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

as the dispersion is generated in a number of n cells the matrix for these n cells is

$$M_n = M_C^n = \begin{pmatrix} \cos n\Phi_C & \beta_C \sin n\Phi_C & D_n \\ \frac{-1}{\beta_C} \sin n\Phi_C & \cos n\Phi_C & D'_n \\ 0 & 0 & 1 \end{pmatrix}$$

$$D_n = \beta_C \sin n\Phi_C * \delta_{\text{supr}} * \sum_{i=1}^n \cos(i\Phi_C - \frac{1}{2}\Phi_C \pm \varphi_m) * \sqrt{\frac{\beta_m}{\beta_C}} - \\ - \cos n\Phi_C * \delta_{\text{supr}} * \sum_{i=1}^n \sqrt{\beta_m \beta_C} * \sin(i\Phi_C - \frac{1}{2}\Phi_C \pm \varphi_m)$$

$$D_n = \sqrt{\beta_m \beta_C} * \sin n\Phi_C * \delta_{\text{supr}} * \sum_{i=1}^n \cos((2i-1)\frac{\Phi_C}{2} \pm \varphi_m) - \sqrt{\beta_m \beta_C} * \delta_{\text{supr}} * \cos n\Phi_C \sum_{i=1}^n \sin((2i-1)\frac{\Phi_C}{2} \pm \varphi_m)$$

remember: $\sin x + \sin y = 2 \sin \frac{x+y}{2} * \cos \frac{x-y}{2}$ $\cos x + \cos y = 2 \cos \frac{x+y}{2} * \cos \frac{x-y}{2}$

$$D_n = \delta_{\text{supr}} * \sqrt{\beta_m \beta_C} * \sin n\Phi_C * \sum_{i=1}^n \cos((2i-1)\frac{\Phi_C}{2}) * 2 \cos \varphi_m - \\ - \delta_{\text{supr}} * \sqrt{\beta_m \beta_C} * \cos n\Phi_C \sum_{i=1}^n \sin((2i-1)\frac{\Phi_C}{2}) * 2 \cos \varphi_m$$

$$D_n = 2\delta_{\text{supr}} * \sqrt{\beta_m \beta_C} * \cos \varphi_m \left\{ \sum_{i=1}^n \cos((2i-1) \frac{\Phi_C}{2}) * \sin n\Phi_C - \sum_{i=1}^n \sin((2i-1) \frac{\Phi_C}{2}) * \cos n\Phi_C \right\}$$

$$D_n = 2\delta_{\text{supr}} * \sqrt{\beta_m \beta_C} * \cos \varphi_m \left\{ \sin n\Phi_C \left\{ \frac{\sin \frac{n\Phi_C}{2} * \cos \frac{n\Phi_C}{2}}{\sin \frac{\Phi_C}{2}} \right\} - \cos n\Phi_C * \left\{ \frac{\sin \frac{n\Phi_C}{2} * \sin \frac{n\Phi_C}{2}}{\sin \frac{\Phi_C}{2}} \right\} \right\}$$

$$D_n = \frac{2\delta_{\text{supr}} * \sqrt{\beta_m \beta_C} * \cos \varphi_m}{\sin \frac{\Phi_C}{2}} \left\{ \sin n\Phi_C * \sin \frac{n\Phi_C}{2} * \cos \frac{n\Phi_C}{2} - \cos n\Phi_C * \sin^2 \frac{n\Phi_C}{2} \right\}$$

set for more convenience $x = n\Phi_C/2$

$$D_n = \frac{2\delta_{\text{supr}} * \sqrt{\beta_m \beta_C} * \cos \varphi_m}{\sin \frac{\Phi_C}{2}} \left\{ \sin 2x * \sin x * \cos x - \cos 2x * \sin^2 x \right\}$$

$$D_n = \frac{2\delta_{\text{supr}} * \sqrt{\beta_m \beta_C} * \cos \varphi_m}{\sin \frac{\Phi_C}{2}} \left\{ 2 \sin x \cos x * \cos x \sin x - (\cos^2 x - \sin^2 x) \sin^2 x \right\}$$

(A2)

$$D_n = \frac{2\delta_{\text{supr}} * \sqrt{\beta_m \beta_C} * \cos \varphi_m * \sin^2 \frac{n\Phi_C}{2}}{\sin \frac{\Phi_C}{2}}$$

and in similar calculations:

$$D'_n = \frac{2\delta_{\text{supr}} * \sqrt{\beta_m \beta_C} * \cos \varphi_m * \sin n\Phi_C}{\sin \frac{\Phi_C}{2}}$$

This expression gives the dispersion generated in a certain number of n cells as a function of the dipole kick δ in these cells.

At the end of the dispersion generating section the value obtained for $D(s)$ and $D'(s)$ has to be equal to the value of the periodic solution:

→equating (A1) and (A2) gives the conditions for the matching of the periodic dispersion in the arc to the values $D = D' = 0$ after the suppressor.

$$D_n = \frac{2\delta_{\text{supr}} * \sqrt{\beta_m \beta_C} * \cos \varphi_m * \sin^2 \frac{n\Phi_C}{2}}{\sin \frac{\Phi_C}{2}} = \delta_{\text{arc}} \sqrt{\beta_m \beta_C} * \frac{\cos \varphi_m}{\sin \frac{\Phi_C}{2}}$$

$$\left. \begin{array}{l} \rightarrow 2\delta_{\text{sup}r} \sin^2(\frac{n\Phi_c}{2}) = \delta_{\text{arc}} \\ \rightarrow \sin(n\Phi_c) = 0 \end{array} \right\} \delta_{\text{sup}r} = \frac{1}{2} \delta_{\text{arc}}$$

and at the same time the phase advance in the arc cell has to obey the relation:

$$n\Phi_c = k * \pi, \quad k = 1, 3, \dots$$