Linear Imperfections

- sources for linear imperfections
- equations of motion with imperfections:
  smooth approximation
- perturbation treatment: driven oscillators and resonances
- transfer matrices with coupling: element and one-turn
- what we have left out (coupling)
- orbit correction for the un-coupled case

Sources for Linear Field Errors

- sources for linear imperfections:
  - -magnetic field errors: b_0, b_1, a_0, a_1
  - -powering errors for dipole and quadrupole magnets
  - -energy errors in the particles ➔ change in normalized strength
  - -roll errors for dipole and quadrupole magnets
  - -feed-down errors from quadrupole and sextupole magnets

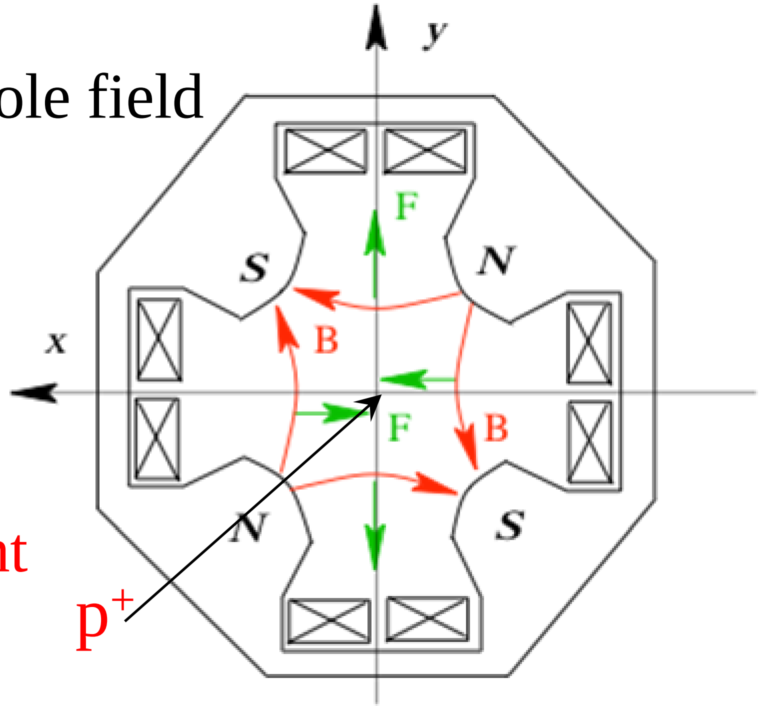
➔ example: feed down from a quadrupole field

$$x = \tilde{x} + \Delta x$$

$$B_y = -g \cdot (\tilde{x} + \Delta x)$$

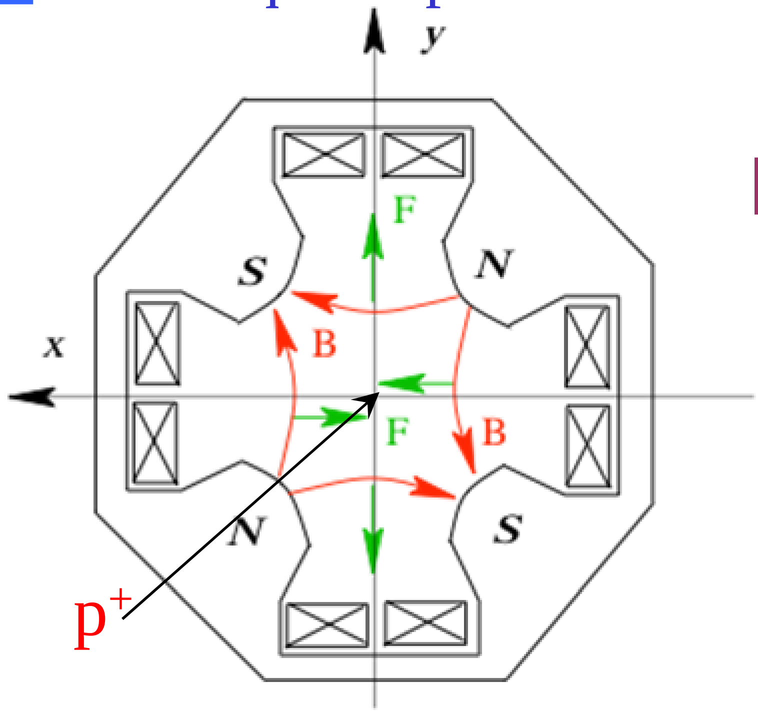
$$B_x = -g \cdot y$$

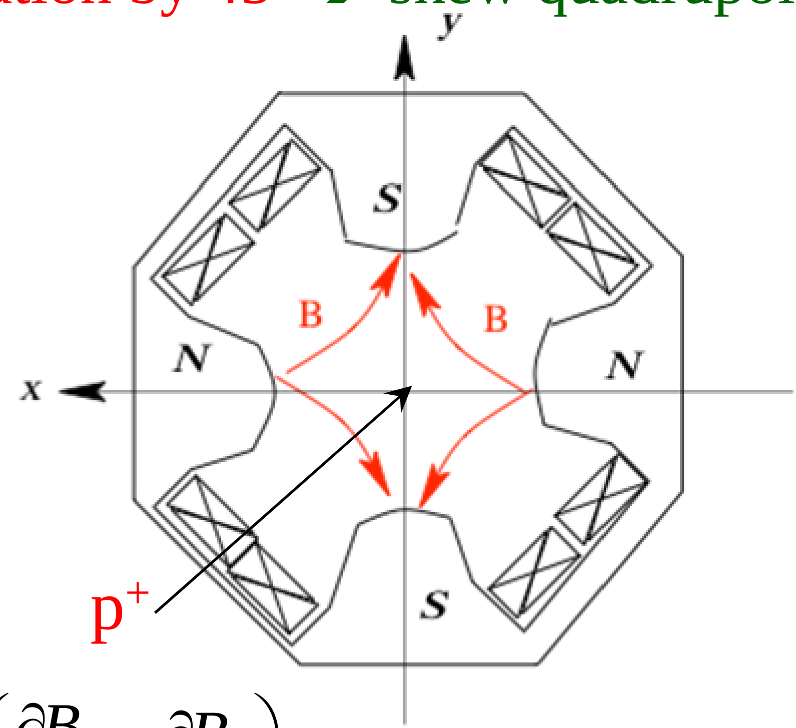
➔ dipole + quadrupole field component



Skew Multipoles: Example Skew Quadrupole

normal quadrupole: ➔ clockwise rotation by 45° ➔ skew quadrupole





$$g = \frac{\partial B_y}{\partial x}$$

$$N = \frac{1}{2}\left(\frac{\partial B_y}{\partial y} - \frac{\partial B_x}{\partial x}\right)$$

$$B_y = g \cdot x \Rightarrow F_x = -q \cdot v \cdot g \cdot x$$

$$B_x = g \cdot y \Rightarrow F_y = +q \cdot v \cdot g \cdot y$$

$$B_y = +N \cdot y \Rightarrow F_x = -q \cdot v \cdot N \cdot y$$

$$B_x = -N \cdot x \Rightarrow F_y = -q \cdot v \cdot N \cdot x$$

Sources for Linear Field Errors

- sources for feed down and roll errors:

-magnet positioning in the tunnel

transverse position → +/- 0.1 mm

roll error → +/- 0.5 mrad

-tunnel movements:

slow drifts

civilization

moon

seasons

civil engineering

-closed orbit errors → beam offset inside magnetic elements

-energy error: → dispersion orbit

Equation of Motion I

- Smooth approximation for Hills equation: $w = x, y$

$$\frac{d^2}{ds^2} w(s) + K(s) \cdot w(s) = 0 \xrightarrow{K(s) = \text{const}} \frac{d^2}{ds^2} w(s) + \omega_0^{\,2} \cdot w(s) = 0$$

(constant β-function and phase advance along the storage ring)

$$\longrightarrow \quad w(s) = A \cdot \sin(\omega_0 \cdot s + \phi_0) \qquad \omega_0 = 2\pi \cdot Q_0 / L$$

(Q is the number of oscillations during one revolution)

- perturbation of Hills equation:

$$\frac{d^2}{ds^2} w(s) + \omega_0^{\,2} \cdot w(s) = F(x(s), y(s), s) / (v \cdot p)$$

in the following the force term will be the Lorenz force of a charged particle in a magnetic field: $F = q \cdot \vec{v} \times \vec{B}$

Equation of Motion I

- perturbation for dipole field errors:

$$\frac{F}{v\cdot p} = -\frac{\Delta B_y}{p}$$

- perturbation for quadrupole field errors:

$$\frac{F_x}{v\cdot p} = -\frac{\Delta g}{p}\cdot x$$

$$\frac{F_y}{v\cdot p} = +\frac{\Delta g}{p}\cdot y$$

- normalized multipole gradients:

$$k_0 = 0.3\cdot\frac{\Delta B[T]}{p[GeV/c]}$$

$$k_1 = 0.3\cdot\frac{g[T/m]}{p[GeV/c]}$$

$$\kappa_1 = 0.3\cdot\frac{N[T/m]}{p[GeV/c]}$$

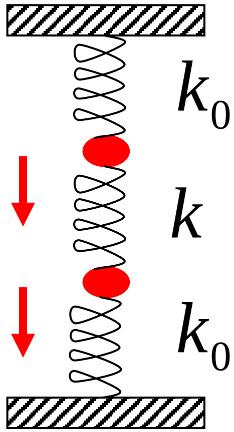
- perturbation of Hills equation: $\frac{d^2}{ds^2}x(s) + \omega_0{}^2\cdot x(s) = \begin{cases} -k_0 \\ -k_1\cdot x(s) \\ -\kappa_1\cdot y(s) \end{cases}$

Coupling I: Identical Coupled Oscillators

- fundamental modes for identical coupled oscillators:

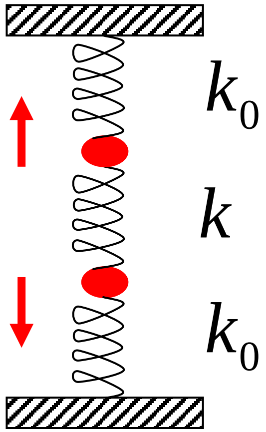
ω mode:

$$q_1(t) = x + y$$



π mode:

$$q_2(t) = x - y$$



$$\frac{d^2}{dt^2} q_1(t) + \omega_\omega^2 \cdot q_1(t) = 0$$

$$\frac{d^2}{dt^2} q_2(t) + \omega_\pi^2 \cdot q_2(t) = 0$$

$$\omega_\omega = \sqrt{k_0}$$

$$\omega_\pi = \sqrt{k_0 + 2k}$$

- weak coupling ($k \ll k_0$): ➔ degenerate mode frequencies

➔ description of motion in unperturbed 'x' and 'y' coordinates

Coupling II: Equation of Motion in Accelerator

- distributed coupling:

$$\frac{d^2}{ds^2}x(s)+\omega_x{}^2\cdot x(s)=-\kappa_1\cdot y(s)$$

$$\frac{d^2}{ds^2}y(s)+\omega_y{}^2\cdot y(s)=-\kappa_1\cdot x(s)$$

- solution by decomposition into 'Eigenmodes':

$$q_1(s)=a\cdot x+b\cdot y \qquad q_2(s)=c\cdot x+d\cdot y$$

With orthogonal condition: $a\cdot c+b\cdot d=0$

➔ $$\frac{d^2}{dt^2}q_1(s)+\omega_1^2\cdot q_1(s)=0 \qquad \frac{d^2}{dt^2}q_2(s)+\omega_2^2\cdot q_2(s)=0$$

Coupling II: Equation of Motion in Accelerator

- take second derivative of q_1 and q_2:

  ➔ expressions for ω_1 and ω_2 as functions of a, b, c, d, ω_x, ω_y

- use Orthogonal condition for calculating a,b,c,d (set b=1=d)

$$a = \frac{\omega_x^2 - \omega_y^2}{2\kappa_1} + \sqrt{1 + \left(\frac{\omega_x^2 - \omega_y^2}{2\kappa_1}\right)^2} \; ; c = \frac{\omega_x^2 - \omega_y^2}{2\kappa_1} - \sqrt{1 + \left(\frac{\omega_x^2 - \omega_y^2}{2\kappa_1}\right)^2}$$

yields: $\frac{d^2}{dt^2} q_1(s) + \omega_1^2 \cdot q_1(s) = 0$ $\qquad \frac{d^2}{dt^2} q_2(s) + \omega_2^2 \cdot q_2(s) = 0$

➔ with:

$$\omega_{1,2}^2 = \tfrac{1}{2} \cdot \left(\omega_x^2 + \omega_y^2\right) \pm \Omega$$

$$\Omega = \sqrt{\kappa_1^2 + \left(\frac{\omega_x^2 - \omega_y^2}{2}\right)^2}$$

Coupled Oscillators Case Study: Case 1

very different unperturbed frequencies: $\left(\frac{\omega_x^2-\omega_y^2}{2\kappa_1}\right)^2 >> 1$

$$\omega_{1,2}^2 = \tfrac{1}{2}\cdot\left(\omega_x^2+\omega_y^2\right) \pm \tfrac{1}{2}\cdot\left(\omega_x^2-\omega_y^2\right)\cdot\sqrt{\left(\frac{2\kappa_1}{\left(\omega_x^2-\omega_y^2\right)}\right)^2+1}$$

expansion of the square root: $\sqrt{1+\varepsilon} \approx 1+\tfrac{1}{2}\varepsilon$

