Short Introduction to

(Classical***) Electromagnetic Theory

Focus: applications to accelerators

Slides at: http://cas.web.cern.ch/schools/constanta-2018

*** good enough for our purpose ... (see later)

The topic cannot be treated rigorously (but accurately) in this lecture, typically 60 hours at the university. The aim is to exhibit the main thoughts and concepts

Details can be found in the proposed reading material, in particular in [1]:

[1] **J.D. Jackson,** *Classical Electrodynamics* **(Wiley, 1998 ..)**

[2] **R.P. Feynman,** *Feynman lectures on Physics*, **Vol2.**

[3] **J. Slater, N. Frank,** *Electromagnetism*, **(McGraw-Hill, 1947, and Dover Books, 1970)**

[4] **A. Wolski,** *Theory of electromagnetic fields*, **Proc. CAS: "RF for accelerators", CERN-2011-007.**

[ ]Part 1:

- **Basic observations**
- **Maxwell's equations**
- **Electromagnetic fields in matter**
- **Multipole expansion of fields**

[ ]Part 2:

- **Lorentz force**
- **Electromagnetic waves**
- **Boundary conditions and Polarisation**
- **Modes in wave guides and cavities**
- **Skin depth and penetration depth**

Variables, units and (CERN) conventions

Maxwell's equations relate Electric and Magnetic fields from charge and current distributions (SI units).

$\vec{E}$	=	**electric field [V/m]**
$\vec{H}$	=	**magnetic field [A/m]**
$\vec{D}$	=	**electric displacement [C/m^2]**
$\vec{B}$	=	**magnetic flux density [T]**
q	=	**electric charge [C]**
ρ	=	**electric charge density [C/m^3]**
$\vec{j}$	=	**current density [A/m^2]**
μ_0	=	**permeability of vacuum, 4** $\pi \cdot 10^{-7}$ **[H/m or N/A^2]**
ϵ_0	=	**permittivity of vacuum, 8.854** $\cdot 10^{-12}$ **[F/m]**
c	=	**speed of light, 2.99792458** $\cdot 10^{8}$ **[m/s]**
e	=	**electronic charge, 1.6021773** $\cdot 10^{-19}$ **[C]**

Some general remarks:

➡ **All classical electromagnetic phenomena are summarized by Maxwell's equations <u>together</u> with Special Relativity and can be deduced from them**

Not appropriate in these lectures, but a simple list of the formulae also won't do, they do not even show what the "words" mean.

More appropriate:

- **Establish (non-relativistic) equations using an empirical and informal approach. Relativistic, i.e. correct equations in lecture on Special Relativity***
- **Get the relevant physics necessary here.**

* **Maxwell's equations look different in a resting and a moving system !!!**

Why do we need Maxwell's equations ?

In Electrodynamics we deal with:

Fields (whatever they are), fluxes, forces ..

Faily abstract and do not say what it really is going between charges. Some early attempts tried to explain it as some kind of "gear wheels" or some "stress" between some kind of material

Need to prepare a set of concepts to conveniently "describe" the effects in mathematical terms in a way that:

> One can understand what an equation means if you have a formulation, by figuring out the characteristics of a solution, without actually solving it.

Maxwell's equations do exactly that job !

Recap vector calculus: define a special vector ∇

called the "gradient":

$$\boxed{\nabla \stackrel{def}{=} \left(\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z}\right)}$$

Can be used like a vector (e.g. in vector and scalar products), for example on a vector $\vec{F}(x,y,z)$ or a scalar function $\phi(x,y,z)$:

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\nabla \phi = \left(\frac{\partial \phi}{\partial x}, \quad \frac{\partial \phi}{\partial y}, \quad \frac{\partial \phi}{\partial z}\right)$$

Consider it an <u>operator-in-waiting</u> (for something to do ..)

Depending how it is used in products the results are very different:

$\nabla \cdot \vec{F}$ **is a scalar (e.g. "density" of a source, see later)**

$\nabla \cdot \phi$ **is a vector (e.g. electric field $\vec{E}$, force)**

$\nabla \times \vec{F}$ **is a pseudo-vector (e.g. magnetic induction $\vec{B}$ (!))**

Work also on matrices (of course) and on itself (e.g.):

$\nabla \cdot \nabla$ **(also written as Δ)**

$$\nabla \times (\nabla \times \vec{F}) \; = \; \nabla \cdot (\nabla \cdot \vec{F}) \; - \; \Delta\vec{F}$$

etc.

Two operations with ∇ acting on <u>vectors</u> have special names:

<u>**DIVERGENCE (scalar product of ∇ with a vector):**</u>

$$\mathrm{div}(\vec{F}) \stackrel{def}{=} \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Physical significance: measure of something "coming out"

<u>**CURL (vector product of ∇ with a vector):**</u>

$$\mathrm{curl}(\vec{F}) \stackrel{def}{=} \nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Physical significance: measure of something "circulating"

Example: Coulomb field of an isolated charge Q

A local charge Q generates an electric field $\vec{E}$ according to :

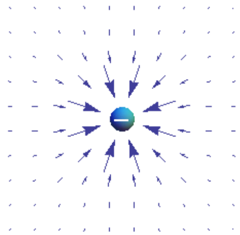
$$\boxed{\vec{E}(x,y,z) \;=\; \frac{Q}{4\pi\epsilon_0}\,\frac{\vec{r}}{r^3}} \qquad \vec{r} \;=\; (x,y,z)$$

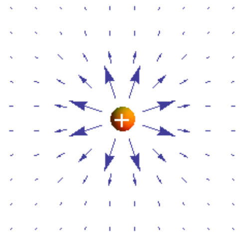
Absolute value depends on : $\left|\frac{\vec{r}}{r^3}\right| \;=\; \frac{1}{r^2}$

Field lines pointing <u>away</u> or <u>towards</u> the charge : $\vec{r}$

Charges are <u>pushed</u> or <u>attracted</u> along the field lines

Expect that div $\vec{E}$ should be relevant ➡





We can do the (non-trivial*) computation of the divergence:

$$\text{div } \vec{E} \;=\; \nabla \vec{E} \;=\; \frac{dE_x}{dx} + \frac{dE_y}{dy} + \frac{dE_z}{dz} \;=\; \frac{\rho}{\epsilon_0}$$

(negative charges) $\nabla \cdot \vec{E} < 0$

(positive charges) $\nabla \cdot \vec{E} > 0$

Divergence related to charge density ρ generating the field** $\vec{E}$

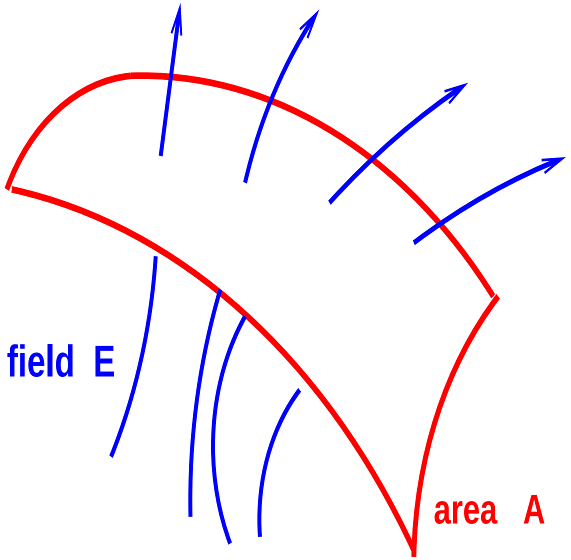
Charge density ρ is charge per volume***: $\rho \;=\; \dfrac{Q}{V} \quad \Longrightarrow \quad \iiint \rho \, dV \;=\; Q$

* see later

** sometime called "source density"

*** becomes important later

How to quantify electric (or magnetic) fields ?



Count field vectors (somehow)

"going" through an area A :

Counting is Integrating

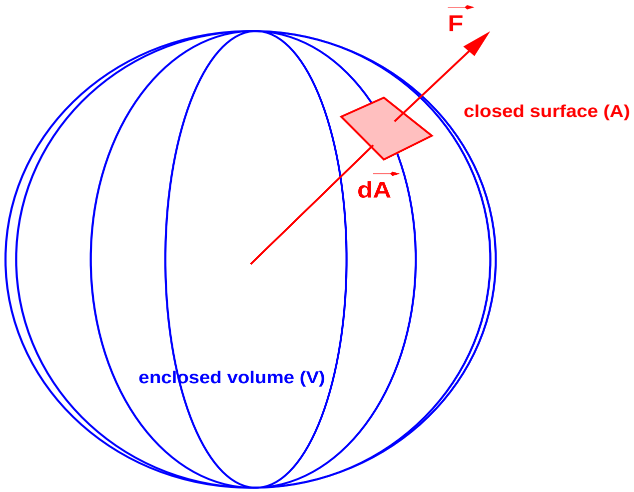
$$\boxed{\Omega \quad = \quad \iint \vec{E} \cdot d\vec{A}}$$

Ω is the flux through the area A

- **Proportional to density of field lines (e.g. stronger field)**
- **Proportional to area within the boundary**

What if the area is closed, i.e. a surface ?

If the shape does not matter - make it a sphere (computaions are easy):



Same definition of flux :

$$\Omega = \iint_A \vec{E} \cdot d\vec{A}$$

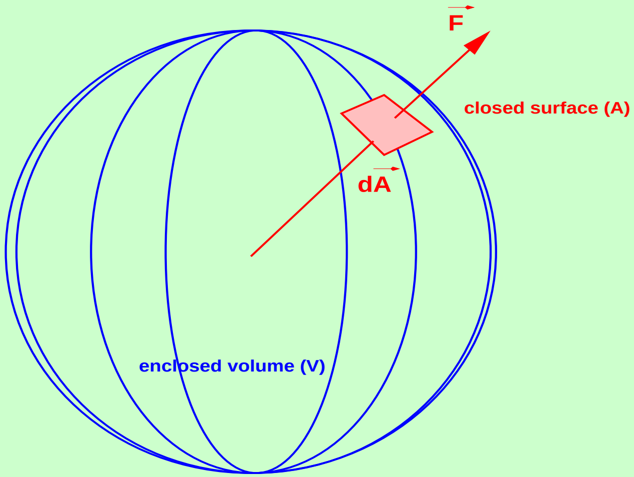
Count how many go in ϕ_{in} and how many go out ϕ_{out}

➡ **Difference is the flux through the sphere**

Measures somehow what is <u>diverging</u> from the enclosed volume ...

div **must be related to what happens inside the volume**

Leading to: Gauss' theorem to evaluate flux integral:



$$\iint_A \vec{E} \cdot d\vec{A} = \iiint_V \nabla \cdot \vec{E} \cdot dV \quad \text{or}$$

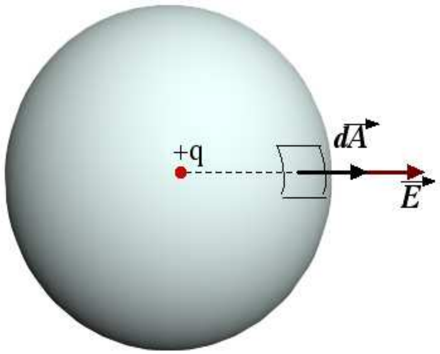
$$\iint_A \vec{E} \cdot d\vec{A} = \iiint_V \text{div } \vec{E} \cdot dV$$

Integral through closed surface (flux) is integral of divergence in the enclosed volume

Surface integral related to the divergence from the enclosed volume

Sum (integral) of all <u>sources inside</u> the volume gives the <u>flux out</u> of this region

Arrive at: Maxwell's first equation using Gauss's



density ρ and charge Q : $\displaystyle\int_V \frac{\rho}{\epsilon_0} \cdot dV \;=\; \frac{Q}{\epsilon_0}$

$$\underbrace{\int_A \vec{E} \cdot d\vec{A} = \int_V \nabla \cdot \vec{E} \cdot dV}_{\text{Gauss theorem}} = \frac{Q}{\epsilon_0} \;=\; \left[\int_V \frac{\rho}{\epsilon_0} \cdot dV\right]$$

Written with <u>charge density</u> ρ we get Maxwell's <u>first</u> equation:

Maxwell (I): $\boxed{\text{div } \vec{E} \;=\; \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}}$

The higher the charge density: The larger the divergence of the field

Simplest possible charge distribution: point charge

$$\vec{E} \quad = \quad \frac{q}{4\pi\epsilon_0} \cdot \frac{\vec{r}}{r^3}$$

What is $\underline{\text{div } \vec{E}}$ **and** $\underline{\text{curl } \vec{E}}$ **for a point charge ?**

First step: compute all derivatives (used for DIV and CURL)

$$\rightarrow \qquad \frac{\partial E_{(x,y,z)}}{\partial(x,y,z)}$$

$$\frac{\partial E_x}{\partial x} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3x^2}{R^5}\right) \quad \frac{\partial E_x}{\partial y} = \frac{-3Q}{4\pi\epsilon_0}\frac{xy}{R^5} \quad \frac{\partial E_x}{\partial z} = \frac{-3Q}{4\pi\epsilon_0}\frac{xz}{R^5}$$

$$\frac{\partial E_y}{\partial x} = \frac{-3Q}{4\pi\epsilon_0}\frac{xy}{R^5} \quad \frac{\partial E_y}{\partial y} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3y^2}{R^5}\right) \quad \frac{\partial E_y}{\partial z} = \frac{-3Q}{4\pi\epsilon_0}\frac{yz}{R^5}$$

$$\frac{\partial E_z}{\partial x} = \frac{-3Q}{4\pi\epsilon_0}\frac{xz}{R^5} \quad \frac{\partial E_z}{\partial y} = \frac{-3Q}{4\pi\epsilon_0}\frac{yz}{R^5} \quad \frac{\partial E_z}{\partial z} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3z^2}{R^5}\right)$$

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$$\boxed{\text{div } \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{Q}{4\pi\epsilon_0}\left(\frac{3}{R^3} - \frac{3}{R^5}(x^2 + y^2 + z^2)\right)}$$

... what a bummer

$$\frac{\partial E_x}{\partial x} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{3x^2}{R^5}\right) \qquad \frac{\partial E_x}{\partial y} = \frac{-3Q}{4\pi\epsilon_0}\frac{xy}{R^5} \qquad \frac{\partial E_x}{\partial z} = \frac{-3Q}{4\pi\epsilon_0}\frac{xz}{R^5}$$

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$$\text{curl } \vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \quad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial z}\right) = (0, \quad 0, \quad 0)$$

(there is nothing circulating around a point charge)

Because there is nothing circulating (curl $\vec{E} = 0$**) one can derive the field $\vec{E}$ from a scalar electrostatic potential $\phi(x,y,z)$, i.e.:**

$$\vec{E} = -\,\text{grad}\ \phi = -\nabla\phi = -(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z})$$

then we have

$$\boxed{\nabla\vec{E} = -\nabla^2\phi = -(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}) = \frac{\rho(x,y,z)}{\epsilon_0}}$$

This is Poisson's equation

All we need is ϕ Example →

A very important example: 3D Gaussian distribution

$$\rho(x,y,z) = \frac{Q}{\sigma_x\sigma_y\sigma_z\sqrt{2\pi}^3}\exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2}\right)$$

($\sigma_x, \sigma_y, \sigma_z$ r.m.s. sizes)

$$\boxed{\phi(x,y,z,\sigma_x,\sigma_y,\sigma_z) = \frac{Q}{4\pi\epsilon_0}\int_0^\infty \frac{\exp(-\frac{x^2}{2\sigma_x^2+t} - \frac{y^2}{2\sigma_y^2+t} - \frac{z^2}{2\sigma_z^2+t})}{\sqrt{(2\sigma_x^2+t)(2\sigma_y^2+t)(2\sigma_z^2+t)}}dt}$$

For the interested: resulting fields given in the backup slides

For a derivation, see e.g. W. Herr, *Beam-Beam Effects*,
in Proceedings CAS Zeuthen, 2003, CERN-2006-002, and references therein.

<u>Very important in practice:</u>

Poisson's equation in Polar coordinates, i.e. 2D (r, φ)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) \; + \; \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \varphi^2} \quad = \quad -\frac{\rho}{\epsilon_0}$$

Poisson's equation in Cylindrical coordinates (r, φ, z)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) \; + \; \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \varphi^2} \; + \; \frac{\partial^2 \phi}{\partial z^2} \quad = \quad -\frac{\rho}{\epsilon_0}$$

Poisson's equation in Spherical coordinates (r, θ, φ)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) \; + \; \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right) \; + \; \frac{1}{r^2 \sin\theta}\frac{\partial^2 \phi}{\partial \varphi^2} \quad = \quad -\frac{\rho}{\epsilon_0}$$

Spherical coordinates sound like a good choice for local charges, e.g. point charges

What are div **and** curl **??**

$$\text{div } E(r,\vec{\theta},\varphi) \;=\; \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) \;+\; \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\;\theta E_\theta) \;+\; \frac{1}{r\sin\theta}\frac{\partial E_\varphi}{\partial\varphi}$$

(very useful !)

$$\text{curl } E(r,\vec{\theta},\varphi) \;=\; \frac{\hat{e_r}}{r\sin\theta}\left(\frac{\partial(E_\varphi\sin\theta)}{\partial\theta} \;-\; \frac{\partial E_\theta}{\partial\varphi}\right) \;+$$

$$\frac{\hat{e_\theta}}{r}\left(\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\varphi} \;-\; \frac{E_\varphi}{\partial r}(rE_\varphi\right) \;+\; \frac{\hat{e_\varphi}}{r}\left(\frac{\partial(rE_\theta)}{\partial r} \;-\; \frac{\partial E_r}{\partial\theta}\right)$$

Note: $\hat{e_\theta},\ \hat{e_r},\ \hat{e_\varphi}$ **are the corresponding orthogonal unit vectors**

(useful, but no so much used !)

