Introduction to Transverse Beam Optics II

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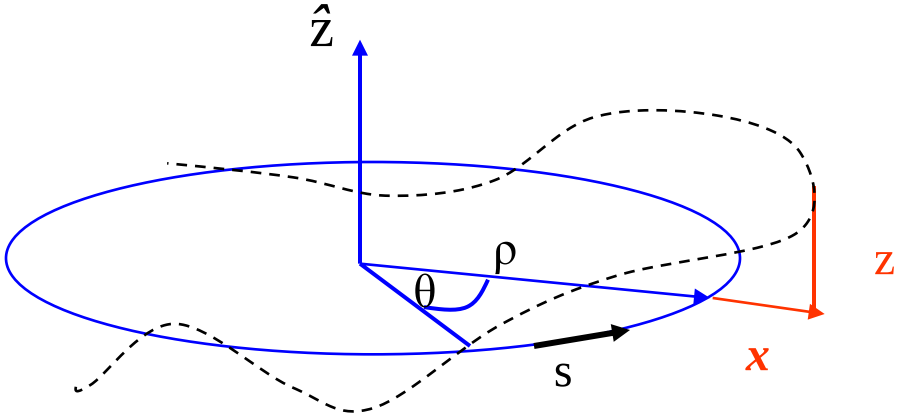
I. Reminder: the ideal world

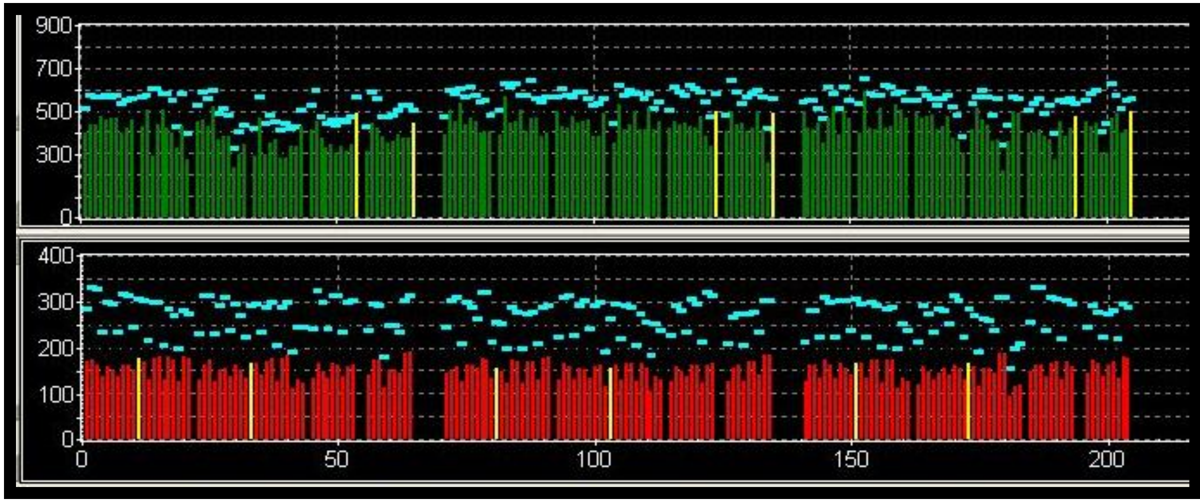
$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_0$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}s \\ -\sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

The Beta Function

Beam parameters of a typical high energy ring: ***Ip = 100 mA***
particles per bunch: $N \approx 10^{11}$

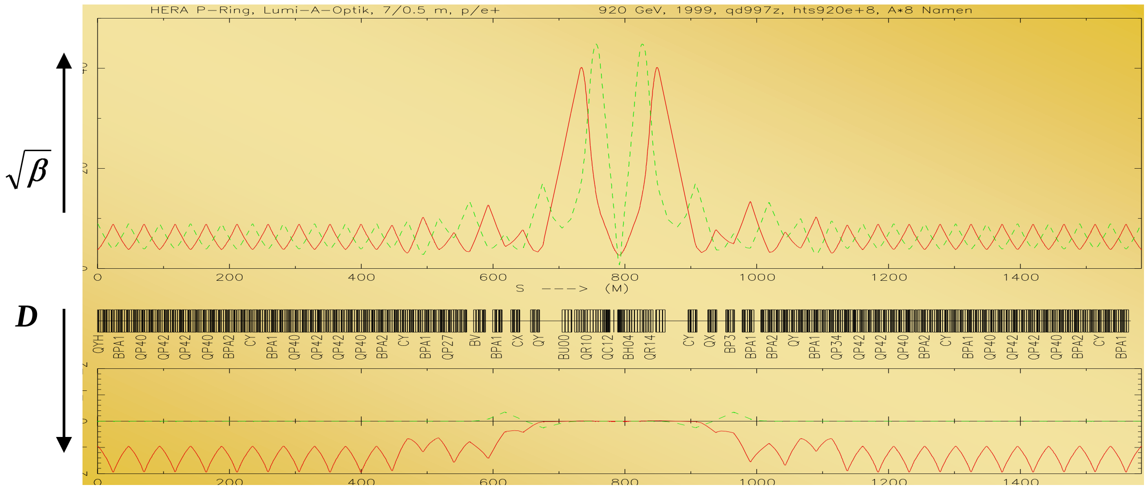


Example: HERA Bunch pattern

... question: do we really have to calculate some 10^{11} single particle trajectories ?

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$$

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$



Beam Emittance and Phase Space Ellipse

equation of motion:

$$x''(s) - k(s)\,x(s) = 0$$

general solution of Hills equation:

$$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s) + \varphi)$$

beam size:

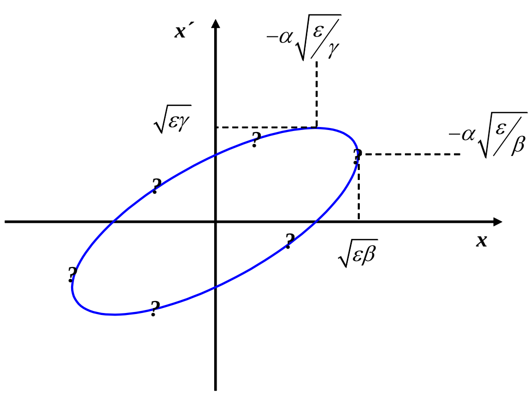
$$\sigma = \sqrt{\varepsilon\beta} \approx \text{"mm"}$$

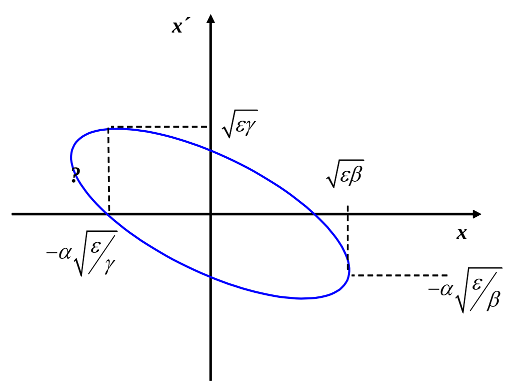
$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

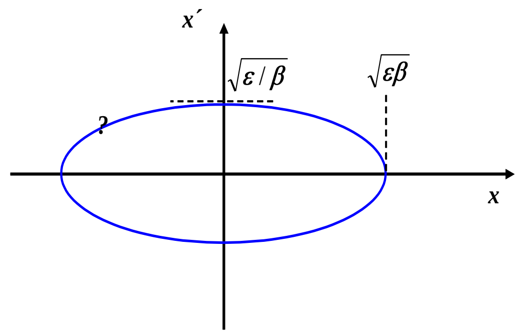
$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

* *ε* is a *constant of the motion … it is independent of „s"*
* *parametric representation of an ellipse in the x x' space*
* *shape and orientation of ellipse are given by α, β, γ*