➔ $\omega_1 = \omega_x + \frac{\kappa_1^2}{\omega_x^2-\omega_y^2} \approx \omega_x$ $\qquad$ $\omega_2 = \omega_y - \frac{\kappa_1^2}{\omega_x^2-\omega_y^2} \approx \omega_y$

➔ 'nearly' uncoupled oscillators $a \approx 1; b = 1; c \approx -1; d = 1$

Coupled Oscillators Case Study: Case 2

almost equal frequencies: $\omega_x = \omega_0 + \frac{1}{2}\Delta \qquad \omega_y = \omega_0 - \frac{1}{2}\Delta$

➔ keep only linear terms in Δ:

➔ $\omega_0 = \frac{1}{2}(\omega_x + \omega_y) \qquad \omega_x^2 + \omega_y^2 \approx 2\omega_0^2 \qquad \omega_x^2 - \omega_y^2 \approx 2\omega_0\Delta$

➔ $$\omega_{1,2}^2 = \omega_0^2 \pm \sqrt{\kappa_1^2 + \Delta^2 \cdot \omega_0^2}$$

$$\omega_{1,2} = \omega_0 \cdot \sqrt{1 \pm \sqrt{\frac{\kappa_1^2}{\omega_0^4} + \frac{\Delta^2}{\omega_0^2}}}$$

expansion of the square root for small coupling and Δ:

➔ $$\omega_{1,2} = \omega_0 \pm \tilde{\Omega}$$

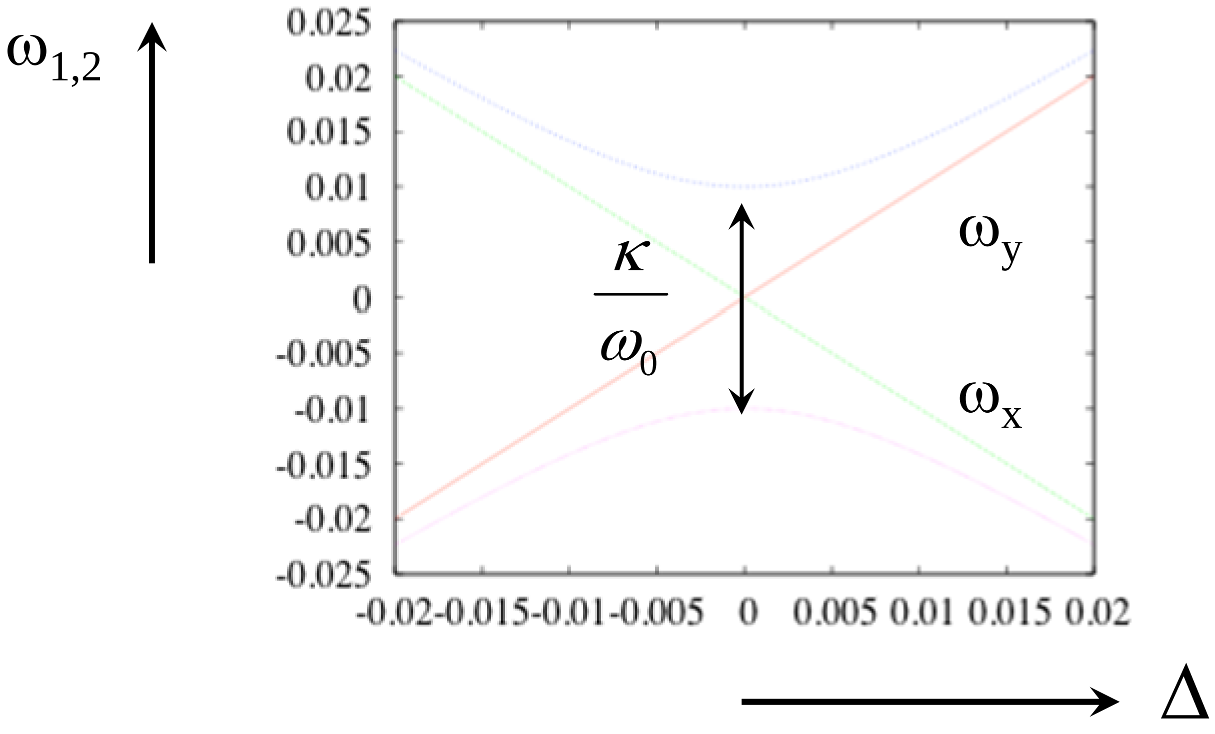
with: $$\tilde{\Omega} = \frac{1}{2} \cdot \sqrt{\frac{\kappa_1^2}{\omega_0^2} + \Delta^2}$$

Coupled Oscillators Case Study: Case 2

- measurement of coupling strength:

$$\omega_{1,2} = \omega_0 \pm \frac{1}{2} \cdot \sqrt{\frac{\kappa_1^2}{\omega_0^2} + \Delta^2}$$

measure the difference in the Eigenmode frequencies while bringing the unperturbed tunes together:



➔ the minimum separation yields the coupling strength!!

Coupled Oscillators Case Study: Case 2

- initial oscillation only in horizontal plane:

$$x(0) = A; \quad x'(0) = 0; \quad y(0) = 0; \quad y'(0) = 0$$

➔ $q_1 = A \cdot \sin(\omega_1 \cdot s + \phi_1)$ and $q_2 = A \cdot \sin(\omega_2 \cdot s + \phi_2)$

with $\omega_{1,2} = \frac{1}{2} \cdot \left(\omega_x + \omega_y\right) \pm \tilde{\Omega}$ and $q_1(t) = x - y$, $q_2(t) = x + y$

sum rules for sin and cos functions:

➔

$$x(s) = A \cdot \cos\left(\tilde{\Omega} \cdot s\right) \cdot \cos\left(\tfrac{1}{2}\left[\omega_1 + \omega_2\right] \cdot s + \tfrac{1}{2}[\phi_1 + \phi_2]\right)$$

$$y(s) = -A \cdot \sin\left(\tilde{\Omega} \cdot s\right) \cdot \sin\left(\tfrac{1}{2}\left[\omega_1 + \omega_2\right] \cdot s + \tfrac{1}{2}[\phi_1 + \phi_2]\right)$$