Try it on a point charge

$$\vec{E} \quad = \quad -\nabla\Psi(r) \quad = \quad \frac{Q}{4\pi\epsilon_0} \cdot \frac{\vec{r}}{r^3}$$

$$\text{div } \vec{E}(r,\theta,\varphi) \; = \; \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) \; + \; \underbrace{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\;\theta E_\theta)}_{\equiv \;\; 0} \; + \; \underbrace{\frac{1}{r\sin\theta}\frac{\partial E_\varphi}{\partial\varphi}}_{\equiv \;\; 0}$$

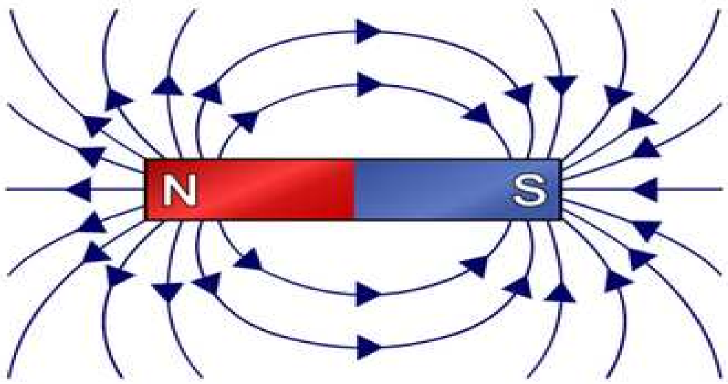
Only the radial component is non-zero:

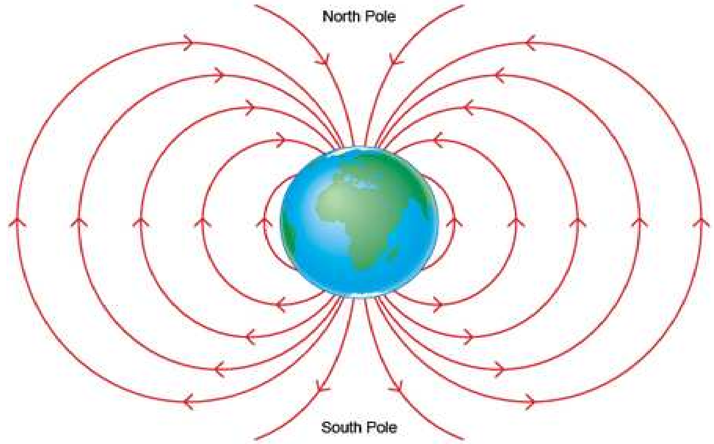
then : $\qquad \text{div } \vec{E}(r,\theta,\varphi) \; = \; \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r)$

$$\Rightarrow \quad \text{div } \vec{E}(r,\theta,\varphi) \; = \; \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2 \cdot Q}{4\pi\epsilon_0 r^2}\right) \; = \; \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{Q}{4\pi\epsilon_0}\right) \; = \quad ???$$

(non-trivial indeed ...)

What about magnetic fields ? ...

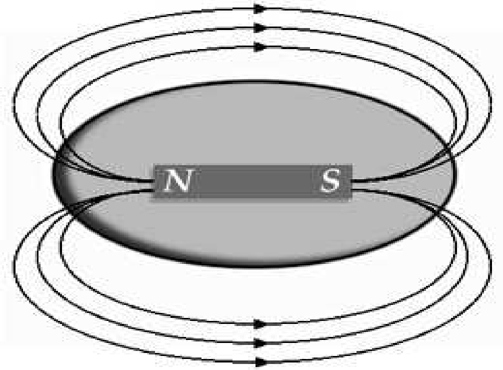




- **They have the direction (by definition): magnetic field lines from North to South**
- **Field lines of $\vec{B}$ are always closed**
- **Magnets (e.g. compass) are pushed or attracted along the field lines**

What about divergence of magnetic fields ?

Enclose it again by a surface - the result is found immediately:



$$\int\int_A \vec{B}\ \mathrm{d}\vec{A} = \int\int\int_V \nabla\vec{B}\ \mathrm{d}V = 0$$

Volume (thus dV) is <u>never</u> $= 0$

$$\nabla\vec{B} = \text{div } \vec{B} = 0$$

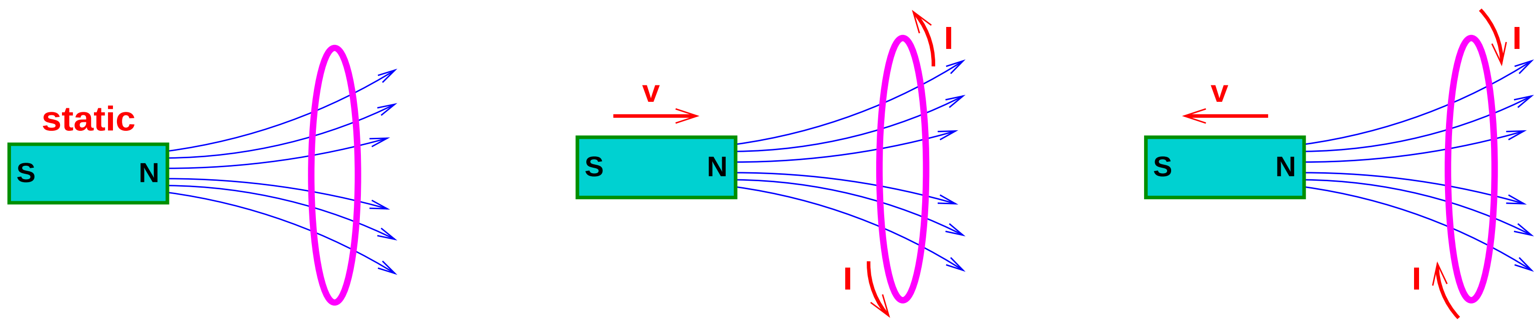
What goes into the closed surface also goes out

Maxwell's second equation: $\boxed{\nabla\vec{B} = \text{div } \vec{B} = 0}$

Physical significance: (probably) no Magnetic Monopoles

Enter Faraday: allow <u>changing</u> flux through an area

static flux : $\Omega = \int\limits_A \vec{B} \cdot \mathrm{d}\vec{A}$ changing flux : $\frac{\partial \Omega}{\partial t} = \int\limits_A \frac{\partial (\vec{B})}{\partial t} \cdot \mathrm{d}\vec{A}$



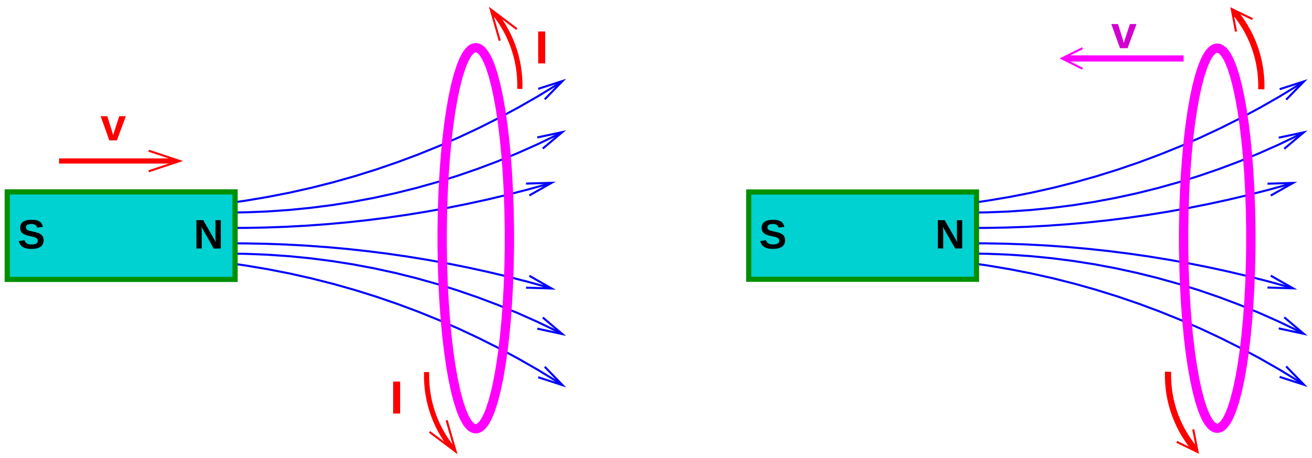
Moving the magnet changes the flux: <u>more</u> or <u>fewer</u> passing through the area (use a conducting coil, e.g. wedding ring) $\Longrightarrow$

Induces a <u>circulating</u> (curling) electric field $\vec{E}$ in the coil which "pushes" charges around the coil $\Longrightarrow$

Moving charges: Current I in the coil (observe its direction ..)

Experimental evidence:

It does not matter whether the magnet or the coil is moved (same direction of induced current):



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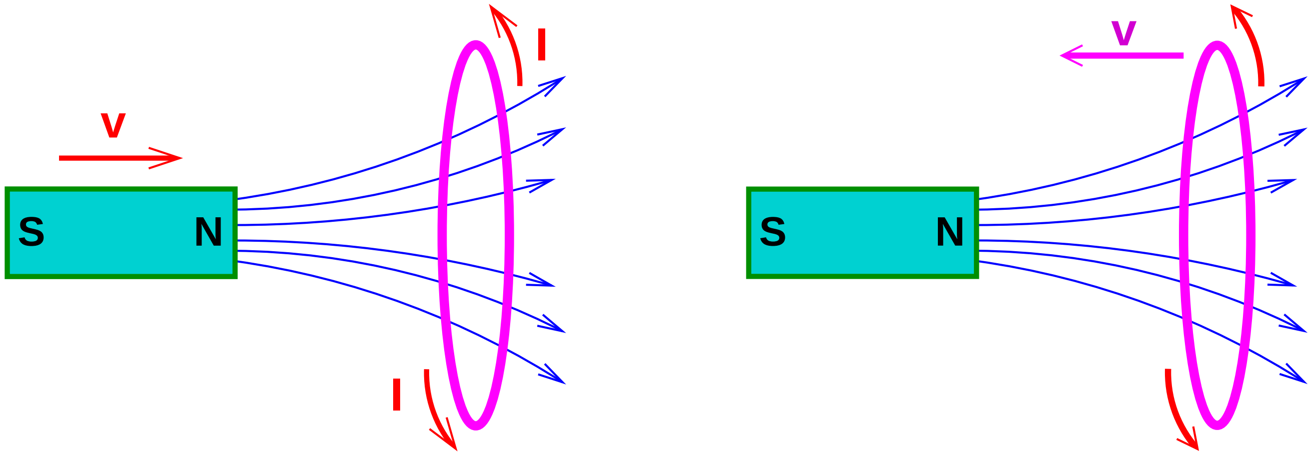


If you think it is obvious - no :

This was the reason for Einstein to develope special relativity !!!

Again: Maxwell different in the two systems (see lecture on Relativity)

A <u>changing</u> flux Ω through an area A produces circular electric field $\vec{E}$, "pushing" charges $\Longrightarrow$ a current I

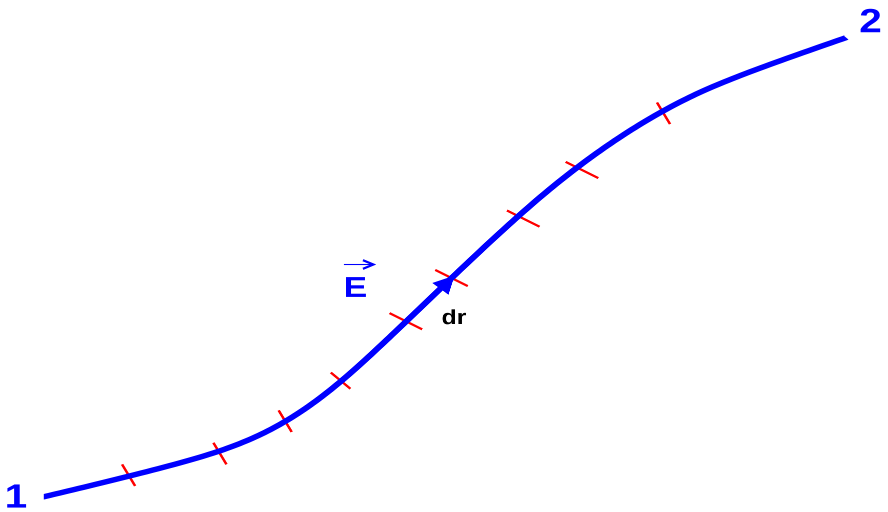
$$-\frac{\partial\Omega}{\partial t} = \frac{\partial}{\partial t}\overbrace{\int_A \vec{B}\cdot d\vec{A}}^{\text{flux }\Omega} = \underbrace{\oint_C \vec{E}\cdot d\vec{r}}_{\text{pushed charges}}$$

Flux can be changed by:

- **Change of magnetic field $\vec{B}$ with time t (e.g. transformers)**
- **Change of area A with time t (e.g. dynamos)**

How to count "pushed charges" $\left[\int_C \vec{E}\cdot d\vec{r}\right.$ is a <u>line integral</u>$\left.\right]$

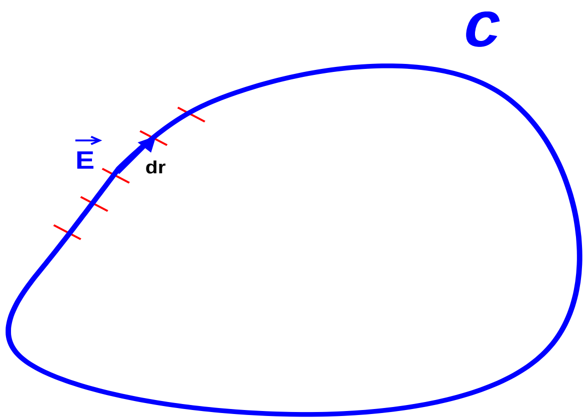
Line integrals sum up "pushes" along lines or curves:



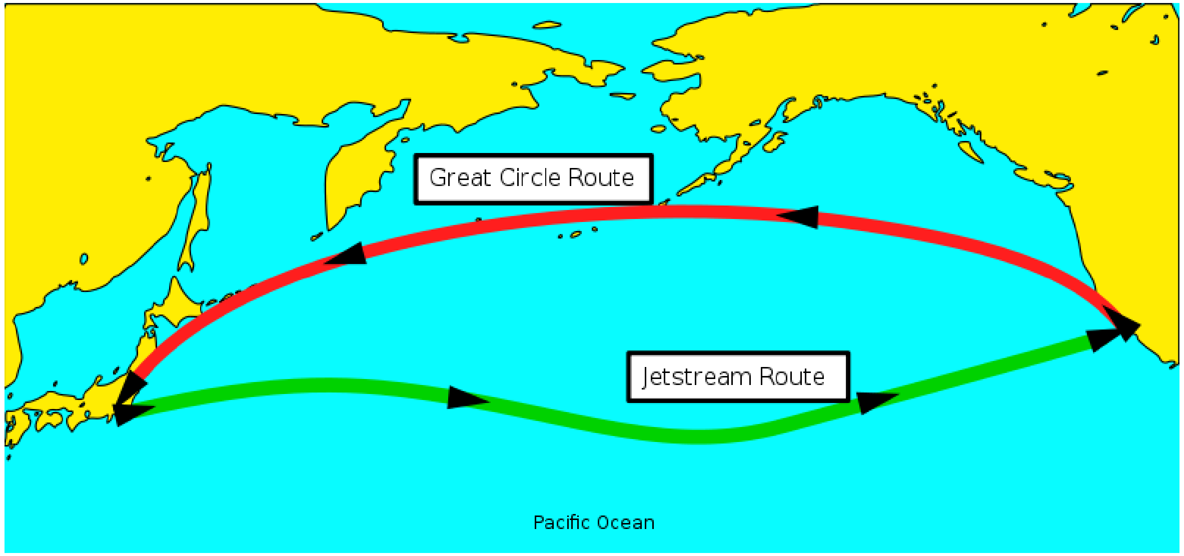
"lines" can be open or closed

Sum along all line elements $d\vec{r}$

$\int_1^2 \vec{E} \cdot d\vec{r}$ or $\int_C \vec{E} \cdot d\vec{r}$



Everyday example ..

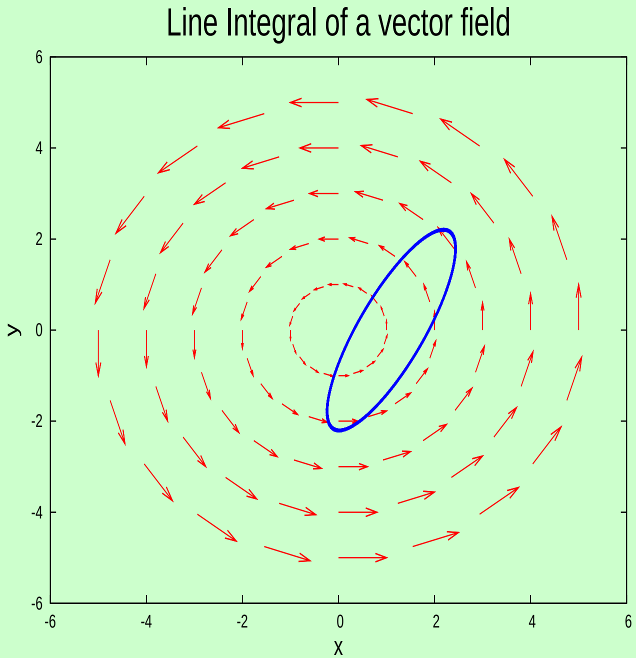


Line integrals: sum up "pushes" along the two Lines/Routes

Optimize: e.g. fuel consumption, time of flight (saves 1 hour !)