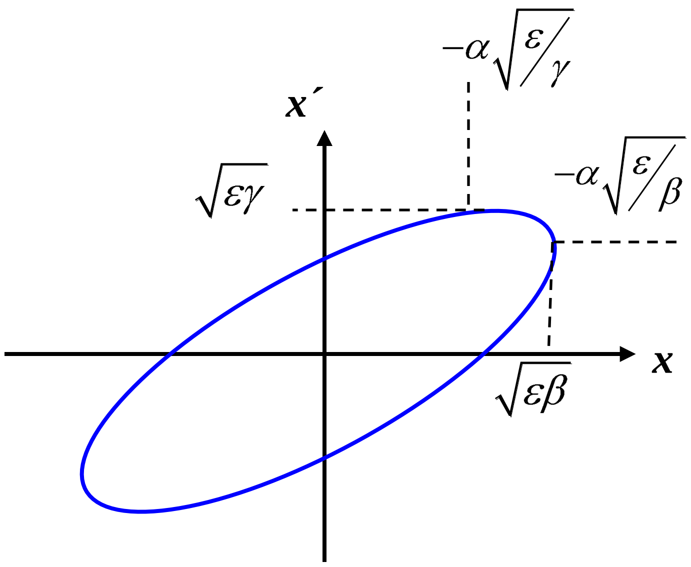




II ... the not so ideal world

13.) Liouville during Acceleration

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$



Beam Emittance ***corresponds to the area covered in the x, x´ Phase Space Ellipse***

Liouville: ***Area in phase space is constant.***

But so sorry ... ε ≠ const !

Classical Mechanics:

phase space **= *diagram of the two canonical variables***
position **&** ***momentum***
x p_x

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad ; \quad L = T - V = kin.\,Energy - pot.\,Energy$$

According to Hamiltonian mechanics:
phase space diagram relates the variables q and p

q = position = x
p = momentum = γmv = mcγβ$_x$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}$$

Liouvilles Theorem: $\int p\,dq = const$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta}$$

where $\beta_x = v_x / c$

$$\int p dq = mc \int \gamma \beta_x dx$$

$$\int p dq = mc\gamma\beta \underbrace{\int x' dx}_{\varepsilon}$$

$$\Rightarrow \quad \varepsilon = \int x' dx \propto \frac{1}{\beta\gamma}$$

the beam emittance shrinks during acceleration $\varepsilon \sim 1/\gamma$

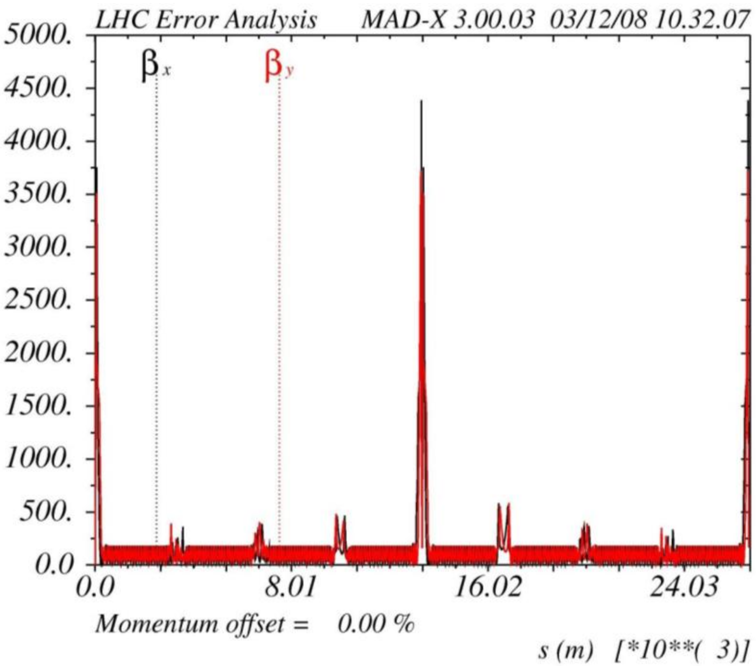
Nota bene:

1.) *A proton machine … or an electron linac … needs the highest aperture at injection energy !!!*
as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.

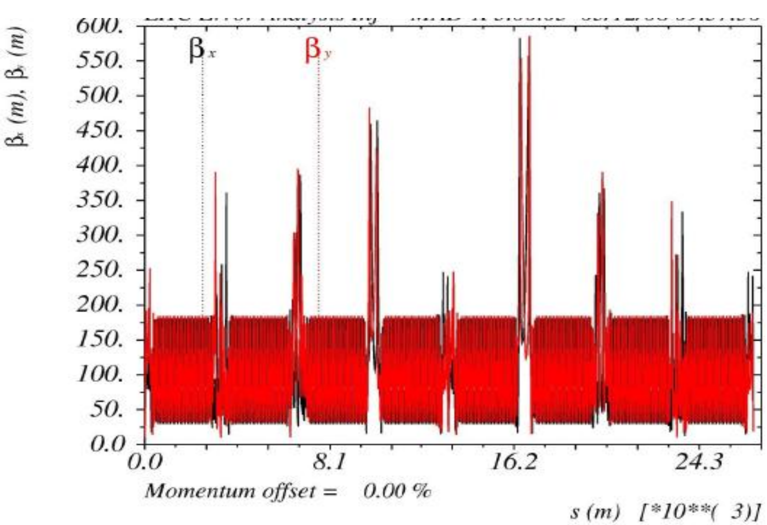
$$\sigma = \sqrt{\varepsilon\beta}$$

2.) *At lowest energy the machine will have the major aperture problems,*
→ *here we have to minimise* $\hat{\beta}$

3.) *we need different beam optics adopted to the energy:*
A Mini Beta concept will only be adequate at flat top.