➔ modulation of the amplitudes

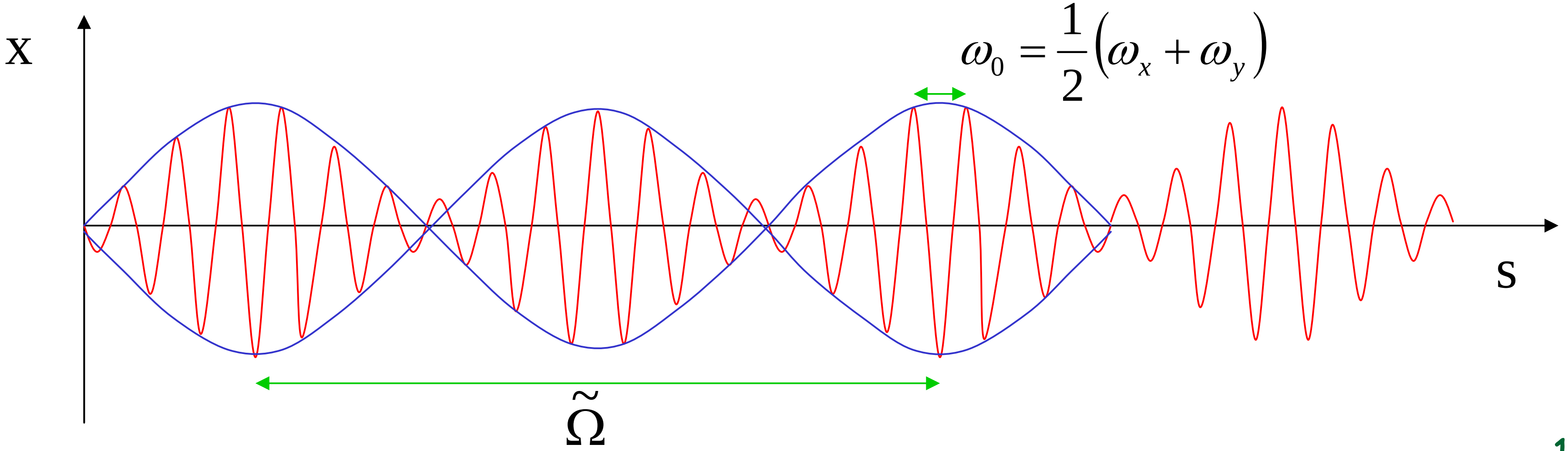
Beating of the Transverse Motion: Case I

- two almost identical harmonic oscillators with weak coupling:

  π-mode and ω=mode frequencies are approximately identical!

  → frequencies can not be distinguished and energy can be exchanged between the two oscillators

- modulation of the oscillation amplitude:



Driven Oscillators

- Perturbation treatment:

substitute the solutions of the homogeneous equation of motion:

$$w(s) = A \cdot \sin(\omega_0 \cdot s + \phi_0)$$

into the right-hand side of the perturbed Hills equation and express the ‘s’ dependence of the multipole terms by their Fourier series (the perturbations must be periodic with one revolution!)

- equation of motion ➔ driven un-damped oscillators:

$$\frac{d^2}{ds^2} w(s) + \omega_w Q^{-1} \frac{d}{ds} w(s) + \omega_w{}^2 w(s) = \sum_{k.l.m} W_{klm} e^{(k \cdot \omega_x \cdot s + l \cdot \omega_y + \frac{2\pi}{L} \cdot m \cdot s + \phi_{klm})}$$

➔ large number of driving frequencies!

Driven Oscillators

- single resonance approximation: $\omega = k\omega_x + l\omega_y + m\frac{2\pi}{L}$

  consider only one perturbation frequency (choose $\omega \approx \omega_0$):

$$\frac{d^2}{ds^2} w(s) + \omega_0 \cdot Q^{-1} \cdot \frac{d}{ds} w(s) + \omega_0{}^2 \cdot w(s) = W(s) \cdot \cos(\omega \cdot s + \phi_0)$$

- general solution: $w(s) = w_{tr}(s) + w_{st}(s)$

- without damping the transient solution is just the HO solution

$$w_{tr}(s) = a \cdot \sin(\omega_0 \cdot s + \phi_0)$$

Driven Oscillators

stationary solution: $$w_{st}(s) = \frac{W(\omega)}{\omega_0^2} \cdot \cos[\omega \cdot s - \alpha(\omega)]$$

where 'ω' is the driving angular frequency!
and W(ω) can become large for certain frequencies!

$$W(\omega) = W_n \cdot \frac{1}{\sqrt{1 - \left(\frac{\omega_n}{\omega_0}\right)^2 + \left(\frac{\omega_n}{Q\omega_0}\right)^2}}$$

resonance condition:

$$\omega_n = \omega_0$$

➔ justification for single resonance approximation:

➔ all perturbation terms with: $\omega_n \neq \omega_0$ de-phase with the transient

➔ no net energy transfer from perturbation to oscillation (averaging)!

Resonances and Perturbation Treatment

example single dipole perturbation: $\frac{F(s)}{v \cdot p} = -k_0 \cdot \delta_L(s - s_0)$

$$\frac{d^2}{ds^2} w(s) + \omega_0{}^2 \cdot w(s) = -l k_0 \cdot \frac{1}{L} \cdot \sum_{n=-\infty}^{\infty} \cos(n \cdot 2\pi \cdot s / L)$$

Fourier series of periodic δ-function

resonance condition: $\omega_0 = n \cdot 2\pi / L \xrightarrow{\omega_0 = 2\pi \cdot Q_0 / L} Q_0 = n$

avoid integer tunes!

$$\Delta CO(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi Q)} \cdot \oint \Delta k_0(t) \cdot \sqrt{\beta(t)} \cdot \cos(|\phi(t) - \phi(s)| - \pi Q) dt$$

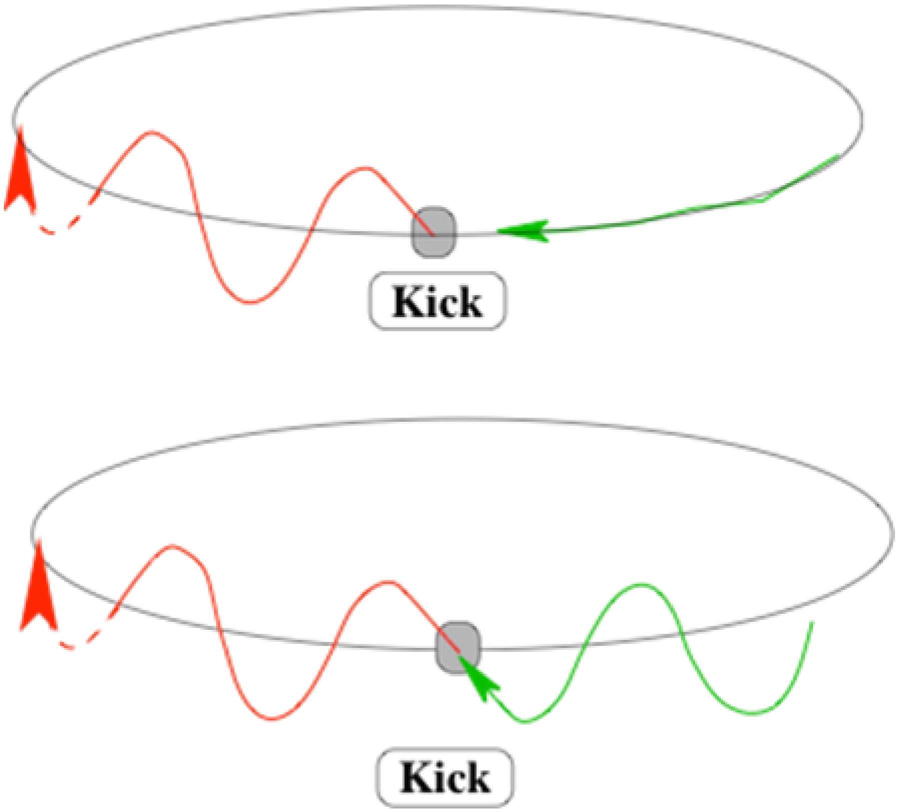
(see general CAS school for more details)

