Free-Electron Lasers

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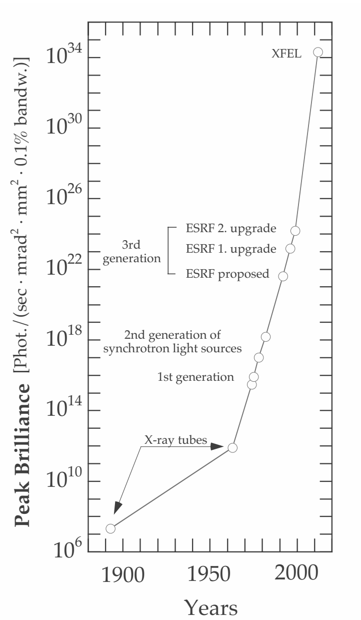
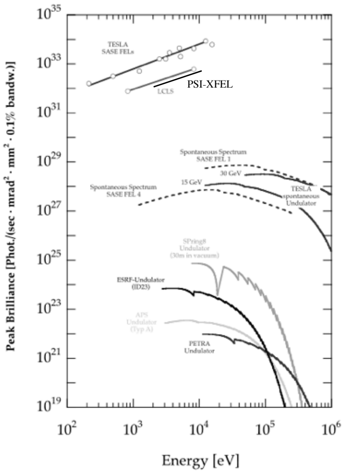
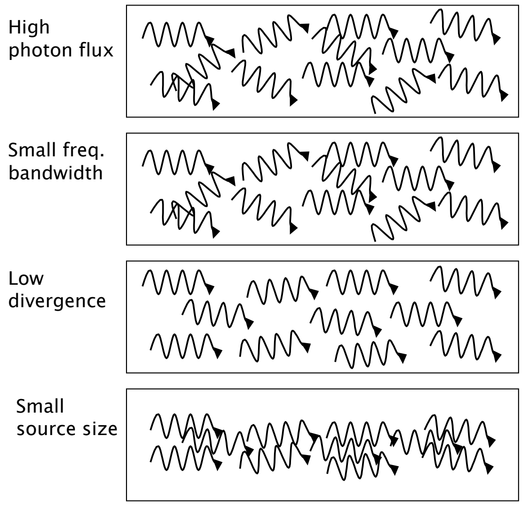
Light Sources

- 1st Generation: Synchrotron radiation from bending magnets in high energy physics storage rings
- 2nd Generation: Dedicated storage rings for synchrotron radiation
- 3rd Generation: Dedicated storage rings with insertion devices (wigglers/undulators)
- 4th Generation: Free-Electron Lasers



FEL as a Brilliant Light Source



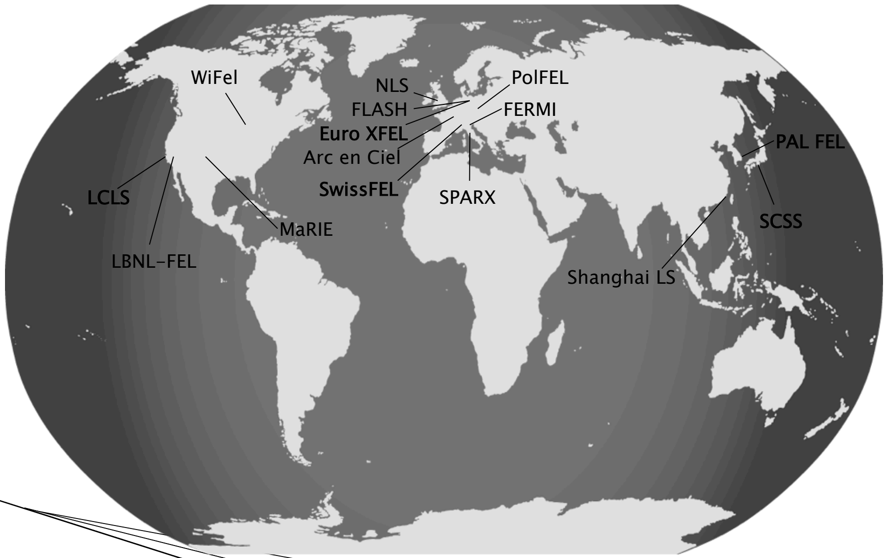


4th Generation Light Sources

- Tunable wavelength, down to 1 Ångstroem
- Pulse Length less then 100 fs
- High Peak Power above 1 GW
- Fully Transverse Coherence
- Transform limited Pulses