LHC mini beta optics at 7000 GeV

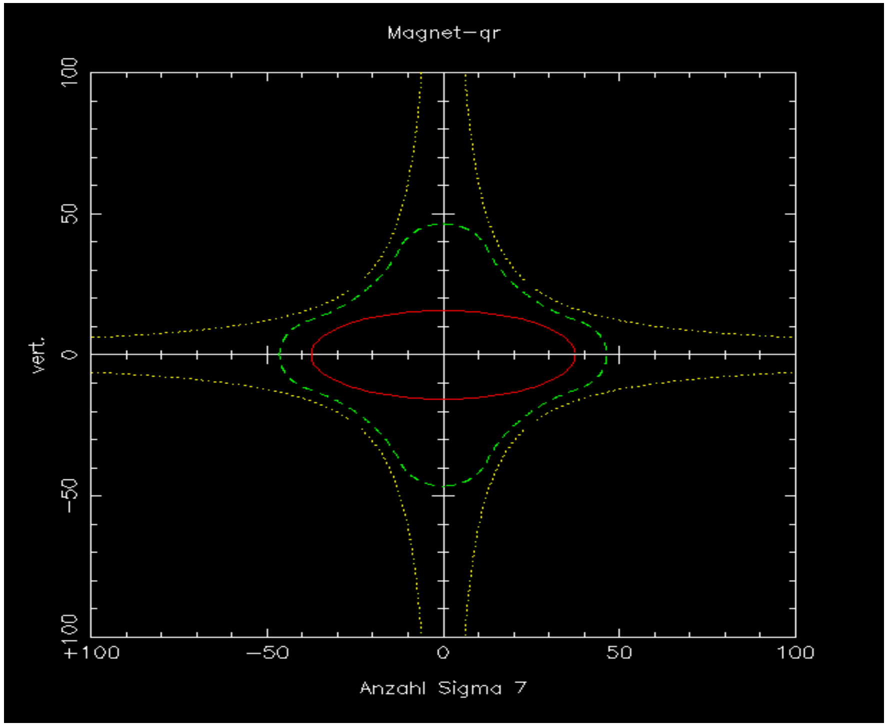


LHC injection optics at 450 GeV

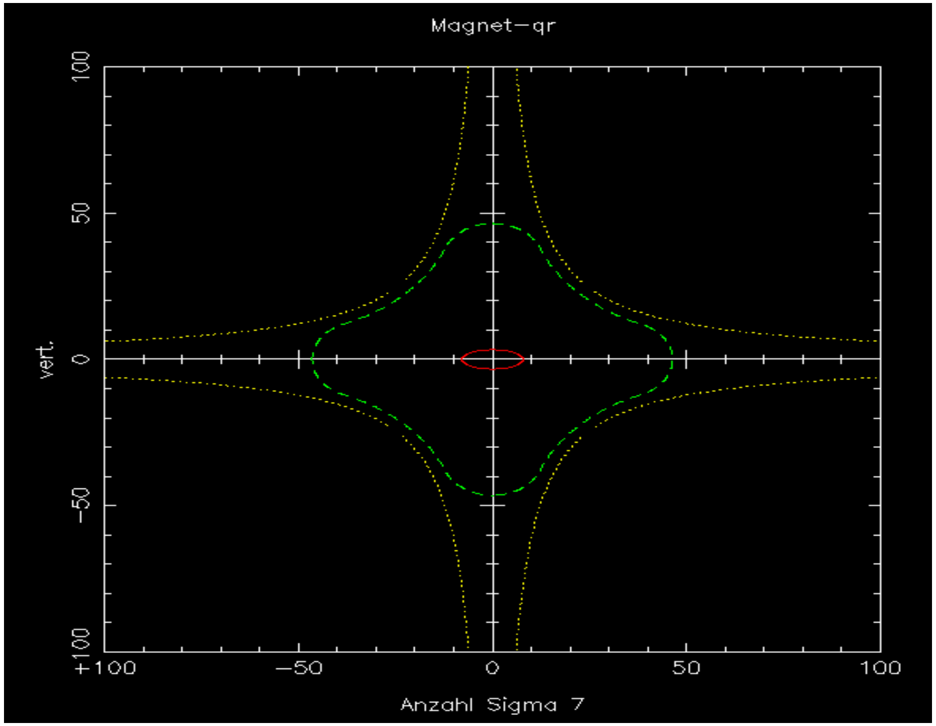
Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$
flat top energy: 920 GeV $\gamma = 980$

emittance $\varepsilon\ (40GeV) = 1.2 * 10^{-7}$
$\varepsilon\ (920GeV) = 5.1 * 10^{-9}$



7 σ beam envelope at E = 40 GeV



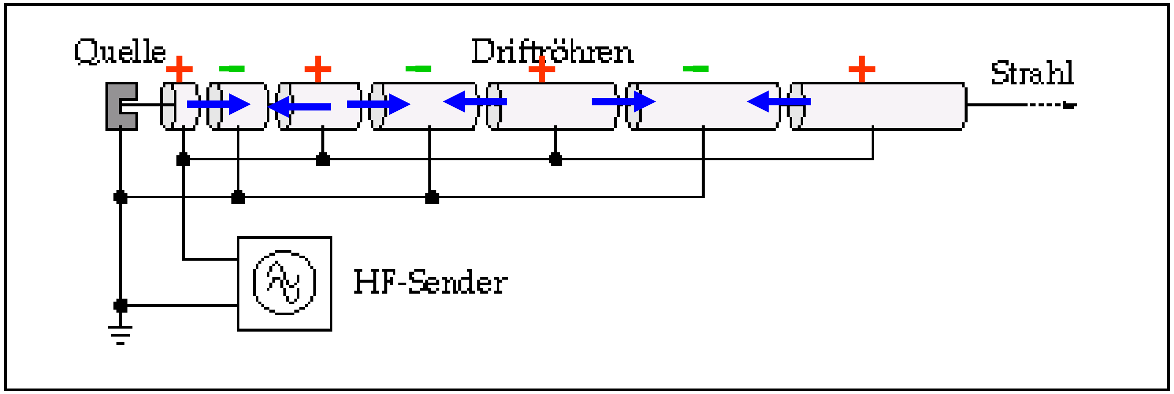
… and at E = 920 GeV

14.) The „ $\Delta p / p \neq 0$“ Problem

Linear Accelerator

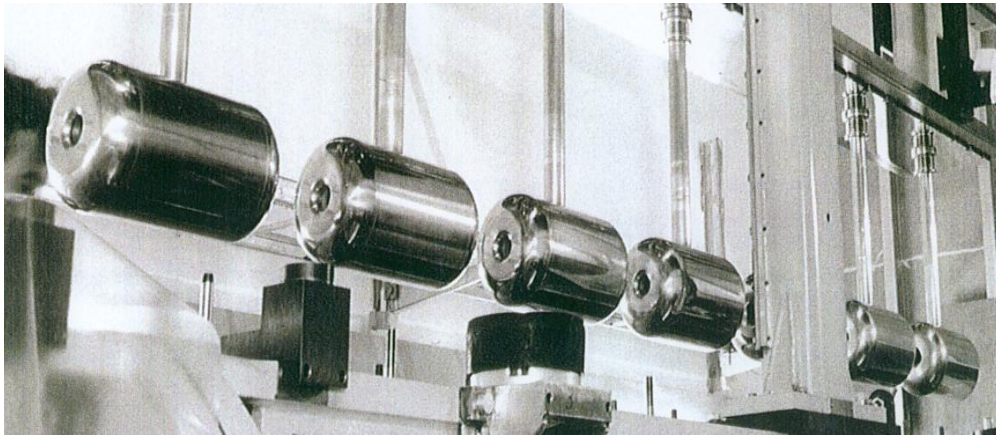
Energy Gain per „Gap“:

$$W = q\, U_0 \sin \omega_{RF}\, t$$



1928, Wideroe

drift tube structure at a proton linac

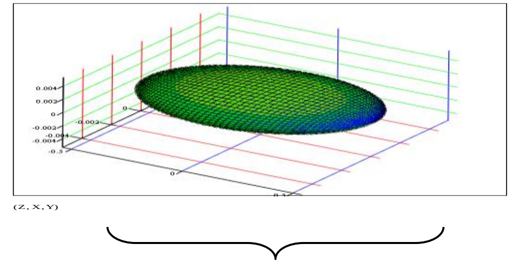


500 MHz cavities in an electron storage ring

* *RF Acceleration:* *multiple application of the same acceleration voltage; brillant idea to gain higher energies ... but changing acceleration voltage*

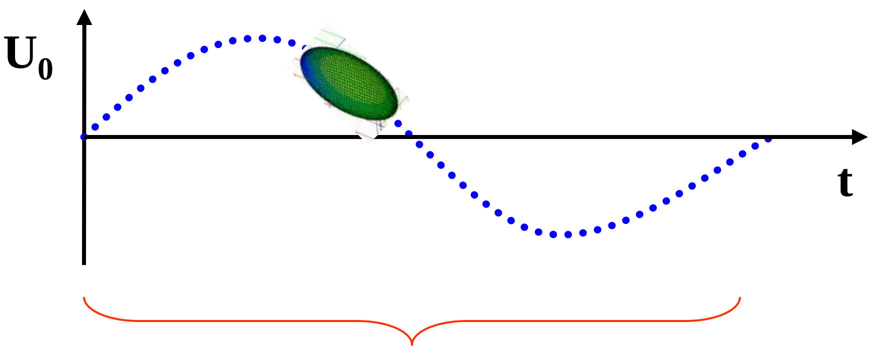
Problem: panta rhei !!!

(Heraklit: 540-480 v. Chr.)



Bunch length of Electrons ≈ 1cm

Example: HERA RF:



$\lambda = 60\ cm$

$$\nu = 500\, MHz$$
$$c = \lambda\, \nu$$
$\Big\}\ \lambda = 60\ cm$

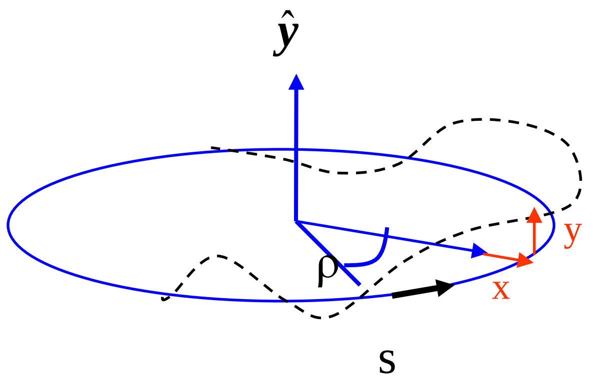
$$\sin(90^o) = 1$$
$$\sin(84^o) = 0.994$$

$$\frac{\Delta U}{U} = 6.0\ \ 10^{-3}$$

typical momentum spread of an electron bunch:

$$\frac{\Delta p}{p} \approx 1.0\ \ 10^{-3}$$

15.) Dispersion: trajectories for $\Delta p / p \neq 0$



***Force* acting on the particle**

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

remember: $x \approx mm\,,\ \rho \approx m\ \ldots$ ***→ develop for small*** x

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = eB_y v$$

consider only linear fields, and change independent variable: $t \to s$

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

$$x'' - \frac{1}{\rho}\left(1-\frac{x}{\rho}\right) = \frac{e\ B_0}{mv} + \frac{e\ x\,g}{mv}$$

$p = p_0 + \Delta p$

… but now take a small momentum error into account !!!

