Injection, extraction and transfer

- Introductory slides:
  - Kickers, septa and normalised phase-space
- Injection methods
  - Single-turn hadron injection
  - Injection errors, filamentation and blow-up
  - Multi-turn hadron injection
  - Charge-exchange H- injection
  - Lepton injection
- Extraction methods
  - Single-turn (fast) extraction
  - Non-resonant and resonant multi-turn (fast) extraction
  - Resonant multi-turn (slow) extraction

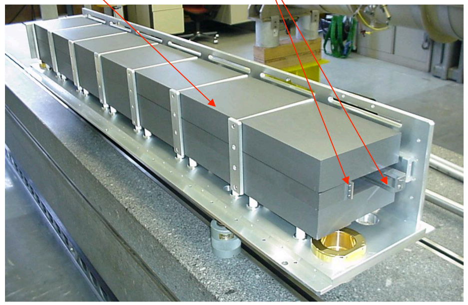
Matthew Fraser, CERN (TE-ABT-BTP)
based on lectures by Brennan Goddard

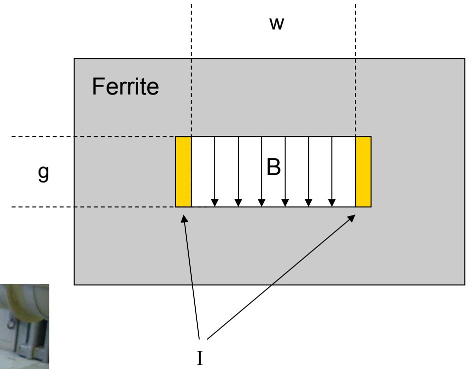
Kicker magnet

Pulsed magnet with very fast rise time (100 ns – few μs)

Ferrite

Conductors





$B = \mu_0 I / g$

L [per unit length] = $\mu_0 w / g$

$dI/dt = V / L$

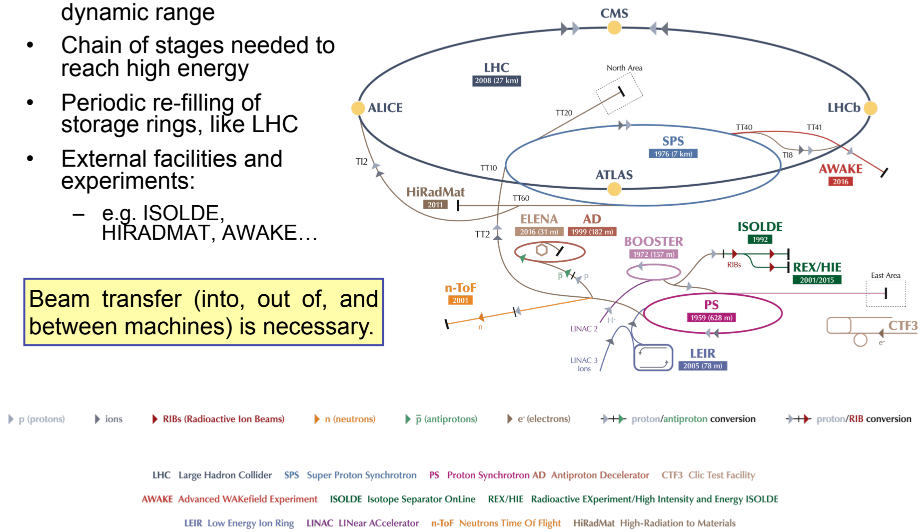
Typically 3 kA in 1 μs rise time

Injection, extraction and transfer

- An accelerator has limited dynamic range
- Chain of stages needed to reach high energy
- Periodic re-filling of storage rings, like LHC
- External facilities and experiments:
  - e.g. ISOLDE, HIRADMAT, AWAKE…

Beam transfer (into, out of, and between machines) is necessary.

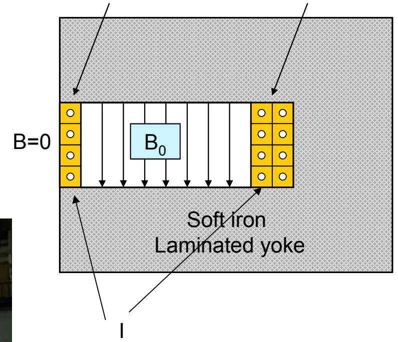
CERN Accelerator Complex



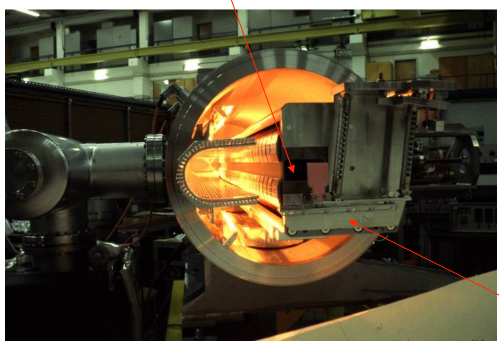
Magnetic septum

Pulsed or DC magnet with thin (2 – 20 mm) septum between zero field and high field region

Typically ~10x more deflection given by magnetic septa, compared to kickers



Septum coil



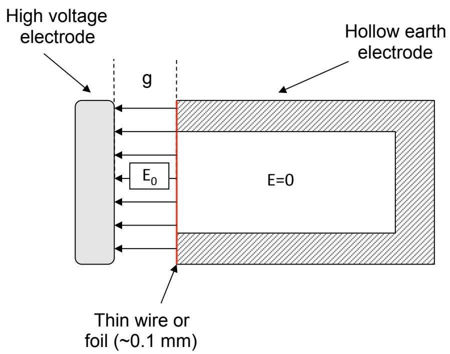
Yoke

$B_o = \mu_0 I / g$

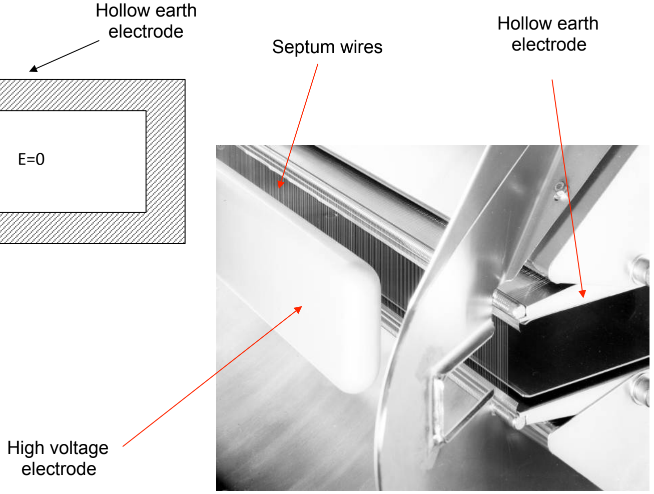
Typically I 5 - 25 kA

Electrostatic septum

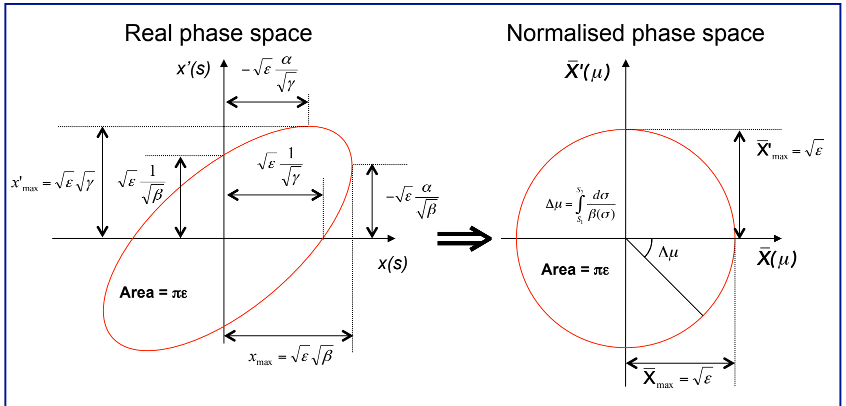
DC electrostatic device with very thin septum between zero field and high field region



$E = V / g$
Typically V = 200 kV
E = 100 kV/cm



Normalised phase space



$$\varepsilon = \gamma \cdot x^2 + 2\alpha \cdot x \cdot x' + \beta \cdot x'^2$$

$$\varepsilon = \bar{X}^2 + \bar{X}'^2$$

Normalised phase space

- Transform real transverse coordinates (*x, x', s*) to normalised co-ordinates $(\bar{X}, \bar{X}', \mu)$ where the independent variable becomes the phase advance μ:

$$\begin{bmatrix} \bar{X} \\ \bar{X}' \end{bmatrix} = \mathbf{N} \cdot \begin{bmatrix} x \\ x' \end{bmatrix} = \sqrt{\frac{1}{\beta(s)}} \cdot \begin{bmatrix} 1 & 0 \\ \alpha(s) & \beta(s) \end{bmatrix} \cdot \begin{bmatrix} x \\ x' \end{bmatrix}$$

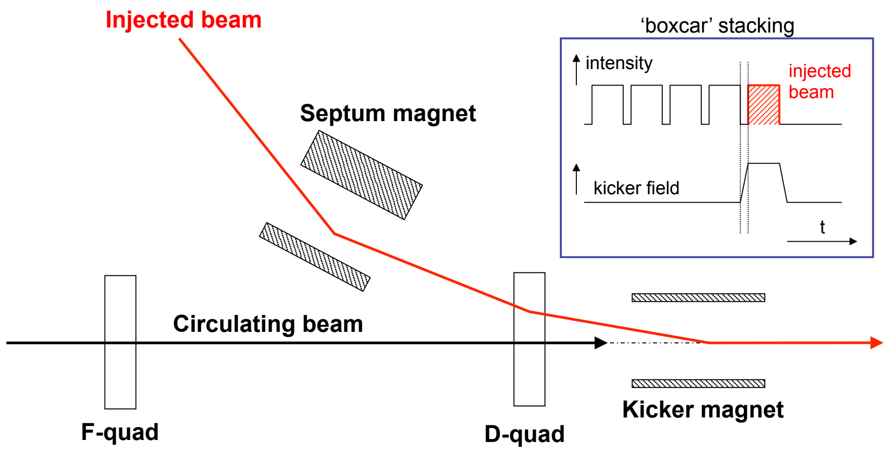
$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\left[\mu(s) + \mu_0\right]$$

$$\mu(s) = \int_0^s \frac{d\sigma}{\beta(\sigma)}$$

$$\bar{X}(\mu) = \sqrt{\frac{1}{\beta(s)}} \cdot x = \sqrt{\varepsilon}\cos\left[\mu + \mu_0\right]$$

$$\bar{X}'(\mu) = \sqrt{\frac{1}{\beta(s)}} \cdot \alpha(s)x + \sqrt{\beta(s)}x' = -\sqrt{\varepsilon}\sin\left[\mu + \mu_0\right] = \frac{d\bar{X}}{d\mu}$$

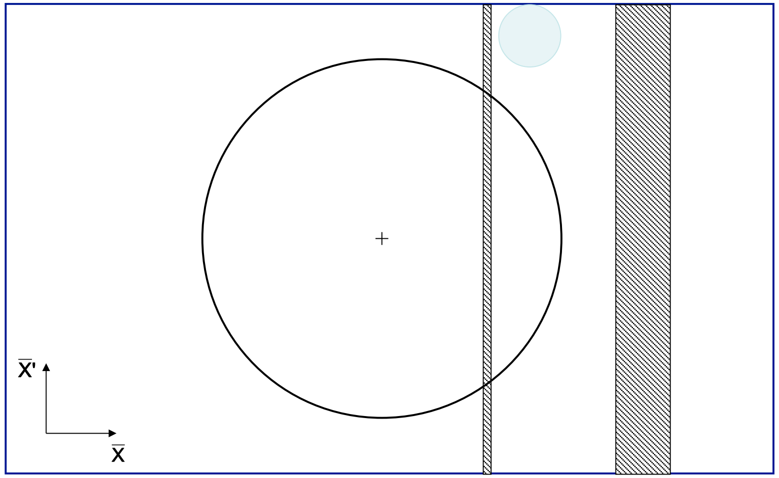
Single-turn injection – same plane



- Septum deflects the beam onto the closed orbit at the centre of the kicker
- Kicker compensates for the remaining angle
- Septum and kicker either side of D quad to minimise kicker strength