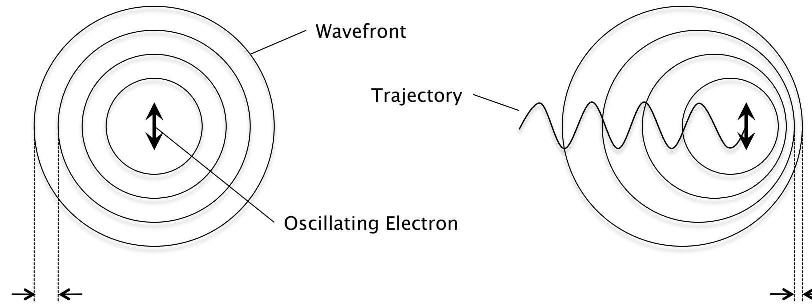
XFELs fulfill all criteria except for the longitudinal coherence (but we are working on it ☺)

X-ray/VUV FEL Projects Around the World



Controlling the Wavelength – The Idea

Dipole Radiation (Antenna) | Dipole Radiation + Doppler Shift



$\lambda = c \cdot T$

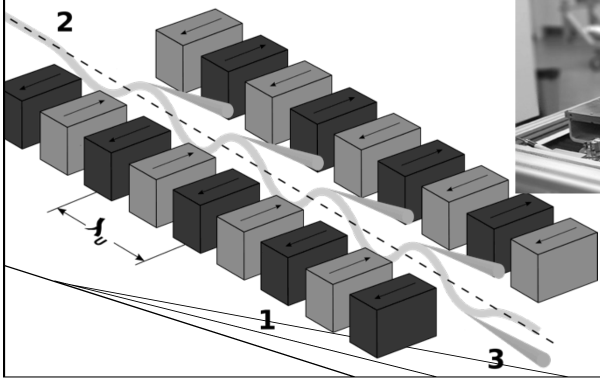
$\lambda = (c - v_z) \cdot T$

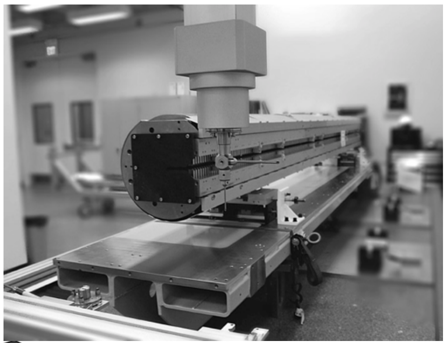
(forward direction)

For relativistic electrons the longitudinal velocity v_z is close to c, resulting in very short wavelength (blue shift of photon energy)

Forcing the Electrons to Wiggle…

- … by injecting them into a period field of an wiggler magnet (also often called undulator).





Wiggler module from the LCLS XFEL

Motion in a Wiggler

- Periodic Field of the Wiggler: $B_y = B_0 \sin(k_u z) \qquad k_u = \frac{2\pi}{\lambda_u}$
- Lorentz-Force: $\vec{F} = e \cdot \vec{v} \times \vec{B}$
- Dominant motion:
  - Electron motion in *z*-direction, field in *y*-direction.
  - Motion in magnetic field preserves total energy:

$$F_x = \frac{d}{dt}p_x = \gamma mc\frac{d}{dt}\beta_x = -ec\beta_z B_0 \sin(k_u z) \quad \xrightarrow{(z = c\beta_z t)} \quad \beta_x = \frac{1}{\gamma}\boxed{\frac{eB_0}{mck_u}}\cos(k_u z)$$

Undulator parameters put into one constant: $K = 0.93 \cdot B_0[\mathrm{T}] \cdot \lambda_u[\mathrm{cm}]$

$$\boxed{\beta_x = \frac{K}{\gamma}\cos(k_u z)}$$

The Undulator Wavelength

- We got everything now to calculate the wavelength:

$$\lambda = cT(1-\beta_z)$$

Period Length $cT = \beta_z \lambda_u \approx \lambda_u$

Long. Velocity $\beta_z = \sqrt{1-\frac{1}{\gamma^2}-\beta_x^2} \approx 1-\frac{1}{2\gamma^2}-\frac{K^2}{2\gamma^2}\sin^2(k_u z)$

- For the average velocity, the square of the sine function is ½. The emitted wavelength is:

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}\right)$$

The Resonant Wavelength

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}\right)$$

- The wavelength can be controlled by
  - **Changing the electron beam energy,**
  - **Varying the magnetic field** (requires K significantly larger than 1)

- To reach 1 Å radiation, it requires an undulator period of 15 mm, a K-value of 1.2 and an energy of 5.8 GeV (γ=11000)

The Free-Electron Lasers are based on undulator radiation and have the same resonant wavelength

'Free-Electron' refers to the fact that unlike quantum lasers the electrons are unbound in the periodic 'potential' of the undulator.