Dispersion:

develop for small momentum error

$$\Delta p << p_0 \Rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{e\ B_0}{p_0}}_{-\frac{1}{\rho}} - \frac{\Delta p}{p_0^2} eB_0 + \underbrace{\frac{xeg}{p_0}}_{k*x} - \underbrace{xeg\frac{\Delta p}{p_0^2}}_{\approx 0}$$

$$x'' + \frac{x}{\rho^2} \approx \frac{\Delta p}{p_0} * \underbrace{\frac{(-eB_0)}{p_0}}_{\frac{1}{\rho}} + k*x = \frac{\Delta p}{p_0} * \frac{1}{\rho} + k*x$$

$$x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0}\frac{1}{\rho} \longrightarrow x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0}\frac{1}{\rho}$$

Momentum spread of the beam adds a term on the r.h.s. of the equation of motion.
→ inhomogeneous differential equation.

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \frac{1}{\rho} \cdot \frac{\Delta p}{p} \end{cases}$$

Normalise with respect to Δp/p:

$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

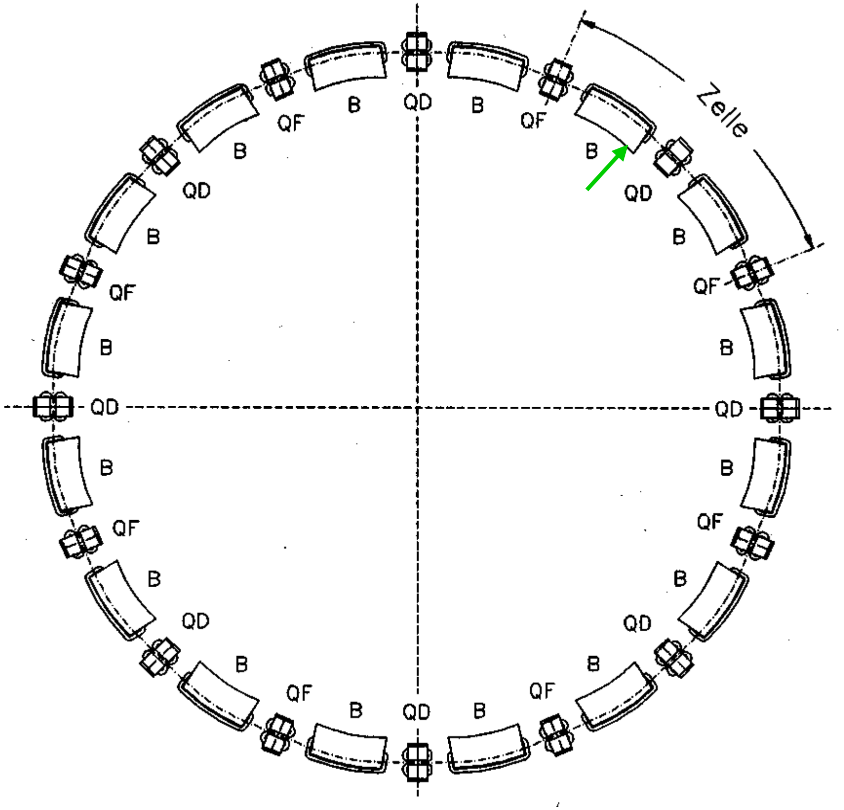
**** is that special orbit, an ideal particle would have for Δp/p = 1***

**** the orbit of any particle is the sum of the well known x_β and the dispersion***

**** as D(s) is just another orbit it will be subject to the focusing properties of the lattice***

Dispersion

Example: homogenous dipole field



bit for $\Delta p/p > 0$

$$= D(s) \cdot \frac{\Delta p}{p}$$

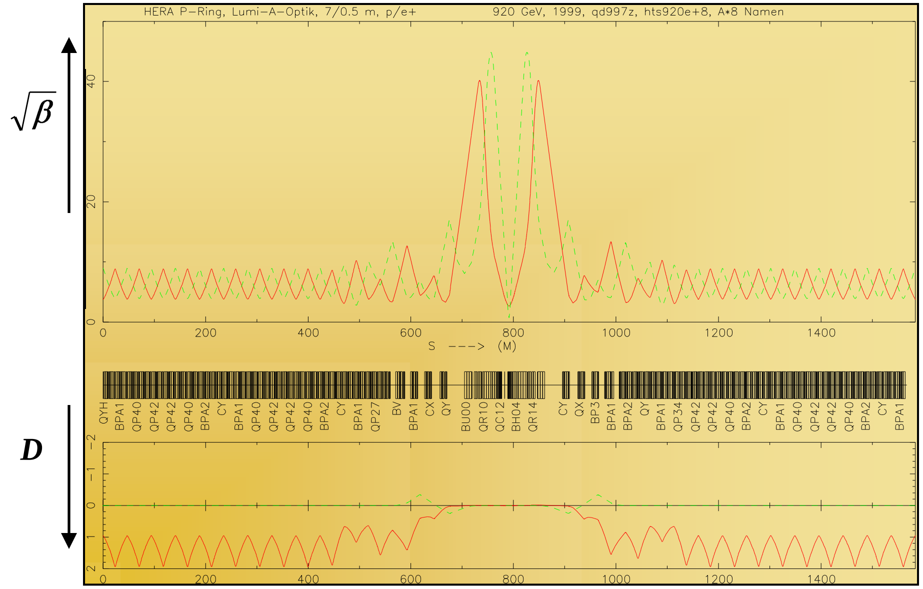
Matrix formalism:

$$x(s) = x_{\beta}(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x_0' + D(s) \cdot \frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$



Example HERA

$$x_\beta = 1 \ldots 2\ mm$$
$$D(s) \approx 1 \ldots 2\ m$$
$$\Delta p/p \approx 1 \cdot 10^{-3}$$

Amplitude of Orbit oscillation

contribution due to Dispersion ≈ beam size

Calculate D, D´

$$D(s) = S(s) \int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s}) d\tilde{s} - C(s) \int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s}) d\tilde{s}$$

Example: Drift

$$M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$D(s) = S(s)\underbrace{\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})\,d\tilde{s}}_{=0} - C(s)\underbrace{\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})\,d\tilde{s}}_{=0}$$

$$M_{Drift} = \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

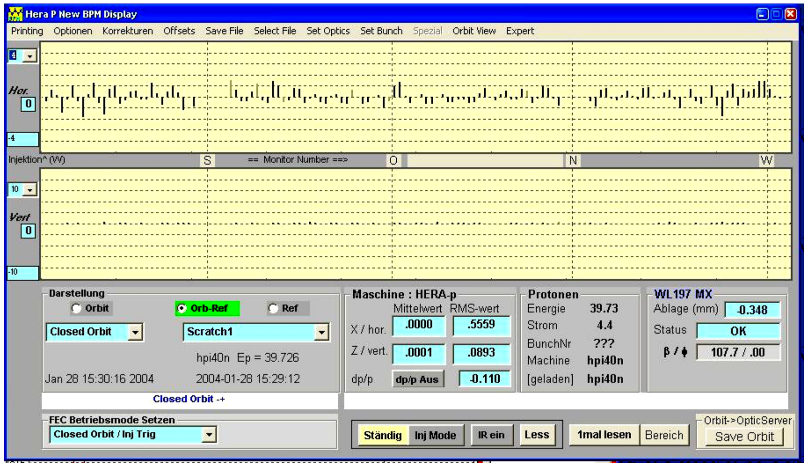
Example: Dipole

$$M_{Dipole} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} \end{pmatrix} \quad \rightarrow$$