Like surface integral, the closed line integral $\int_C \vec{E} \cdot d\vec{r}$ **can be re-written:**

Used in the following: Stoke's theorem



$$\oint_C \vec{E} \cdot d\vec{r} \;=\; \iint_A \nabla \times \vec{E} \cdot d\vec{A} \quad \text{or}$$

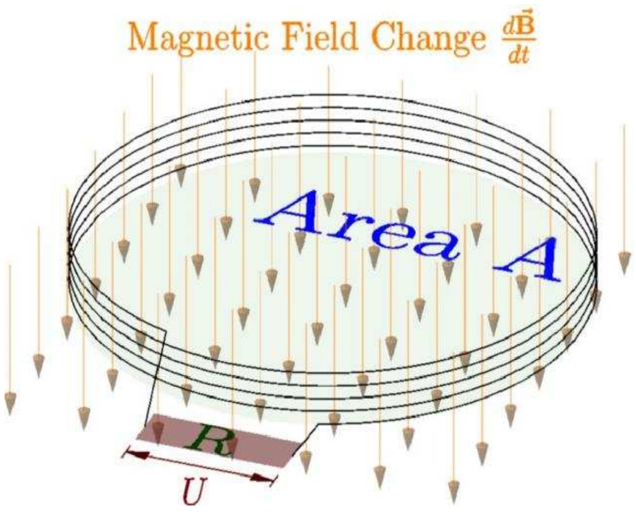
$$\oint_C \vec{E} \cdot d\vec{r} \;=\; \iint_A \text{curl } \vec{E} \cdot d\vec{A}$$

obviously : div $\vec{E} \;=\; 0$

Summing up all vectors inside the area: net effect is the sum along the closed curve

➡ **measures something that is circulating ("curling") inside and how strongly**

Use this theorem for a coil enclosing a closed area



$$\int_A -\frac{\partial \vec{B}}{\partial t} d\vec{A} = \underbrace{\int_A \nabla \times \vec{E}\ d\vec{A} = \oint_C \vec{E} \cdot d\vec{r}}_{\text{Stoke's formula}}$$

$$\underbrace{\int_A -\frac{\partial \vec{B}}{\partial t} d\vec{A} = \int_A \nabla \times \vec{E}\ d\vec{A}}_{\text{same Integration}}$$

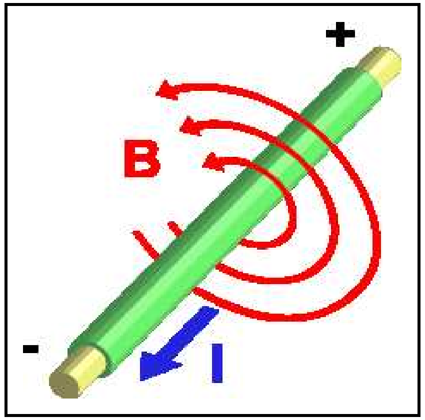
Re-written: changing magnetic field through an area induces circulating electric field around the area (Faraday)

Maxwell' 3rd equation

$$\boxed{-\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} = \text{curl } \vec{E}}$$

Next: Maxwell's fourth equation (part 1) ...

From Ampere's law, for example current density $\vec{j}$:



$$\int_A \nabla \times \vec{B} \;\; d\vec{A} \;=\; \oint_C \vec{B} \cdot d\vec{r} \;=\; \int_A \mu_0 \vec{j} \;\; d\vec{A}$$

$\vec{j}$: "amount" of charges through area $\vec{A}$

$$\int_A \mu_0 \vec{j} \;\; d\vec{A} \;=\; \mu_0 \; I \qquad \text{(total current)}$$

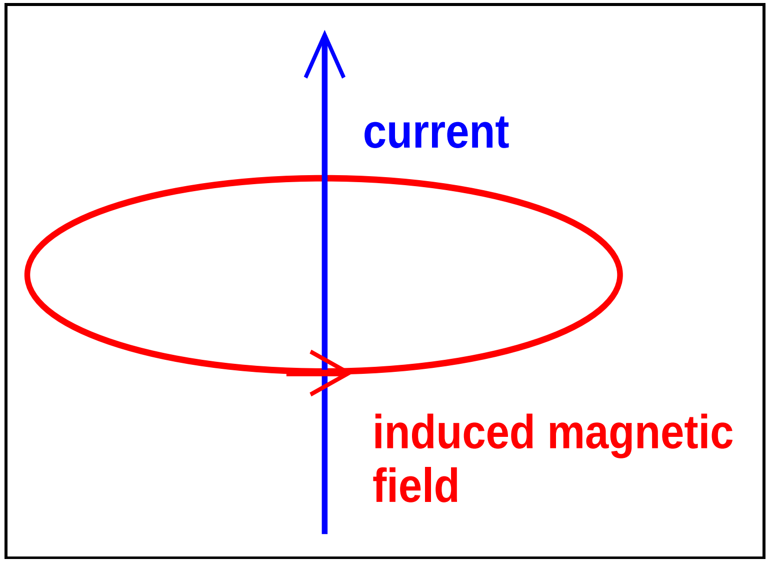
<u>Static</u> electric current induces circular magnetic field (e.g. in magnets)

Using the same argument as before (the same integral formula):

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{j}}$$

An important application:

For a static electric current I in a single (infinitely long) wire we get Biot-Savart law (using the area of a circle $A = r^2 \cdot \pi$, we can easily do the integral):



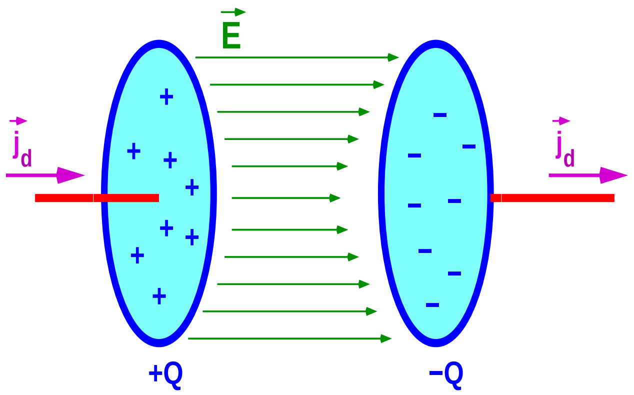
$$\vec{B} = \frac{\mu_0}{4\pi} \oint \vec{j} \cdot \frac{\vec{r} \cdot d\vec{r}}{r^3}$$

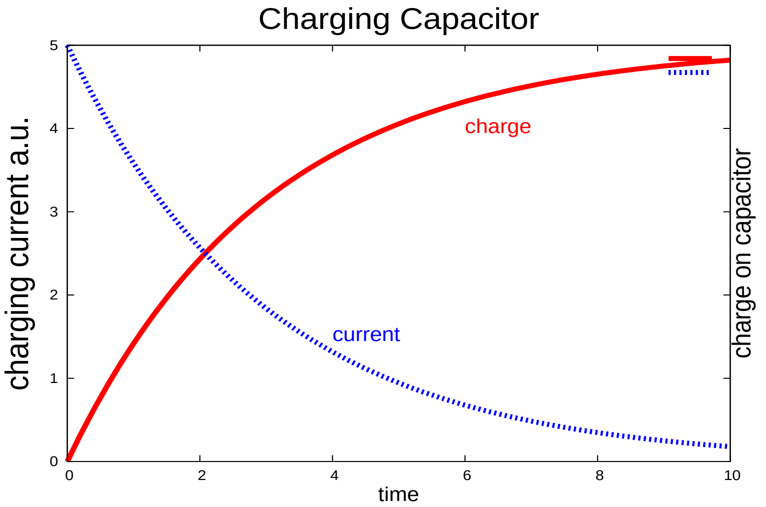
$$\vec{B} = \frac{\mu_0}{2\pi} \frac{\vec{j}}{r}$$

Application: magnetic field calculations in wires

Maxwell's fourth equation (part 2) ...

Charging capacitor: Current enters left plate - leaves from right plate, builds up an electric field between plates → produces a "current" <u>during</u> the charging process

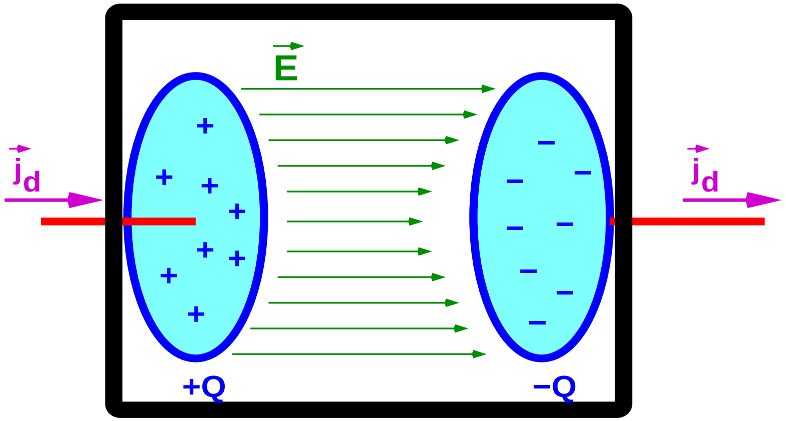




$$Q(t) \;=\; C \cdot V_b \cdot (1 \;-\; \exp(-t/RC)) \qquad \textbf{and} \qquad I(t) \;=\; \frac{V_b}{R}\; \exp(-t/RC)$$

Part 2: Maxwell's fourth equation

Charging capacitor: Current enters left plate - leaves from right plate, builds up an electric field between plates ➡ produces a "current" <u>during</u> the charging process



Displacement Current :

$$\vec{j}_d = \frac{d\vec{E}}{dt}$$

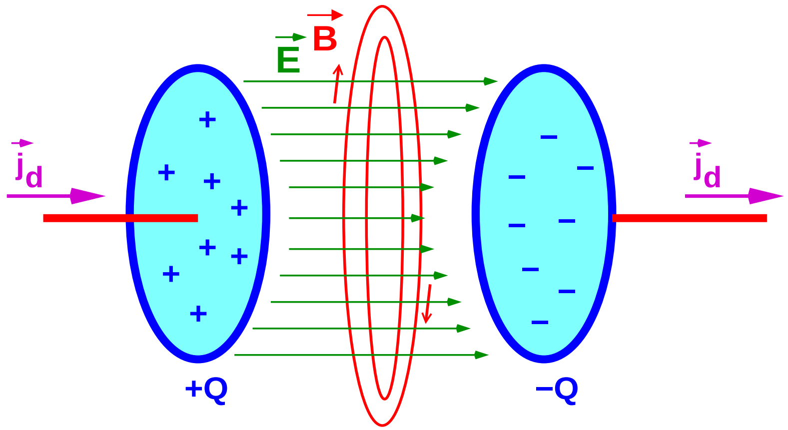
This is <u>not</u> a current from charges moving through a wire

This is a "current" from <u>time varying electric fields</u>

Once charged: fields are constant, (displacement) "current" stops

Cannot distinguish the origin of a current - apply Ampere's law to j_d

- Displacement current j_d produces magnetic field, just like "real currents" do ...



- Time varying electric field induces time varying circular magnetic field (using the current density $\vec{j}_d$)

$$\nabla \times \vec{B} = \mu_0 \vec{j}_d = \epsilon_0 \mu_0 \frac{d\vec{E}}{dt}$$

Magnetic fields $\vec{B}$ can be generated by two different currents:

$\nabla \times \vec{B} = \mu_0 \vec{j}$ **(electrical current)**

$\nabla \times \vec{B} = \mu_0 \vec{j}_d = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$ **(displacement current)**

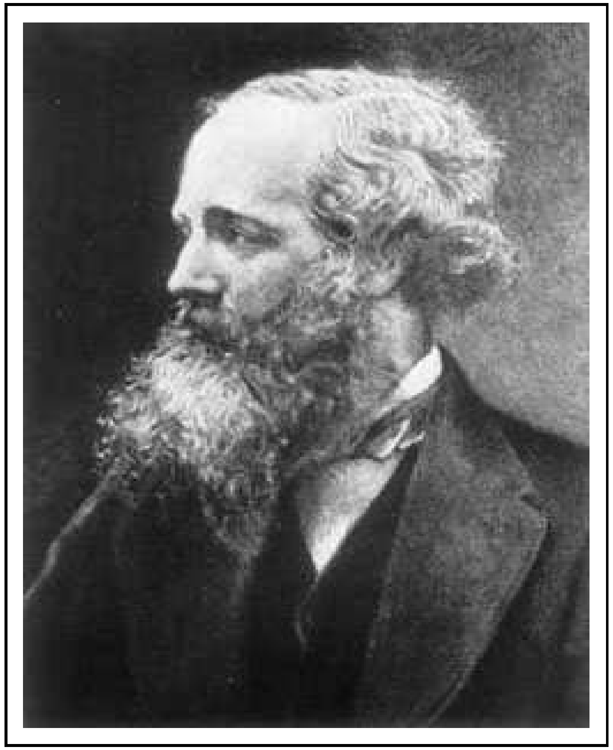
or putting them together to get Maxwell's fourth equation:

$$\boxed{\nabla \times \vec{B} = \mu_0 (\vec{j} + \vec{j}_d) = \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}}$$

or in integral form:

$$\int_A \nabla \times \vec{B} \cdot d\vec{A} = \int_A \left(\mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{A}$$

SUMMARY: MAXWELL'S EQUATIONS



$$\int_A \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

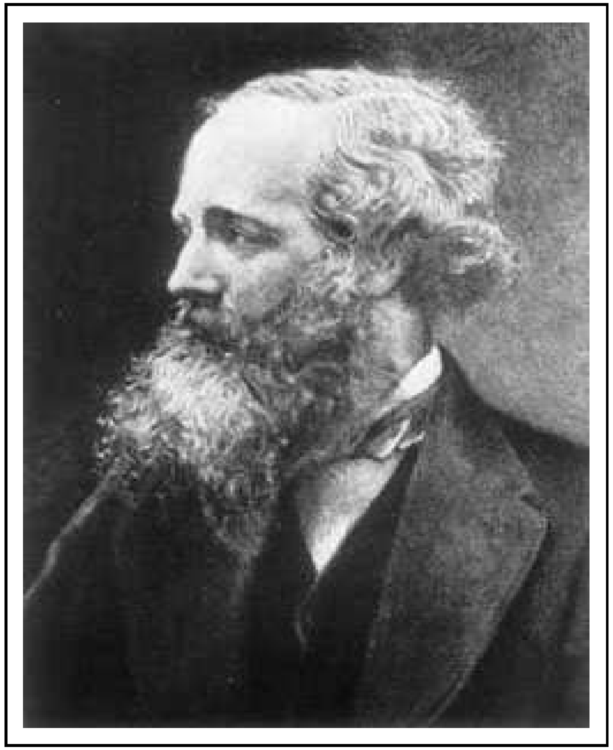
$$\int_A \vec{B} \cdot d\vec{A} = 0$$

$$\oint_C \vec{E} \cdot d\vec{r} = -\int_A \left(\frac{d\vec{B}}{dt}\right) \cdot d\vec{A}$$

$$\oint_C \vec{B} \cdot d\vec{r} = \int_A \left(\mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}\right) \cdot d\vec{A}$$

Written in Integral form

SUMMARY: MAXWELL'S EQUATIONS



$$\nabla \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} \; + \; \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}$$

Written in Differential form (my preference)

V.G.F.A.Q:

Why :

$$\text{div } \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\text{curl } \vec{E} = -\frac{d\vec{B}}{dt}$$

$$\text{div } \vec{B} = 0$$

$$\text{curl } \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}$$

Why Not :

$$\int_A \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\oint_C \vec{E} \cdot d\vec{r} = -\int_A \left(\frac{d\vec{B}}{dt} \right) \cdot d\vec{A}$$

$$\int_A \vec{B} \cdot d\vec{A} = 0$$

$$\oint_C \vec{B} \cdot d\vec{r} = \int_A \left(\mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt} \right) \cdot d\vec{A}$$

$\text{div } \vec{E} = \frac{\rho}{\epsilon_0}$ something $(\vec{E})$ spreading out

$$\int_A \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

???

$\text{curl } \vec{E} = -\frac{d\vec{B}}{dt}$ something $(\vec{E})$ circulating

$$\oint_C \vec{E} \cdot d\vec{r} = -\int_A \left(\frac{d\vec{B}}{dt} \right) \cdot d\vec{A}$$

???

Maxwell's Equations - compact

1. Electric fields $\vec{E}$ are generated by charges and proportional to total charge

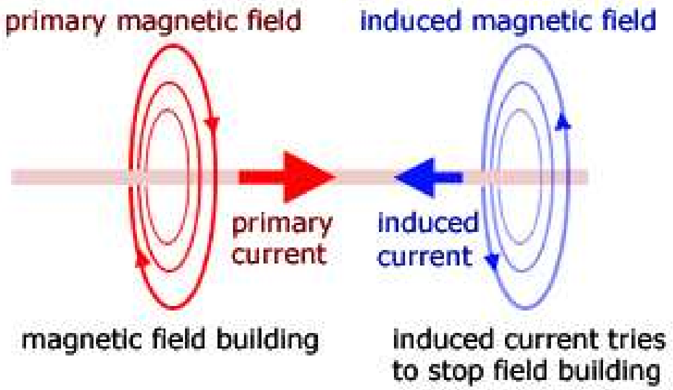
2. Magnetic monopoles do (probably) not exist

3. Changing magnetic flux generates circular electric fields/currents

4.1 Changing electric flux generates circular magnetic fields

4.2 Static electric current generates circular magnetic fields

Changing fields: Powering and self-induction



- If the current is not static:
- Primary magnetic flux $\vec{B}$ changes with changing current

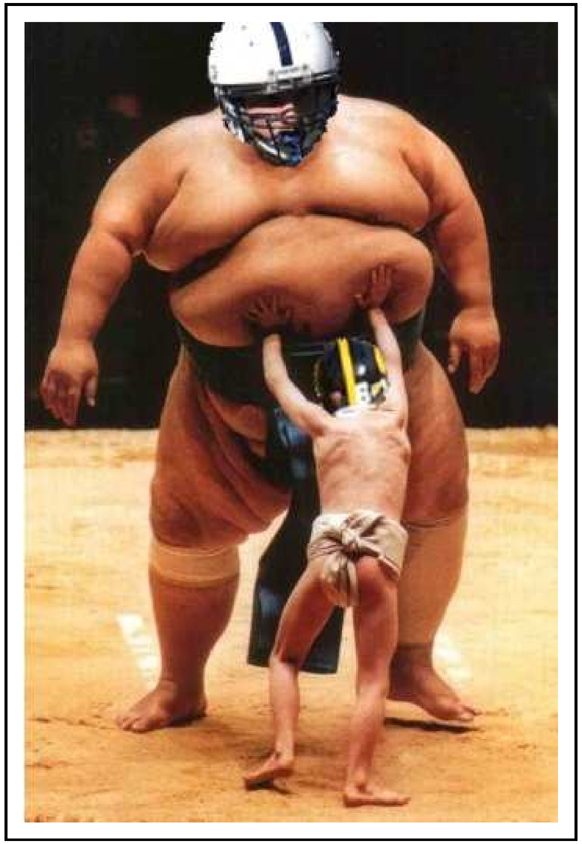
➡ Induces an electric field, resulting in a current and induced magnetic field $\vec{B}_i$

➡ Induced current will oppose a change of the primary current

➡ If we want to change a current to ramp a magnet ...