Resonances and Perturbation Treatment

integer resonance for dipole perturbations:

assume:

Q = integer



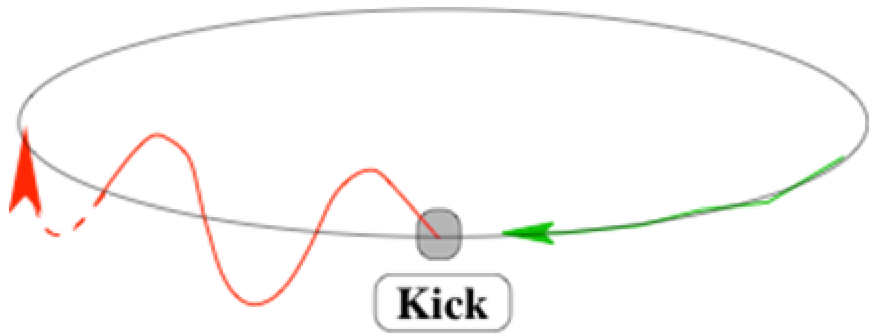
➔ dipole perturbations add up on consecutive turns! ➔ Instability

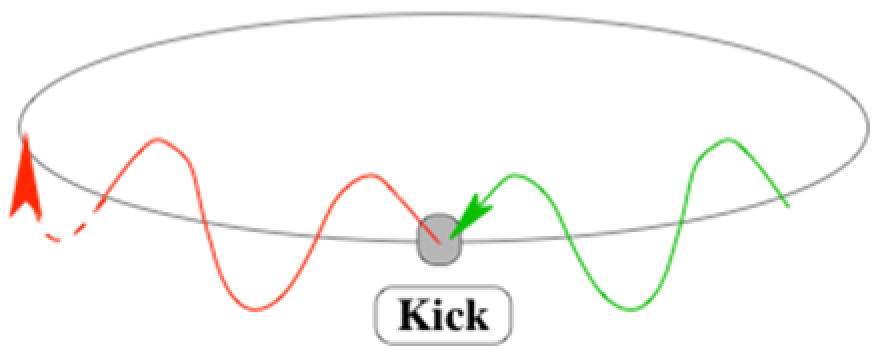
Resonances and Perturbation Treatment

- integer resonance for dipole perturbations:

assume:

Q = integer/2





➔dipole perturbations compensate on consecutive turns!

➔ stability

Resonances and Perturbation Treatment

example single quadrupole perturbation:

with: $$\frac{F(s)}{v \cdot p} = -k_1 \cdot x \qquad w_0(s) = A \cdot \cos(\omega_{0,x} \cdot s + \phi_0)$$

→ $$\frac{d^2}{ds^2} w(s) + \omega_{x,0}{}^2 \cdot w(s) = -A \cdot \frac{lk_1}{2L} \sum_{n=-\infty}^{\infty} \cos([2\pi \cdot n / L \pm \omega_{0,x}] \cdot s \pm \phi_0)$$

→ resonance condition: $$2 \cdot \omega_0 = n \cdot 2\pi / L \xrightarrow{\omega_0 = 2\pi \cdot Q_0 / L} Q_0 = \frac{n}{2}$$

→ avoid half integer tunes!

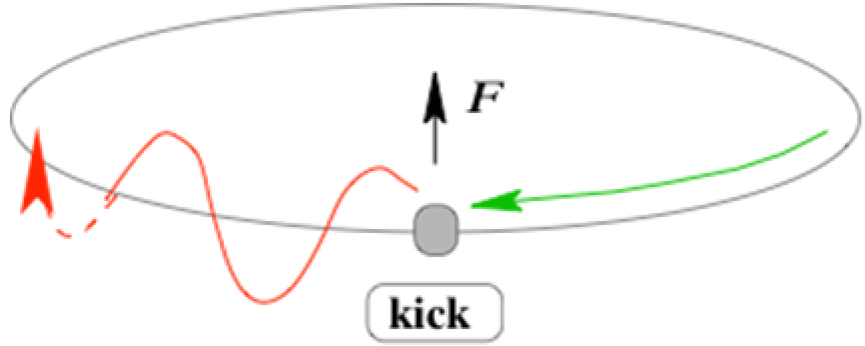
$$\frac{\Delta\beta(s)}{\beta_0(s)} = \frac{1}{2\sin(2\pi Q)} \cdot \oint \Delta k_1(t) \cdot \beta(t) \cdot \cos(2\,|\phi(t) - \phi(s)| - 2\pi Q) dt$$

→ (see general CAS school for more details)

Resonances and Perturbation Treatment

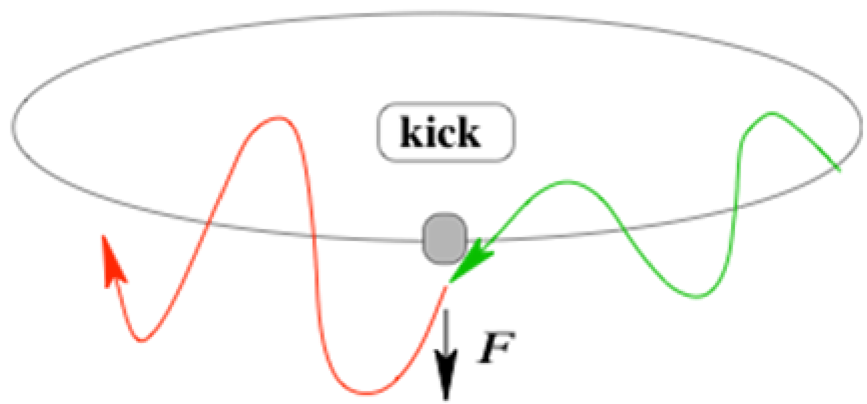
half integer resonance for quadrupole perturbations:

assume:
Q = integer + 0.5



feed down error:

$$B_x = b_1 \cdot y \Rightarrow F_y = +q \cdot v \cdot b_1 \cdot y$$



➔quadrupole perturbations add up on consecutive turns!
➔ Instability

Resonances and Perturbation Treatment

- example single skew quadrupole perturbation:

with: $$\frac{F_x(s)}{v \cdot p} = -\kappa_1 \cdot y \qquad y_0(s) = A \cdot \cos(\omega_{0,y} \cdot s + \phi_0)$$