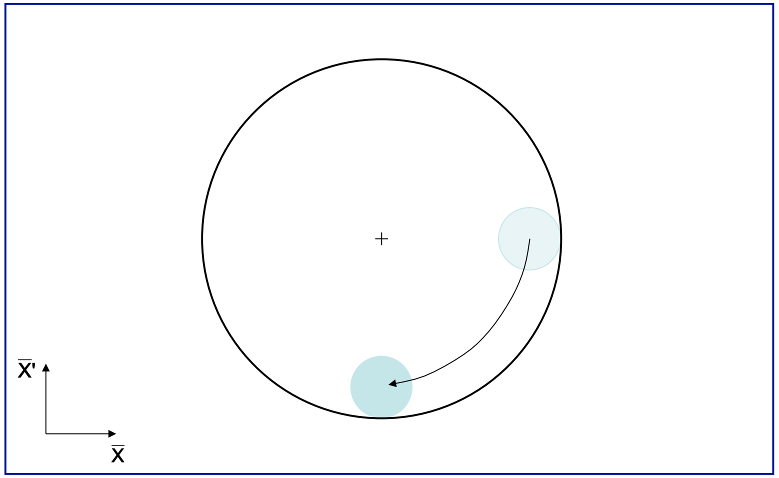
Single-turn injection

Normalised phase space at centre of idealised septum



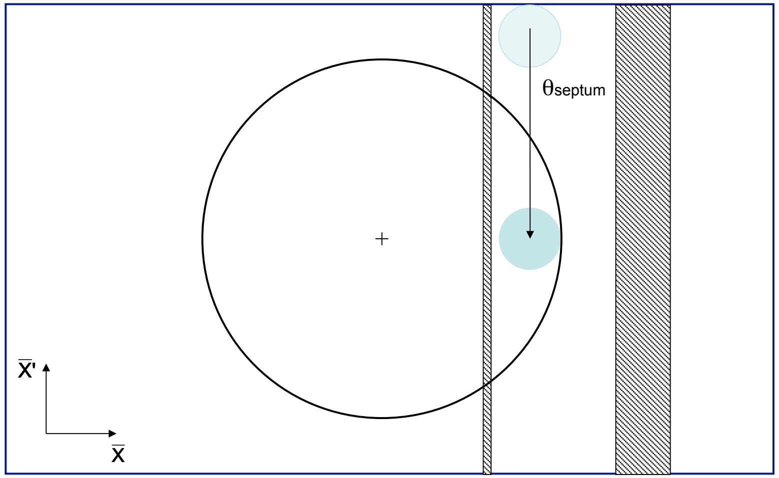
Single-turn injection

μ/2 phase advance to kicker location



Single-turn injection

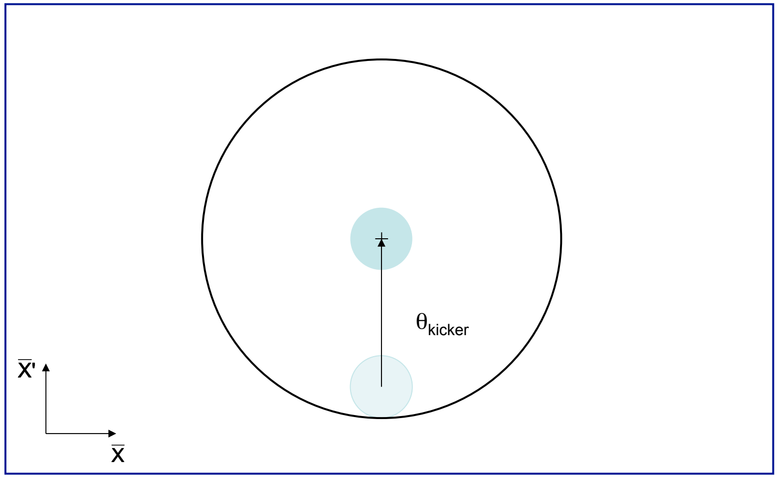
Normalised phase space at centre of idealised septum



Single-turn injection

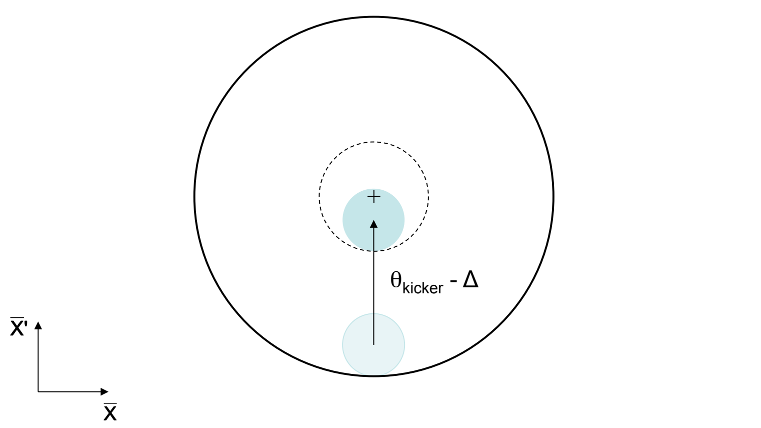
Normalised phase space at centre of idealised kicker

Kicker deflection places beam on central orbit:



Injection oscillations

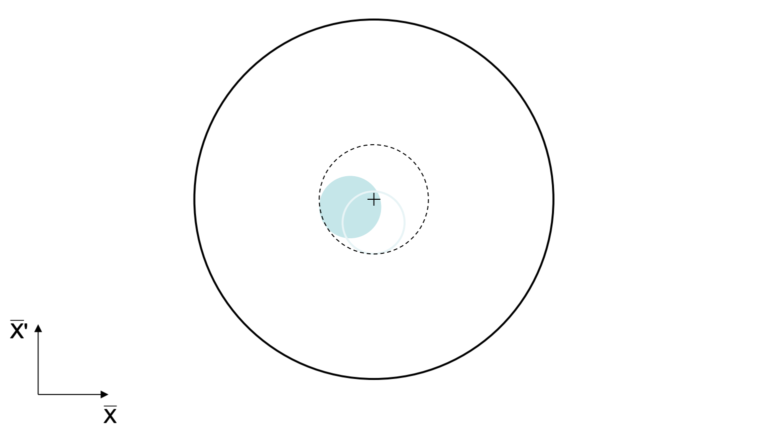
For imperfect injection the beam oscillates around the central orbit, e.g. kick error, Δ:



Injection oscillations

For imperfect injection the beam oscillates around the central orbit, e.g. kick error, Δ:

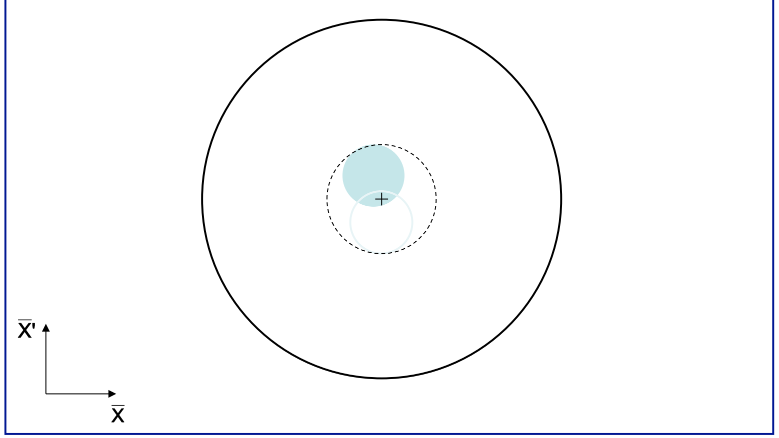
After 1 turn…



Injection oscillations

For imperfect injection the beam oscillates around the central orbit, e.g. kick error, Δ:

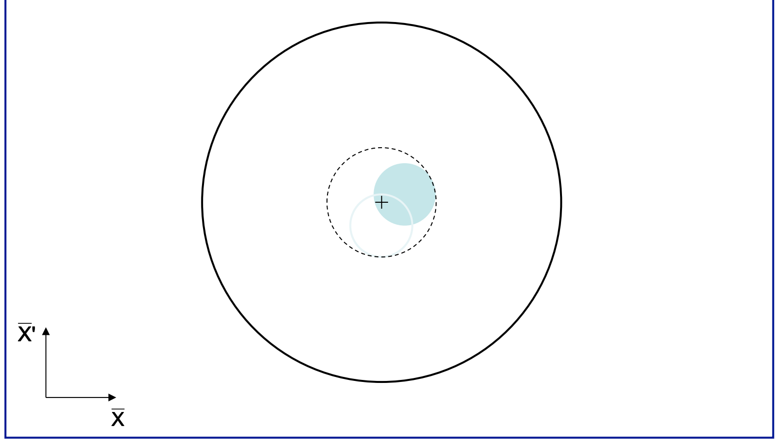
After 2 turns…



Injection oscillations

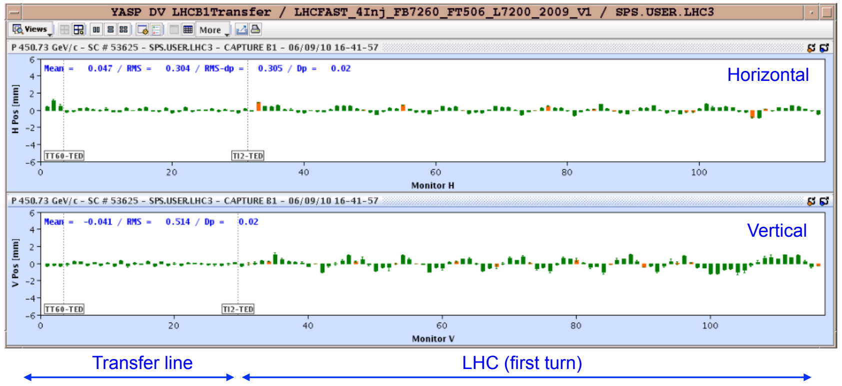
For imperfect injection the beam oscillates around the central orbit, e.g. kick error, Δ:

After 3 turns etc…

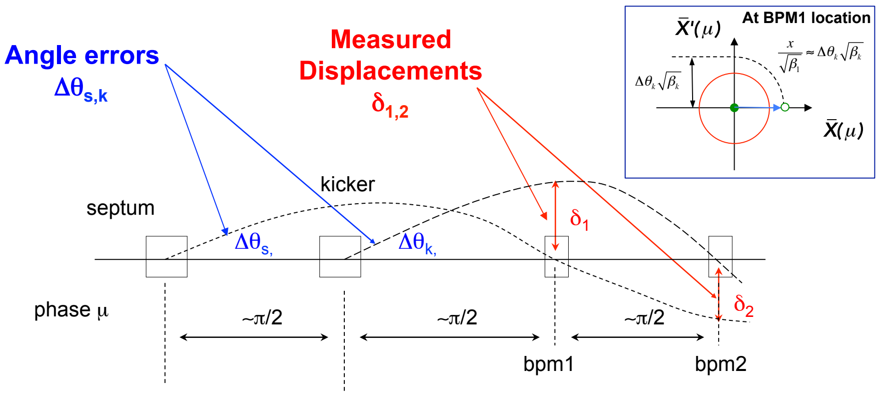


Injection oscillations

- Betatron oscillations with respect to the Closed Orbit:



Injection errors



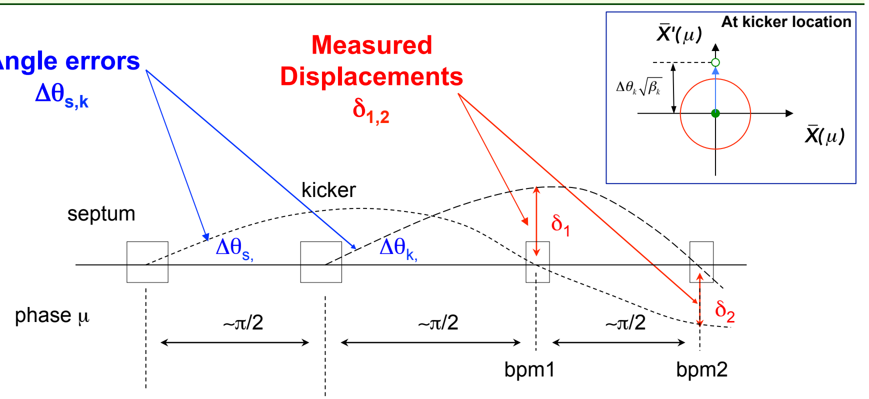
$$\delta_1 = \Delta\theta_s \sqrt{(\beta_s\beta_1)} \sin(\mu_1 - \mu_s) + \Delta\theta_k \sqrt{(\beta_k\beta_1)} \sin(\mu_1 - \mu_k)$$

$$\approx \Delta\theta_k \sqrt{(\beta_k\beta_1)}$$

$$\delta_2 = \Delta\theta_s \sqrt{(\beta_s\beta_2)} \sin(\mu_2 - \mu_s) + \Delta\theta_k \sqrt{(\beta_k\beta_2)} \sin(\mu_2 - \mu_k)$$

$$\approx -\Delta\theta_s \sqrt{(\beta_s\beta_2)}$$

Injection errors



$$\delta_1 = \Delta\theta_s \sqrt{(\beta_s\beta_1)} \sin(\mu_1 - \mu_s) + \Delta\theta_k \sqrt{(\beta_k\beta_1)} \sin(\mu_1 - \mu_k)$$

$$\approx \Delta\theta_k \sqrt{(\beta_k\beta_1)}$$

$$\delta_2 = \Delta\theta_s \sqrt{(\beta_s\beta_2)} \sin(\mu_2 - \mu_s) + \Delta\theta_k \sqrt{(\beta_k\beta_2)} \sin(\mu_2 - \mu_k)$$

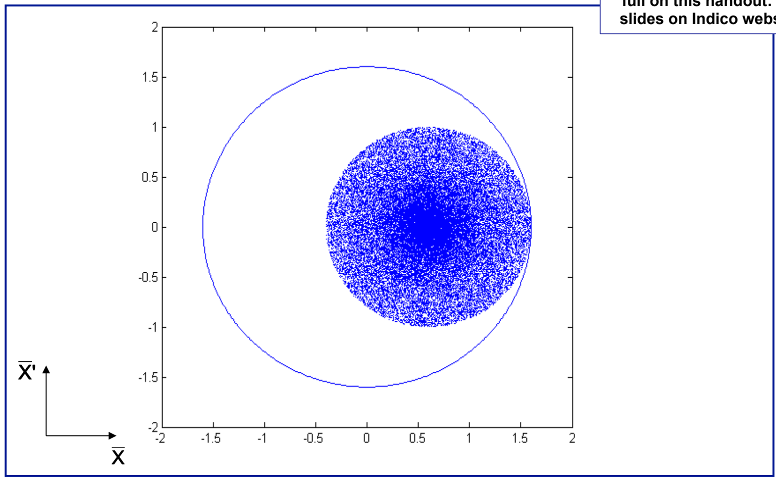
$$\approx -\Delta\theta_s \sqrt{(\beta_s\beta_2)}$$

Filamentation

- Non-linear effects (e.g. higher-order field components) introduce amplitude-dependent effects into particle motion
- Over many turns, a phase-space oscillation is transformed into an emittance increase
- So any residual transverse oscillation will lead to an emittance blow-up through filamentation
  - Chromaticity coupled with a non-zero momentum spread at injection can also cause filmentation, often termed *chromatic decoherence*
  - "Transverse damper" systems are used to damp injection oscillations - bunch position measured by a pick-up, which is linked to a kicker

Filamentation

Animation not included in full on this handout: see slides on Indico website!



Filamentation

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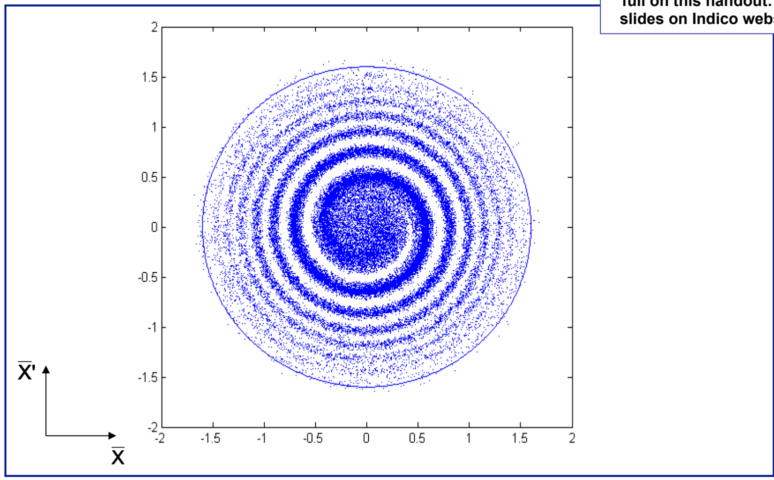

Filamentation

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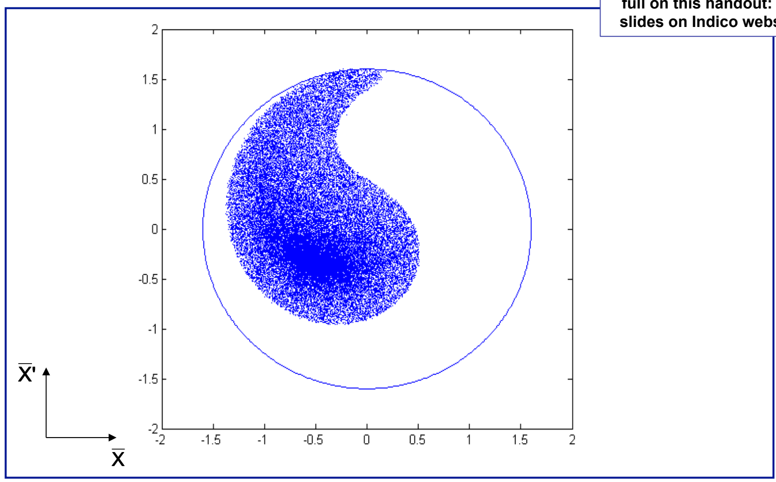

Filamentation

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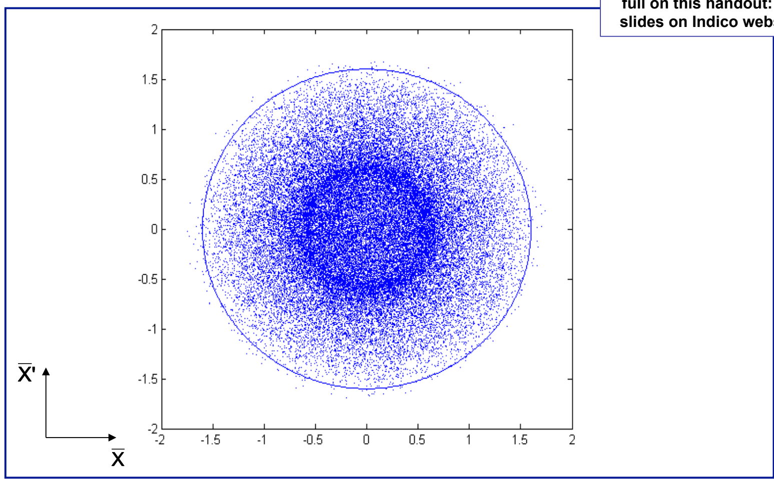
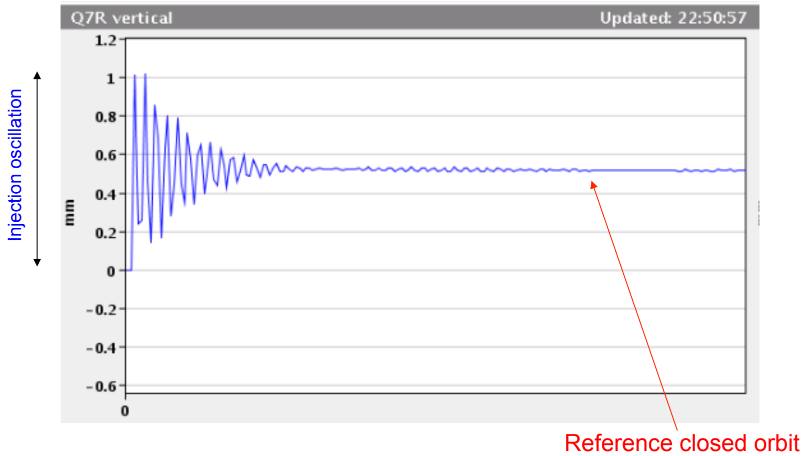


Filamentation

- Residual transverse oscillations lead to an *effective* emittance blow-up through filamentation:



Blow-up from steering error

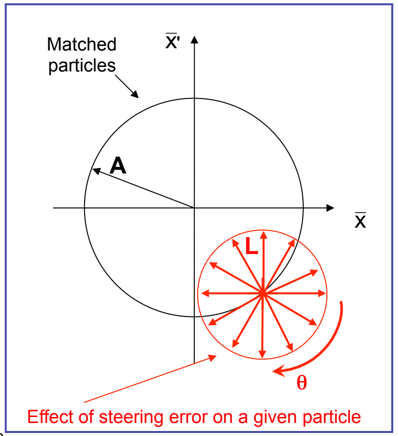
- Consider a collection of particles with max. amplitudes A
- The beam can be injected with an error in angle and position.
- For an injection error Δa, in units of σ = √(βε), the mis-injected beam is offset in normalised phase space by an amplitude L = Δa√ε
- Any given point on the matched ellipse is randomised over all phases after filamentation due to the steering error
- For a general particle distribution, where A_i denotes amplitude in normalised phase of particle i:

$$\varepsilon_{matched} = \langle A_i^2 \rangle / 2$$

- After filamentation:

$$\varepsilon_{diluted} = \varepsilon_{matched} + \frac{L^2}{2}$$

See appendix for derivation



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Blow-up from steering error

- Consider a collection of particles with max. amplitudes A
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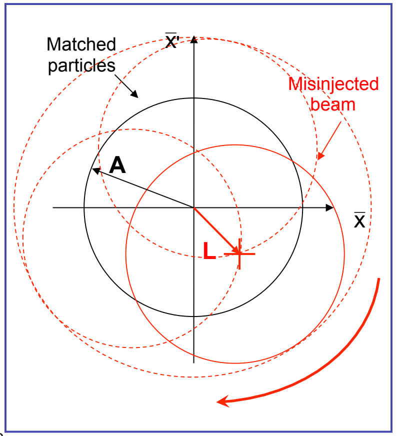

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Blow-up from steering error

- A numerical example….
- Consider an offset Δa = 0.5σ for injected beam:

$$L = \Delta a \sqrt{\varepsilon_{matched}}$$

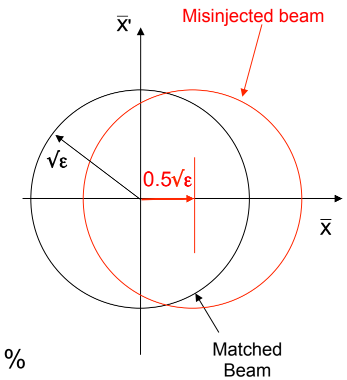
$$\varepsilon_{diluted} = \varepsilon_{matched} + \frac{L^2}{2}$$

$$= \varepsilon_{matched} \left[1 + \frac{\Delta a^2}{2} \right]$$

$$= \varepsilon_{matched} \left[1.125 \right]$$

- For nominal LHC beam:

…allowed growth through LHC cycle ~10 %



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Multi-turn injection

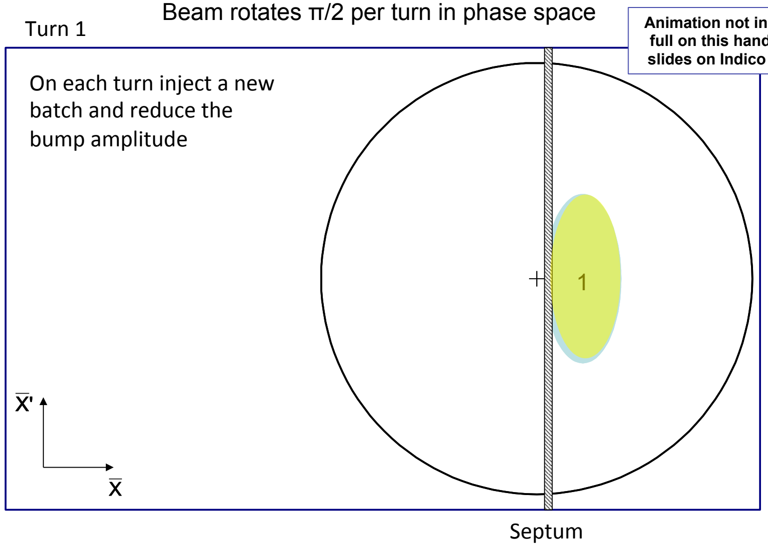
- For hadrons the beam density at injection can be limited either by space charge effects or by the injector capacity
- If we cannot increase charge density, we can sometimes fill the horizontal phase space to increase overall injected intensity.
  - If the acceptance of the receiving machine is larger than the delivered beam emittance we can accumulate intensity

Multi-turn injection for hadrons

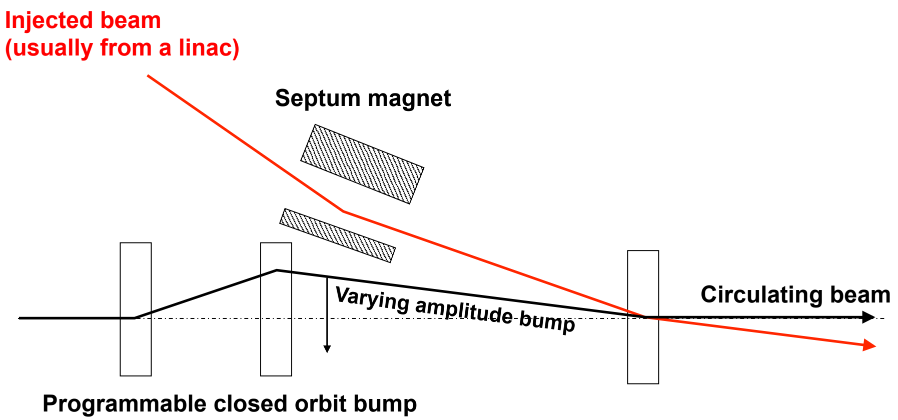
Example: CERN PSB injection, high intensity beams, fractional tune $Q_h \approx 0.25$

Beam rotates π/2 per turn in phase space

Animation not included in full on this handout: see slides on Indico website!



Multi-turn injection for hadrons



- No kicker but fast programmable bumpers
- Bump amplitude decreases and a new batch injected turn-by-turn
- Phase-space "painting"

Multi-turn injection for hadrons

Example: CERN PSB injection, high intensity beams, fractional tune $Q_h \approx 0.25$

Beam rotates π/2 per turn in phase space

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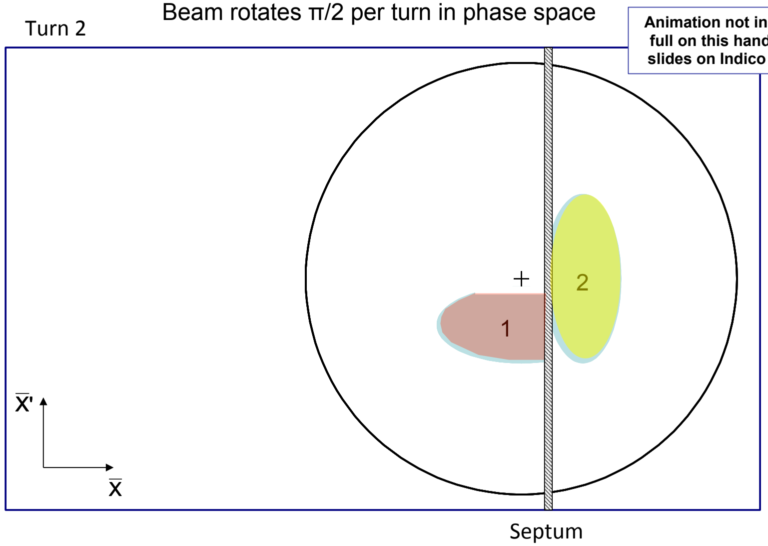


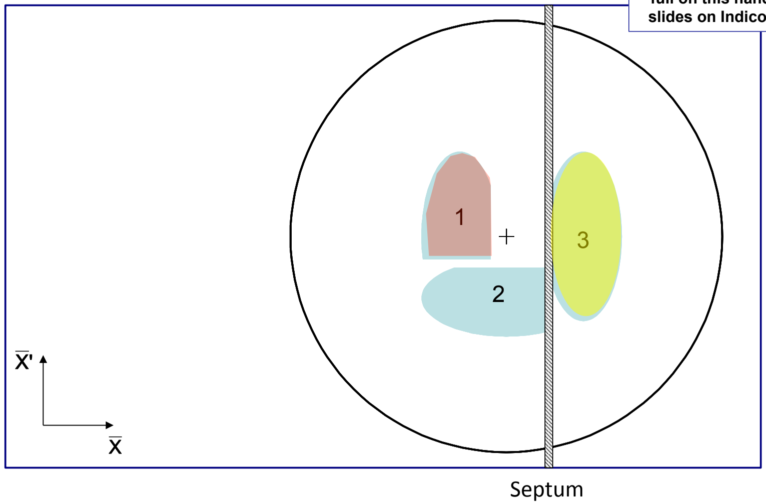
Multi-turn injection for hadrons

Example: CERN PSB injection, high intensity beams, fractional tune $Q_h \approx 0.25$

Beam rotates π/2 per turn in phase space

Animation not included in full on this handout: see slides on Indico website!

Turn 3



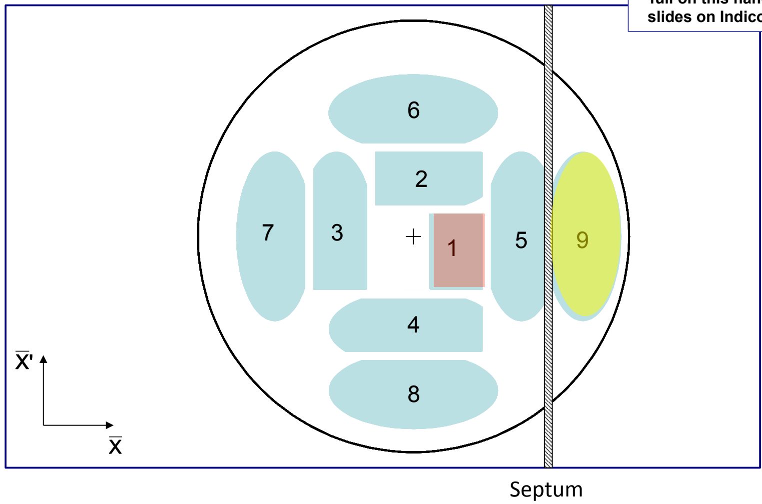
Multi-turn injection for hadrons

Example: CERN PSB injection, high intensity beams, fractional tune $Q_h \approx 0.25$

Beam rotates π/2 per turn in phase space

Animation not included in full on this handout: see slides on Indico website!

Turn 9



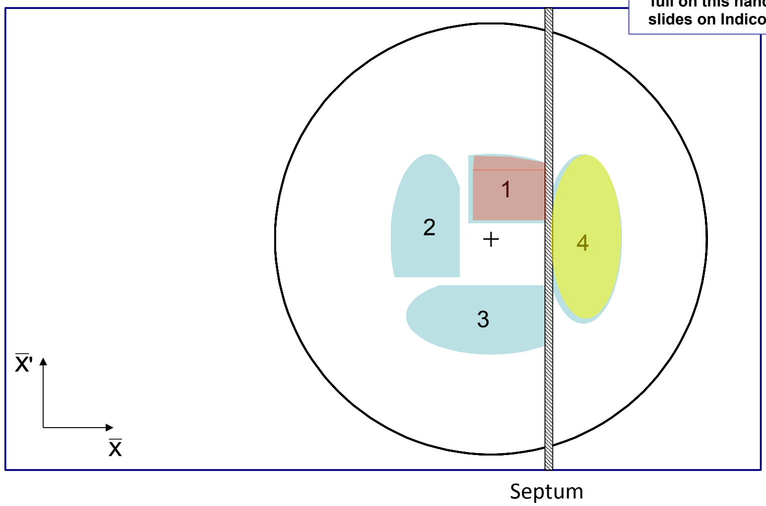
Multi-turn injection for hadrons

Example: CERN PSB injection, high intensity beams, fractional tune $Q_h \approx 0.25$

Beam rotates π/2 per turn in phase space

Animation not included in full on this handout: see slides on Indico website!

Turn 4

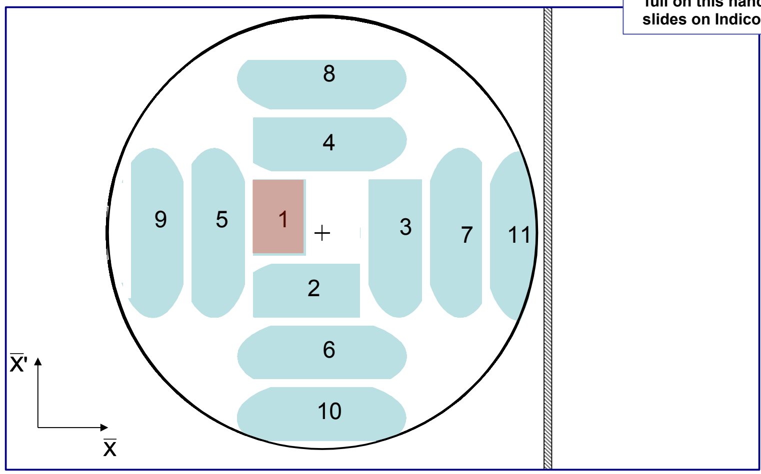


Multi-turn injection for hadrons

Phase space has been "**painted**"

Animation not included in full on this handout: see slides on Indico website!