Have to overcome this counteraction, applying a sufficient Voltage:

if pushed ➡ push harder



Ramp rate determines required Voltage :

$$\mathrm{U} = -\mathrm{L}\,\frac{\partial I}{\partial t}$$

Inductance L in Henry H

Example :

Required ramp rate : 10 A/s

With L = 15.1 H per powering sector

Required Voltage is ≈ 150 V

Surprise - as is always the case:

Units:	**Gauss law**	**Ampere/Maxwell**
SI	$\nabla \vec{E} = \frac{\rho}{\epsilon_0}$	$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}$
Electro-static ($\epsilon_0 = 1$)	$\nabla \vec{E} = 4\pi\rho$	$\nabla \times \vec{B} = \frac{4\pi}{c^2}\vec{j} + \frac{1}{c^2}\frac{d\vec{E}}{dt}$
Electro-magnetic ($\mu_0 = 1$)	$\nabla \vec{E} = 4\pi c^2 \rho$	$\nabla \times \vec{B} = 4\pi \vec{j} + \frac{1}{c^2}\frac{d\vec{E}}{dt}$
Gauss cgs	$\nabla \vec{E} = 4\pi\rho$	$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{d\vec{E}}{dt}$
Lorentz	$\nabla \vec{E} = \rho$	$\nabla \times \vec{B} = \frac{1}{c}\vec{j} + \frac{1}{c}\frac{d\vec{E}}{dt}$

Also: $\vec{B}^{Gauss} = \sqrt{\frac{4\pi}{\mu_0}}\vec{B}^{SI}$ $\qquad \rho^{Gauss} = \frac{\rho^{SI}}{\sqrt{4\pi\epsilon_0}}$ **and so on .....**

That's not all ➡ Electromagnetic fields in material

In vacuum:

$$\vec{D} = \epsilon_0 \cdot \vec{E}, \qquad \vec{B} = \mu_0 \cdot \vec{H}$$

In a material:

$$\vec{D} = \epsilon_r \cdot \epsilon_0 \cdot \vec{E} \;=\; \epsilon_0 \vec{E} \;+\; \vec{P}$$

$$\vec{B} = \mu_r \cdot \mu_0 \cdot \vec{H} \;=\; \mu_0 \vec{H} \;+\; \vec{M}$$

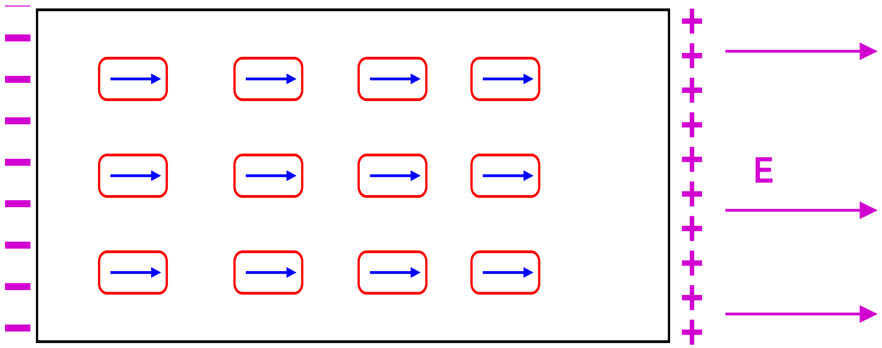
Origin: $\vec{P}$**olarization** and $\vec{M}$**agnetization**

$\epsilon_r(\vec{E}, \vec{r}, \omega)$ ➡ ϵ_r **is relative permittivity** $\approx \quad [1 - 10^5]$

$\mu_r(\vec{H}, \vec{r}, \omega)$ ➡ μ_r **is relative permeability** $\approx \quad [0(!) - 10^6]$

(i.e.: linear, isotropic, non-dispersive)

Polarization $\vec{P}$: displacement of charges in non-conducting material



Appears as electric dipole

$\vec{P} \; = \; \xi_e \cdot \vec{E}$ $\qquad$ (ξ_e **is electric susceptibility)**

Dielectric displacement follows: $\vec{D} \; = \; (1 \; + \; 4\pi\xi_e)\vec{E} \; = \; \epsilon \cdot \vec{E}$

Magnetism: occurence of circular currents of atomic electrons

Classification of magnetic material properties:

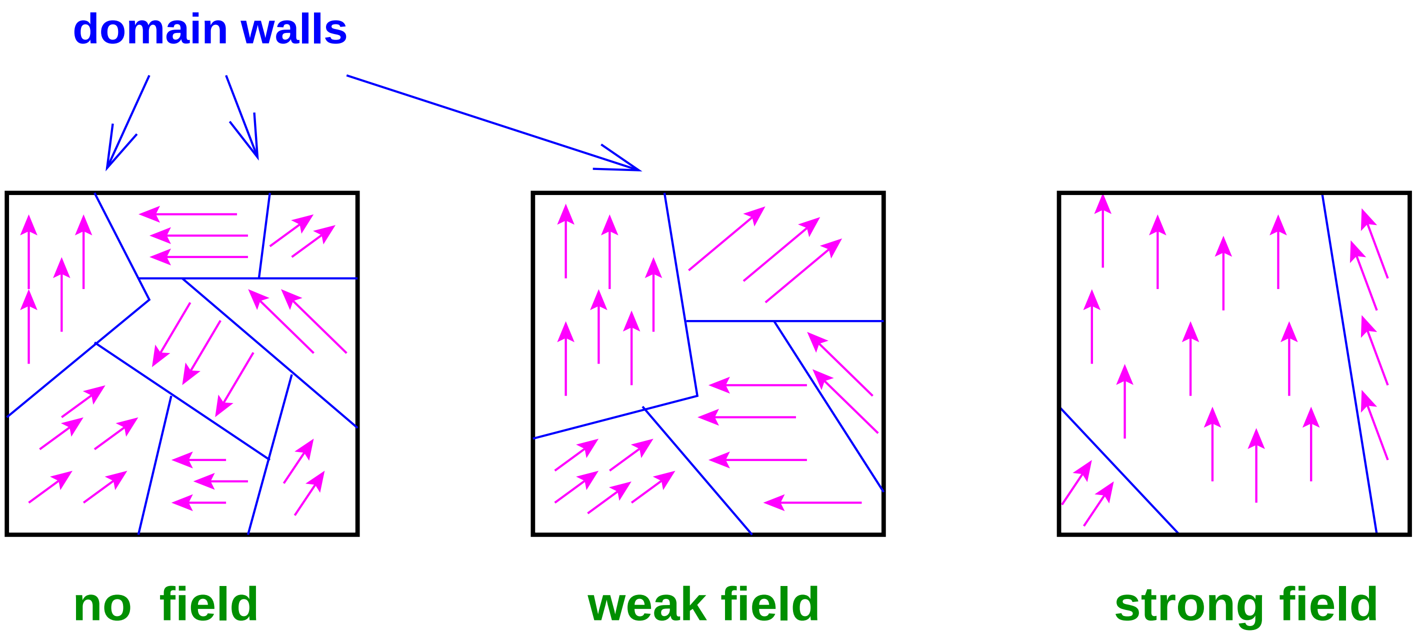
Diamagnetism	**repelled**	$\mu < 1$	$\xi_m < 0$	**typical:** $\xi_m \approx -10^{-7}$
Paramagnetism	**aligned**	$\mu > 1$	$\xi_m > 0$	**typical:** $\xi_m \approx +10^{-7}$
Ferromagnetism	**aligned**	$\mu \gg 1$	$\xi_m \gg 0$	**typical:** $\xi_m \approx +10^{6}$

Diamagnetism: Atoms and molecules without magnetic moment

Paramagnetism: Atoms and molecules with magnetic moment

Ferromagnetism: Saturation magnetization occur within microscopic domains (Weiss domains)

Magnetism in ferromagnetic: not rigorous - just to get an idea ...



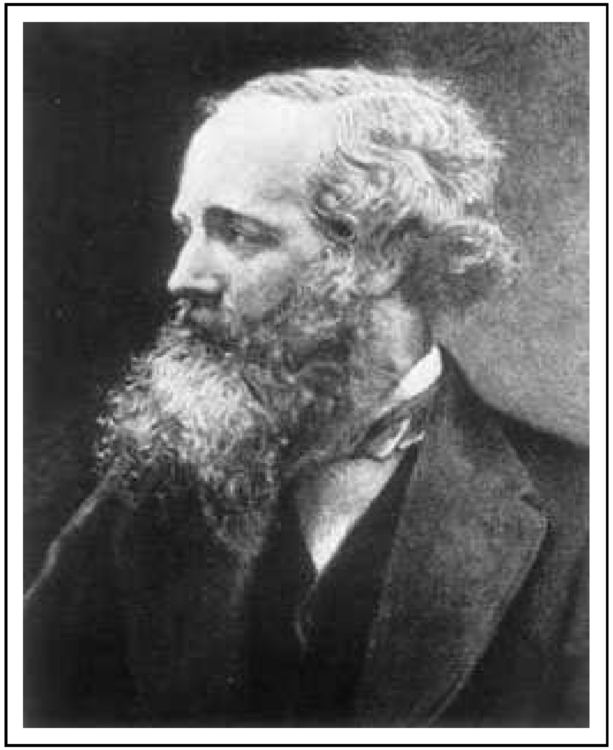
Unmagnetized Iron: spontaneous magnetization around seeds, leading to "Weiss domains"

Randomly oriented domains cancel out

With external fields: domain walls <u>move</u> - domains in direction of external field <u>"grow"</u>! They do <u>not</u> align (although this is sometimes said)!

Magnetic field follows : $\vec{B} = (1 + 4\pi\xi_m)\vec{H} = \mu \cdot \vec{H}$

Once more: Maxwell's Equations



$$\nabla \vec{D} = \rho$$

$$\nabla \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

$$\nabla \times \vec{H} = \vec{j} + \frac{d\vec{D}}{dt}$$

(a.k.a. Macroscopic Maxwell equations)

Something on potentials (needed in lecture on Relativity):

Electric fields can be written using a (scalar) potential Ψ:

$$\vec{E} \;=\; -\vec{\nabla}\Psi$$

Since div $\vec{B} = 0$**, we can write $\vec{B}$ using a (vector) potential $\vec{A}$:**

$$\vec{B} \;=\; \vec{\nabla} \times \vec{A} \;=\; \text{curl } \vec{A}$$

combining Maxwell(I) + Maxwell(III):

$$\vec{E} \;=\; -\vec{\nabla}\Psi \;-\; \frac{\partial \vec{A}}{\partial t}$$

Fields can be written as derivatives of scalar and vector potentials $\Psi(x, y, z)$ and $\vec{A}(x, y, z)$

(<u>absolute values</u> of potentials Ψ and $\vec{A}$ can <u>not</u> be measured ..)

The Coulomb potential of a static charge q is written as:

$$\Psi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{|\vec{r} - \vec{r_q}|} \qquad \left[\text{ or } \int \frac{1}{4\pi\epsilon_0} \cdot \frac{\rho(\vec{r_q})}{|\vec{r} - \vec{r_q}|} \right]$$

where $\vec{r}$ is the observation point* and $\vec{r}_q$ the location of the charge

The vector potential is linked to the current $\vec{j}$:

$$\nabla^2 \vec{A} = \mu_0 \vec{j}$$

The knowledge of the potentials allows the computation of the fields ➡ see lecture on relativity (fields of moving charges)

* **a shameless lie: one cannot observe/measure a potential (only fields) !**

Accelerator magnets ➡ Multipole expansion

If Ψ is periodic[*] in θ, the components in cylindrical coordinates are:

$$B_r(\theta, r) = -\left(\frac{\partial \Psi}{\partial r}\right) = \sum_{n=1}^{\infty} C(n) \left(\frac{r}{R_{ref}}\right)^{n-1} \sin(n(\theta - \alpha_n)) \quad \text{and}$$

$$B_\theta(\theta, r) = -\left(\frac{1}{r}\right)\left(\frac{\partial \Psi}{\partial \theta}\right) = \sum_{n=1}^{\infty} C(n) \left(\frac{r}{R_{ref}}\right)^{n-1} \cos(n(\theta - \alpha_n))$$

$C(n)$ **is the strength of the 2n-pole component of the total field**

R_{ref} **is a reference radius (LHC: 17 mm for 28 mm aperture, typical ratio)**

α_n **is a constant (related to the orientation of the 2n component)**

[*] **a good assumption for accelerator magnets**

The n-th component in $B_r(r,\theta)$ has n South poles and n North poles as a function of the azimuthal angle θ

This implies (for any n):

$$\theta_S \quad = \quad \frac{\pi}{2n} + \alpha_n; \qquad \frac{5\pi}{2n} + \alpha_n; \qquad \frac{9\pi}{2n} + \alpha_n; \ldots \qquad \text{(South poles)}$$

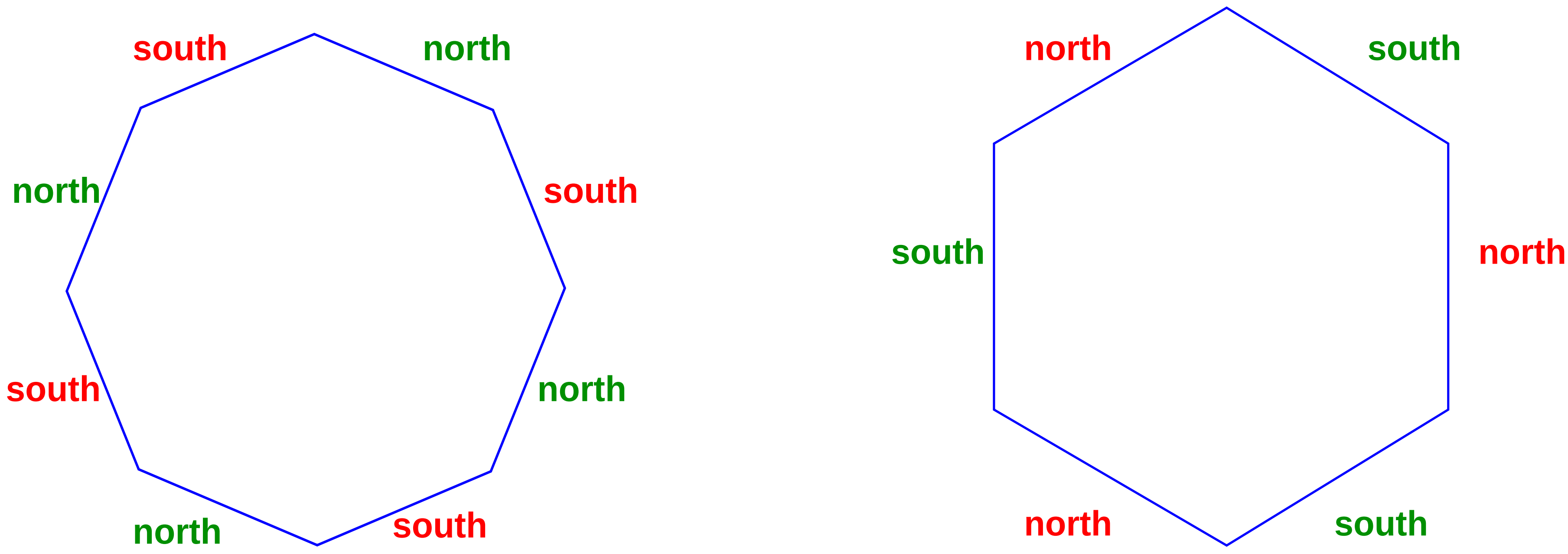
$$\theta_N \quad = \quad \frac{3\pi}{2n} + \alpha_n; \qquad \frac{7\pi}{2n} + \alpha_n; \qquad \frac{11\pi}{2n} + \alpha_n; \ldots \qquad \text{(North poles)}$$

A focusing Quadrupole ($n = 2,\ \alpha_2 = 0$):

$$\theta_S \quad = \quad 45\,\text{deg}; \qquad 225\,\text{deg}; \qquad 405\,\text{deg}; \ldots.. \qquad \text{(South poles)}$$

$$\theta_N \quad = \quad 135\,\text{deg}; \qquad 315\,\text{deg}; \qquad 495\,\text{deg}; \ldots.. \qquad \text{(North poles)}$$

(What if: $\alpha_2 = \frac{\pi}{2}$ or $\alpha_2 = \frac{\pi}{4}$ **???)**



A little exercise: what are n**,** Θ**,** α_n **?**

If Cartesian coordinates are preferred:

$$\boxed{\begin{aligned} B_x(\theta, r) &= B_r \cos\theta - B_\theta \sin\theta = \sum_{n=1}^{\infty} C(n) \left(\frac{r}{R_{ref}}\right)^{n-1} \sin[(n-1)\theta - n\alpha_n)] \\ B_y(\theta, r) &= B_r \sin\theta + B_\theta \cos\theta = \sum_{n=1}^{\infty} C(n) \left(\frac{r}{R_{ref}}\right)^{n-1} \cos[(n-1)\theta - n\alpha_n)] \end{aligned}}$$

or defined as complex field* :

$$\boxed{\vec{B}(z) = B_y(x, y) + iB_x(x, y) = \sum_{n=1}^{\infty} [C(n) \exp(-in\alpha_n)] \left(\frac{z}{R_{ref}}\right)^{n-1}}$$

then one gets a "physical" picture:

$$C(n) \exp(-in\alpha_n) = (2n \;\; pole)_{\text{normal}} + i(2n \;\; pole)_{\text{skew}} = b_n + ia_n$$

$$\boxed{\textbf{Example:} \quad \textbf{for} \quad \alpha_2 = \frac{\pi}{4} \neq 0 \quad \Rightarrow \quad b_2 = 0, \quad a_2 = 1}$$

* **Euler:** $z = x + iy = r \cdot \exp(i\theta) = r(\cos(\theta) + i \sin(\theta))$

Finally replacing $C(n)\exp(-in\alpha_n)$ by $(b_n + ia_n)$ (and some confusion):

$$B_y + iB_x = \sum_{n=0}^{\infty} (b_n + i \cdot a_n) \left(\frac{r}{R_{ref}} \right)^{n} \qquad \text{(U.S. convention)}$$

$$B_y + iB_x = \sum_{n=1}^{\infty} (b_n + i \cdot a_n) \left(\frac{r}{R_{ref}} \right)^{n-1} \qquad \text{(European and LHC convention)}$$

more physical significance (and more confusion):

$$b_{n+1} = \frac{R_{ref}^n}{n!} \left(\frac{\partial^n B_y}{\partial x^n} \right) \qquad [\ = b_n\ (U.S.)\ ]$$

$$a_{n+1} = \frac{R_{ref}^n}{n!} \left(\frac{\partial^n B_x}{\partial x^n} \right) \qquad [\ = a_n\ (U.S.)\ ]$$

Lorentz force on charged particles

So far: Lorentz force is added to Maxwell equations !