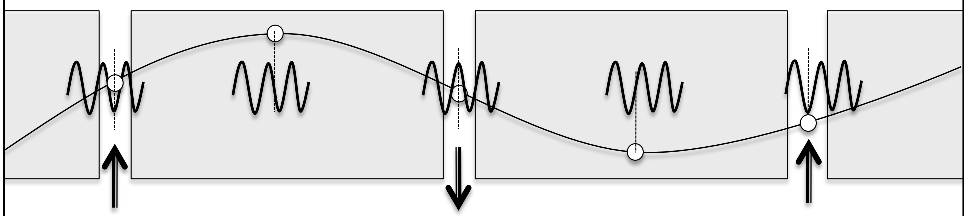
Interaction with Co-Propagating Field

- The wiggling electrons emit radiation. Some of it co-propagates along the undulator.
- The transverse oscillation allows the coupling of the
- Electrons, absorbing more photons than emitting, become faster and tend to group with electrons, which are emitting more photons than absorbing.

The FEL exploits a collective process, which ends with an almost fully coherent emission at the resonant wavelength.

Co-Propagation of Electrons and Field

- The transverse oscillation allows to couple with a co-propagating field $\vec{v}_\perp \cdot \vec{E}_\perp$
- The electron moves either with or against the field line
- After half undulator period the radiation field has slipped half wavelength. Both, velocity and field, have changed sign and the direction of energy transfer remains.



The net energy change can be accumulated over many period.

Step I : Energy Modulation

- Energy change:

$$mc^2 \frac{d}{dz}\gamma = \frac{K}{\gamma}\sin(k_u z)\cdot E_0 e^{i(kz-\omega t+\phi)} = \frac{KE_0}{2\gamma i}\left[e^{i[(k+k_u)z-\omega t+\phi)} - e^{i[(k-k_u)z-\omega t+\phi)}\right]$$

transverse motion *plane wave*

$$z = c\beta_z t + z_0 + \Delta z \sin(2k_u z)$$

- Ponderomotive Phase: $\theta = \left[(k+k_u)\beta_z - k\right]ct + \theta_0$

=0

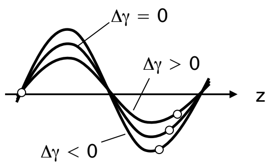
Resonance Condition $\beta_z = \frac{k}{k+k_u} \Rightarrow \lambda = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}\right.$

- Averaging over an undulator period:
  - Second exp-function drops out ($<\exp(i\theta-2ik_u z)>=0$)
  - Longitudinal oscillation smears out position → reduced coupling f_c

$$\frac{d}{dz}\gamma = \frac{f_c K}{\gamma}\frac{E_0}{mc^2}\sin(\theta+\phi)$$

Step II: Longitudinal Motion

- For a given wavelength λ there is one energy γ_r, where the electron stays in phase with radiation field.
- Electrons with energies above the resonant energy, move faster ($d\theta/dz > 0$), while energies below will make the electrons fall back ($d\theta/dz < 0$).
- For ***small energy deviation*** from the resonant energy, the change in phase is linear with $\Delta\gamma=(\gamma-\gamma_r)$.



$$\frac{d}{dz}\theta = (k+k_u)\beta_z - k$$

$$= k_u - k\frac{1+K^2/2}{2(\gamma_r+\Delta\gamma)^2} = 2k_u\frac{\Delta\gamma}{\gamma_r}$$

Analogy to Pendulum

FEL Equations

$$\frac{d}{dz}\theta \propto \frac{\Delta\gamma}{\gamma_r} \qquad \frac{d}{dz}\frac{\Delta\gamma}{\gamma_r} \propto \frac{KE_0}{\gamma_r^2}\sin(\theta+\phi)$$