$$D(s) = \rho \cdot (1 - \cos\frac{l}{\rho})$$

$$D'(s) = \sin\frac{l}{\rho}$$

Dispersion is visible

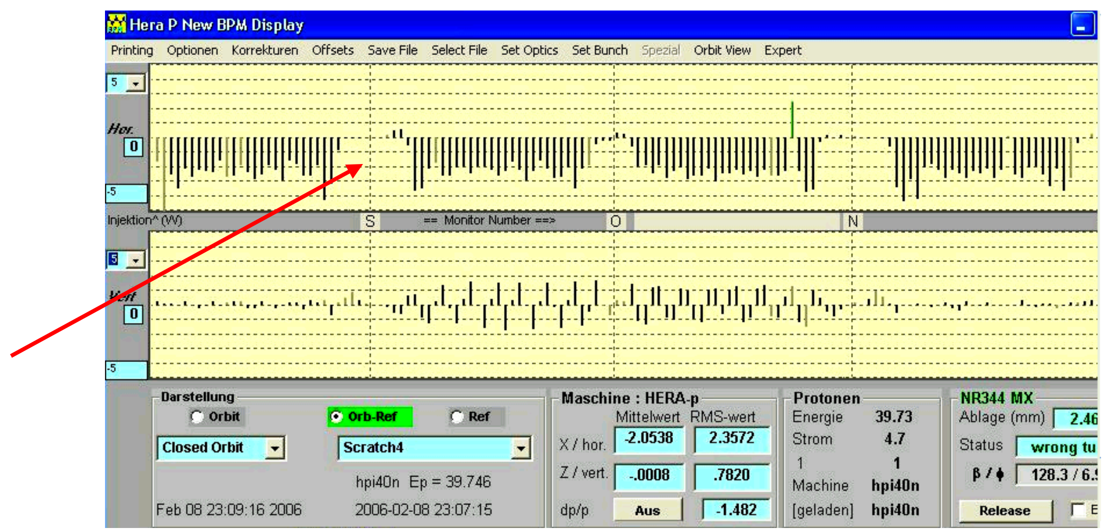


HERA Standard Orbit

HERA Dispersion Orbit

dedicated energy change of the stored beam

→ closed orbit is moved to a dispersions trajectory



$$x_D = D(s) * \frac{\Delta p}{p}$$

Attention: at the Interaction Points we require D=D´= 0

16.) Momentum Compaction Factor:

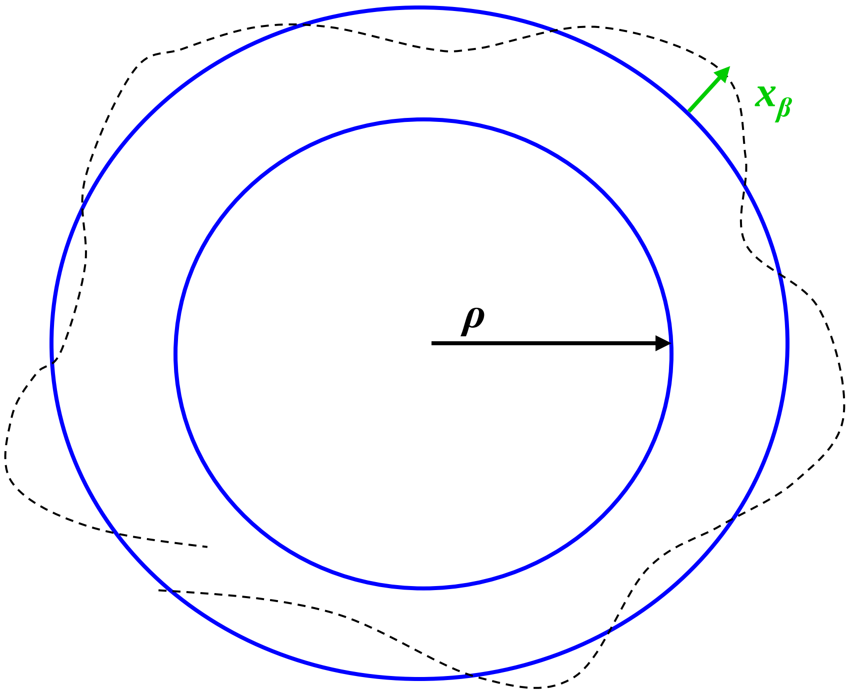
The dispersion function relates the momentum error of a particle to the horizontal orbit coordinate.

inhomogeneous differential equation

$$x'' + K(s) * x = \frac{1}{\rho} \frac{\Delta p}{p}$$

general solution

$$x(s) = x_\beta(s) + D(s) \frac{\Delta p}{p}$$



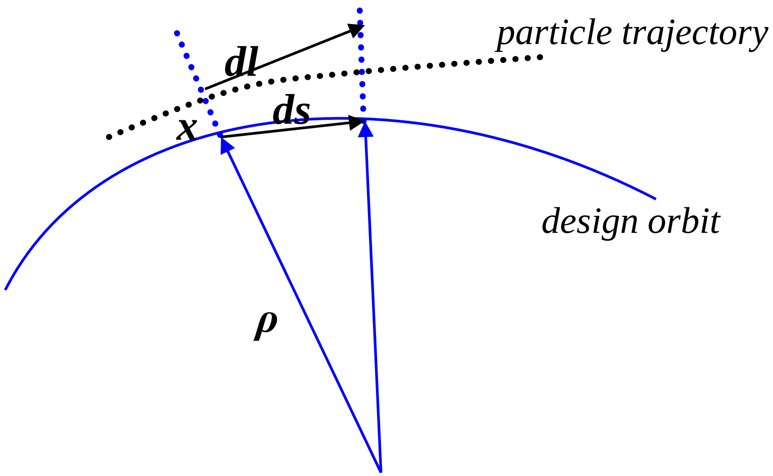
But it does much more:

it changes the length of the off - energy - orbit !!

particle with a displacement x to the design orbit
→ path length dl ...

$$\frac{dl}{ds} = \frac{\rho + x}{\rho}$$

$$\rightarrow \; dl = \left(1 + \frac{x}{\rho(s)}\right) ds$$



circumference of an off-energy closed orbit

$$l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}}{\rho(s)}\right) ds$$

remember:

$$x_{\Delta E}(s) = D(s) \frac{\Delta p}{p}$$

$$\delta l_{\Delta E} = \frac{\Delta p}{p} \oint \left(\frac{D(s)}{\rho(s)}\right) ds$$

**** The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.***

Definition:

$$\frac{\delta l_\varepsilon}{L} = \alpha_{cp} \frac{\Delta p}{p}$$

$$\rightarrow \quad \alpha_{cp} = \frac{1}{L} \oint \left(\frac{D(s)}{\rho(s)} \right) ds$$

For first estimates assume: $\frac{1}{\rho} = const$

$$\int_{dipoles} D(s)ds = \Sigma \, (l_{dipoles}) * \langle D \rangle_{dipole}$$

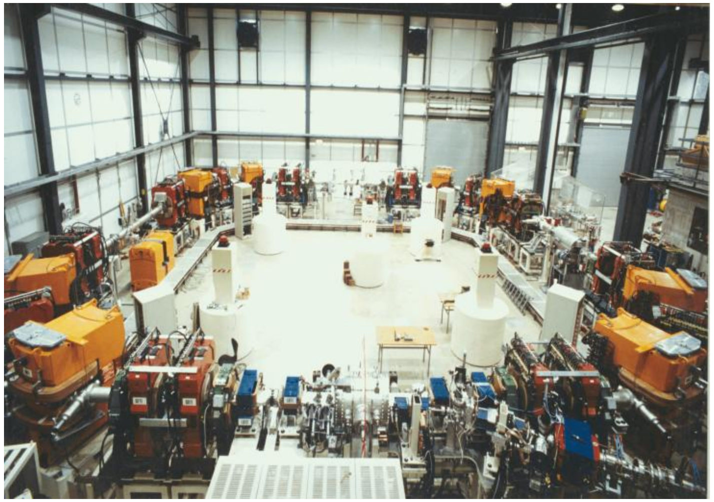
$$\alpha_{cp} = \frac{1}{L} l_{dipoles} \langle D \rangle \frac{1}{\rho} = \frac{1}{L} 2\pi\rho \langle D \rangle \frac{1}{\rho} \quad \rightarrow \quad \alpha_{cp} \approx \frac{2\pi}{L} \langle D \rangle \approx \frac{\langle D \rangle}{R}$$

Assume: $v \approx c$

$$\rightarrow \frac{\delta T}{T} = \frac{\delta l_\varepsilon}{L} = \alpha_{cp} \frac{\Delta p}{p}$$

α_{cp} combines via the dispersion function the momentum spread with the longitudinal motion of the particle.