From experience and experiments, can not be derived/understood without Relativity (but then it comes out easily, see lecture !)

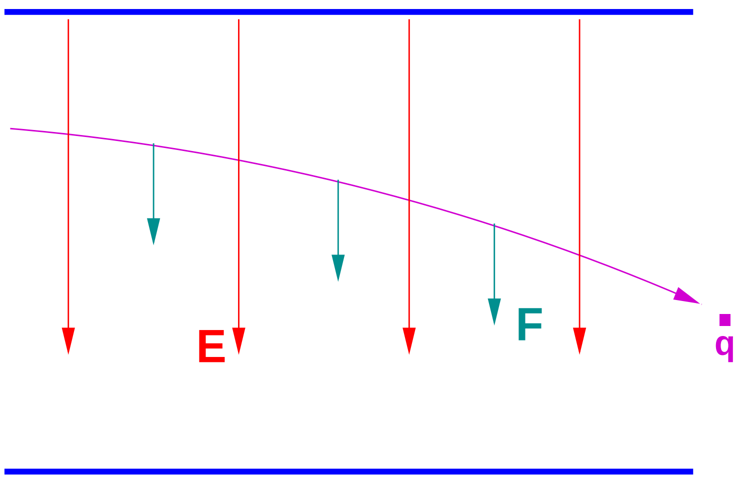
Moving ($\vec{v}$) charged (q) particles in electric ($\vec{E}$) and magnetic ($\vec{B}$) fields experience the Lorentz force $\vec{f}$:

$$\vec{f} \;=\; q \cdot (\vec{E} \;+\; \vec{v} \;\times\; \vec{B})$$

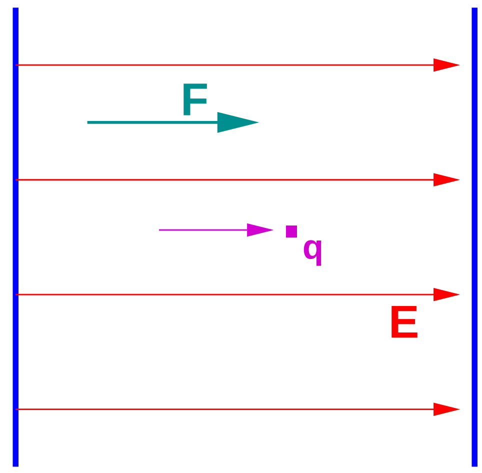
for the equation of motion we get (using Newton's law);

$$\frac{d}{dt}(m\vec{v}) \;=\; \vec{f} \;=\; q \cdot (\vec{E} \;+\; \vec{v} \;\times\; \vec{B})$$

Motion in electric fields



$\vec{v} \perp \vec{E}$



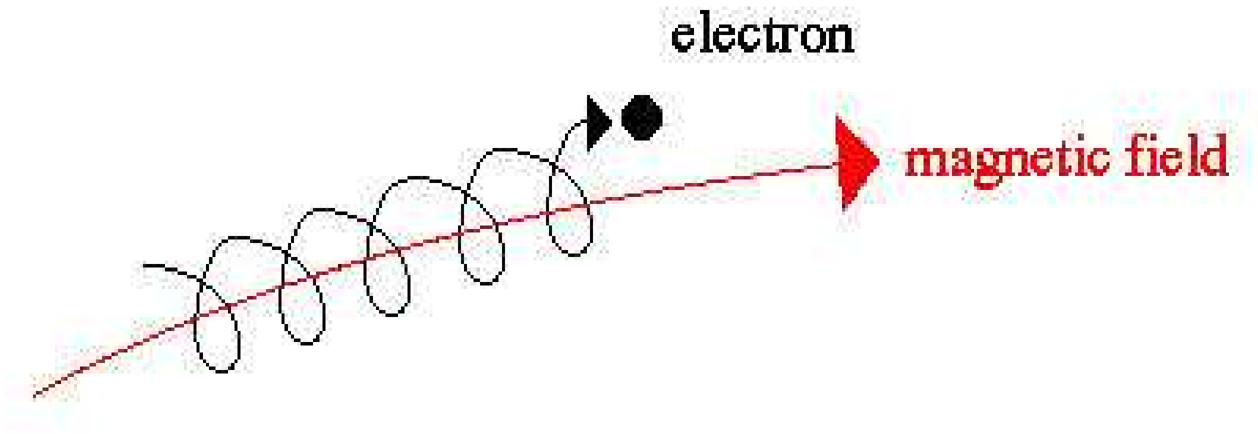
$\vec{v} \parallel \vec{E}$

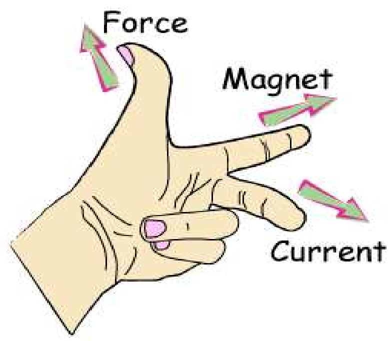
Assume no magnetic field:

$$\frac{d}{dt}(m\vec{v}) \quad = \quad \vec{f} \quad = \quad q \cdot \vec{E}$$

Force always in direction of field $\vec{E}$, also for particles at rest.

Motion in magnetic fields





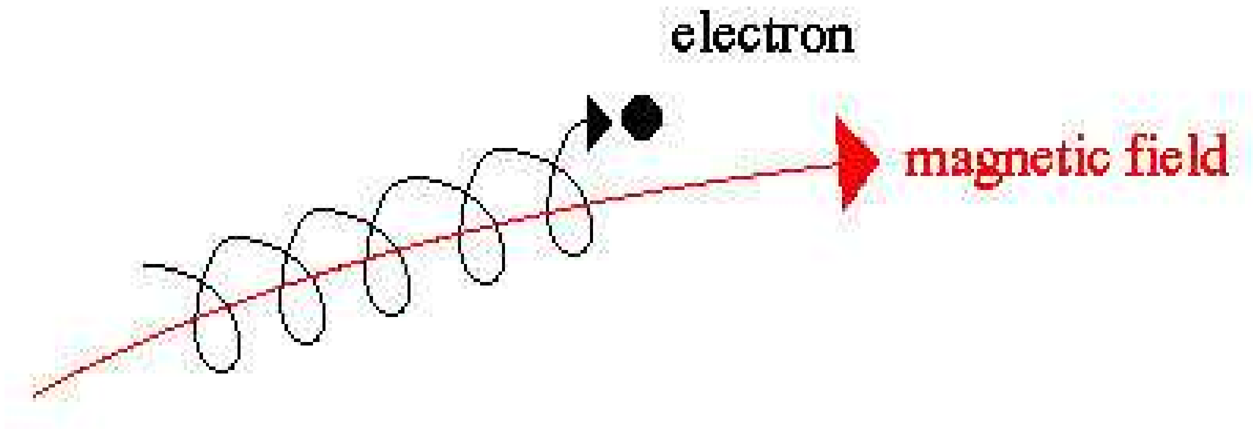
Without electric field : $$\frac{d}{dt}(m\vec{v}) \;=\; \vec{f} \;=\; q \cdot \vec{v} \times \vec{B}$$

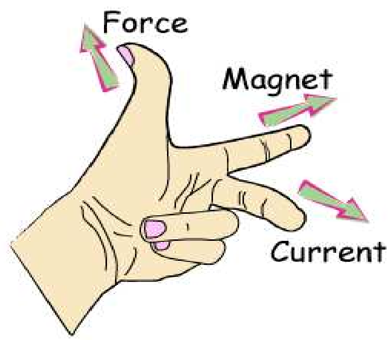
Force is perpendicular to both, $\vec{v}$ and $\vec{B}$

<u>No</u> force on particles at rest - do we understand that ?

Or is it just a empirical story to get the right answer ?

Motion in magnetic fields





Without electric field : $$\frac{d}{dt}(m\vec{v}) \;=\; \vec{f} \;=\; q \cdot \vec{v} \times \vec{B}$$

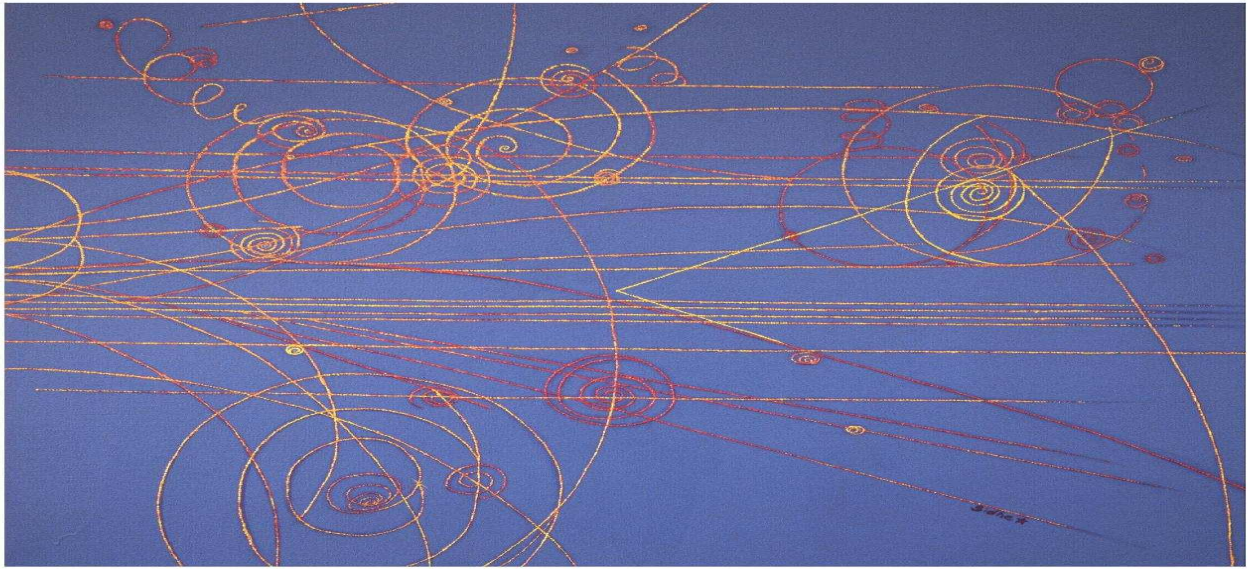
Force is perpendicular to both, $\vec{v}$ and $\vec{B}$

No force on particles at rest - do we understand that ?

Or is it just a empirical story to get the right answer ?

Yes, but see next lecture ...

Particle motion in magnetic fields - made visible



Magnetic field perpendicular to motion

Bending radius depends on <u>momentum</u> and <u>charge</u>

- **Direction of the magnetic field $\vec{B}$???**

Practical units:

$$B\ [T] \cdot \rho\ [m] \quad = \quad \frac{p\ [eV/c]}{c\ [m/s]}$$

Example LHC:

B = 8.33 T, p = 7^{12} **eV/c** $\rightarrow$ ρ **= 2804 m**

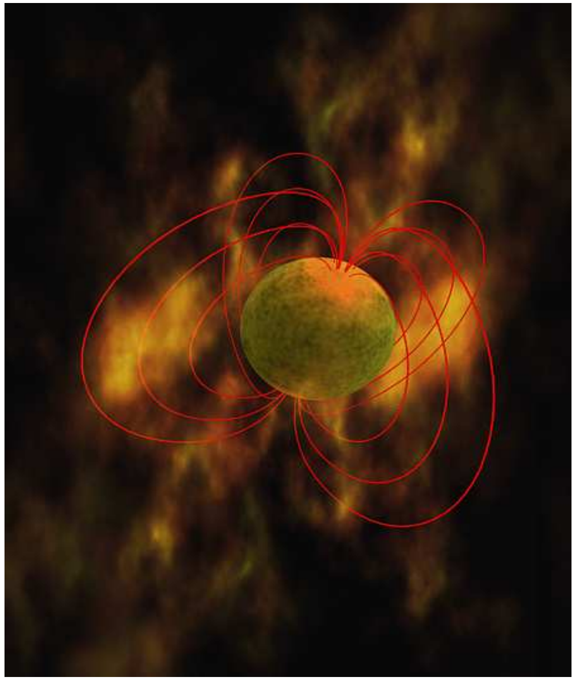
More - bending angle α **of a dipole magnet of length** L**:**

$$\alpha \;=\; \frac{B\ [T] \cdot L\ [m] \cdot 0.3}{p\ [GeV/c]}$$

Example LHC:

B = 8.33 T, p = 7000 GeV/c, L = 14.3 m $\rightarrow$ α **= 5.11 mrad**

... and some <u>really</u> strong magnetic fields



Example : CXOUJ164710.2 – 45516

Diameter : 10 – 20 km

Field : $\approx 10^{12}$ Tesla

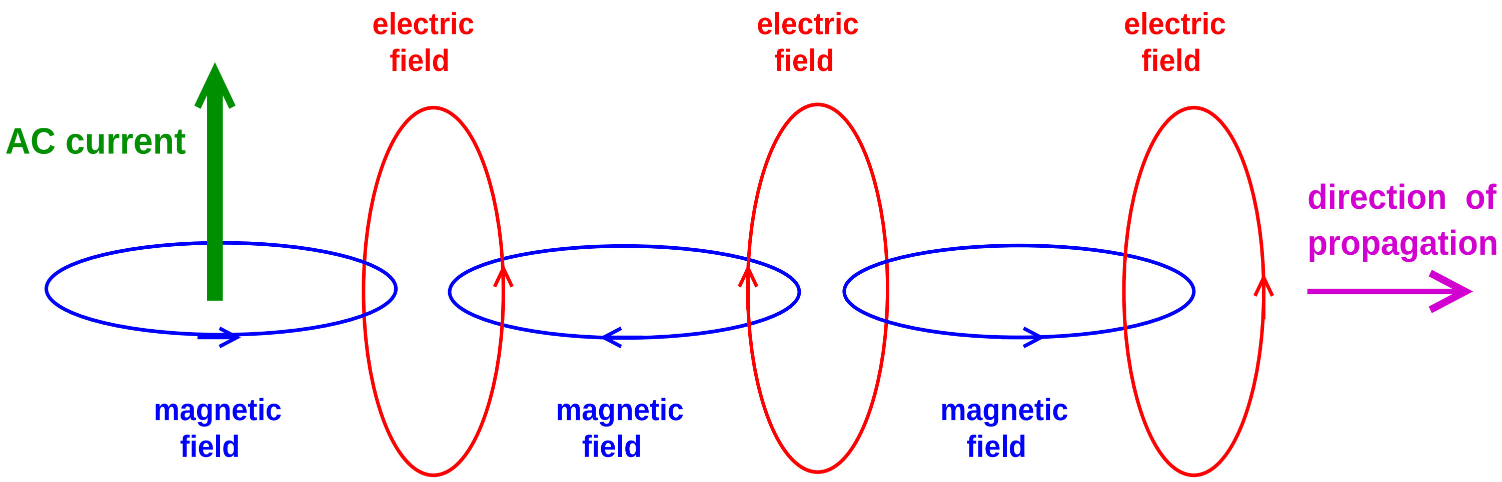
As accelerator : $\approx 10^{12}$ TeV

Very fast time varying electromagnetic fields - γ-ray bursts up to 10^{40} W (sun: $\approx 4 \cdot 10^{26}$ W)

<u>Punchline:</u>

Grand Unification (world formula) may show up near 10^{12} TeV

Time Varying Fields - (Maxwell 1864)



Time varying magnetic fields produce circular electric fields

Time varying electric fields produce circular magnetic fields

- Can produce self-sustaining, propagating fields (i.e. waves)
- Rather useful picture in the classical framework, but ...

In vacuum: only fields, no charges ($\rho = 0$), no current ($j = 0$) ...