Frequency

$$\Omega \propto \frac{\sqrt{KE_0}}{\gamma_r}$$

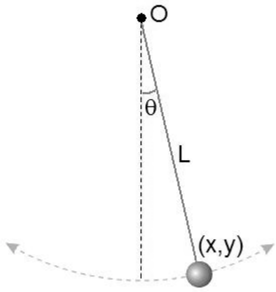
Pendulum Equations

$$\frac{d}{dt}\theta = I \qquad \frac{d}{dt}I = \frac{g}{L}\sin\theta$$

Frequency

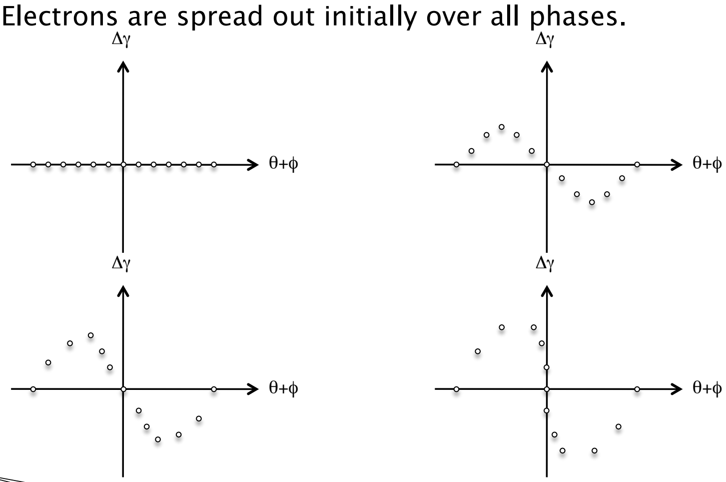
$$\Omega \propto \sqrt{\frac{g}{L}}$$

- Stable Fix Point: $\Delta\gamma=0$, $\theta+\phi=0$
- Instable Fix Point: $\Delta\gamma=0$, $\theta+\phi=\pi$
- Oscillation gets faster with growing E-Field (Pendulum: shorter length L)



Motion in Phasespace

- Wavelength typically much smaller than bunch length.
- Electrons are spread out initially over all phases.



Electrons are bunched on same phase after quarter rotation

Microbunching

Transverse position

3D Simulation for FLASH FEL over 4 wavelengths

Frame moving with electron beam through 15 m undulator

Wiggle motion is too small to see. The 'breathing' comes from focusing to keep beam small.

Slice of electron bunch (4 wavelengths)

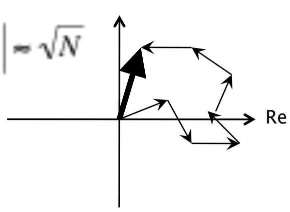
Microbunching has periodicity of FEL wavelength. All electrons emit coherently.

Coherent Emission

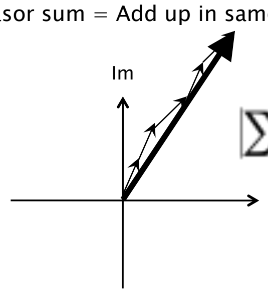
- The electrons are spread out over the bunch length with its longitudinal position δz_j. The position adds a phase $\phi_j = k\delta z_j$ to the emission of the photon.
- The total signal is: $E(t) \propto \sum_j e^{i(kz_j - \omega t)} = e^{i(k\langle z\rangle - \omega t)} \cdot \sum_j e^{ik\delta z_j}$

Electrons spread over wavelength:
Phasor sum = random walk in 2D

$\left|\sum e^{i\phi}\right| \approx \sqrt{N}$



Electrons bunched within wavelength:
Phasor sum = Add up in same direction



$\left|\sum e^{i\phi}\right| \approx N$

Power ~ |E|² -> Possible Enhancement: N

Complete Picture: Evolving Radiation Field

- Because the degree of coherence grows the field does not stay constant (Pendulum is only valid for adiabatic changes)
- The change in the radiation field is given by Maxwell equation:

$$\left[\nabla_\perp^2 + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\right]\frac{E_0}{ck}e^{ikz-i\omega t+i\phi} = \mu_0 \sum_j ec\vec{\beta}\cdot\delta(\vec{r}_j - \vec{r})$$

Slow Varying Envelope Approximation: $\partial_z E_0 << k E_0$

Period-averaged motion: $\left\langle \beta_x e^{-ik+i\omega t}\delta(\vec{r}_j - \vec{r})\right\rangle = \frac{f_c K}{2\gamma_j}e^{-i\theta_j}$

$$\left[\nabla_\perp^2 + 2ik\left(\frac{\partial}{\partial z} + \frac{\partial}{c\partial t}\right)\right]u = i\frac{e^2\mu_0}{2m}\sum_j \frac{f_c K}{\gamma_j}e^{-i\theta_j} \qquad \left(u = \frac{eE_0}{mc^2k}e^{i\phi}\right)$$

The Levels of the FEL Model

$$\left[\nabla_\perp^2 + 2ik\left(\frac{\partial}{\partial z} + \frac{\partial}{c\partial t}\right)\right]u = i\frac{e^2\mu_0}{2m}\sum_j \frac{f_c K}{\gamma_j}e^{-i\theta_j}$$

Diffraction

Evolution along Undulator

Evolution along Bunch (Slippage)