Turn 15



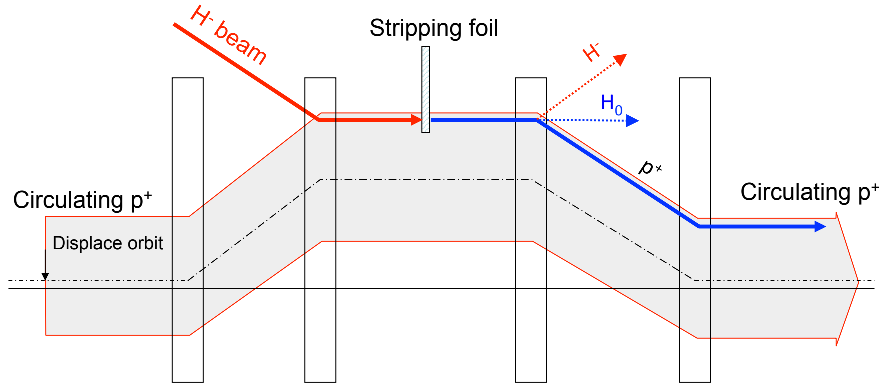
In reality, filamentation (often space-charge driven) occurs to produce a quasi-uniform beam

Charge exchange H- injection

- Multi-turn injection is essential to accumulate high intensity
- Disadvantages inherent in using an injection septum:
  - Width of several mm reduces aperture
  - Beam losses from circulating beam hitting septum:
    - typically 30 – 40 % for the CERN PSB injection at 50 MeV
  - Limits number of injected turns to 10 - 20
- Charge-exchange injection provides elegant alternative
  - Possible to “cheat” Liouville’s theorem, which says that emittance is conserved….
  - Convert H^- to p^+ using a thin stripping foil, allowing injection into the same phase space area

Charge exchange H- injection

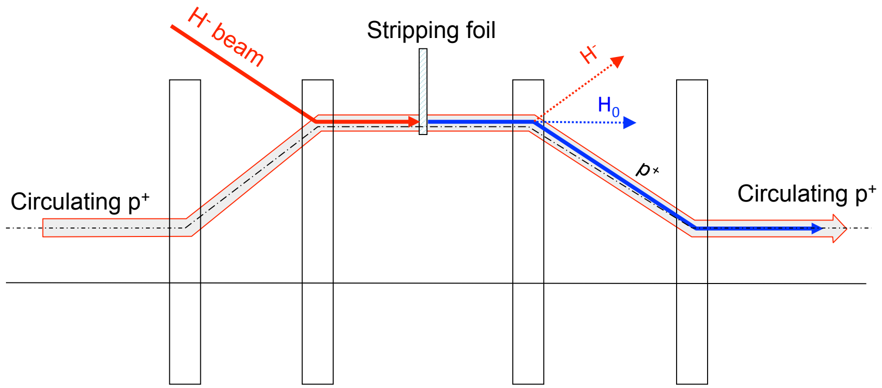
End of injection process with painting



Injection chicane dipoles

Charge exchange H- injection

Start of injection process

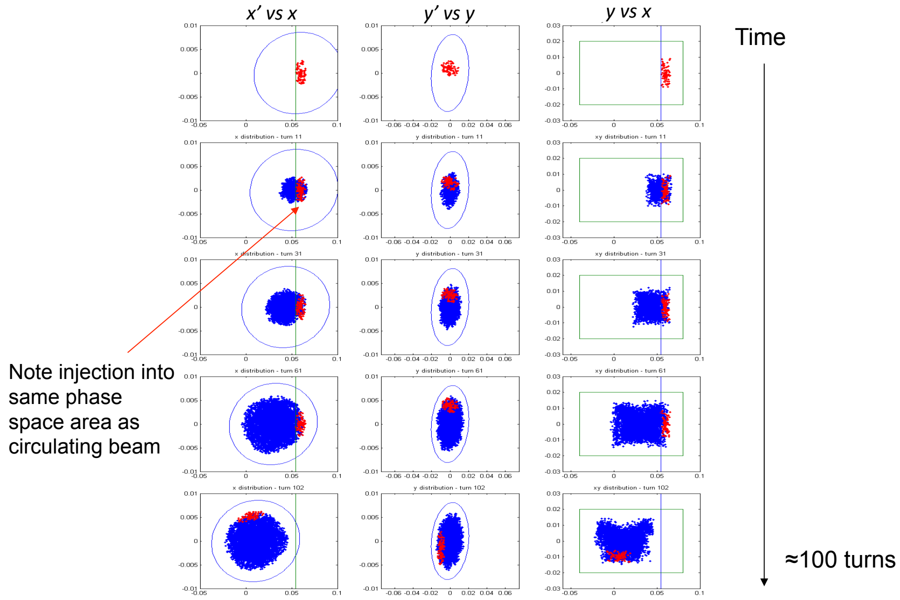


Injection chicane dipoles

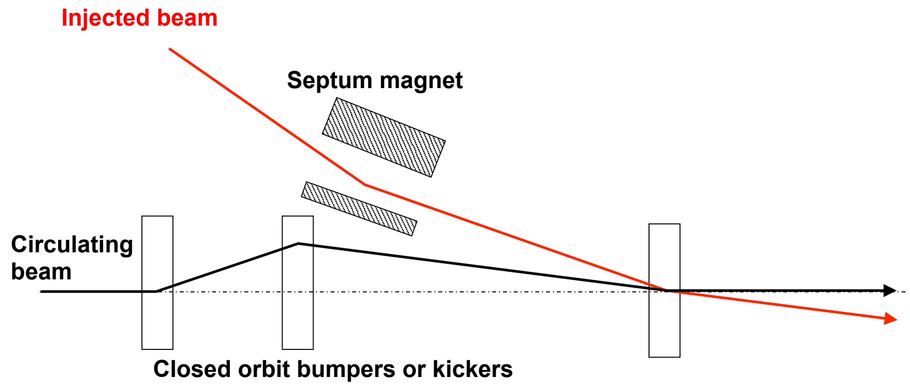
Charge exchange H- injection

- Paint uniform transverse phase space density by modifying closed orbit bump and steering injected beam
- Foil thickness calculated to double-strip most ions (≈99%)
  - 50 MeV – 50 μg.cm^{-2}
  - 800 MeV – 200 μg.cm^{-2} (≈ 1 μm of C!)
- Carbon foils generally used – very fragile
- Injection chicane reduced or switched off after injection, to avoid excessive foil heating and beam blow-up
- Longitudinal phase space can also be painted turn-by-turn:
  - Variation of the injected beam energy turn-by-turn (linac voltage scaled)
  - Chopper system in linac to match length of injected batch to bucket

H- injection - painting



Betatron lepton injection



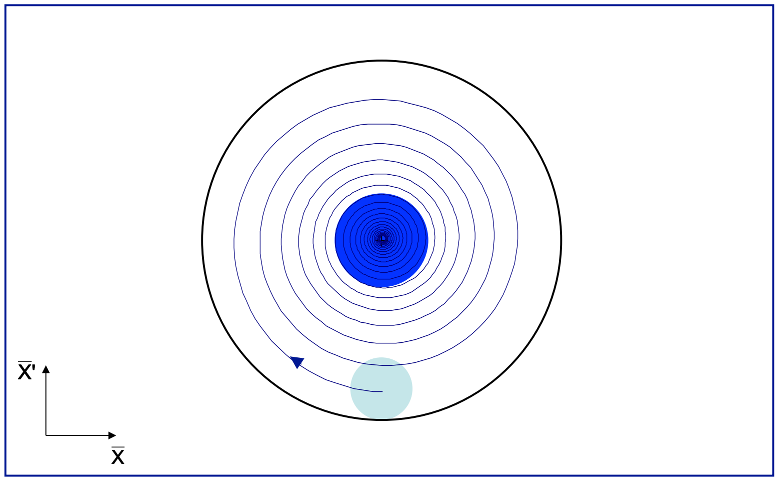
- Beam is injected with an angle with respect to the closed orbit
- Injected beam performs damped betatron oscillations about the closed orbit

Lepton injection

- Single-turn injection can be used as for hadrons; however, lepton motion is strongly damped (different with respect to proton or ion injection).
  - Synchrotron radiation
    - see *Electron Beam Dynamics lectures by L. Rivkin*
- Can use transverse or longitudinal damping:
  - Transverse - Betatron accumulation
  - Longitudinal - Synchrotron accumulation

Betatron lepton injection

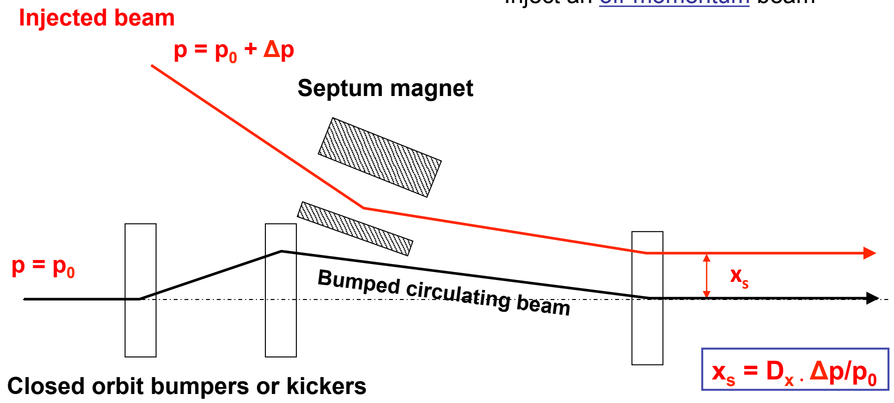
Injected bunch performs damped betatron oscillations



In LEP at 20 GeV, the damping time was about 6’000 turns (0.6 seconds)

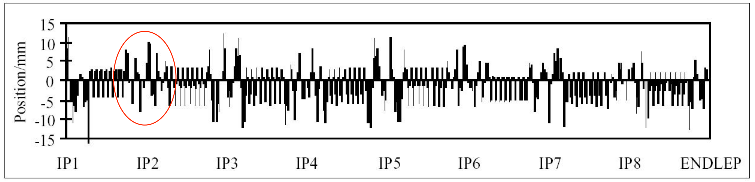
Synchrotron lepton injection

Inject an off-momentum beam

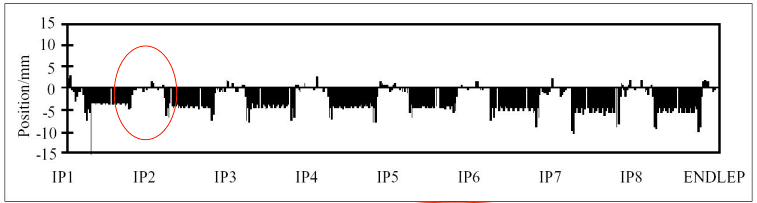


- Beam injected parallel to circulating beam, onto dispersion orbit of a particle having the same momentum offset $\Delta p/p$
- Injected beam makes damped synchrotron oscillations at Q_s but does not perform betatron oscillations

Synchrotron lepton injection in LEP



Optimized Horizontal First Turn Trajectory for Betatron Injection of Positrons into LEP.

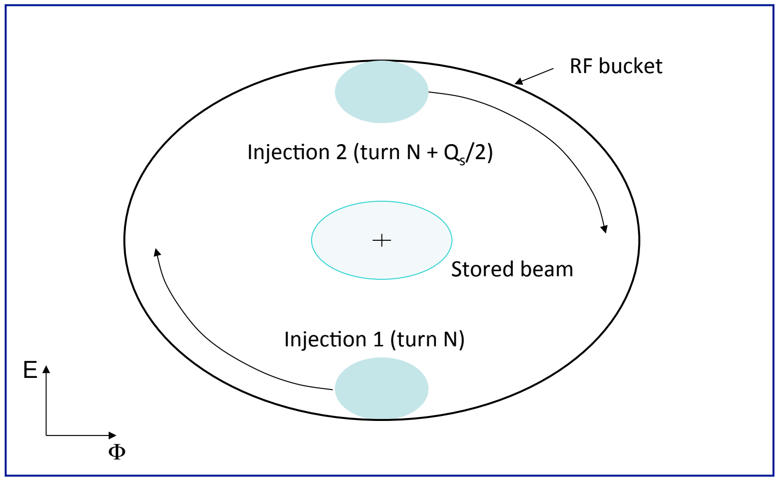


Optimized Horizontal First Turn Trajectory for Synchrotron Injection of Positrons with ΔP/P at -0.6%

Synchrotron injection in LEP gave improved background for LEP experiments due to small orbit offsets in zero dispersion straight sections

Synchrotron lepton injection

Double batch injection possible….