From Maxwell's (III and IV) and some educated guess:

$$\nabla \times (\nabla \times \vec{B}) \quad = \quad \nabla^2 \vec{E} \quad = \quad - \nabla \times (\frac{\partial \vec{B}}{\partial t}) \quad = \quad - \frac{\partial}{\partial t}(\nabla \times \vec{B})$$

$$\Longrightarrow \quad \nabla^2 \vec{E} \quad = \quad \frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \quad = \quad \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\boxed{\nabla^2 \vec{E} \;=\; \mu_0 \cdot \epsilon_0 \cdot \frac{\partial^2 \vec{E}}{\partial t^2} \;=\; \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}}$$

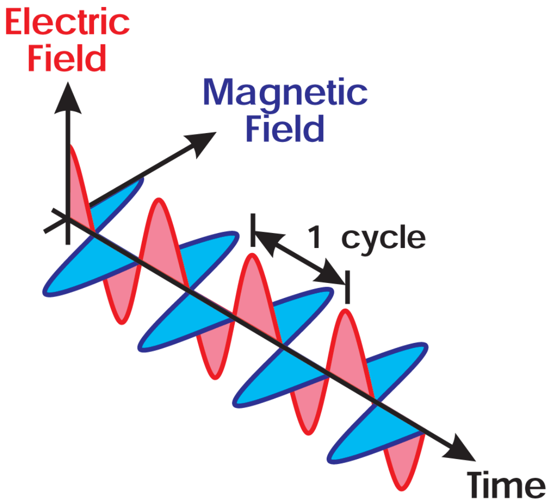
(correspondingly for $\vec{B}$)

➡ **Equation for a wave with speed:** $c = \dfrac{1}{\sqrt{\mu_0 \cdot \epsilon_0}}$

It has a different form for a resting and a moving system !!!
see lecture on Special Relativity why and how it saves the day ...

Electromagnetic waves

$$\vec{E} = \vec{E_0} e^{i(\omega t - \vec{k}\cdot\vec{x})}$$
$$\vec{B} = \vec{B_0} e^{i(\omega t - \vec{k}\cdot\vec{x})}$$



Important quantities :

$\vec{k}$ = (propagation vector)

λ = (wave length, 1 cycle)

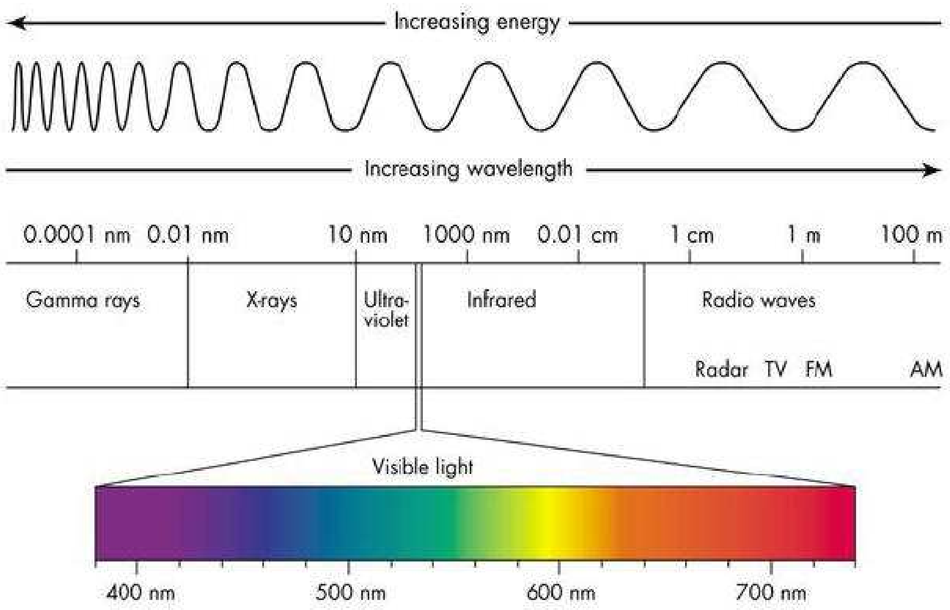
ω = (frequency $\cdot\, 2\pi$)

$\vec{k}\cdot\vec{k} = \dfrac{\omega^2}{c^2}$ (Dispersion Relation)

Magnetic and electric fields are transverse to direction of propagation:
$\vec{E} \perp \vec{B} \perp \vec{k}$

Short wave length $\rightarrow$ high frequency $\rightarrow$ high energy

Spectrum of Electromagnetic waves



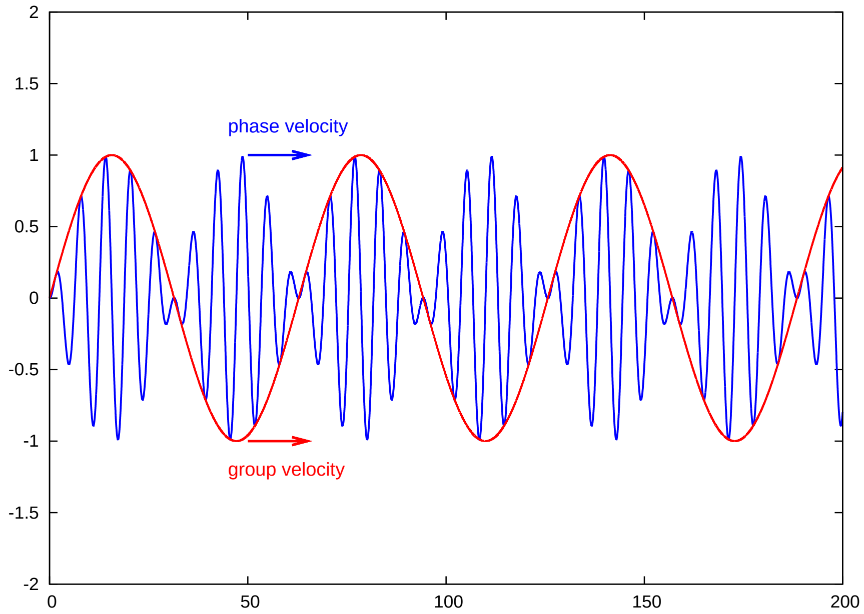
Example: yellow light ➡ $\approx\ 5\cdot10^{14}$ **Hz (i.e.** $\approx$ **2 eV !)**

LEP (SR) ➡ $\leq\ 2\cdot10^{20}$ **Hz (i.e.** $\approx$ **0.8 MeV !)**

gamma rays ➡ $\geq\ 3\cdot10^{20}$ **Hz (1 MeV to 10 TeV !)**

(For estimates using temperature: 2.71 K $\approx$ **0.00023 eV)**

Modulated wave: Group velocity and Phase velocity



carrier wave :

$$v_p = \frac{\omega}{k} \quad \text{(phase velocity)}$$

wave packet :

$$v_g = \frac{\partial \omega}{\partial k} \quad \text{(group velocity)}$$

Wave packet can be considered as superposition of a number of harmonic waves

- **Carrier wave moves at phase velocity, can be larger the c**
- **Wave packet moves at group velocity, carries information**

v_p **and** v_g **may or may not be equal depends on "dispersion relation"**

Energy in electromagnetic waves (in brief, details in [2, 3, 4]):

define Poynting vector : $\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$ (in direction of propagation)

"Energy flux": energy crossing a unit area, per second $[\frac{J}{m^2 s}]$

In free space: energy is shared between electric and magnetic field

The energy density U **would be:**

$$U_{EB} = \frac{1}{2} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right)$$

Some example: $B = \mathbf{5 \cdot 10^{-5}}$ **T (= 0.5 Gauss)**

➡ $U_B \approx$ **1 mJ/m**^{3} **(corresponds to** $\underline{\approx 10^{16} \gamma s}$ **at visible light)**

➡ **Classical Electrodynamics is a <u>very</u> good approximation ..**

Waves interacting with material

Need to look at the behaviour of electromagnetic fields at <u>boundaries between different materials</u> (air-glass, air-water, vacuum-metal, ...).

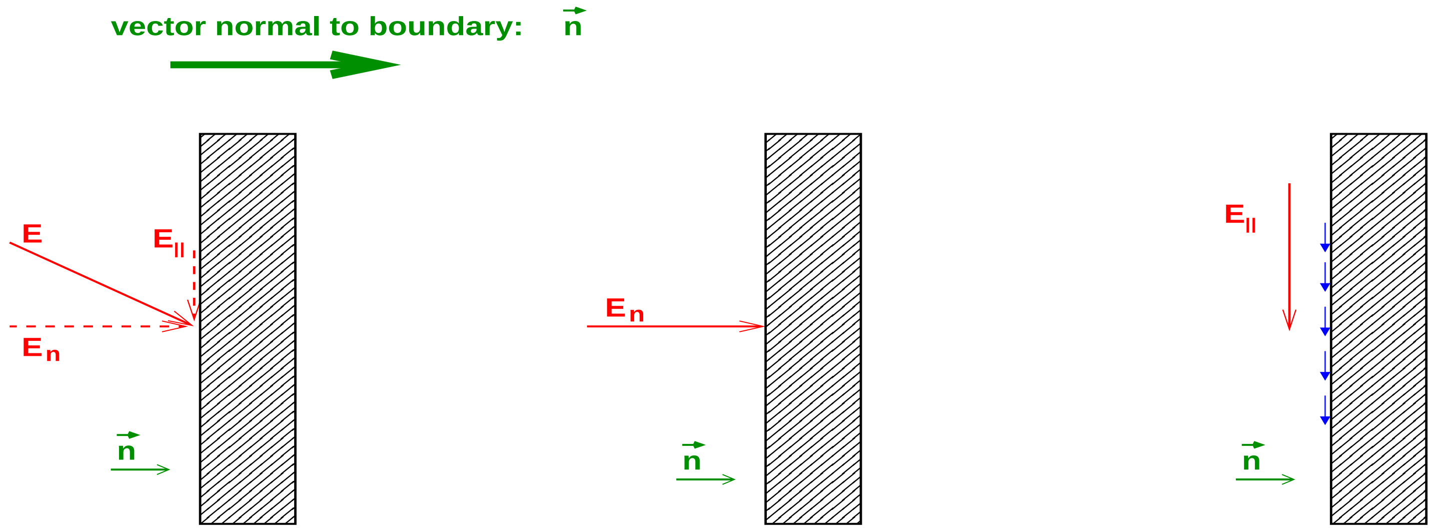
Have to consider two particular cases:

- **Ideal conductor (i.e. no resistance), apply to:**
  - **RF cavities**
  - **Wave guides**
- **Conductor with finite resistance, apply to:**
  - **Penetration and attenuation of fields in material (skin depth)**
  - **Impedance calculations**

Can be derived from Maxwell's equations, here only the results !

Boundary conditions: air/vacuum and conductor

A simple case ($\vec{E}$-fields on a conducting surface):



Field parallel to surface $E_{||}$ cannot exist (it would move charges and we get a surface current): $E_{||} \; = \; 0$

- **Only a field normal (orthogonal) to surface E_n is possible**

Extreme case: surface of ideal conductor

For an ideal conductor (i.e. no resistance) the tangential electric field must vanish Corresponding conditions for normal magnetic fields. We must have:

$$\vec{E}_t = 0, \qquad \vec{B}_n = 0$$

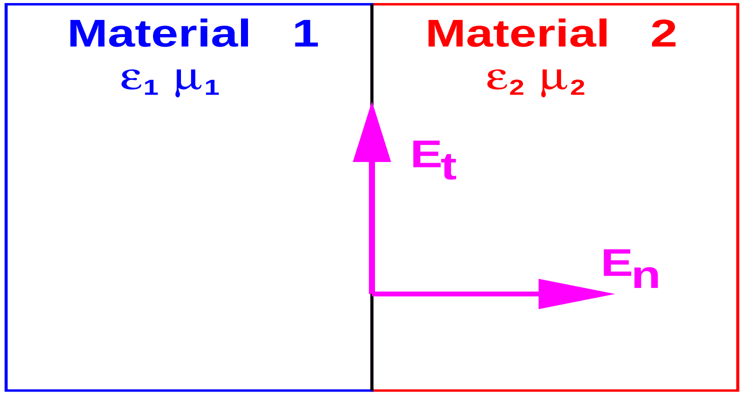
This implies:

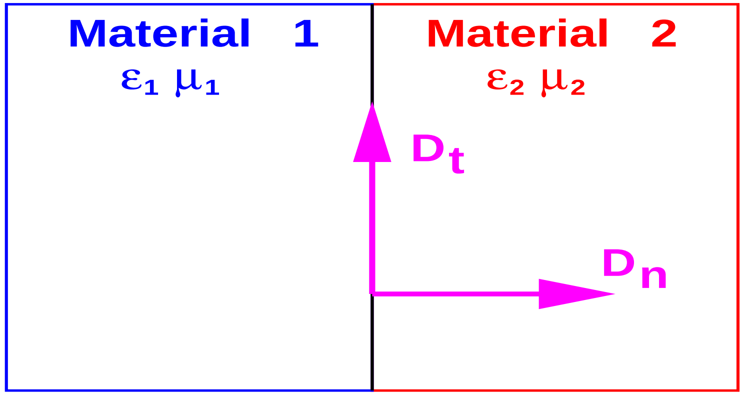
- Fields at any point in the conductor are zero.
- Only some field patterns are allowed in waveguides and RF cavities

A very nice lecture in R.P.Feynman, Vol. II

Now for Boundary Conditions between two different regions →

Boundary conditions for electric fields





Assuming no surface charges (proof e.g. [3, 5])*:

From curl $\vec{E} = 0$**:**

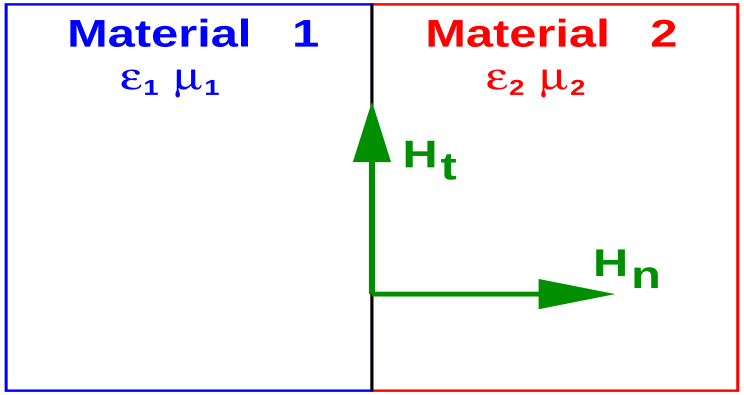
➡ **tangential $\vec{E}$-field continuous across boundary** $(E_t^1 \; = \; E_t^2)$

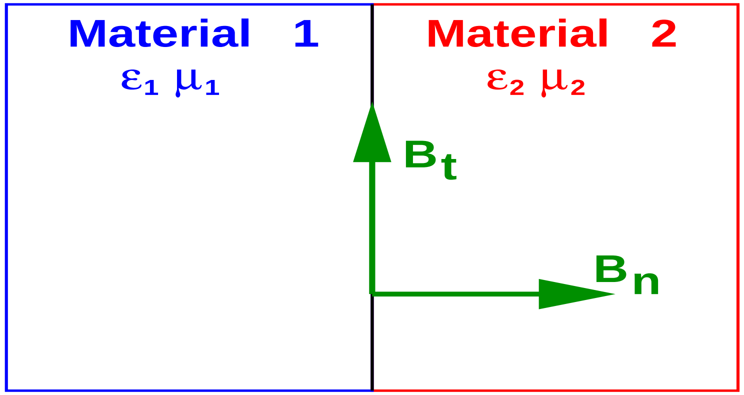
From div $\vec{D} = \rho$**:**

➡ **normal $\vec{D}$-field continuous across boundary** $(D_n^1 \; = \; D_n^2)$

* **with surface charges, see backup slides**

Boundary conditions for magnetic fields





Assuming no surface currents (proof e.g. [3, 5])*:

From curl $\vec{H} = \vec{j}$**:**

➡ **tangential $\vec{H}$-field continuous across boundary** $(H_t^1 \;=\; H_t^2)$

From div $\vec{B} = 0$**:**

➡ **normal $\vec{B}$-field continuous across boundary** $(B_n^1 \;=\; B_n^2)$

* with surface current, see backup slides

Summary: boundary conditions for fields

Electromagnetic fields at boundaries between different materials with different permittivity and permeability ($\epsilon_1, \epsilon_2, \mu_1, \mu_2$).

$$(E_t^1 \;=\; E_t^2), \quad (E_n^1 \;\neq\; E_n^2) \qquad \text{or :} \qquad (\vec{E}_2 - \vec{E}_1) \;\times\; \vec{n} \;=\; 0$$

$$(D_t^1 \;\neq\; D_t^2), \quad (D_n^1 \;=\; D_n^2) \qquad \text{or :} \qquad (\vec{D}_2 - \vec{D}_1) \;\cdot\; \vec{n} \;=\; \rho_{surface}$$

$$(H_t^1 \;=\; H_t^2), \quad (H_n^1 \;\neq\; H_n^2) \qquad \text{or :} \qquad (\vec{H}_2 - \vec{H}_1) \;\times\; \vec{n} \;=\; \vec{J}_{surface}$$

$$(B_t^1 \;\neq\; B_t^2), \quad (B_n^1 \;=\; B_n^2) \qquad \text{or :} \qquad (\vec{B}_2 - \vec{B}_1) \;\cdot\; \vec{n} \;=\; 0$$

where $\vec{n}$ is the unit vector pointing into medium 2
(derivation deserves its own lectures, just believe it)

They determine: reflection, refraction and refraction index n.