→ $$\frac{d^2}{ds^2} x(s) + \omega_{x,0}{}^2 \cdot x(s) = -A \cdot \frac{l\kappa_1}{2L} \sum_{n=-\infty}^{\infty} \cos([2\pi \cdot n / L \pm \omega_{0,y}] \cdot s \pm \phi_0)$$

resonance condition:

→ $$\omega_{x,0} \pm \omega_{y,0} = n \cdot 2\pi / L \xrightarrow{\omega = 2\pi \cdot Q / L} Q_x \pm Q_y = n$$

→ **avoid sum and difference resonances!**

→ difference resonance ➔ stable with energy exchange
sum resonance ➔ instability as for externally driven dipole

Resonances and Perturbation Treatment: Case 1

coupling with: $Q_x >> Q_y \quad or \quad Q_x << Q_y$

➔ drive and response oscillation de-phase quickly
no energy transfer between motion in ‘x’ and ‘y’ plane

➔ small amplitude of ‘stationary’ solution: $W(\omega) = W_0 \cdot \frac{1}{\sqrt{[1-(\frac{\omega}{\omega_0})^2]^2 + (\frac{\omega}{Q\omega_0})^2}}$

➔ no damping of oscillation in ‘x’ plane due to coupling

➔ coupling is weak ➔ tune measurement in one plane will show both tunes in both planes but with unequal amplitudes

➔ tune measurement is possible for both planes

Resonances and Perturbation Treatment: Case 2

coupling with: $Q_x \approx Q_y$

➔ drive and response oscillation remain in phase and energy can be exchanged between motion in 'x' and 'y' plane:

➔ large amplitude of 'stationary' solution: $W(\omega) = W_0 \cdot \frac{1}{\sqrt{[1-(\frac{\omega}{\omega_0})^2]^2 + (\frac{\omega}{Q\omega_0})^2}}$

➔ damping of oscillation in 'x' plane and growth of oscillation amplitude in 'y' plane

➔ 'x' and 'y' motion exchange role of driving force!

➔ each plane oscillates on average with: $\frac{1}{2}\left(Q_x + Q_y\right)$

➔ Impossible to separate tune in 'x' and 'y' plane!

Exact Solution for Transport in Skew Quadrupole

- coupled equation of motion: $x'' + \kappa_1 \cdot y = 0$ and $y'' + \kappa_1 \cdot x = 0$

- can be solved by linear combinations of 'x' and 'y':

$$(x+y)'' + \kappa_1 \cdot (x+y) = 0 \quad \text{and} \quad (x-y)'' - \kappa_1 \cdot (x-y) = 0$$

➔ solution as for focusing and defocusing quadrupole

- transport matrix for 'x-y' and 'x+y' coordinates for $\kappa_1 > 0$:

$$\begin{pmatrix} x-y \\ x'-y' \end{pmatrix}_{end} = \begin{pmatrix} \cos(l\sqrt{\kappa_1}) & \dfrac{\sin(l\sqrt{\kappa_1})}{\sqrt{\kappa_1}} \\ \sqrt{\kappa_1}\cdot\sin(l\sqrt{\kappa_1}) & \cos(l\sqrt{\kappa_1}) \end{pmatrix} \cdot \begin{pmatrix} x-y \\ x'-y' \end{pmatrix}_{ini}$$

$$\begin{pmatrix} x+y \\ x'+y' \end{pmatrix}_{end} = \begin{pmatrix} \cosh(l\sqrt{\kappa_1}) & \dfrac{\sinh(l\sqrt{\kappa_1})}{\sqrt{\kappa_1}} \\ \sqrt{\kappa_1}\cdot\sinh(l\sqrt{\kappa_1}) & \cosh(l\sqrt{\kappa_1}) \end{pmatrix} \cdot \begin{pmatrix} x+y \\ x'+y' \end{pmatrix}_{ini}$$

Transport Map with Coupling

- transport map for skew quadrupole: $\vec{z}_{end} = \underline{M}_{sq} \cdot \vec{z}_{ini}$

with: $\vec{z} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}$ and $\underline{M}_{sq} = \begin{pmatrix} a & b & c & d \\ -\kappa_1 d & a & -\kappa_1 b & c \\ c & d & a & b \\ -\kappa_1 b & c & -\kappa_1 d & a \end{pmatrix}$

- transport map for linear elements without coupling:

$\vec{z}_{end} = \underline{M}_{l} \cdot \vec{z}_{ini}$ with $\underline{M}_{l} = \begin{pmatrix} m_{11} & m_{12} & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{43} & m_{44} \end{pmatrix}$

Transport Map with Coupling

- coefficients for the transport map for skew quadrupole:

with:

$$a = [\cos(\sqrt{N}s) + \cosh(\sqrt{N}s)]/2$$
$$b = [\sin(\sqrt{N}s) + \sinh(\sqrt{N}s)]/2\sqrt{N}$$
$$c = [\cos(\sqrt{N}s) - \cosh(\sqrt{N}s)]/2$$
$$d = [\sin(\sqrt{N}s) - \sinh(\sqrt{N}s)]/2\sqrt{N}$$

One-Turn Map with Coupling

- one-turn map around the whole ring:

$$\vec{z}(s_0 + L) = \underline{T}(s_0) \cdot \vec{z}(s_0)$$ with: $$\underline{T} = \prod_i \underline{M}_i$$

- notation:

$$\underline{T} = \begin{pmatrix} \underline{M} & \underline{n} \\ \underline{m} & \underline{N} \end{pmatrix}$$ with: $\underline{M}, \underline{N}, \underline{m}, \underline{n}$ being 2x2 matrices

➔ 16 parameters in total

- $\underline{\text{T}}$ is a symplectic 4x4 matrix

$${}^t\underline{T} \cdot \underline{S} \cdot \underline{T} = \underline{S}$$ with: $$\underline{S} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

determines n*(n-1)/2 parameters for a n x n matrix

Parametrization of One-Turn Map with Coupling

- uncoupled system: parameterization by Courant-Snyder variables

$\underline{T}$ is a 2 x 2 matrix ➔ 4 parameters

$\underline{T}$ is symplectic ➔ determines 1 parameter

➔ 3 independent parameters

$$\underline{T} = \underline{I} \cdot \cos(\mu) + \underline{J} \cdot \sin(\mu)$$

with:

$$\underline{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \underline{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \qquad \gamma = \frac{1+\alpha^2}{\beta}$$