Longitudinal damping time in LEP was ~3'000 turns (2x faster than transverse)

Injection - summary

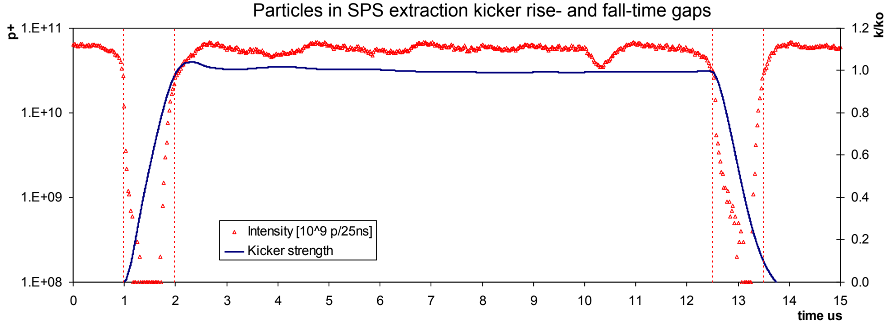
- Several different techniques using kickers, septa and bumpers:
  - Single-turn injection for hadrons
    - Boxcar stacking: transfer between machines in accelerator chain
    - Angle / position errors ⇒ injection oscillations
    - Uncorrected errors ⇒ filamentation ⇒ emittance increase
  - Multi-turn injection for hadrons
    - Phase space painting to increase intensity
    - H- injection allows injection into same phase space area
  - Lepton injection: take advantage of damping
    - Less concerned about injection precision and matching

Extraction

- Different extraction techniques exist, depending on requirements
  - Fast extraction: ≤1 turn
  - Non-resonant multi-turn extraction: few turns
  - Resonant low-loss multi-turn extraction: few turns
  - Resonant multi-turn extraction: many thousands of turns
- Usually higher energy than injection ⇒ stronger elements ($\int B.dl$)
  - At high energies many kicker and septum modules may be required
  - To reduce kicker and septum strength, beam can be moved near to septum by closed orbit bump

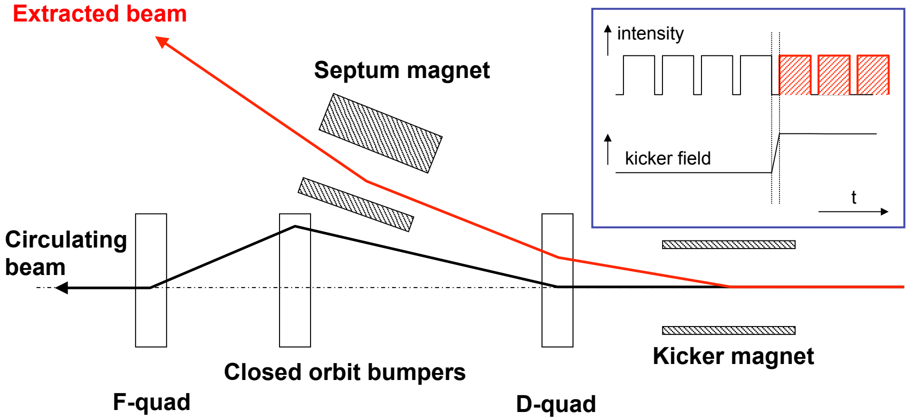
Fast single turn extraction

- For transfer of beams between accelerators in an injector chain
- For secondary particle production (e.g. neutrinos, radioactive beams)
- Septum deflection may be in the other plane to the kicker deflection
  - Lambertson septum to be discussed tomorrow…
- Losses from transverse scraping or from particles in extraction gap:
  - Fast extraction from SPS to CNGS:



Fast single turn extraction

Entire beam kicked into septum gap and extracted over a single turn



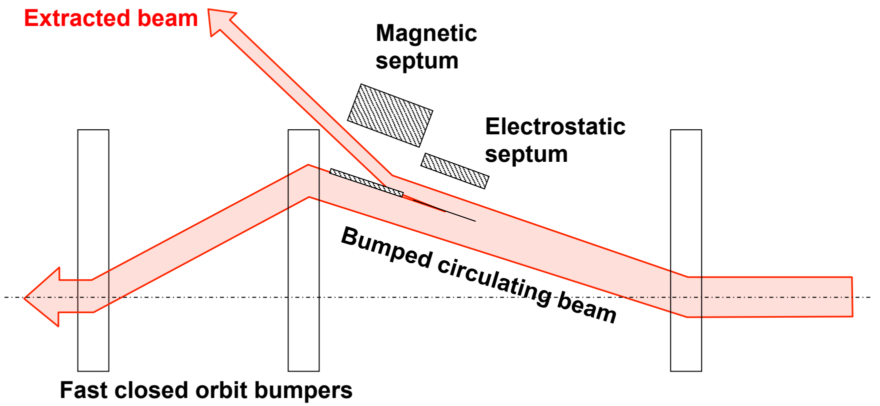
- Bumpers move circulating beam close to septum to reduce kicker strength
- Kicker deflects the entire beam into the septum in a single turn
- Most efficient (lowest deflection angles required) for π/2 phase advance between kicker and septum

Multi-turn extraction

- Some filling schemes require a beam to be injected in several turns to a larger machine…
- And very commonly Fixed Target physics experiments and medical accelerators often need a quasi-continuous flux of particles…
- Multi-turn extraction…
  - Non-resonant multi-turn ejection (few turns) for filling
    - e.g. PS to SPS at CERN for high intensity proton beams (>2.5 10^{13} protons)
  - Resonant extraction (ms to hours) for experiments

Non-resonant multi-turn extraction

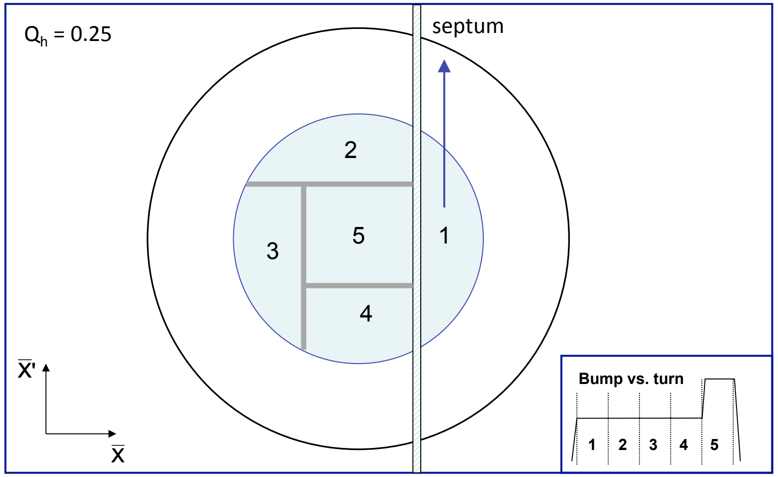
Beam bumped to septum; part of beam 'shaved' off each turn



- Fast bumper deflects the whole beam onto the septum
- Beam extracted in a few turns, with the machine tune rotating the beam
- Intrinsically a high-loss process: thin septum essential
- Often combine thin electrostatic septa with magnetic septa

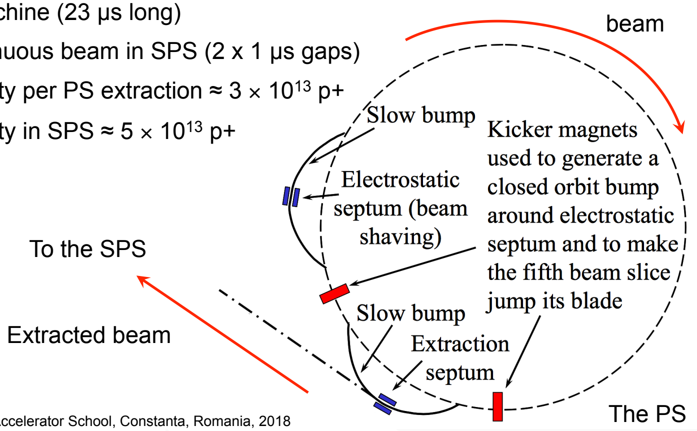
Non-resonant multi-turn extraction

CERN PS to SPS: 5-turn continuous transfer – 1st turn



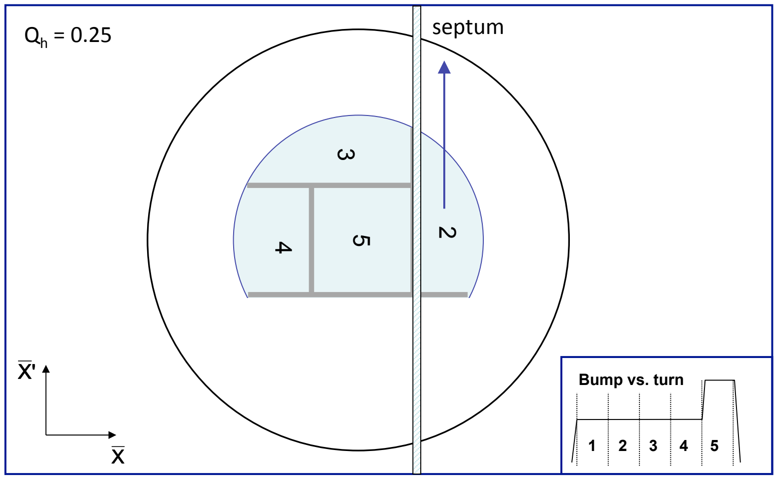
Non-resonant multi-turn extraction

- Example system: CERN PS to SPS Fixed-Target 'continuous transfer'.
  - Accelerate beam in PS to 14 GeV/c
  - Empty PS machine (2.1 μs long) in 5 turns into SPS
  - Do it again
  - Fill SPS machine (23 μs long)
  - Quasi-continuous beam in SPS (2 x 1 μs gaps)
  - Total intensity per PS extraction ≈ 3 × 10^{13} p+
  - Total intensity in SPS ≈ 5 × 10^{13} p+



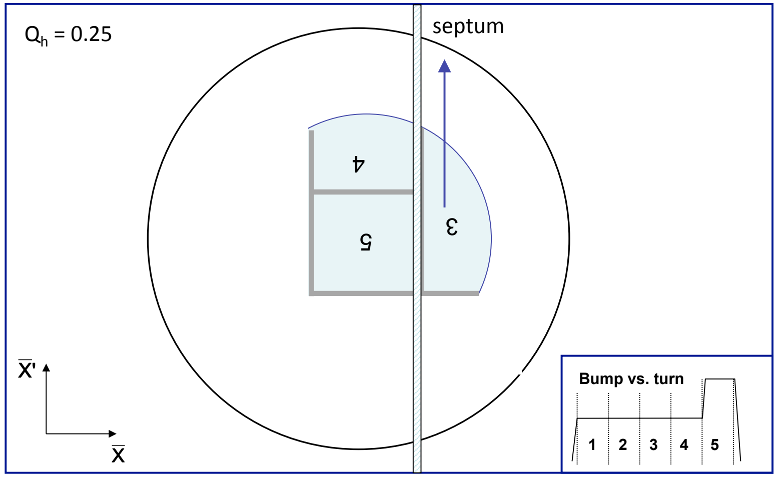
Non-resonant multi-turn extraction

CERN PS to SPS: 5-turn continuous transfer – 2nd turn



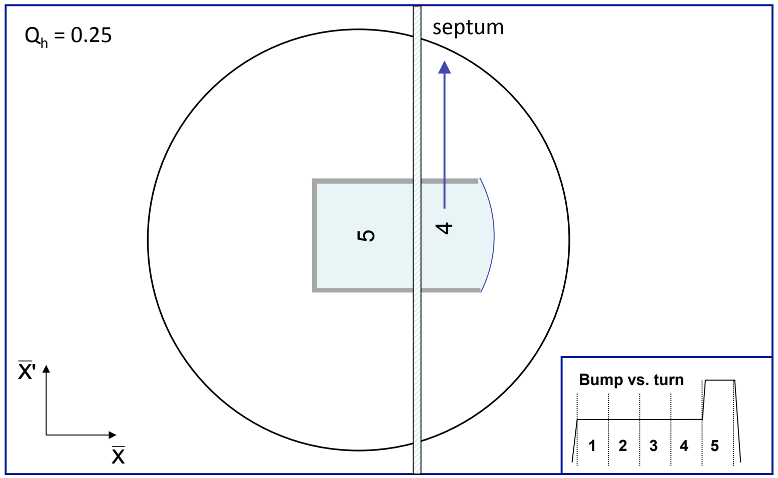
Non-resonant multi-turn extraction

CERN PS to SPS: 5-turn continuous transfer – 3rd turn



Non-resonant multi-turn extraction

CERN PS to SPS: 5-turn continuous transfer – 4th turn



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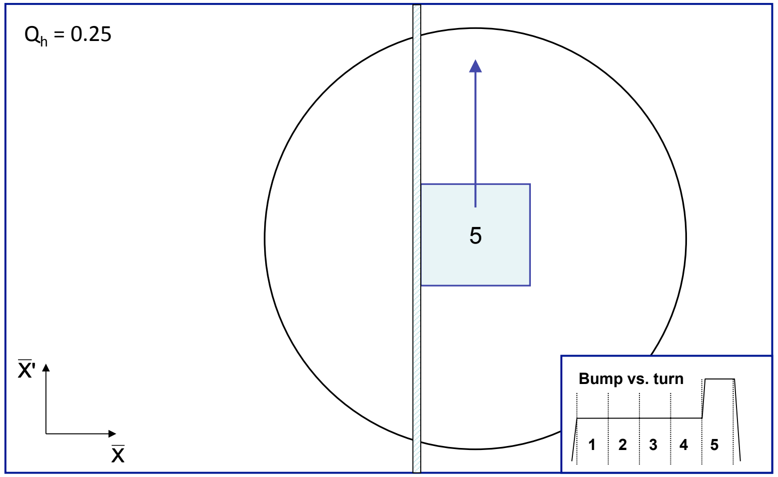
Non-resonant multi-turn extraction