Source Term

1D Model: Ignore Diffraction Effects

Steady-State Model: Ignore Time Dependence

To start-easy: 1D, steady-state FEL Model

Scaling of 1D FEL Equations

$$\frac{d}{dz}\theta = 2k_u \frac{\gamma - \gamma_r}{\gamma_r} \quad \xrightarrow{\hat{z} = 2k_u \rho z} \quad \frac{d}{d\hat{z}}\theta = \frac{\gamma - \gamma_0}{\rho\gamma_0} + \frac{\gamma_0 - \gamma_r}{\rho\gamma_r} = \eta + \Delta$$

Δ=Detuning
η=Deviation
$\gamma_0 \approx \gamma_r$

$$\frac{d}{dz}\gamma = -k\frac{f_c K}{2\gamma_0}\left(ue^{i\theta} + c.c.\right) \quad \xrightarrow{\eta = (\gamma - \gamma_0)/\rho\gamma_0} \quad \frac{d}{d\hat{z}}\eta = -\left(Ae^{i\theta} + c.c.\right) \qquad A = \frac{kf_c K}{4\gamma_0^2 k_u \rho^2} u$$

$$\frac{d}{dz}u = \frac{e^2\mu_0 n_e}{4km}\frac{f_c K}{\gamma_0}\left\langle e^{-i\theta}\right\rangle \quad \xrightarrow[I_A = 4\pi mc/e\mu_0]{en_e = I/2\pi c\sigma_x^2} \quad \frac{d}{d\hat{z}}A = \rho^{-3}\left[\frac{1}{\gamma_0^3}\left(\frac{f_c K}{4k_u\sigma_x}\right)^2 \frac{I}{I_A}\right]\left\langle e^{-i\theta}\right\rangle$$

Choose ρ to eliminate constants

$$\rho = \frac{1}{\gamma_0}\left[\left(\frac{f_c K}{4k_u\sigma_x}\right)^2 \frac{I}{I_A}\right]^{\frac{1}{3}} \quad \Rightarrow \quad \begin{aligned} \theta' &= \eta + \Delta \\ \eta' &= -(Ae^{i\theta} + c.c.) \\ A' &= \left\langle e^{-i\theta}\right\rangle \end{aligned}$$

How to solve the Problem?

- 2N Ordinary and 1 partial differential equation.
- Electrons are represented by the density function f(θ,η,z), fulfilling Liouville's theorem:

$$\frac{d}{dz}f(\theta,\eta,z) = \left[\frac{\partial}{\partial z} + \theta'\frac{\partial}{\partial\theta} + \eta'\frac{\partial}{\partial\eta}\right]f(\theta,\eta,z) = 0 \qquad \begin{aligned} \theta' &= \eta + \Delta \\ \eta' &= -(Ae^{i\theta} + c.c.) \end{aligned}$$

(Vlasov Equation)

- Fourier series expansion:

$$f(\theta,\eta,z) = f_0(\eta) + f_1(\eta,z)e^{i\theta} + \ldots$$

Initial distribution (f_0) — Current modulation (f_1)

- Equation with terms proportional to $e^{i\theta}$:

$$\frac{\partial}{\partial z}f_1 + i(\Delta + \eta)f_1 - A\frac{\partial}{\partial\eta}f_0 = 0$$

Including the field equation…

- Field evolution is given by density function:

$$\frac{d}{dz}A = \langle e^{-i\theta} \rangle = \iint f(\theta,\eta,z)e^{-i\theta}\,d\eta\,d\theta = \int f_1(\eta,z)\,d\eta$$

- The formal solution for f_1 is:

$$\frac{\partial}{\partial z}f_1 + i(\Delta+\eta)f_1 - A\frac{\partial}{\partial \eta}f_0 = 0 \Longrightarrow f_1(z) = \int_0^z A\frac{\partial f_0}{\partial \eta}e^{-i(\Delta+\eta)(z-z')}dz'$$

- Problem reduce to integro-differential equation:

$$\frac{d}{dz}A = \int_{-\infty}^{\infty}\int_0^z A\frac{\partial f_0}{\partial \eta}e^{-i(\Delta+\eta)(z-z')}dz'\,d\eta$$

Solving the FEL Equations

- Laplace transformation: (Fourier transformation does not work) :

$$\tilde{g}(p) = \int_0^\infty g(z)e^{-pz}dz$$

- FEL Equation:

$$\int_0^\infty\left(\frac{d}{dz}A(z)\right)e^{-pz}dz = \int_{-\infty}^{\infty}d\eta\frac{\partial f_0}{\partial \eta}\int_0^\infty dz\,e^{i(ip-\eta-\Delta)z}\int_0^z dz'\,A(z')e^{i(\eta+\Delta)z'}$$