17.) Tune and Quadrupoles



Question: ***what will happen, if you do not make too many mistakes and your particle performs one complete turn ?***

Transfer Matrix from point „0“ in the lattice to point „s“:

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\ \cos\psi_s + \alpha_0 \sin\psi_s & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\ \cos\psi_s - \alpha_s \sin\psi_s \end{pmatrix}$$

Matrix for one complete turn

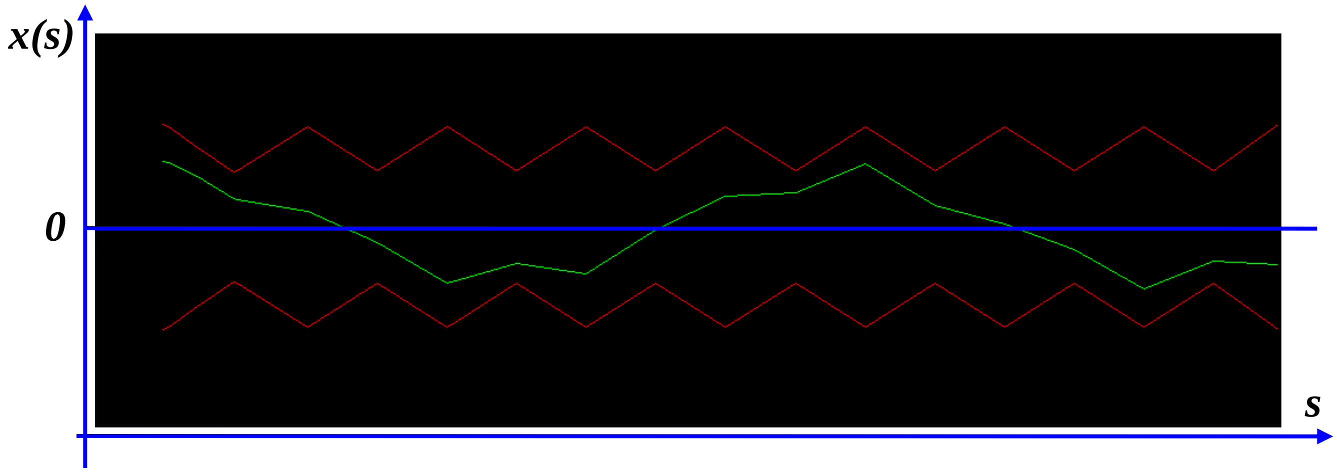
the Twiss parameters are periodic in L:

$$\beta(s+L) = \beta(s)$$
$$\alpha(s+L) = \alpha(s)$$
$$\gamma(s+L) = \gamma(s)$$

$$M_{turn} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos\psi_{turn} + \alpha\sin\psi_{turn} & \beta\sin\psi_{turn} \\ -\gamma\sin\psi_{turn} & \cos\psi_{turn} - \alpha\sin\psi_{turn} \end{pmatrix}$$

***Definition:** phase advance of the particle oscillation per revolution in units of 2π is called tune*

$$Q = \frac{\Delta\psi_{turn}}{2\pi} = \frac{\mu}{2\pi}$$



Quadrupole Error in the Lattice

optic perturbation described by thin lens quadrupole

$$M_{dist} = M_{\Delta k} * M_0 = \begin{pmatrix} 1 & 0 \\ \Delta Kds & 1 \end{pmatrix} * \begin{pmatrix} \cos\psi_{turn} + \alpha \sin\psi_{turn} & \beta \sin\psi_{turn} \\ -\gamma \sin\psi_{turn} & \cos\psi_{turn} - \alpha \sin\psi_{turn} \end{pmatrix}$$

quad error — ***ideal storage ring***

$$M_{dist} = \begin{pmatrix} \cos\psi_{turn} + \alpha \sin\psi_{turn} & \beta \sin\psi_{turn} \\ \Delta Kds\,(\cos\psi_{turn} + \alpha \sin\psi_{turn}) - \gamma \sin\psi_{turn} & \Delta Kds * \beta \sin\psi_{turn} + \cos\psi_{turn} - \alpha \sin\psi_{turn} \end{pmatrix}$$

rule for getting the tune

$$Trace(M) = 2\cos\psi = 2\cos\psi_0 + \Delta Kds\beta \sin\psi_0$$

$$\psi = \psi_0 + \Delta\psi$$

Quadrupole error → Tune Shift

$$\cos(\psi_0 + \Delta\psi) = \cos\psi_0 + \frac{\Delta K ds \beta \sin\psi_0}{2}$$

remember the old fashioned trigonometric stuff and assume that the error is small !!!

$$\cos\psi_0 * \underbrace{\cos\Delta\psi}_{\approx 1} - \sin\psi_0 * \underbrace{\sin\Delta\psi}_{\approx \Delta\psi} = \cos\psi_0 + \frac{\Delta K ds \beta \sin\psi_0}{2}$$

$$\Delta\psi = \frac{\Delta K ds \beta}{2}$$

and referring to Q instead of ψ: $\psi = 2\pi Q$

$$\Delta Q = \int_{s0}^{s0+L} \frac{\Delta K(s)\beta(s)ds}{4\pi}$$

a quadrupol error leads to a shift of the tune:

$$\Delta Q = \int_{s0}^{s0+L} \frac{\Delta K(s)\beta(s)ds}{4\pi} \approx \frac{\Delta K l_{quad}\bar{\beta}}{4\pi}$$

! *the tune shift is proportional to the β-function at the quadrupole*

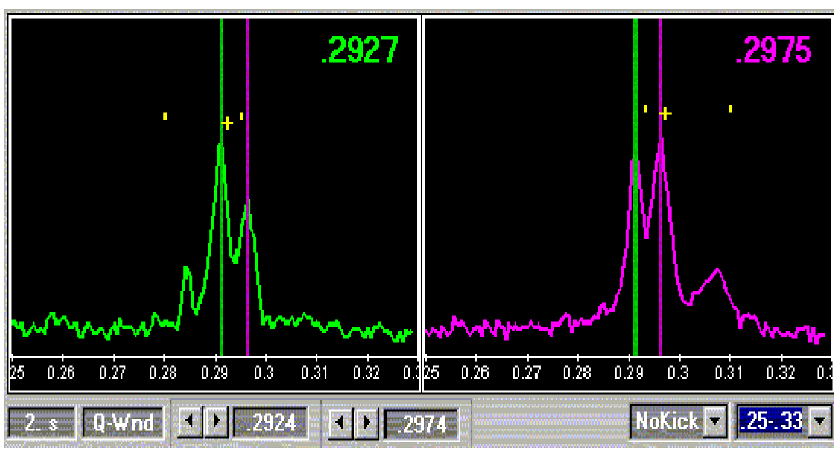
!! *field quality, power supply tolerances etc are much tighter at places where β is large*