CERN PS to SPS: 5-turn continuous transfer – 5th turn



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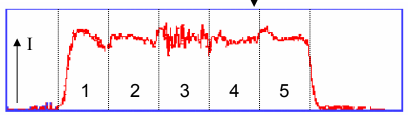
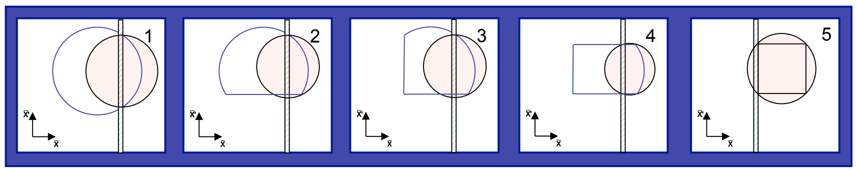
Non-resonant multi-turn extraction

- CERN PS to SPS: 5-turn continuous transfer
  - Losses impose thin (ES) septum…
    …a second magnetic septum is needed
  - Still about 15 % of beam lost in PS-SPS CT
  - Difficult to get equal intensities per turn
  - Different trajectories for each turn
  - Different emittances for each turn





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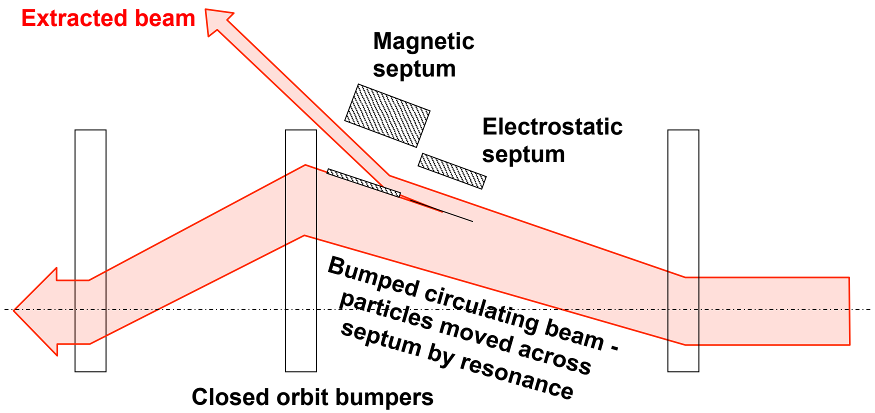
Resonant low-loss multi-turn extraction

- Adiabatic capture of beam in stable "islands"
  - Use non-linear fields (sextupoles and octupoles) to create islands of stability in phase space
  - A slow (adiabatic) tune variation to cross a resonance and to drive particles into the islands (capture) with the help of transverse excitation (using damper)
  - Variation of field strengths to separate the islands in phase space
- Several big advantages:
  - Losses reduced significantly (no particles at the septum in transverse plane)
  - Phase space matching improved with respect to existing non-resonant multi-turn extraction - 'beamlets' have similar emittance and optical parameters

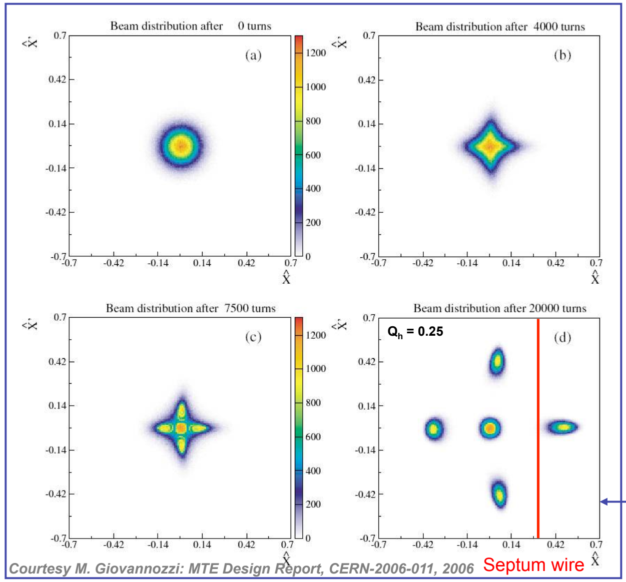
Resonant multi-turn extraction

Non-linear fields excite resonances that drive the beam slowly across the septum



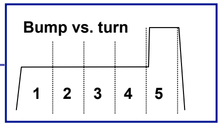
- Slow bumpers move the beam near the septum
- Tune adjusted close to n^{th} order betatron resonance
- Multipole magnets excited to define stable area in phase space, size depends on $\Delta Q = Q - Q_r$

Resonant low-loss multi-turn extraction



Courtesy M. Giovannozzi: MTE Design Report, CERN-2006-011, 2006

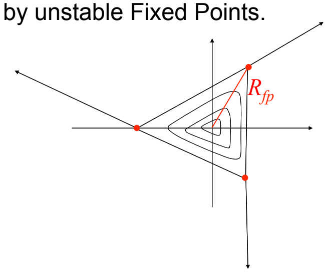
a. Unperturbed beam
b. Increasing non-linear fields
c. Beam captured in stable islands
d. Islands separated and beam bumped across septum – extracted in 5 turns



Resonant multi-turn extraction

- 3^{rd} order resonances – see *lectures by Y. Papaphilippou*
  - Sextupole fields distort the circular normalised phase space particle trajectories.
  - Stable area defined, delimited by unstable Fixed Points.

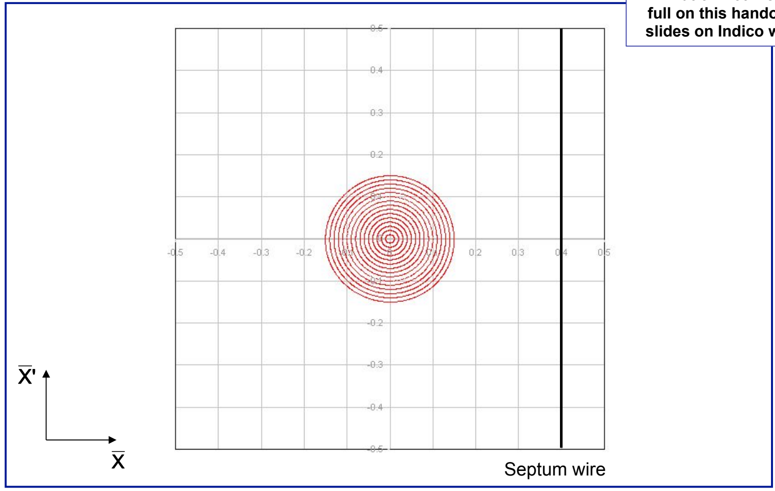
$$R_{fp}^{1/2} \propto \Delta Q \cdot \frac{1}{k_2}$$



  - Sextupole magnets arranged to produce suitable phase space orientation of the stable triangle at thin electrostatic septum
  - Stable area can be reduced by…
    - Increasing the sextupole strength, or…
    - Fixing the sextupole strength and scanning the machine tune Q_h (and therefore the resonance) through the tune spread of the beam
    - Large tune spread created with RF gymnastics (large momentum spread) and large chromaticity

Third-order resonant extraction

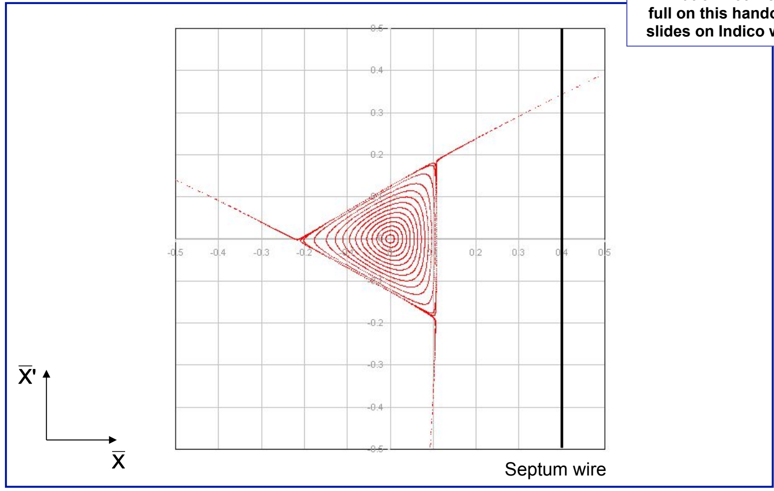
Animation not included in full on this handout: see slides on Indico website!



- Particles distributed on emittance contours
- ΔQ large – no phase space distortion

Third-order resonant extraction

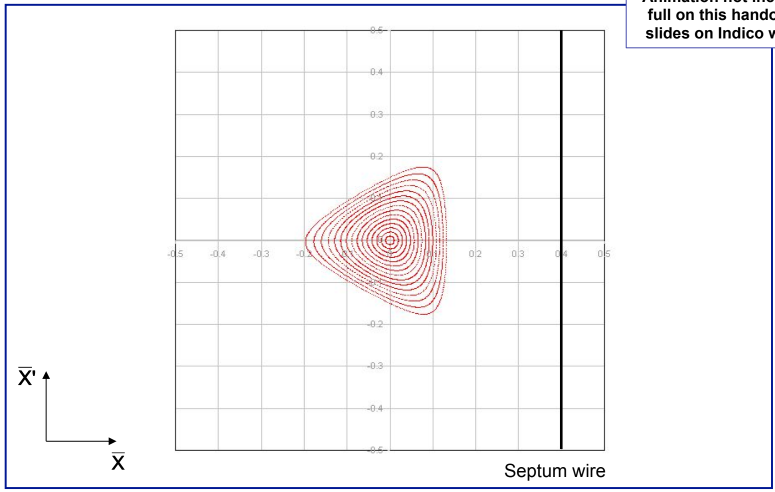
Animation not included in full on this handout: see slides on Indico website!



- ΔQ small enough that largest amplitude particle trajectories are unstable
- Unstable particles follow separatrix branches as they increase in amplitude

Third-order resonant extraction

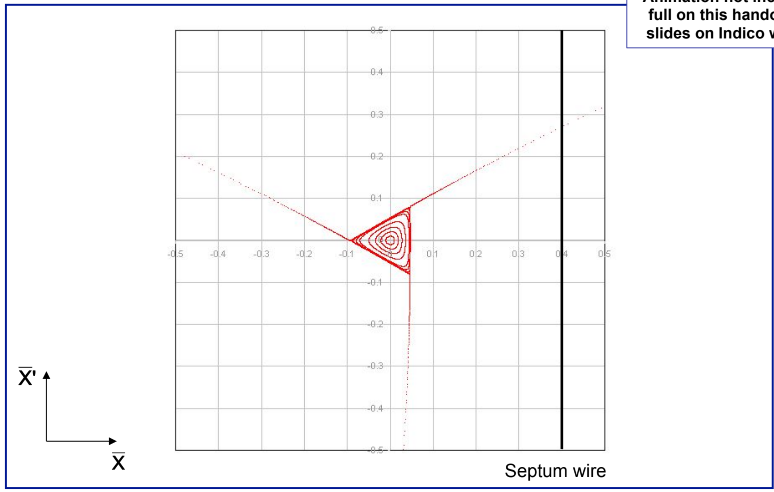
Animation not included in full on this handout: see slides on Indico website!