- Integration over *z* by parts on both sides:

$$A(0) + p\tilde{A}(p) = -\int_{-\infty}^{\infty}d\eta\frac{\partial f_0}{\partial z}\int_0^\infty\frac{A(z)e^{-pz}}{i(ip-\eta-\Delta)}dz$$

- Formal Solution:

$$\tilde{A}(p) = \frac{A(0)}{p - \int_{-\infty}^{\infty}\frac{\partial f_0}{\partial \eta}\frac{d\eta}{p+i\eta+i\Delta}}$$

Dispersion Equation

- Laplace Transformation:

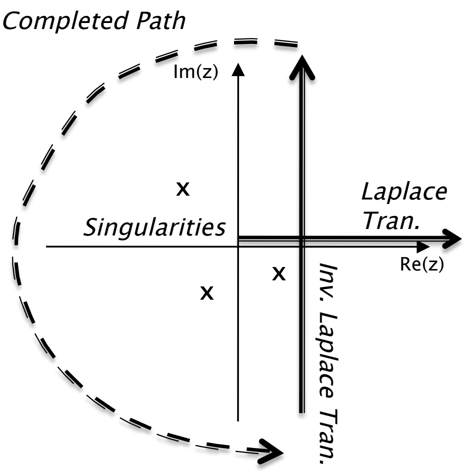
$$\tilde{A}(p) = \frac{A(0)}{p - \int_{-\infty}^{\infty} \frac{\partial f_0}{\partial \eta} \frac{d\eta}{p + i\eta + i\Delta}}$$

- Inverse Laplace Transformation:

$$A(z) = \frac{1}{2\pi i} \int_{p_0 - i\infty}^{p_0 + i\infty} \tilde{A}(p) e^{pz} dp$$

Dispersion Equation

$$p - \int_{-\infty}^{\infty} \frac{\partial f_0}{\partial \eta} \frac{d\eta}{p + i\eta + i\Delta} = 0$$



The 1D Solution

- No detuning (Δ=0) and no energy spread ($f_0=\delta(\eta)$)

$$p - \int_{-\infty}^{\infty} \frac{\partial f_0}{\partial \eta} \frac{d\eta}{p + i\eta + i\Delta} = 0 \quad \Rightarrow \quad p^3 = i$$

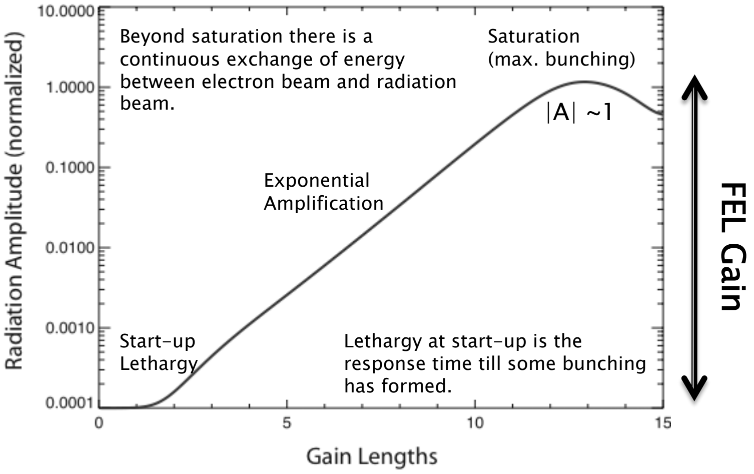
- 3 Solutions to: $A(z) = A(0)e^{pz}$

Oscillating	*Growing*	*Decaying*
$p = -i$	$p = \frac{\sqrt{3}}{2} + \frac{i}{2}$ ***FEL Mode***	$p = -\frac{\sqrt{3}}{2} + \frac{i}{2}$

- Gain Length (e-folding length):

$$P(z) \propto |A(z)|^2 \propto e^{2pz} = e^{2\sqrt{3}k_u \rho z} \Longrightarrow L_g = \frac{\lambda_u}{4\pi\sqrt{3} \cdot \rho}$$

The Generic Amplification Process



Efficiency and Bandwidth

- Energy conservation and efficiency:

$$|A|^2 + \langle \eta \rangle = const \quad \Rightarrow \quad \langle \eta \rangle_{sat} = |A_0|^2 + \langle \eta \rangle_0 - |A_{sat}|^2 \approx -1$$

$$\Longrightarrow \quad \frac{\langle \Delta\gamma \rangle}{\gamma} \approx \rho \quad \Longrightarrow \quad \boxed{P_{FEL} \approx \rho P_{beam}}$$