Waves in material → **Index of refraction: n**

Speed of electromagnetic waves in vacuum: $c = \frac{1}{\sqrt{\mu_0 \cdot \epsilon_0}}$

$$\text{n} = \frac{\text{Speed of light in vacuum}}{\text{Speed of light in material}} = \frac{c}{v_p}$$

For water n ≈ 1.33

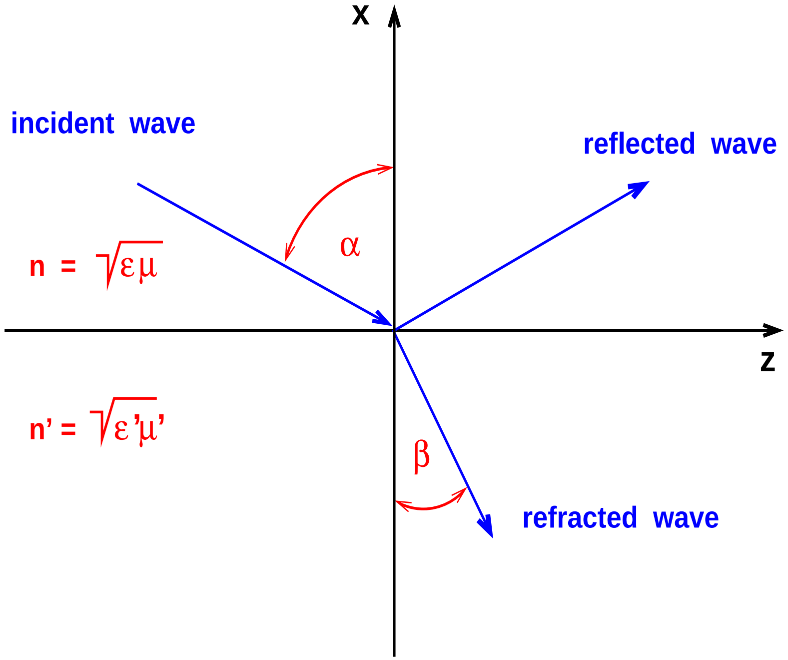
Depends on wavelength

n $\approx 1.32 - 1.39$

<u>some others</u>	**ice:**	**1.31**
	alcohol:	**1.36**
	sugar solution:	**1.49**
	glass:	**1.51**

(propose an experiment !)

Reflection and refraction angles related to the refraction index n and n':



$$\frac{\sin\alpha}{\sin\beta} = \frac{n'}{n} = \tan\alpha_B$$

If light is incident under angle α_B [3]:
Reflected light is linearly polarized perpendicular to plane of incidence

(Application: fishing ➡ air-water gives $\alpha_B \approx 53^o$)

Polarization of EM waves (Classical Picture !):

The solutions of the wave equations imply monochromatic plane waves:

$$\vec{E} = \vec{E_0} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \vec{B} = \vec{B_0} e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

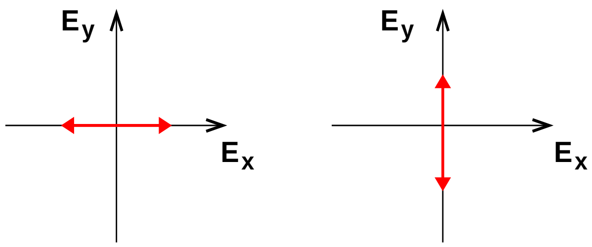
As defined, consider only electric field, re-written using unit vectors in the plane transverse to propagation: $\vec{\epsilon_1} \perp \vec{\epsilon_2}$

Two Components: $\vec{E_1} = \vec{\epsilon_1} E_1 \; e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \vec{E_2} = \vec{\epsilon_2} E_2 \; e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

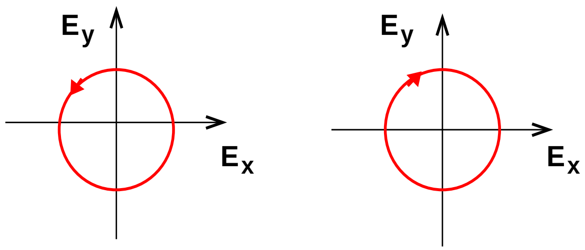
$$\Rightarrow \vec{E} = (\vec{E_1} + \vec{E_2}) = (\vec{\epsilon_1} E_1 + \vec{\epsilon_2} E_2) \; e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

With a relative phase ϕ between the two directions:

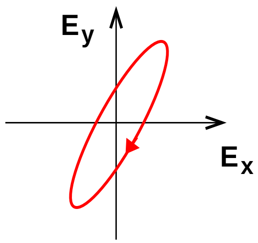
$$\vec{E} = \vec{\epsilon_1} E_1 \; e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \vec{\epsilon_2} E_2 \; e^{i(\vec{k}\cdot\vec{r}-\omega t+\phi)}$$



linear : $E_2 = 0$ or $E_1 = 0$



circular : $\phi \pm \frac{\pi}{2}$



elliptical : ϕ e.g. $\frac{\pi}{4}$

Polarized light - why is it interesting:

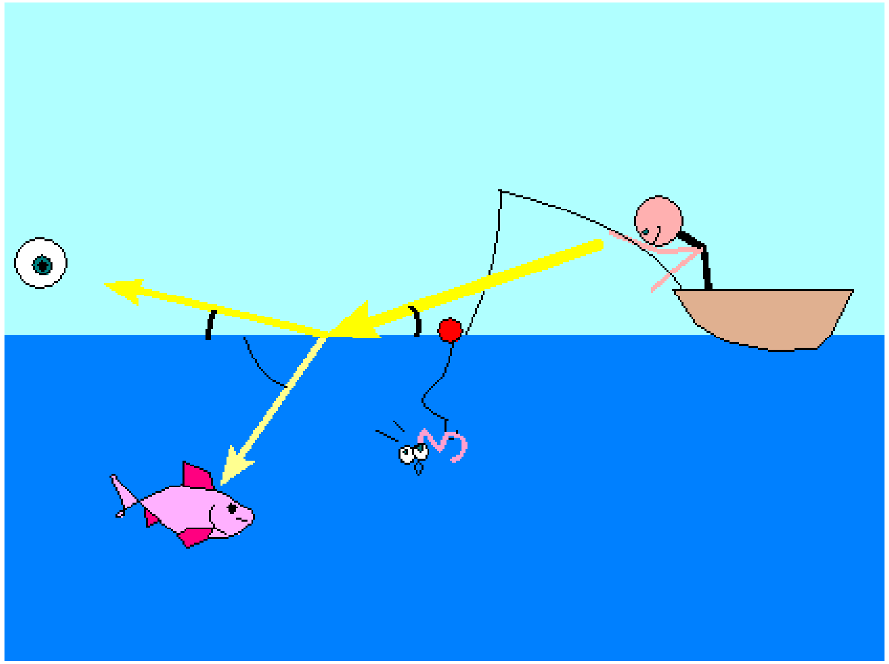
Produced (amongst others) in Synchrotron light machines (linearly and circularly polarized light, adjustable) blue sky (!), reflections (e.g. water surface)

Accelerator and other applications:

- Polarized light reacts differently with charged particles
- Radio communication, capacity doubling
- Beam diagnostics, medical diagnostics (blood sugar, ..)
- Inverse FEL
- 3-D motion pictures, LCD display, cameras (reduce glare)
- Outdoor activities (e.g. Fishing, driving a car through a rainy night, ...)
- ...

Less classical, in Quantum Electro Dynamics: photons with spin +1 or -1

Practical application:



- **Some of the light is reflected**
- **Some of the light is transmitted and refracted**

For α_B, can be largely supressed using polarised glasses

Extreme case: ideal conductor

For an ideal conductor (i.e. no resistance) we must have:

$$\vec{E}_{\|} \;=\; 0, \qquad \vec{B}_n \;=\; 0$$

otherwise the surface current becomes infinite

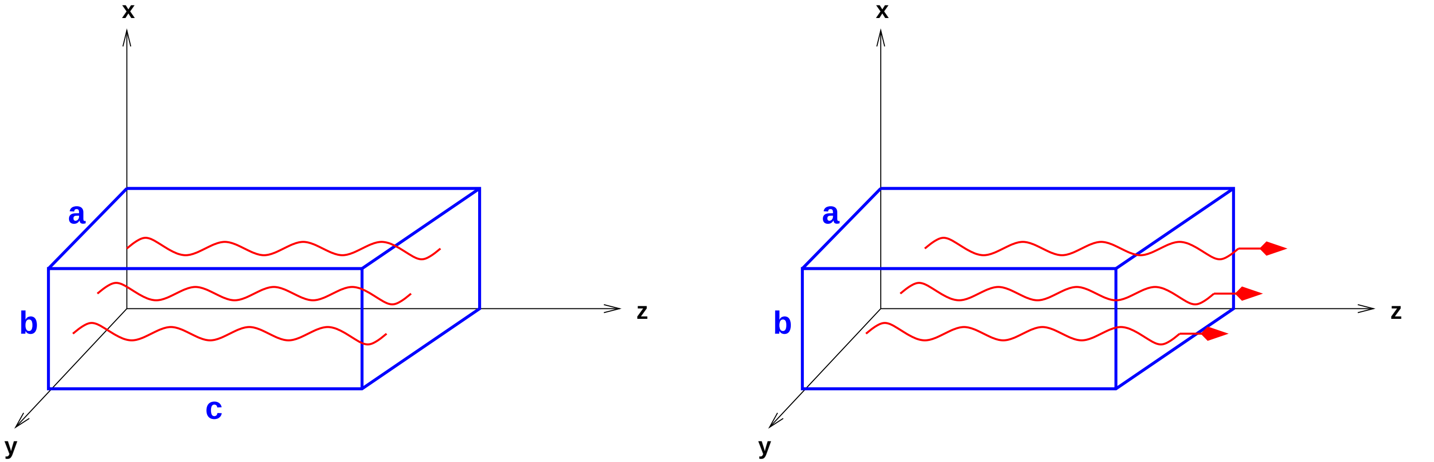
This implies:

- **All energy of an electromagnetic wave is reflected from the surface of an <u>ideal</u> conductor.**
- **Fields at any point <u>in</u> the <u>ideal</u> conductor are zero.**
- **Only some fieldpatterns are allowed in waveguides and RF cavities**

A very nice lecture in R.P.Feynman, Vol. II

Examples: cavities and wave guides

Rectangular, conducting cavities and wave guides (schematic) with dimensions $a \times b \times c$ and $a \times b$:



- **RF cavity, fields can persist and be stored (reflection !)**
- **Plane waves can propagate along wave guides, here in z-direction**

(here just the basics, many details in "RF Systems" by Frank Tecker)

Fields in RF cavities - as reference

Assume a rectangular RF cavity ($a,\ b,\ c$), ideal conductor.

Without derivations, the components of the fields are:

$$E_x = E_{x0} \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot \sin(k_z z) \cdot e^{-i\omega t}$$

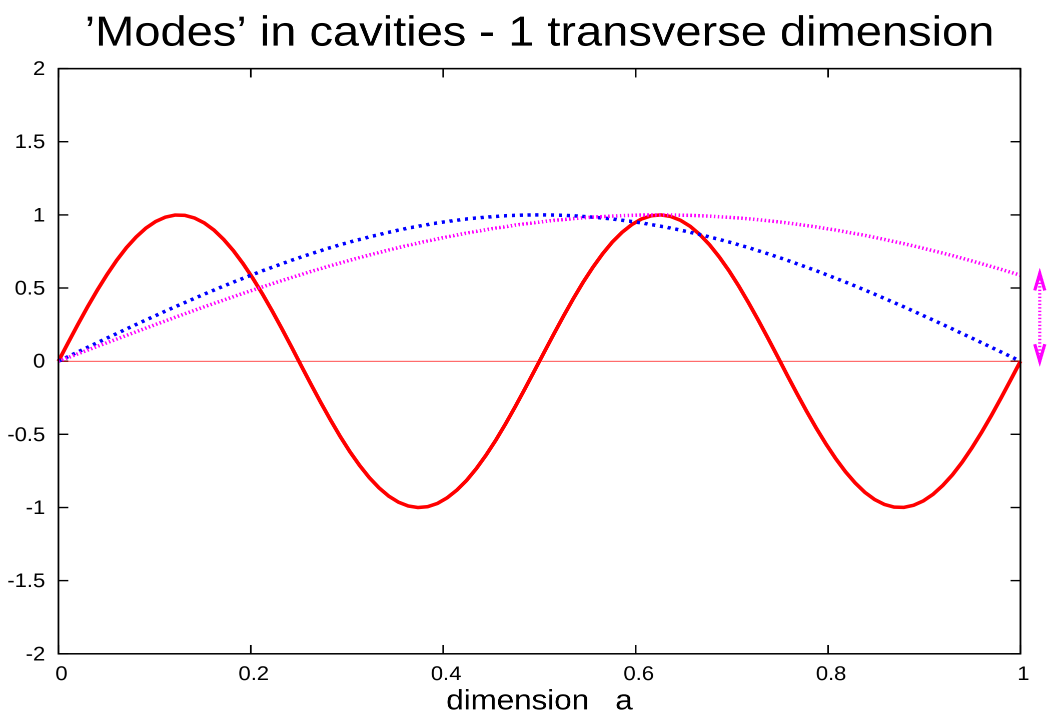
$$E_y = E_{y0} \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot \sin(k_z z) \cdot e^{-i\omega t}$$

$$E_z = E_{z0} \cdot \sin(k_x x) \cdot \sin(k_y y) \cdot \cos(k_z z) \cdot e^{-i\omega t}$$

$$B_x = \frac{i}{\omega}(E_{y0}k_z - E_{z0}k_y) \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot \cos(k_z z) \cdot e^{-i\omega t}$$

$$B_y = \frac{i}{\omega}(E_{z0}k_x - E_{x0}k_z) \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot \cos(k_z z) \cdot e^{-i\omega t}$$

$$B_z = \frac{i}{\omega}(E_{x0}k_y - E_{y0}k_x) \cdot \cos(k_x x) \cdot \cos(k_y y) \cdot \sin(k_z z) \cdot e^{-i\omega t}$$



No electric field at boundaries, wave must have "nodes" = zero fields at the boundaries

Only modes which 'fit' into the cavity are allowed

In the example: $\frac{\lambda}{2} = \frac{a}{4}$, $\quad \frac{\lambda}{2} = \frac{a}{1}$, $\quad \frac{\lambda}{2} = \frac{a}{0.8}$

(then either "sin" or "cos" is 0)

Consequences for RF cavities

Field must be zero at conductor boundary, only possible if:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

and for k_x, k_y, k_z **we can write, (then they all fit):**

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b}, \qquad k_z = \frac{m_z \pi}{c},$$

The integer numbers m_x, m_y, m_z **are called mode numbers, important for design of cavity !**

$\rightarrow$ **half wave length** $\lambda/2$ **must always fit exactly the size of the cavity.**

(For cylindrical cavities: use cylindrical coordinates)

Wave guides and cavities are more likely to be circular:

Derivation using the Laplace equation in cylindrical coordinates, for a derivation see e.g. [2, 3]:

$$
\begin{aligned}
E_r &= E_0\ \frac{k_z}{k_r} J_n'(k_r) \cdot \cos(n\theta) \cdot \sin(k_z z) \cdot e^{-i\omega t} \\
E_\theta &= E_0\ \frac{nk_z}{k_r^2 r} J_n(k_r) \cdot \sin(n\theta) \cdot \sin(k_z z) \cdot e^{-i\omega t} \\
E_z &= E_0\ J_n(k_r r) \cdot cos(n\theta) \cdot sin(k_z z) \cdot e^{-i\omega t}
\end{aligned}
$$

$$
\begin{aligned}
B_r &= iE_0\ \frac{\omega}{c^2 k_r^2 r} J_n(k_r r) \cdot \sin(n\theta) \cdot \cos(k_z z) \cdot e^{-i\omega t} \\
B_\theta &= iE_0\ \frac{\omega}{c^2 k_r r} J_n'(k_r r) \cdot \cos(n\theta) \cdot \cos(k_z z) \cdot e^{-i\omega t} \\
B_z &= 0
\end{aligned}
$$

2mm Homework: write it down for wave guides ..

Accelerating circular cavities

For accelerating cavities we need longitudinal electric field component $E_z \neq 0$ and magnetic field purely transverse.

$$
\begin{aligned}
E_r &= 0 \\
E_\theta &= 0 \\
E_z &= E_0 \; J_0(p_{01}\frac{r}{R}) \cdot e^{-i\omega t}
\end{aligned}
$$

$$
\begin{aligned}
B_r &= 0 \\
B_\theta &= -iE_0 \; J_1(p_{01}\frac{r}{R}) \cdot e^{-i\omega t} \\
B_z &= 0
\end{aligned}
$$

(p_{nm} is the mth zero of J_n, e.g. $p_{01} \approx$ 2.405)

This would be a cavity with a TM$_{010}$ mode: $\omega_{010} = p_{01} \cdot \frac{c}{R}$

Similar considerations lead to (propagating) solutions in (rectangular) wave guides:

$$E_x = E_{x0} \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$E_y = E_{y0} \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$E_z = i \cdot E_{z0} \cdot \sin(k_x x) \cdot \sin(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$B_x = \frac{1}{\omega}(E_{y0} k_z - E_{z0} k_y) \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$B_y = \frac{1}{\omega}(E_{z0} k_x - E_{x0} k_z) \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$B_z = \frac{1}{i \cdot \omega}(E_{x0} k_y - E_{y0} k_x) \cdot \cos(k_x x) \cdot \cos(k_y y) \cdot e^{i(k_z z - \omega t)}$$

This part is new: $e^{i(k_z z)}$ $\Longrightarrow$ **something moving in z direction**

In z direction: No Boundary - No Boundary Condition ...

Consequences for wave guides

Similar considerations as for cavities, no field at boundary.

We must satisfy again the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2} \qquad \Longrightarrow \qquad k_z = \sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}$$

This leads to modes like (no boundaries in direction of propagation, only k_x and k_y have to "fit"):

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b}, \qquad k_z = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{m_x \pi}{a}\right)^2 - \left(\frac{m_y \pi}{b}\right)^2}$$

The numbers m_x, m_y are called mode numbers for waves in wave guides !

Consequences for wave guides

Similar considerations as for cavities, no field at boundary.

We must satisfy again the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2} \qquad \Longrightarrow \qquad k_z = \sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}$$

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The numbers m_x, m_y are called mode numbers for waves in wave guides !

Should we worry about k_z ?? Argument of root can become negative .. !