- Sextupole magnets produce a triangular stable area in phase space
- ΔQ decreasing – phase space distortion for largest amplitudes

Third-order resonant extraction

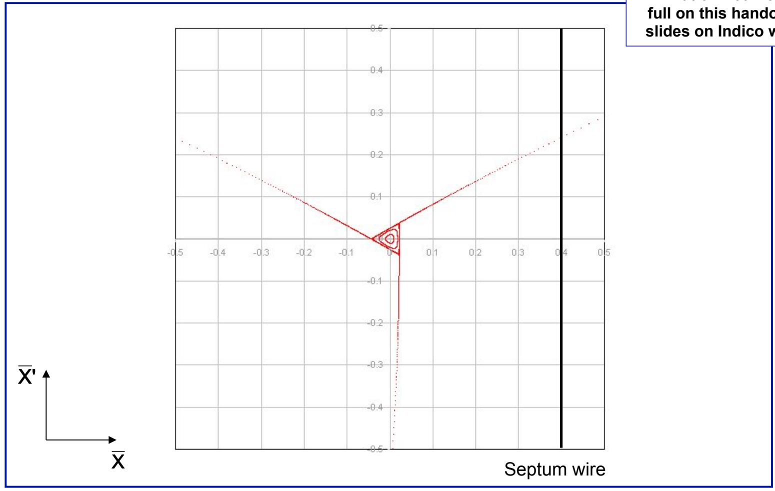
Animation not included in full on this handout: see slides on Indico website!



- As the stable area shrinks, the circulating beam intensity drops since particles are being continuously extracted

Third-order resonant extraction

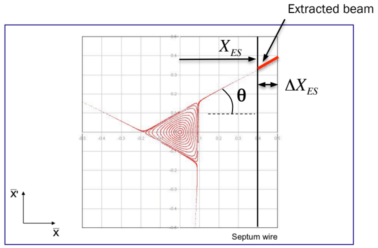
Animation not included in full on this handout: see slides on Indico website!



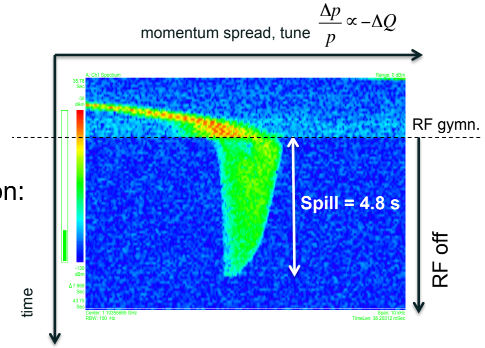
- As the stable area shrinks, the circulating beam intensity drops since particles are being continuously extracted

Third-order resonant extraction

- On resonance, sextupole kicks add-up driving particles over septum
  - Distance travelled in these final three turns is termed the "spiral step," ΔX_{ES}
  - Extraction bump trimmed in the machine to adjust the spiral step

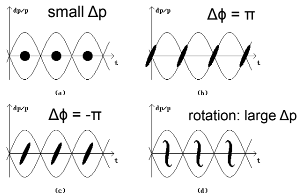


$$\Delta X_{ES} \propto |k_2| \frac{X_{ES}^2}{\cos\theta}$$



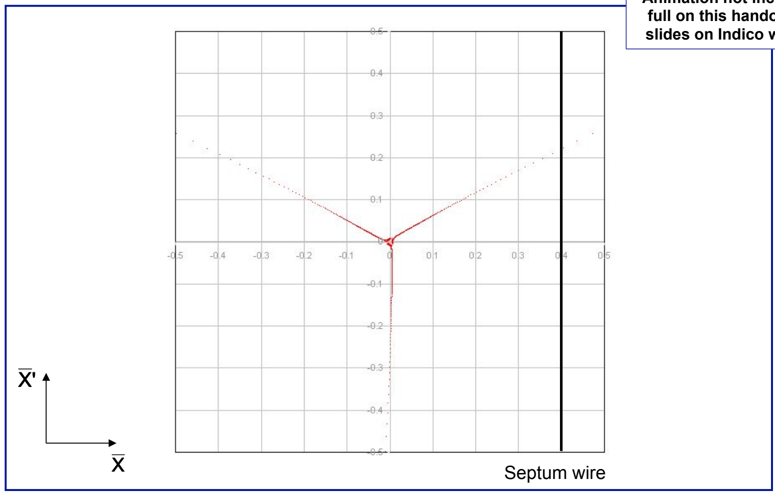
Schottky measurement during spill, courtesy of T. Bohl

- RF gymnastics before extraction:



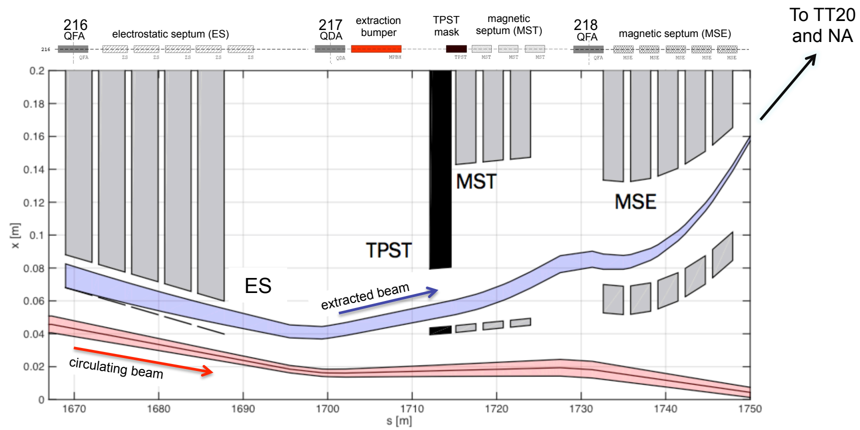
Third-order resonant extraction

Animation not included in full on this handout: see slides on Indico website!



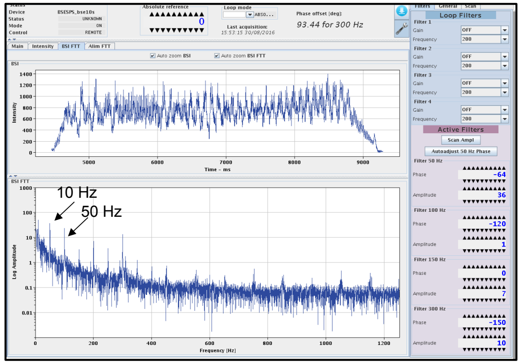
- As ΔQ approaches zero, the particles with very small amplitude are extracted

Slow extraction channel: SPS



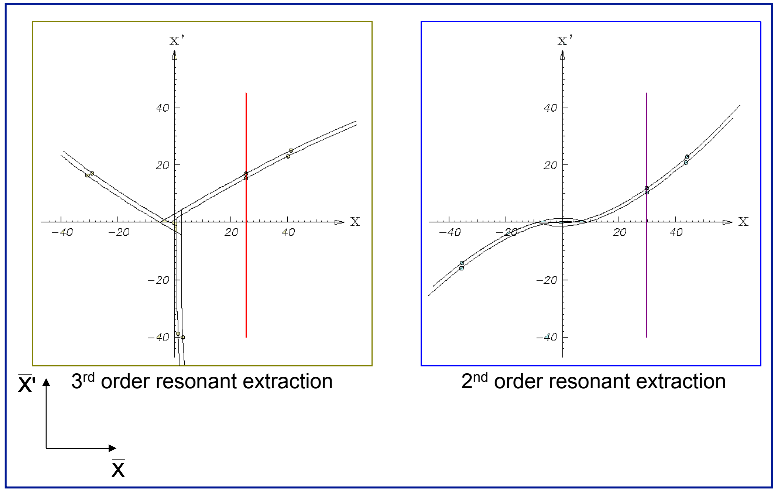
Slow extracted spill quality

- The slow-extraction is a resonant process and it amplifies the smallest imperfections in the machine:
  - e.g. spill intensity variations can be explained by ripples in the current of the quads (mains: *n* x 50 Hz) at the level of a few ppm!
  - Injection of *n* x 50 Hz signals in counter-phase on dedicated quads can be used to compensate



A recent example of a spill at SPS to the North Area with large n x 50 Hz components and another noise source at 10 Hz

Resonant extraction separatrices



3^{rd} order resonant extraction

2^{nd} order resonant extraction

- Amplitude growth for 2^{nd} order resonance much faster than 3^{rd} – shorter spills (≈milliseconds vs. seconds)
- Used where intense pulses are required on target – e.g. neutrino production

Second-order resonant extraction

- An extraction can also be made over a few hundred turns
- 2^{nd} and 4^{th} order resonances
  - Octupole fields distort the regular phase space particle trajectories
  - Stable area defined, delimited by two unstable Fixed Points
  - Beam tune brought across a 2^{nd} order resonance ($Q \rightarrow 0.5$)
  - Particle amplitudes quickly grow and beam is extracted in a few hundred turns

Extraction - summary

- Several different techniques:
  - Single-turn fast extraction:
    - for Boxcar stacking (transfer between machines in accelerator chain), beam abort
  - Non-resonant multi-turn extraction
    - slice beam into equal parts for transfer between machine over a few turns.
  - Resonant low-loss multi-turn extraction
    - create stable islands in phase space: slice off over a few turns.
  - Resonant multi-turn extraction
    - create stable area in phase space ⇒ slowly drive particles into resonance ⇒ long spill over many thousand turns.

Thank you for your attention

Appendix

Blow-up from steering error

- The new particle coordinates in normalised phase space are:

$$\bar{X}_{error} = \bar{X}_0 + L\cos\theta$$

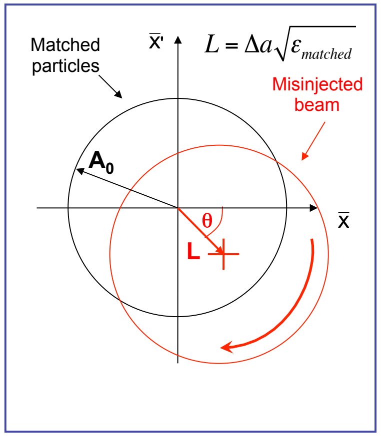
$$\bar{X}'_{error} = \bar{X}'_0 + L\sin\theta$$

- For a general particle distribution, where A_i denotes amplitude in normalised phase of particle i:

$$\boldsymbol{A}_i^2 = \bar{X}_{0,i}^2 + \bar{X}'^2_{0,i}$$

- The emittance of the distribution is:

$$\varepsilon_{matched} = \left\langle \boldsymbol{A}_i^2 \right\rangle / 2$$



Blow-up from steering error

- So we plug in the new coordinates:

$$\boldsymbol{A}_{error}^2 = \bar{X}_{error}^2 + \bar{X}'^2_{error}$$
$$= (\bar{X}_0 + L\cos\theta)^2 + (\bar{X}'_0 + L\sin\theta)^2$$
$$= \bar{X}_0^2 + \bar{X}'^2_0 + 2L(\bar{X}_0\cos\theta + \bar{X}'_0\sin\theta) + L^2$$

$\cos^2\theta + \sin^2\theta = 1$

- Taking the average over distribution:

$$\left\langle \boldsymbol{A}_{error}^2 \right\rangle = \left\langle \boldsymbol{A}_0^2 \right\rangle + 2L(\left\langle \bar{X}_0\cos\theta \right\rangle + \left\langle \bar{X}'_0\sin\theta \right\rangle) + \left\langle L^2 \right\rangle$$
$$= 2\varepsilon_{matched} + L^2$$

0 0

- Giving the diluted emittance as:

$$\varepsilon_{diluted} = \varepsilon_{matched} + \frac{L^2}{2}$$
$$= \varepsilon_{matched}\left[1 + \frac{\Delta a^2}{2}\right]$$