- Bandwidth:

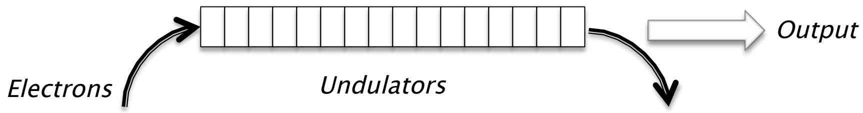
$$\omega = \frac{2ck_u}{1 + K^2/2} \cdot \gamma^2 \quad \Longrightarrow \quad \boxed{\frac{\Delta\omega}{\omega} = 2\rho}$$

For a given wavelength the beam can be detuned by $\Delta\gamma=\rho\gamma$ and still amplify the signal.

For a given energy the wavelength can be detuned by $\Delta\omega=2\rho\omega$ and still be amplified.

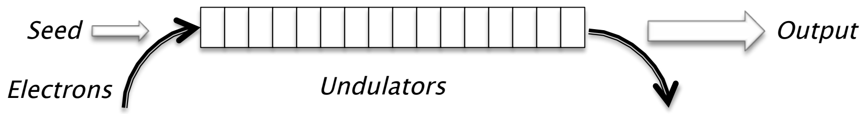
FEL Modes

- SASE FEL (Self-Amplified Spontaneous Emission)



Disadvantage: Output is noisy

- FEL Amplifier (starts with an input signal)



Disadvantage: Seed source may not exist

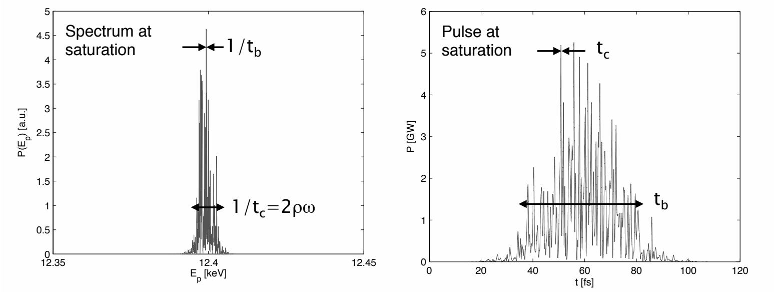
Typical Growth of SASE Pulse

Simulation for FLASH FEL

SASE FELs

- FEL starts with the broadband signal of spontaneous radiation (almost a white noise signal)
- Within the FEL bandwidth $\Delta\omega=2\rho\omega$ the noise is amplified
- Spikes in spectrum and time profile.

SwissFEL: Simulation for 1 Angstrom radiation



Coorperation Length:

$$L_c = \frac{1}{2k\rho} = \frac{\lambda}{4\pi\rho}$$

The Importance of the FEL Parameter ρ

- Typical values $\rho = 10^{-2} - 10^{-4}$

$$\rho = \frac{1}{\gamma_0}\left[\left(\frac{f_c K}{4k_u\sigma_x}\right)^2 \frac{I}{I_A}\right]^{\frac{1}{3}}$$

- Scaling

Gain length

$$L_g = \frac{\lambda_u}{4\pi\sqrt{3}\cdot\rho}$$

Efficiency

$$P_{FEL} \approx \rho P_{beam}$$

SASE Spike Length

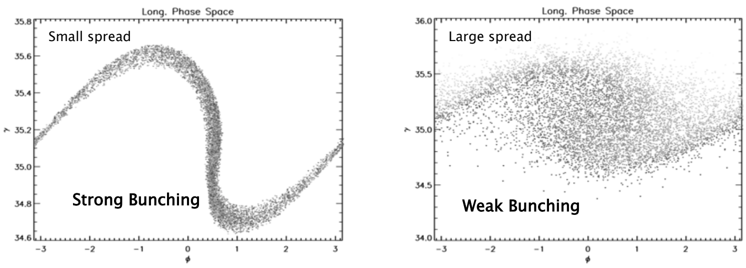
$$L_c = \frac{\lambda}{4\pi\rho}$$

Bandwidth

$$\frac{\Delta\omega}{\omega} = 2\rho$$

Electron Beam Requirements: Energy Spread

- Only electrons within the FEL bandwidth can contribute to FEL gain.
- FEL process is a quarter rotation in the separatrix of the FEL. If separatrix is filled homogeneously, no bunching and thus coherent emission can be achieved.