Propagation without losses requires k_z to be real, i.e.:

$$\frac{\omega^2}{c^2} > \left(\frac{m_x \pi}{a}\right)^2 + \left(\frac{m_y \pi}{b}\right)^2 \quad \Rightarrow \quad \boxed{\omega > \pi c \sqrt{\left(\frac{m_x}{a}\right)^2 + \left(\frac{m_y}{b}\right)^2}}$$

defines a (minimum) <u>cut-off frequency</u> ω_c:

$$\boxed{\omega_c = \frac{\pi \cdot c}{a}}$$

(with a the <u>longest</u> side length of the wave guide)

- **<u>Above</u> cut-off frequency: propagation without loss**
- **<u>At</u> cut-off frequency: standing wave**
- **<u>Below</u> cut-off frequency: attenuated wave (it does not "fit in").**

(if bored : try to compute v_p and v_g for the limits : $\omega \rightarrow \infty$ and $\omega \rightarrow \omega_c$)

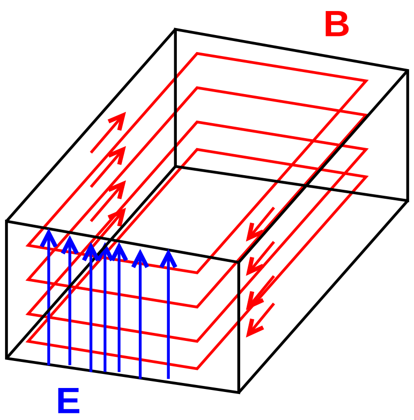
Classification of modes:

Transverse electric modes (TE): $E_z = 0 \quad H_z \neq 0$

Transverse magnetic modes (TM): $E_z \neq 0 \quad H_z = 0$

Transverse electric-magnetic modes (TEM): $E_z = 0 \quad H_z = 0$

(Not all of them can be used for acceleration ... !)



Note (here a TE mode) :

Electric field lines end at boundaries

Magnetic field lines appear as "loops"

Other case: finite conductivity

Starting from Maxwell equation:

$$\nabla \times \vec{B} = \mu \vec{j} \; + \; \mu\epsilon \frac{d\vec{E}}{dt} \; = \; \underbrace{\overbrace{\sigma \cdot \vec{E}}^{\vec{j}}}_{Ohm's\ law} + \mu\epsilon \frac{d\vec{E}}{dt}$$

Wave equations:

$$\vec{E} \; = \; \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \qquad \vec{B} \; = \; \vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

We want to know k with this new contribution:

$$\boxed{k^2 \; = \; \frac{\omega^2}{c^2} \; - \; \underbrace{i\omega\sigma\mu}_{new}}$$

Consequence → Skin Depth

Electromagnetic waves can now penetrate <u>into</u> the conductor !

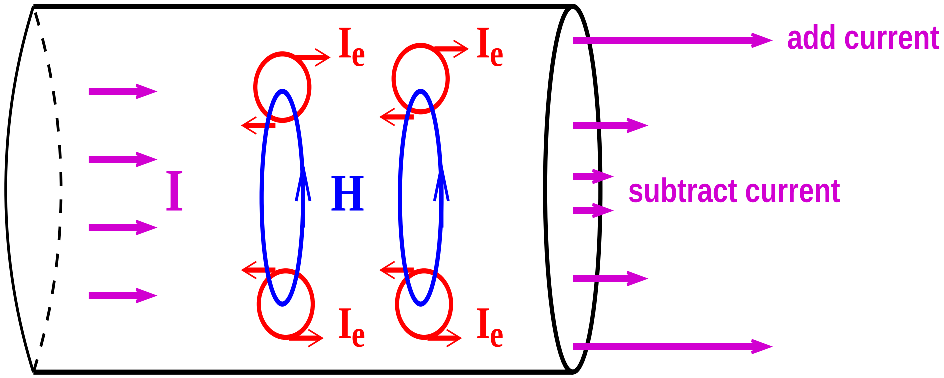
For a good conductor $\sigma \gg \omega\epsilon$:

$$k^2 \approx -i\omega\mu\sigma \quad \longrightarrow \quad k \approx \sqrt{\frac{\omega\mu\sigma}{2}}(1+i) = \frac{1}{\delta}(1+i)$$

$$\boxed{\delta = \sqrt{\frac{2}{\omega\mu\sigma}}}$$ **is the <u>Skin Depth</u>**

- **High frequency waves "avoid" penetrating into a conductor, flow near the surface**
- **Penetration depth small for large conductivity**

"Explanation" - inside a conductor (very schematic)



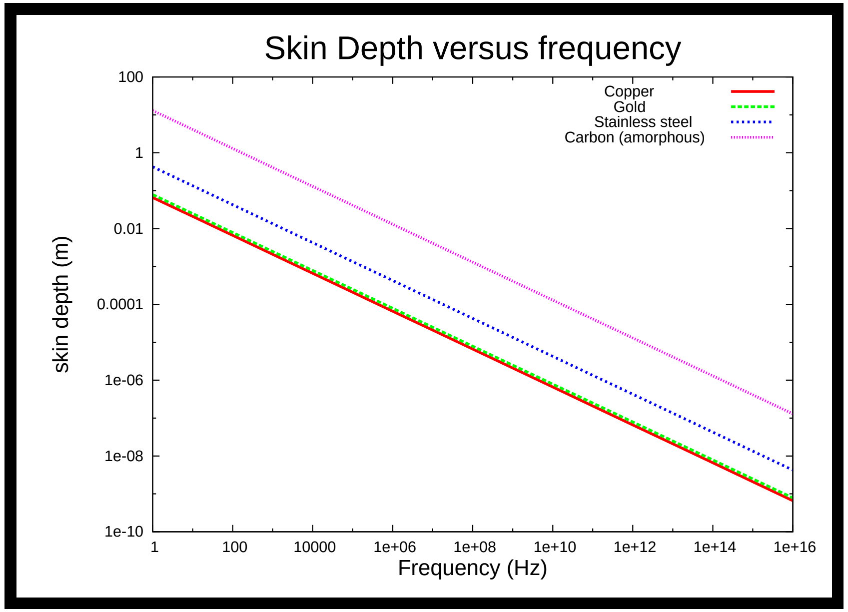
Eddy currents $\mathbf{I}_E$ **from changing** $\vec{H}$**-field:** $\nabla \times \vec{E} \;=\; \mu_0 \dfrac{d\vec{H}}{dt}$

Cancel current flow in the centre of the conductor $\quad \mathbf{I} - \mathbf{I_e}$

Enforce current flow near the "skin" (surface) $\quad \mathbf{I} + \mathbf{I_e}$

Q: Why are high frequency cables thin ??

Attenuated waves - penetration depth



Waves incident on conducting material are attenuated

Basically by the Skin depth :

(attenuation to 1/e)

$$\boxed{\delta = \sqrt{\frac{2}{\omega\mu\sigma}}}$$

Wave form:

$$e^{i(kz-\omega t)} = e^{i((1+i)z/\delta-\omega t)} = e^{\frac{-z}{\delta}} \cdot e^{i(\frac{z}{\delta}-\omega t)}$$

Values of δ can have a very large range ..

- **Skin depth Copper ($\sigma \approx 6 \cdot 10^7$ S/m):**

  2.45 GHz: $\delta \approx$ 1.5 μm, 50 Hz: $\delta \approx$ 10 mm

- **Penetration depth Glass (strong variation, σ typically $6 \cdot 10^{-13}$ S/m):**

  2.45 GHz: $\delta >$ km

- **Penetration depth Sushi (strong variation, σ typically $3 \cdot 10^{-2}$ S/m):**

  2.45 GHz: $\delta \approx$ 6 cm

- **Penetration depth Seawater ($\sigma \approx 4$ S/m):**

  76 Hz: $\delta \approx$ 25 - 30 m (Design an antenna !!, <u>very</u> low bandwidth)

Done list:

1. **Review of basics and write down Maxwell's equations**
2. **Electromagnetic fields in vacuum and in material**
3. **Add Lorentz force and motion of particles in EM fields**
4. **Electromagnetic waves in vacuum**
5. **Electromagnetic waves in conducting media**
   - **Waves in RF cavities**
   - **Waves in wave guides**
   - **Important concepts: mode numbers, cut-off frequency, skin depth**

But still a few (important) problems to sort out →

What next:

- **We have to deal with moving charges in accelerators**
- **Applied to moving charges Maxwell's equations are not compatible with observations of electromagnetic phenomena**
- **Electromagnetism and laws of classical mechanics are inconsistent**
- **Ad hoc introduction of Lorentz force**

Needed: A formulation to solve these problems

The fix: Special Relativity (next lecture)

- **Formulate Maxwell's equations relativistically invariant**
- **Problems solved (easily !) ...**

- BACKUP SLIDES -

Relativity and electrodynamics

- Back to the start: electrodynamics and Maxwell equations
- Life made easy with four-vectors ..
- Strategy: one $+$ three

Write potentials and currents as four-vectors:

$$\Psi,\ \vec{A} \quad \Rightarrow \quad A^{\mu} \ = \ (\frac{\Psi}{c},\ \vec{A})$$

$$\rho,\ \vec{j} \quad \Rightarrow \quad J^{\mu} \ = \ (\rho \cdot c,\ \vec{j})$$

What about the transformation of current and potentials ?

Transform the four-current like:

$$\begin{pmatrix} \rho' c \\ j'_x \\ j'_y \\ j'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho c \\ j_x \\ j_y \\ j_z \end{pmatrix}$$

It transforms via: $J'^{\mu} = \Lambda \; J^{\mu}$ **(always the same** Λ**)**

Ditto for: $A'^{\mu} = \Lambda \; A^{\mu}$ **(always the same** Λ**)**

Note: $\partial_\mu J^\mu = \dfrac{\partial \rho}{\partial t} + \vec{\nabla}\vec{j} = 0$ **(charge conservation)**

Electromagnetic fields using potentials:

Magnetic field: $\vec{B} = \nabla \times \vec{A}$

e.g. the x-component:

$$B_x = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

Electric field: $\vec{E} = -\nabla\Psi - \frac{\partial \vec{A}}{\partial t}$

e.g. for the x-component:

$$E_x = -\frac{\partial A_0}{\partial x} - \frac{\partial A_1}{\partial t} = -\frac{\partial A_t}{\partial x} - \frac{\partial A_x}{\partial t}$$

→ **after getting all combinations ..**

Electromagnetic fields described by field-tensor $F^{\mu\nu}$:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & \frac{-E_x}{c} & \frac{-E_y}{c} & \frac{-E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

It transforms via: $F'^{\mu\nu} = \Lambda\ F^{\mu\nu}\ \Lambda^T$ **(same Λ as before)**

(Warning: There are different ways to write the field-tensor $F^{\mu\nu}$,
I use the convention from [1, 3, 5])

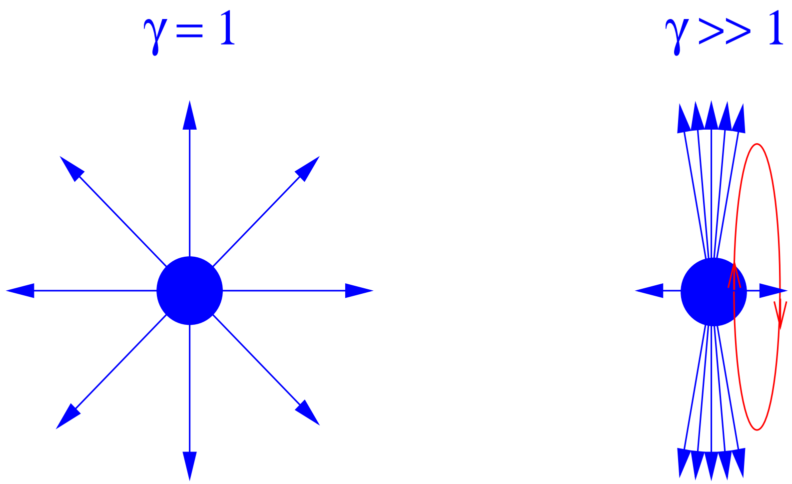
Transformation of fields into a moving frame (x-direction):

Use Lorentz transformation of $F^{\mu\nu}$ and write for components:

$$
\begin{aligned}
E'_x &= E_x & B'_x &= B_x \\
E'_y &= \gamma(E_y - v \cdot B_z) & B'_y &= \gamma(B_y + \tfrac{v}{c^2} \cdot E_z) \\
E'_z &= \gamma(E_z + v \cdot B_y) & B'_z &= \gamma(B_z - \tfrac{v}{c^2} \cdot E_y)
\end{aligned}
$$

Fields perpendicular to movement are transformed

Example Coulomb field: (a charge moving with constant speed)



- **In rest frame purely electrostatic forces**
- **In moving frame $\vec{E}$ transformed and $\vec{B}$ appears**

How do the fields look like ?

Needed to compute e.g. radiation of a moving charge, wake fields, ...

For the static charge we have the Coulomb potential (see lecture on Electrodynamics) and $\vec{A} = 0$

Transformation into the new frame (moving in x-direction) with our transformation of four-potentials:

$$\frac{\Psi'}{c} = \gamma(\frac{\Psi}{c} - A_x) = \gamma\frac{\Psi}{c}$$

$$A'_x = \gamma\left(A_x - \frac{v\Psi}{c^2}\right) = -\gamma\frac{v}{c^2}\Psi = -\frac{v}{c^2}\Psi'$$

i.e. all we need to know is Ψ' $\Rightarrow$

$$\Psi'(\vec{r}) = \gamma\Psi(\vec{r}) = \gamma \cdot \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{|\vec{r} - \vec{r}_q|}$$

After transformation of coordinates, e.g. $x = \gamma(x' - vt')$
The resulting potentials can be used to compute the fields.

Watch out !!

We have to take care of causality:
The field observed at a position $\vec{r}$ at time t was caused at an <u>earlier</u> time $t_r < t$ at the location $\vec{r}_0(t_r)$

$$\Psi(\vec{r},t) \; = \; \frac{qc}{|\vec{R}|c - \vec{R}\vec{v}} \qquad \vec{A}(\vec{r},t) \; = \; \frac{q\vec{v}}{|\vec{R}|c - \vec{R}\vec{v}}$$

The potentials $\Psi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ depend on the state at <u>retarted</u> time t_r, not t
$\vec{v}$ is the velocity at time t_r and $\vec{R} \; = \; \vec{r} - \vec{r}_0(t_r)$ relates the retarted position to the observation point.

Q: Can we also write a <u>Four-Maxwell</u> ?

Re-write Maxwell's equations using four-vectors and $F^{\mu\nu}$:

$$\nabla \vec{E} = \frac{\rho}{\epsilon_0} \qquad \textbf{and} \qquad \nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \qquad \xrightarrow{1+3}$$

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 J^\nu \quad \text{(Inhomogeneous Maxwell equation)}}$$

$$\nabla \vec{B} = 0 \qquad \textbf{and} \qquad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \xrightarrow{1+3}$$

$$\boxed{\partial_\gamma F^{\mu\nu} + \partial_\mu F^{\nu\lambda} + \partial_\nu F^{\lambda\mu} = 0 \qquad \text{(Homogeneous Maxwell equation)}}$$

We have Maxwell's equation in a very compact form, transformation between moving systems very easy

How to use all that stuff ??? **Look at first equation:**

$$\partial_\mu F^{\mu\nu} \quad = \quad \mu_0 J^\nu$$

Written explicitly (Einstein convention, sum over μ):

$$\partial_\mu F^{\mu\nu} \;=\; \sum_{\mu=0}^{3} \partial_\mu F^{\mu\nu} \;=\; \partial_0 F^{0\nu} \;+\; \partial_1 F^{1\nu} \;+\; \partial_2 F^{2\nu} \;+\; \partial_3 F^{3\nu} \quad = \quad \mu_0 J^\nu$$

Choose e.g. $\nu = 0$ and replace $F^{\mu\nu}$ by corresponding elements:

$$\begin{array}{lllllll}
\partial_0 F^{00} & + \partial_1 F^{10} & + \partial_2 F^{20} & + \partial_3 F^{30} & = & \mu_0 J^0 & \\
0 & + \partial_x \dfrac{E_x}{c} & + \partial_y \dfrac{E_y}{c} & + \partial_z \dfrac{E_z}{c} & = & \mu_0 J^0 & = \quad \mu_0 \; c\rho
\end{array}$$

This corresponds exactly to:

$$\vec{\nabla} \cdot \; \vec{E} \quad = \quad \frac{\rho}{\epsilon_0} \qquad\qquad (c^2 \; = \; \epsilon_0 \mu_0)$$

➡ **For $\nu = 1, 2, 3$ you get Ampere's law**

For example in the x-plane ($\nu = 1$) and the S frame:

$$\partial_y B_z \;-\; \partial_z B_y \;-\; \partial_t \frac{E_x}{c} \;=\; \mu_0 J^x$$

after transforming ∂^γ and $F^{\mu\nu}$ to the S' frame:

$$\partial'_y B'_z \;-\; \partial'_z B'_y \;-\; \partial'_t \frac{E'_x}{c} \;=\; \mu_0 J'^x$$

Now Maxwell's equation have the identical form in S and S'

(In matter: can be re-written with $\vec{D}$ and $\vec{H}$ using "magnetization tensor")

Finally: since $$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$$

$$\partial_{\mu} F^{\mu\nu} = \mu_0 J^{\nu}$$

$$\partial_{\gamma} F^{\mu\nu} + \partial_{\mu} F^{\nu\lambda} + \partial_{\nu} F^{\lambda\mu} = 0$$

We can re-write them two-in-one in a new form:

$$\boxed{\frac{\partial^2 A^{\mu}}{\partial x_{\nu} \partial x^{\nu}} = \mu_0 J^{\mu}}$$

This contains all four Maxwell's equations, and the only one which stays the same in all frames !!

There are <u>no</u> separate electric and magnetic fields, just a frame dependent manifestation of a single electromagnetic field

Quite obvious in Quantum ElectroDynamics !