Parametrization of One-Turn Map with Coupling

rotated coordinate system:

➔ using a linear combination of the horizontal and vertical position vectors the matrix can be put in 'symplectic rotation' form

$$\underline{T} = \begin{pmatrix} \underline{I}\cos(\phi) & \underline{D}^{-1}\sin(\phi) \\ -\underline{D}\sin(\phi) & \underline{I}\cos(\phi) \end{pmatrix} \cdot \begin{pmatrix} \underline{A}_1 & \underline{0} \\ \underline{0} & \underline{A}_2 \end{pmatrix} \cdot \begin{pmatrix} \underline{I}\cos(\phi) & \underline{-D}^{-1}\sin(\phi) \\ \underline{D}\sin(\phi) & \underline{I}\cos(\phi) \end{pmatrix}$$

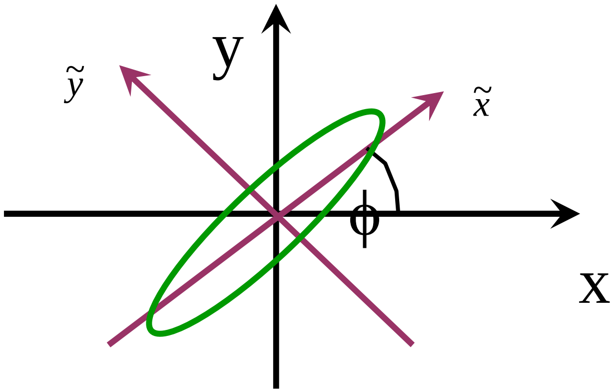
or: $\underline{T} = \underline{R} \cdot \underline{U} \cdot \underline{R}^{-1}$ with: $\underline{A}_i = \underline{I}_i \cdot \cos(\mu_i) + \underline{J}_i \cdot \sin(\mu_i); i = 1{,}2$

$\underline{D}$ is a symplectic 2x2 matrice ➔ 3 independent parameters

➔ total of 10 independent parameters for the One-Turn map

One-Turn Map with Coupling

- rotated coordinate system:



$$\tan(2\phi) = \frac{-\sqrt{\det\left(m + n^{+}\right)}}{\frac{1}{2} Tr\,(M - N)}$$

with:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{+} \xrightarrow{adjoint-matrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- rotated coordinate system:

➔ new Twiss functions and phase advances for the rotated coordinates

$$\underline{A}_i = \underline{I} \cdot \cos(\mu_i) + \underline{J}_i \cdot \sin(\mu_i) \qquad \underline{J}_i = \begin{pmatrix} \alpha_i & \beta_i \\ -\gamma_i & -\alpha_i \end{pmatrix}$$

$$\cos(\mu_1) - \cos(\mu_2) = \left[\tfrac{1}{2} Tr\left(M - N\right)\right]^2 + \det(m + n^{+})$$

Summary One-Turn Map with Coupling

- coupling changes the Twiss functions and tune values in the horizontal and vertical planes

➔ a global coupling correction is required for a restoration of the uncoupled tune values (can not be done by QF and QD adjustments)

- coupling changes the orientation of the beam ellipse along the ring

➔ a local coupling correction is required for a restoration of the uncoupled oscillation planes

(➔ mixing of horizontal and vertical kicker elements and correction dipoles)

What We Have Left Out

- β-beat:

  skew quadrupole perturbations generate β-beat similar to normal quadrupole perturbations

- dispersion beat:

  skew quadrupole perturbations generate vertical dispersion

- integer tune split and super symmetry

  the (1,-1) coupling resonance in storage rings with super symmetry can be strongly suppressed by an integer tune split

- general definition of the coupling coefficients:

$$\kappa = \frac{\kappa_1}{\omega} \xrightarrow{\omega = 1/\beta} \qquad \kappa_{r,\pm} = \frac{1}{2\pi} \cdot \oint \kappa_1(s) \cdot \sqrt{\beta_x(s)\beta_y(s)} \cdot e^{i(\phi_x(s) \pm \phi_y(s) + \frac{2\pi}{L} r \cdot s)} ds$$

Orbit Correction

- deflection angle:

$$\theta_i = -\frac{0.3 \cdot \Delta B_y[T] \cdot l}{p[GeV/c]} = \Delta X'(s_i)$$

- trajectory response:

$$\Delta Z(s) = \sqrt{\beta_i \cdot \beta(s)} \cdot \theta_i \cdot \sin(\phi(s) - \phi_i)$$

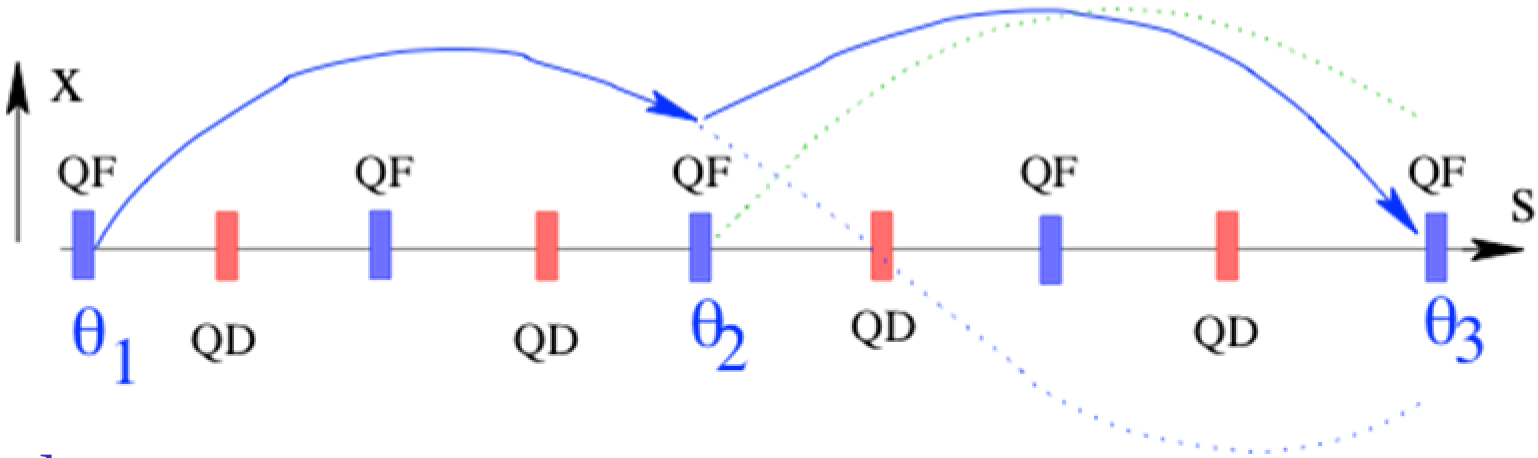
$$\Delta Z'(s) = \sqrt{\beta_i / \beta(s)} \cdot \theta_i \cdot \cos(\phi(s) - \phi_i)$$