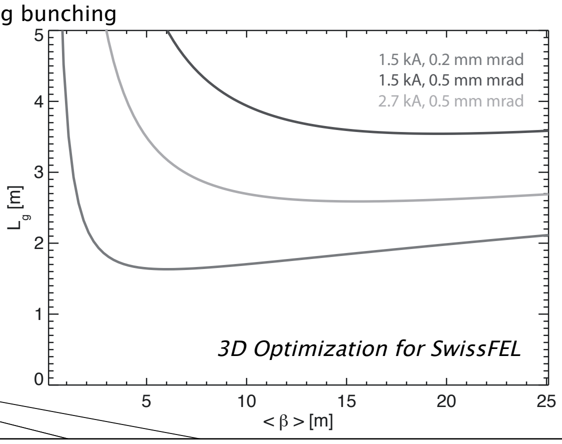
Energy Spread Constraint: $\frac{\sigma_\gamma}{\gamma} << \rho$

Optimizing the Focusing

- Decreasing the β-function (increase focusing), increases the FEL parameter ρ.
- Stronger focusing:
  - Larger kinetic energy of betatron oscillation
  - Less kinetic energy for longitudinal motion
  - Smearing out of growing bunching

From 1D Theory:

$$\beta_{opt} \approx 3\sqrt{\frac{\varepsilon_n}{\gamma}\frac{4\pi}{\lambda}}L_g$$



Quick Overview of 3D Theory

- New free Parameter: ***Diffraction Parameter***

$$B = \frac{z_R}{z_{FEL}}$$

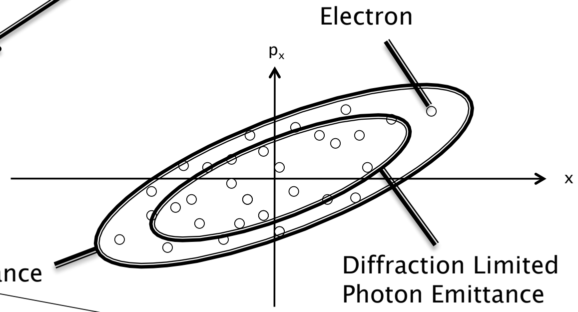
z_R ← Rayleigh Length

z_{FEL} ← Scaling length of the FEL ($2k_u\rho$)

- Assuming electron size as radiation source size:

$$z_R = 2r_0^2 k = \frac{4\pi}{\lambda} \cdot \frac{\varepsilon_n}{\gamma} \beta$$

"Soft" constraint for emittance to be smaller than photon emittance for all electrons to contribute on the emission process



FEL Eigenmodes + Mode Competion

- Growth rates for FEL eigenmodes (r,ϕ–decomposition):

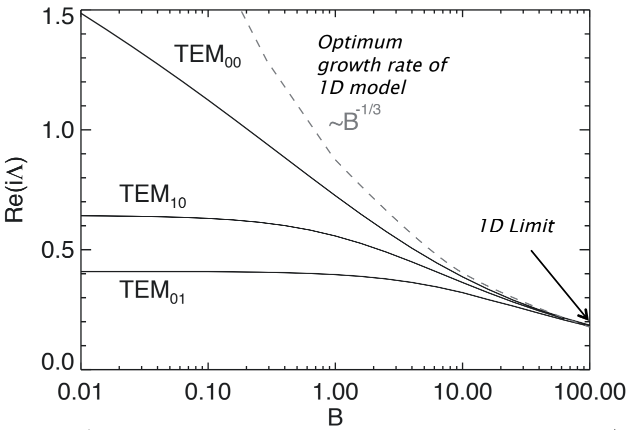
Optimum for SASE:

Only one dominating mode at saturation

Transverse coherence

$$B \approx 7$$

$$\frac{\varepsilon_n}{\gamma} \approx \frac{\lambda}{2\pi}$$

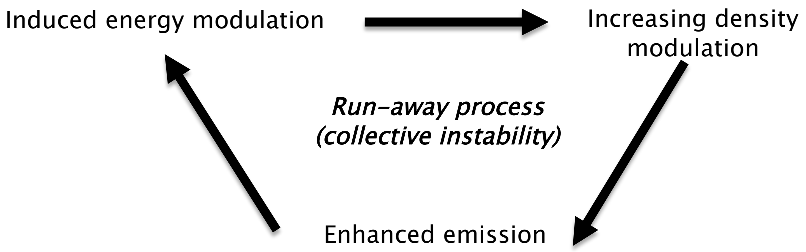


Increased gain length due to strong diffraction

Mode competition and reduced coherence

Summary

FEL Process



- FEL utilized the strong coherent emission in the collective instability with the tuning ability of the wavelength.
- Instability can only occur with a beam with low energy spread and emittance.