!!! *mini beta quads: β ≈ 1900*

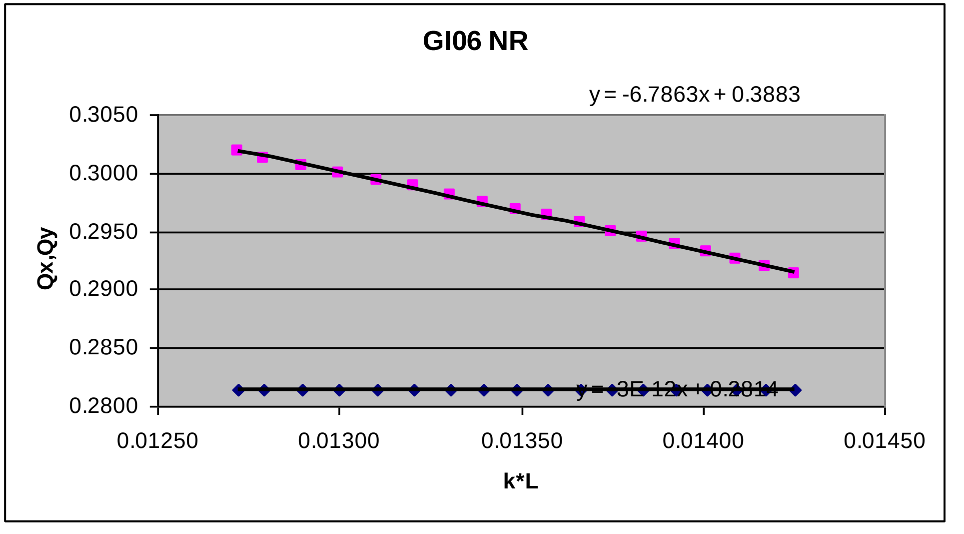
arc quads: β ≈ 80

!!!! *β is a measure for the sensitivity of the beam*

Example: measurement of β in a storage ring:



tune spectrum ...

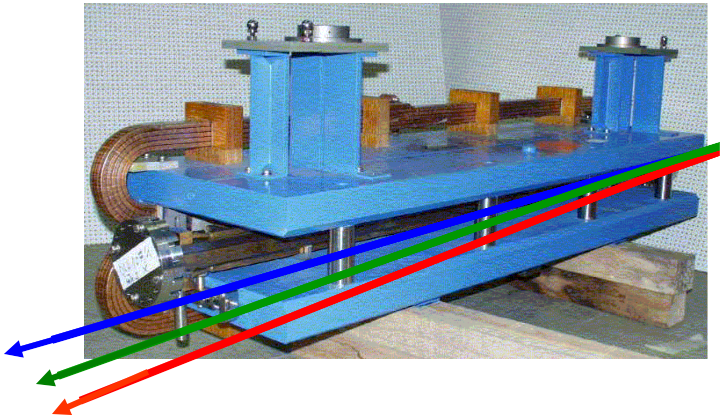


tune shift as a function of a gadient change

18.) Chromaticity:
A Quadrupole Error for $\Delta p/p \neq 0$

Influence of external fields on the beam: *prop. to magn. field & prop. zu 1/p*

dipole magnet $$\alpha = \frac{\int B\,dl}{p/e}$$



$$x_D(s) = D(s)\,\frac{\Delta p}{p}$$

focusing lens $$k = \frac{g}{p/e}$$

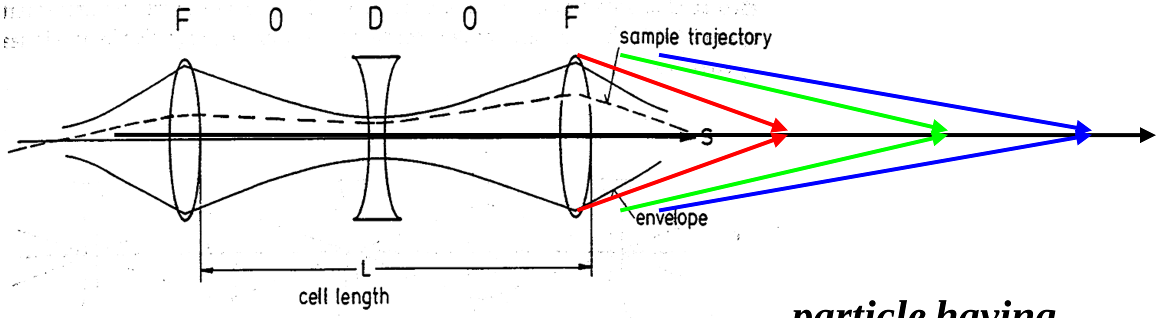


Figure 29: FODO cell

particle having ...
to high energy
to low energy
ideal energy

Chromaticity: Q'

$$k = \frac{g}{p/e} \qquad\qquad p = p_0 + \Delta p$$

in case of a momentum spread:

$$\boldsymbol{k} = \frac{\boldsymbol{eg}}{\boldsymbol{p}_0 + \Delta \boldsymbol{p}} \approx \frac{\boldsymbol{e}}{\boldsymbol{p}_0}\left(1 - \frac{\Delta \boldsymbol{p}}{\boldsymbol{p}_0}\right) \boldsymbol{g} = \boldsymbol{k}_0 + \Delta \boldsymbol{k}$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

… which acts like a quadrupole error in the machine and leads to a tune spread:

$$\Delta \boldsymbol{Q} = -\frac{1}{4\pi} \frac{\Delta \boldsymbol{p}}{\boldsymbol{p}_0} \boldsymbol{k}_0 \beta(\boldsymbol{s}) \boldsymbol{ds}$$

definition of chromaticity:

$$\Delta \boldsymbol{Q} = \boldsymbol{Q}' \frac{\Delta \boldsymbol{p}}{\boldsymbol{p}}$$

Problem: chromaticity is generated by the lattice itself !!

ξ is a number indicating the size of the tune spot in the working diagram,
ξ is always created if the beam is focussed
→ it is determined by the focusing strength k of all quadrupoles

$$Q' = \frac{-1}{4\pi} * \oint k(s)\beta(s)\, ds$$

k = quadrupole strength
β = betafunction indicates the beam size … and even more the sensitivity of the beam to external fields

Example: HERA

HERA-p: ***Q' = -70 … -80***
Δ p/p = 0.5 *10⁻³ → $\Delta p/p = 0.5 \cdot 10^{-3}$
Q = 0.257 … 0.337

→ Some particles get very close to resonances and are lost

Correction of Q':

1.) sort the particles acording to their momentum

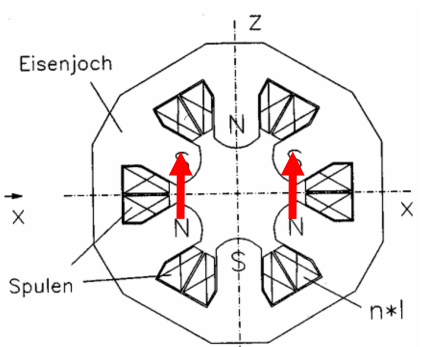
$$x_D(s) = D(s)\frac{\Delta p}{p}$$

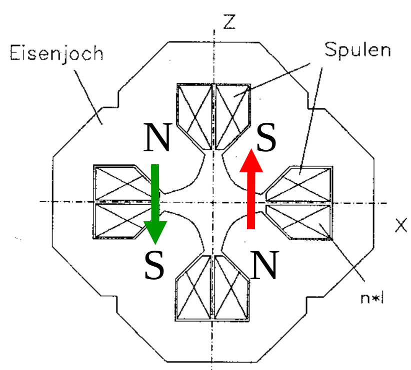
2.) apply a magnetic field that rises quadratically with x (sextupole field)

$$\left.\begin{aligned} B_x &= \tilde{g}xz \\ B_z &= \frac{1}{2}\tilde{g}(x^2 - z^2) \end{aligned}\right\} \qquad \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \tilde{g}x$$

linear rising „gradient“:

Sextupole Magnets:





k_1 normalised quadrupole strength

k_2 normalised sextupole strength

$$k_1(sext) = \frac{\tilde{g}x}{p/e} = k_2 * x$$

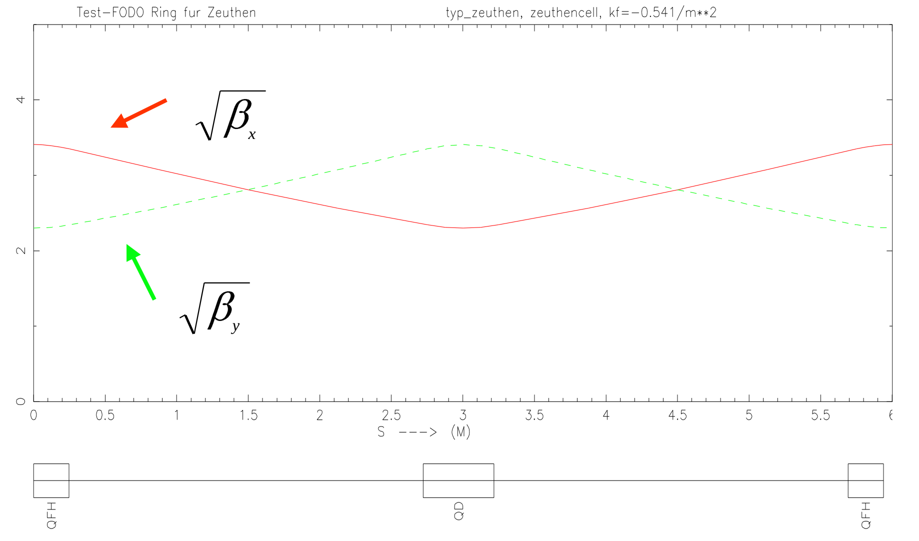
$$k_1(sext) = k_2 * D * \frac{\Delta p}{p}$$

corrected chromaticity:

$$Q'_x = \frac{-1}{4\pi} * \oint k_1(s)\beta(s)\,ds + \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F - \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

Chromaticity in the FoDo Lattice

$$Q' = \frac{-1}{4\pi} * \oint k(s)\beta(s)\, ds$$



β-Function in a FoDo

$$\hat{\beta} = \frac{(1+\sin\frac{\mu}{2})L}{\sin\mu}$$

$$\check{\beta} = \frac{(1-\sin\frac{\mu}{2})L}{\sin\mu}$$

$$Q' = \frac{-1}{4\pi} N * \frac{\hat{\beta}-\breve{\beta}}{f_Q}$$

$$Q' = \frac{-1}{4\pi} N * \frac{1}{f_Q} * \left\{ \frac{L(1+\sin\frac{\mu}{2}) - L(1-\sin\frac{\mu}{2})}{\sin\mu} \right\}$$

using some TLC transformations ... ξ can be expressed in a very simple form:

$$Q' = \frac{-1}{4\pi} N * \frac{1}{f_Q} * \frac{2L \sin\frac{\mu}{2}}{\sin\mu}$$

$$Q' = \frac{-1}{4\pi} N * \frac{1}{f_Q} * \frac{L \sin\frac{\mu}{2}}{\sin\frac{\mu}{2}\cos\frac{\mu}{2}}$$

remember ...

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$Q'_{cell} = \frac{-1}{4\pi\, f_Q} * \frac{L \tan\frac{\mu}{2}}{\sin\frac{\mu}{2}}$$

putting ...

$$\sin\frac{\mu}{2} = \frac{L}{4 f_Q}$$

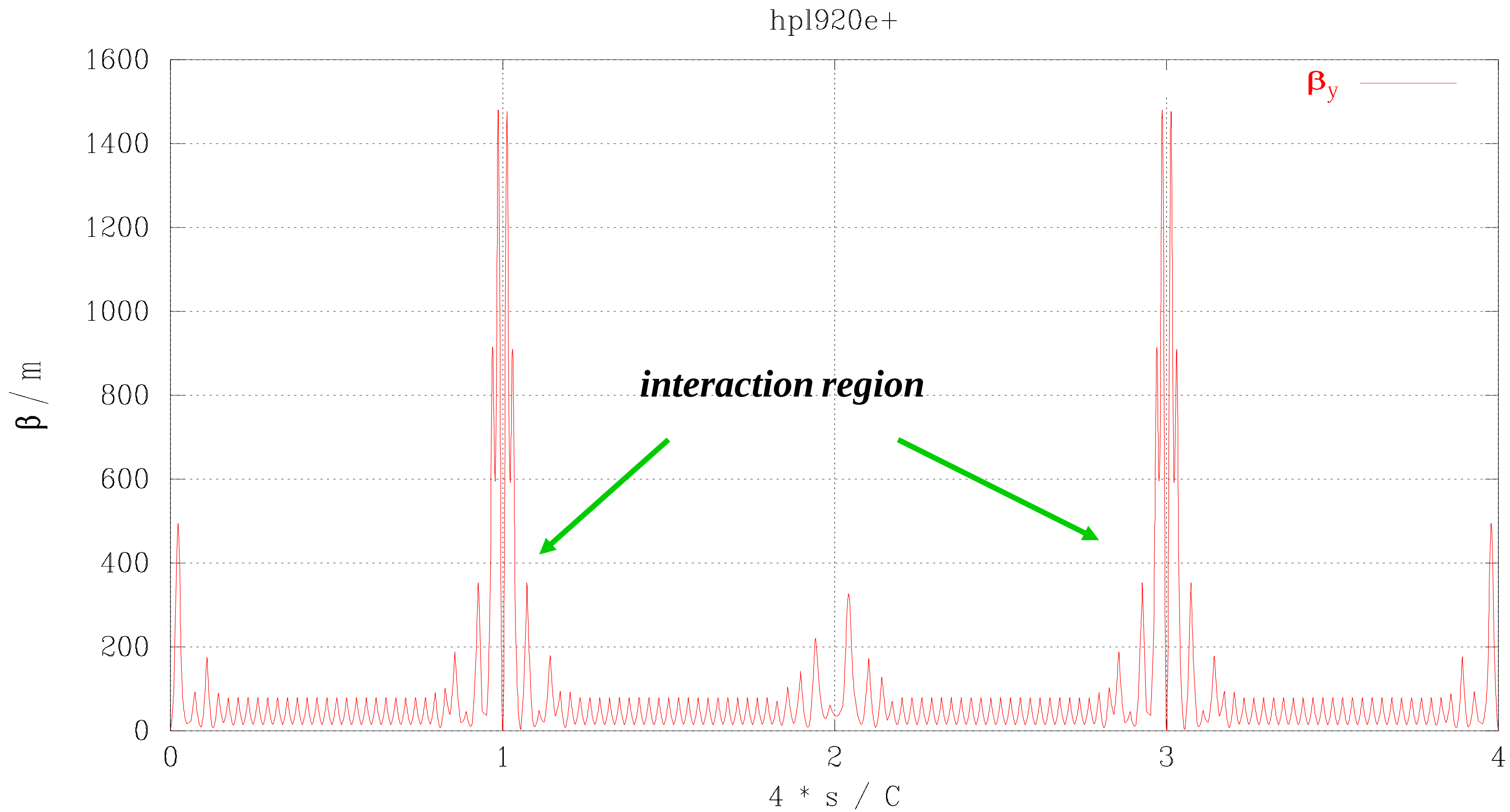
$$Q'_{cell} = \frac{-1}{\pi} * \tan\frac{\mu}{2}$$

contribution of one FoDo Cell to the chromaticity of the ring:

Chromaticity

$$Q' = -\frac{1}{4\pi}\oint K(s)\beta(s)ds$$

question: main contribution to ξ in a lattice … ?



19.) Resume´:

beam emittance: $$\varepsilon \propto \frac{1}{\beta\gamma}$$

beta function in a drift: $$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

… and for α = 0 $$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

particle trajectory for Δp/p ≠ 0
inhomogenious equation: $$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0}\frac{1}{\rho}$$

… and its solution: $$x(s) = x_\beta(s) + D(s)\cdot\frac{\Delta p}{p}$$

momentum compaction: $$\frac{\delta l_\varepsilon}{L} = \alpha_{cp}\frac{\Delta p}{p} \qquad \alpha_{cp} \approx \frac{2\pi}{L}\langle D\rangle \approx \frac{\langle D\rangle}{R}$$

quadrupole error: $$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta K(s)\beta(s)ds}{4\pi}$$

chromaticity: $$Q' = -\frac{1}{4\pi}\oint K(s)\beta(s)ds$$