- closed orbit bump

compensate the trajectory response with additional dipole fields further down-stream ➔ 'closure' of the perturbation within one turn

Orbit Correction

- 3 corrector bump:



- closure

$$\theta_2 = -\frac{\sqrt{\beta_1}}{\sqrt{\beta_2}} \cdot \frac{\sin(\Delta\phi_{3-1})}{\sin(\Delta\phi_{3-2})} \cdot \theta_1 \qquad \theta_3 = \left(\frac{\sin(\Delta\phi_{3-1})}{\tan(\Delta\phi_{3-2})} - \cos(\Delta\phi_{3-1}) \right) \cdot \frac{\sqrt{\beta_1}}{\sqrt{\beta_2}} \cdot \theta_1$$

- limits

sensitive to BPM errors; large number of correctors

SVD Algorithm I

- linear relation between corrector setting and BPM reading:

$$\overrightarrow{COR} = (c_1, c_2, ..., c_m)$$ ➔ vector of corrector strengths

$$\overrightarrow{BPM} = (b_1, b_2, ..., b_n)$$ ➔ vector of all BPM data

$$\overrightarrow{BPM} = \underline{A} \cdot \overrightarrow{COR}$$ $\underline{A}$ being a n x m matrix

- global correction: $$\overrightarrow{COR} = \underline{A}^{-1} \cdot \overrightarrow{BPM}$$

problem ➔ $\underline{A}$ is normally not invertible
(it is normally not even a square matrix)!

solution ➔ minimize the norm: $$\left\| \overrightarrow{BPM} - \underline{A} \cdot \overrightarrow{COR} \right\|$$

SVD Algorithm II

- solution:

  ➔ find a matrix $\underline{B}$ such that $$\left\| \overrightarrow{BPM} - \underline{A} \cdot \underline{B} \cdot \overrightarrow{COR} \right\|$$

  attains a minimum with $\underline{B}$ being a m x n matrix and:

$$\|x\| = \left(\sum_{i}^{m} |x_i|^p \right)^{1/p}$$

- singular value decomposition (SVD):

  any matrix can be written as: $$\underline{A} = \underline{O}_1 \cdot \underline{D} \cdot \underline{O}_2$$

  where $\underline{O}_1$ and $\underline{O}_2$ are orthogonal matrices and $\underline{D}$ is diagonal

$$\underline{O}^{-1} = \underline{O}^{t}$$

SVD Algorithm III

- diagonal form:

$$\underline{D} = \begin{pmatrix} \sigma_{11} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \sigma_{22} & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots & 0 \\ 0 & \cdots & 0 & \sigma_{kk} & 0 & \cdots & 0 \end{pmatrix} \qquad k \le \min(n,m)$$

- define a pseudo inverse matrix:

$$\hat{\underline{D}} = \begin{pmatrix} 1/\sigma_{11} & 0 & 0 & 0 \\ 0 & 1/\sigma_{11} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1/\sigma_{11} \\ 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \quad \Rightarrow \quad \underline{D} \cdot \hat{\underline{D}} = \underline{1}_k = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$\underline{1}_k$ being the k x k unit matrix

SVD Algorithm IV

correction matrix:

- define the 'correction' matrix: $\underline{B} = \underline{O}_2^t \cdot \underline{\widehat{D}} \cdot \underline{O}_1^t$

- $$\underline{A} \cdot \underline{B} = \left(\underline{O}_1 \cdot \underline{D} \cdot \underline{O}_2\right) \cdot \left(\underline{O}_2^t \cdot \underline{\widehat{D}} \cdot \underline{O}_1^t\right) = \underline{1}_k$$

main properties:

- SVD allows you to adjust k corrector magnets $k \leq \min(n, m)$
- if k = m = n one obtains a zero orbit (using all correctors)
- for m = n SVD minimizes the norm (using all correctors)
- the algorithm is not stable if $\underline{D}$ has small Eigenvalues
  - can be used to find redundant correctors!

Harmonic Filtering

- Unperturbed solution (smooth approximation):

$$x'' + \frac{2\pi}{L} \cdot Q^2 \cdot x = 0 \quad \rightarrow \quad x(s) = A \cdot e^{i \cdot \frac{2\pi}{L} \cdot Q \cdot s}$$

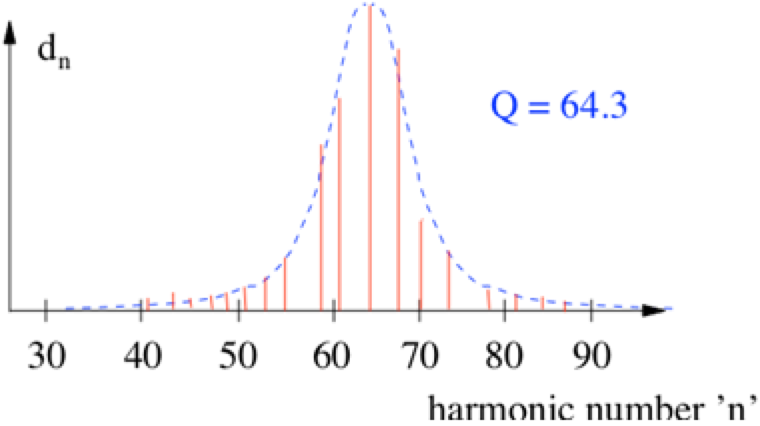
- orbit perturbation

$$x'' + \frac{2\pi}{L} \cdot Q^2 \cdot x = F(s)$$

- periodicity:

$$F(s) = \sum_n f_n \cdot e^{i \cdot \frac{2\pi}{L} \cdot n \cdot s} \qquad CO(s) = \sum_n d_n \cdot e^{i \cdot \frac{2\pi}{L} \cdot n \cdot s}$$

$$\rightarrow \quad d_n = \frac{f_n}{\left(\frac{2\pi}{L} Q\right)^2 - \left(\frac{2\pi}{L} Q\right)^2}$$



spectrum peaks around Q = n ➔ small number of relevant terms!

Most Effective Corrector

- the orbit error is dominated by a few large perturbations:

  ➔ minimize the norm: $\left\| \overrightarrow{BPM} - \underline{A} \cdot \underline{B} \cdot \overrightarrow{COR} \right\|$

  using only a small set of corrector magnets

- brut force: select all possible corrector combinations

  ➔ time consuming but god result

- selective: use one corrector at the time + keep most effective

  ➔ much faster but has a finite chance to miss best solution and can generate π bumps

- MICADO: selective + cross correlation between orbit residues and remaining correcotr magnets

Example for Measured & Corrected Orbit Data

- LEP: