# LANDAU DAMPING

**L. Palumbo**

**SAPIENZA UNIVERSITY ROME & LNF-INFN**

*L. Palumbo and M. Migliorati, Landau damping in Particle Accelerators, 2011.*

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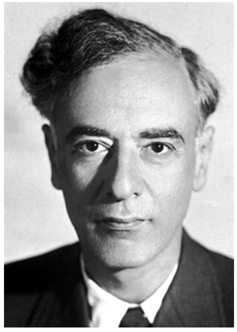
**Landau damping** is a physical effect named after his discoverer, the Russian physicist **Lev Davidovich Landau**, who studied in 1946 the wave propagation in a plasma.

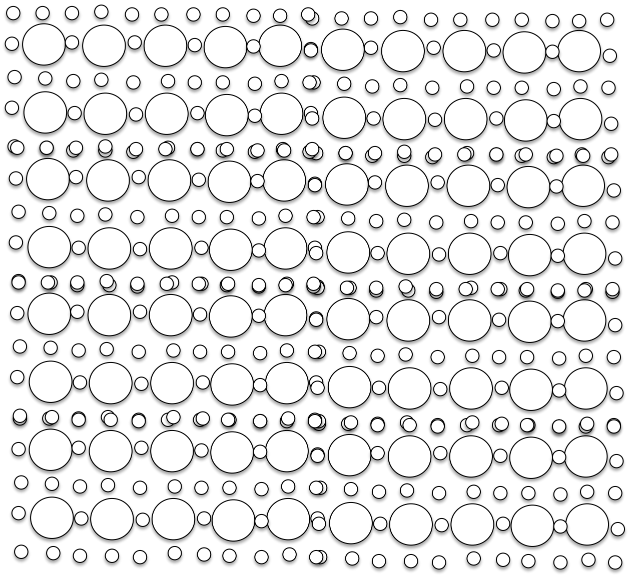


According to Landau theory, an initial perturbation of longitudinal charge density in plasma waves is prevented from developing because of a natural stabilizing mechanism.

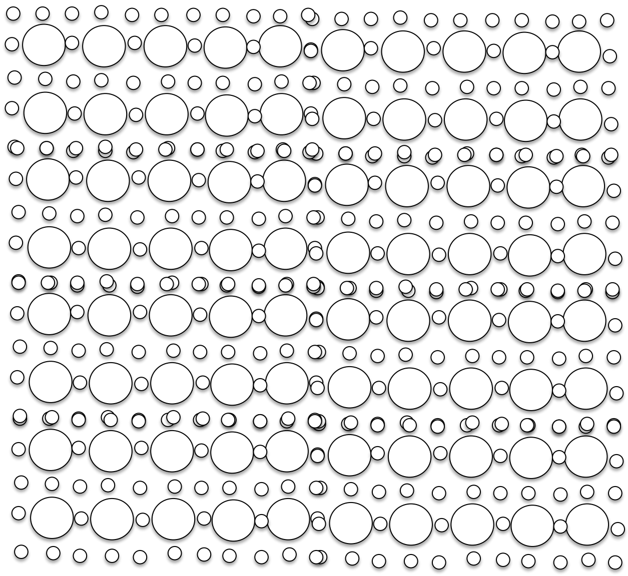
## 1. Plasma oscillation

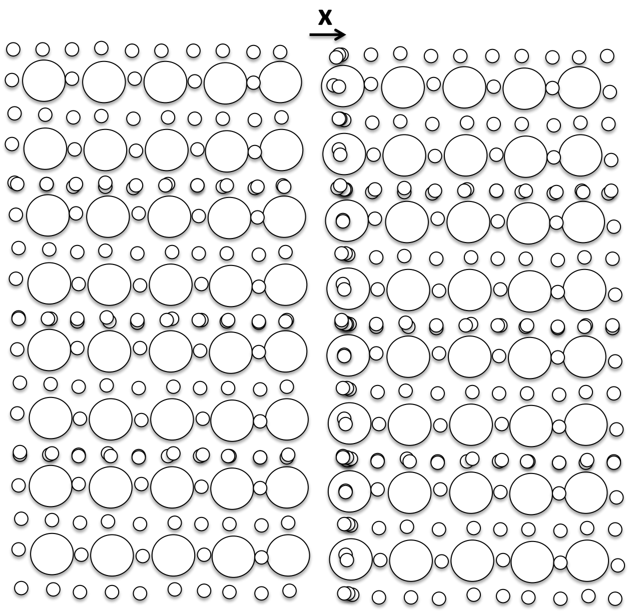
- A cold plasma of ionized gas consists of ions and free electrons distributed over a region in space. The positive ions are very much heavier than the electrons, so that we can neglect their motion in comparison to that of electrons.

- The plasma at the equilibrium, being neutral, is characterized by the same local density $n_0$ [1/m$^3$] for both electrons and ions.



- If, for some reason, electrons are displaced from their equilibrium position, the local density changes producing electrical forces that tend to restore the equilibrium.
- As in any classical harmonic oscillator, the electrons gain kinetic energy, and instead of coming to rest, they start oscillating back and forth, at a frequency called ”plasma frequency”.

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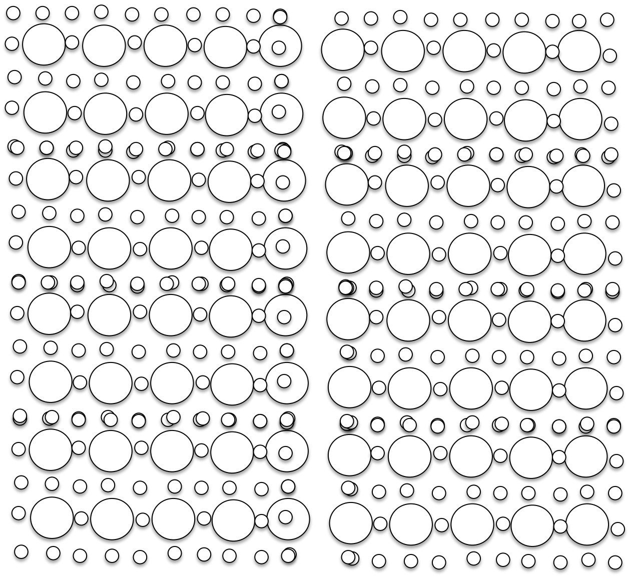
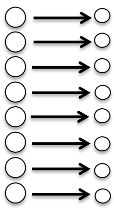




$$\rho = n_0 e \qquad \sigma = \rho x$$

$$E_x = \frac{\sigma}{\varepsilon_0} \qquad F_x = -eE_x = -\frac{n_0 e^2}{\varepsilon_0} x$$

$$m_e \ddot{x} = -\frac{n_0 e^2}{\varepsilon_0} x$$

$$\omega_{plasma} = \frac{n_0 e^2}{m_e \varepsilon_0}$$

## 2. Dispersion relation for plasma waves

We consider now the more general case of a charge density with a distibution function depending on the position and velocity such that:

$$\int f(x,v_x,t)\,dx\,dv_x = N$$

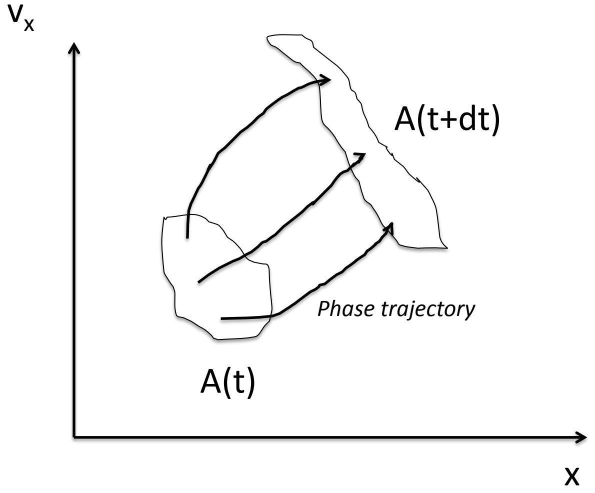
If the charges are not in a state of equilibrium, we will observe a time evolution of the distribution under the effect of the self electric field.

Such a system can be studied by means of the methods developed by Boltzmann to describe the behavior of systems far from the thermodynamical equilibrium.

We have to study the motion of an ensemble of N particles characterized by a distribution function $f(x,v_x,t)$ under the action of self forces.

The fundamental equation which describes the kinematics of this ensemble is the continuity equation for the density of the particles in the phase space.

It states the conservation of the number particles in any phase space volume during the motion.




The phase space area enclosing a number of particle at time *t* can be distorted at time *t+dt* but it remanins constant. For an infinitesimal area dA=dx $dv_x$ we have:

$$dN = f(x, v_x, t)dxdv_x = f(x + v_x dt,\ v_x + a_x dt,\ t + dt)dxdv_x$$

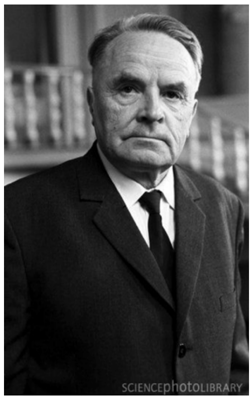
where $a_x = \dfrac{F_x}{m_e}$

If we expand at the first order the RHS term, simplifying the common terms, we get:

$$\boxed{\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + \frac{F_x}{m_e}\frac{\partial f}{\partial v_x} = 0}$$ ***(Boltzmann Equation)***

An important contribution to the comprehension of plasma waves came first from the work of the russian physicist **Anatoly Alexandrovich Vlasov.**

- In 1937, **Vlasov** showed that Boltzmann equation is suitable for a decription of plasma dynamics only if we consider the **long range collective forces** existing in the plasma.
- Thus, a system of equations, known today as Vlasov-Poisson equation, was suggested for the correct description to take into account the collective forces through a **self-consistent field**.




The electric field is derived from the scalar potential

$$E_x = -\frac{\partial \phi}{\partial x}$$

which in turns is related to the net local density:

$$\frac{\partial^2 \phi}{\partial x^2} = -\frac{\rho}{\varepsilon_0} = -\frac{e}{\varepsilon_0}\left(n_0 - \int f dv_x\right)$$

We assume now that for the system of charges there is an equilibrium state $f_o(v_x)$ with a proper velocity distribution, and we consider a smal perturbation $f_1(x, v_{x,} t)$ around that equilibrium:

$$f(x, v_x, t) = f_0(v_x) + f_1\left(x, v_x, t\right)$$

Since $f_o$ doesn't depend on time and position, neglecting the second order terms, from the Bolzmann equation we have:

$$\frac{\partial f_1}{\partial t} + v_x \frac{\partial f_1}{\partial x} - \frac{e}{m_e} E_x \frac{\partial f_0}{\partial v_x} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{e}{\varepsilon_0} \int f_1 dv_x$$

Vlasov-Poisson Equations

These two coupled equations tell us that a density perturbation produces an electric field which acts back on the perturbation, both evolve in the time.

This mechanism can sustain plasma vaves propagating in the medium. In order to find a self consistent solution, Vlasov expanded the unknown functions $f_1$ and $\phi$ through the double Fourier transforms:

$$f_1(x, v_x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}_1(k, v_x, \omega) e^{i(kx - \omega t)} dk d\omega$$

$$\phi(x, v_x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\phi}(k, v_x, \omega) e^{i(kx - \omega t)} dk d\omega$$

which applied to the Bolzmann-Poisson equation produce the well known **Dispersion Relation** for plasma waves:

$$1 + \frac{e^2}{\varepsilon_0 m_e k} \int \frac{\partial f_0 / \partial v_x}{\omega - k v_x} dv_x = 0$$

**A. A. Vlasov (1937): "On Vibration Properties of Electron Gas" (in Russian).**

**ЦИОННЫХ СВОЙСТВАХ ЭЛЕКТРОН**

***А. А. Власов***

$$\frac{4\pi e^2}{k}\int\limits_{-\infty}^{+\infty}\frac{\xi\,\partial\Phi_0/\partial\varepsilon}{k\xi-\omega}\,d\xi\,d\eta\,d\zeta=1.$$

$$\boxed{1+\frac{e^2}{\varepsilon_0 m_e k}\int\frac{\partial f_0/\partial v_x}{\omega-kv_x}dv_x=0}$$

- It provides a relationship between the wave number $k=2\pi/\lambda$ and the frequency $\omega$ of the wave in the plasma
- It depends on the slope of the equilibrium distribution w.r.t. the velocity.
- Mathematically, the integral shows a singular point (zero of the denominator) at $\omega=kv_x$. Vlasov overcame this difficulty calculating the Principal Value of the integral.

**Example**
Maxwell distribution of a warm plasma at temperature T

$$f_0(v_x) = \frac{n_0}{(2\pi k_B T/m_e)^{1/2}} \exp\left(-\frac{m_e v_x^2}{2k_B T}\right)$$

$k_B$= is the Boltzmann constant

Note that for $T \rightarrow 0, f_o(v_x) \rightarrow n_o$ (cold plasma)

$$\boxed{\omega_{plasma} = \frac{n_0 e^2}{m_e \varepsilon_0}}$$

For a given wavelength, the frequency of the plasma wave depends on the "plasma frequency" $\omega_p$ and on the average kinetic energy of the electrons (T)

$$\omega_r^2 \simeq \omega_p^2 \left(1 + 3k^2 \frac{k_B T}{\omega_p^2 m_e}\right)$$

According to Vlasov results, plasma waves can be excited and can persist forever in a interplay between pertubation and self-fields. Vlasov theory doesn't predict any damping effect.

**In a very original paper of 1946 Landau proposed a new method of solution of Vlasov-Poisson equations putting the basis of the theory of plasma oscillations and instabilities.**
He demonstrated that the problem had to be considered as an initial value or Cauchy problem, with a perturbation ***$f_1(x, v_x, t)$*** known at t = 0.
To this end he adopted the Laplace transform for the time domain and used the Fourier transform only for the space domain.

Accordingly, the perturbation and the electric field are first Fourier-transformed (space x) as follows:

$$\tilde{f}_1(k, v_x, t) = \int_{-\infty}^{\infty} f_1(x, v_x, t)\, e^{-ikx} dx$$

$$\tilde{E}_x(k, t) = \int_{-\infty}^{\infty} E_x(x, t)\, e^{-ikx} dx$$

And then Laplace-transformed (time t):

$$\mathcal{F}_1(v_x, k, p) = \int_0^{\infty} \tilde{f}_1(v_x, k, t) e^{-pt} dt$$

$$\mathcal{E}_x(k, p) = \int_0^{\infty} \tilde{E}_x(k, t) e^{-pt} dt$$

Applying the properties of the Laplace transforms, Vlasov-Poisson equation become:

$$p\mathcal{F}_1 + ikv_x\mathcal{F}_1 = \frac{e}{m_e}\mathcal{E}_x\frac{\partial f_0}{\partial v_x} + \tilde{f}_1(t=0)$$

$$ik\mathcal{E}_x(k, p) = -\frac{e}{\varepsilon_0}\int \mathcal{F}_1 dv_x$$

where we note the presence of the initial condition.

Solution of the above coupled equations gives the general expression of the transformed (k,p) electric field:

$$\mathcal{E}_x(k, p) = -\frac{e/\varepsilon_0}{ik\epsilon(k, p)}\int \frac{\tilde{f}_1(t=0)}{p + ikv_x} dv_x$$

which depends on *ε(k,p)*, the plasma dielectric funcion.

$$\epsilon(k,p) = 1 + \frac{e^2}{\varepsilon_0 m_e k} \int \frac{\partial f_0/\partial v_x}{ip - kv_x} dv_x$$

Landau showed that the asymptotic time behaviour of the electric field depends on the solutions of *ε(k,p)=0*. He also pointed out that this condition corresponds to the Vlasov's dispersion relation when *p=-iω*. He could also overcome the "divergence" problem applying the integration theory in the complex plane, getting:

$$1 + \frac{e^2}{\varepsilon_0 m_e k} \left[ \underbrace{\boxed{P.V. \int \frac{\partial f_0/\partial v_x}{\omega - kv_x} dv_x}}_{\text{VLASOV}} - \underbrace{\boxed{\frac{i\pi}{k} \left( \frac{\partial f_0}{\partial v_x} \right)_{v_x = \omega/k}}}_{\text{LANDAU}} \right] = 0$$

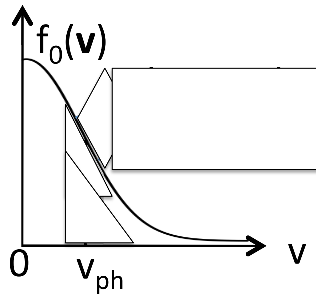
If we consider the generic harmonic of the field:

$$\tilde{E}_x(k, v_x, \omega) e^{\omega_i t} e^{i(kx - \omega_r t)}$$

The imaginary term $\omega_i$ produces **(Landau)** damping or antidampig effect, depending on the sign of the slope of the distribution function.

The propagation constants **k** and $\omega_r$ are still derived by the real part of the Dispersion Relation **(Vlasov)**.

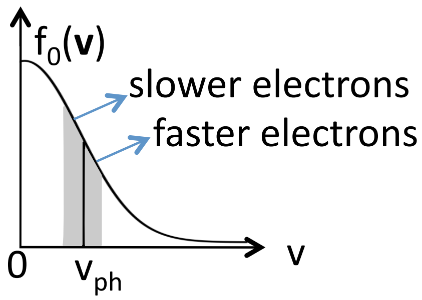
For the Maxwell distribution discussed before, we have:




$$\omega_r^2 \simeq \omega_p^2 \left(1 + 3k^2 \frac{k_B T}{\omega_p^2 m_e}\right)$$

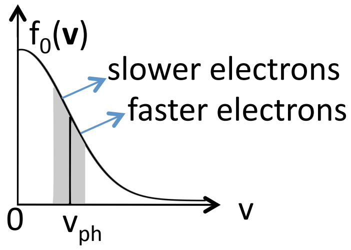
$$\omega_i \simeq \frac{\pi}{2} \frac{\omega_p e^2}{\varepsilon_0 m_e k^2} \left(\frac{\partial f_0}{\partial v_x}\right)_{v_x=\omega/k}$$

The phase velocity of the plasma wave is $v_{ph} = \frac{\omega_r}{k}$
is the phase velocity of the wave is derived by the solutions of the dispersion relation.




Consider a perturbation in the electron distribution such that a plasma wave propagates with a phase velocity $v_{ph} = \frac{\omega_r}{k}$

Let us imagine plasma waves as waves in the sea, and the electrons as surfers trying to catch the wave, all moving in the same direction.




Electrons slightly faster than $v_{ph}$ are decelerated by the wave electric field and yield energy to the wave. Electrons slightly slower than $v_{ph}$ are accelerated by the wave electric field and gain energy from the wave.

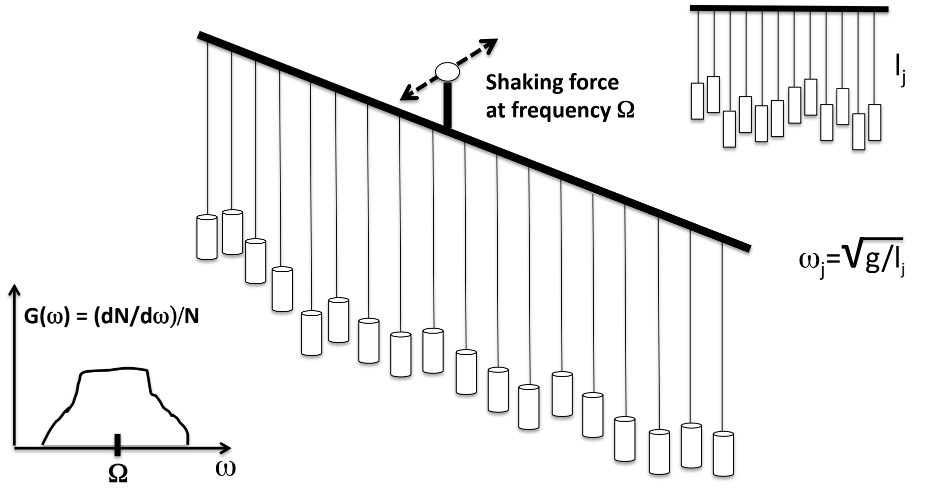
Since $v_{ph}$ is in the negative slope of the velocity distribution function, the number of “faster” electrons is less than the number of “slower” ones. Hence, there are more particles gaining energy from the wave than losing to the wave. The balance is a net energy loss which leads to wave damping.

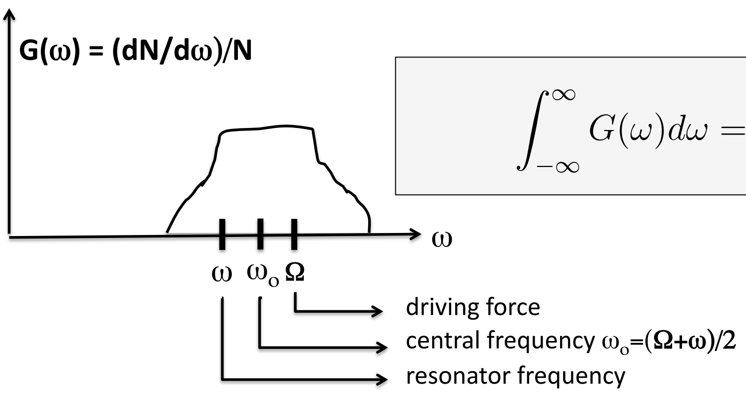
## 3. Mechanical System Model

The demonstration given by Landau was purely mathematical, an experimental behaviour was observed only 18 years later. The basic physical mechanism behind was not well understood, and still today several papers are devoted to a better comprehension of Landau Damping.

We wonder how is it possible that for a collisionless, lossless system there exists a physical solution for the oscillations characterized by an exponential decay corresponding to a damping.

To this end we consider a system of N uncoupled ideal linear oscillators, with a normalized frequency distribution G(ω).

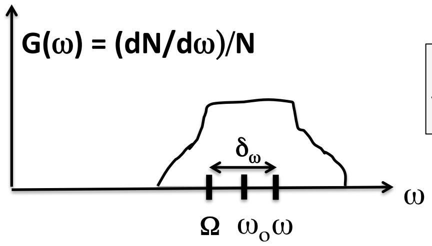



$$\int_{-\infty}^{\infty} G(\omega)d\omega = 1$$

$$\ddot{x} + \omega^2 x = A\cos\Omega t$$

$$\begin{cases} x(t=0) = 0 \\ \dot{x}(t=0) = 0 \end{cases}$$

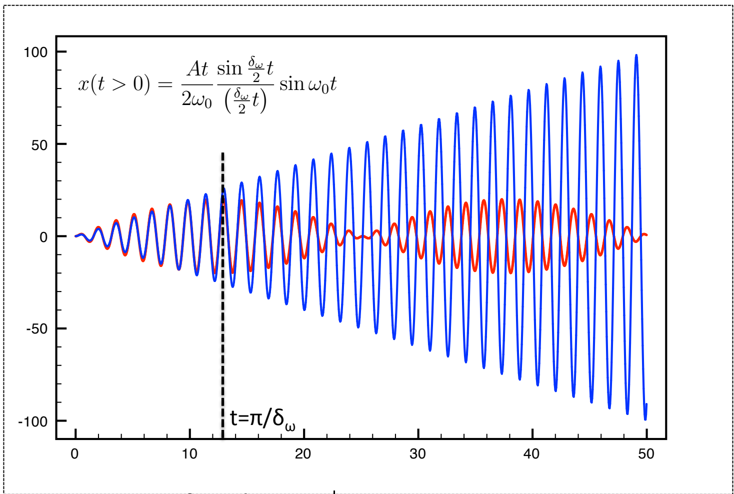



$$x(t>0) = \frac{A}{\Omega^2 - \omega^2}(\cos\omega t - \cos\Omega t)$$

Define now $|\Omega - \omega| = \delta_\omega$, and assume that $\delta_\omega \ll \omega_o$

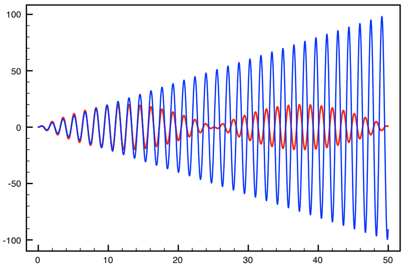
$$x(t>0) \cong \frac{A}{2\omega_0\delta_\omega}\left[\cos\left(\omega_0 + \frac{\delta_\omega}{2}\right)t - \cos\left(\omega_0 - \frac{\delta_\omega}{2}\right)t\right]$$

$$x(t>0) \cong \frac{A}{2\omega_0\delta_\omega}\sin(\omega_0 t)\sin\left(\frac{\delta_\omega}{2}t\right)$$




$\delta\omega = |\Omega - \omega| = 0$

$\delta\omega = |\Omega - \omega| \neq 0$




The amplitude of "on resonance" oscillator with $\delta_\omega$=0, blue curve, growths linearly with time. The oscillator with $\delta_\omega \neq 0$, red curve, reaches a maximum amplitude after a time $\mathbf{t=\pi/\delta_\omega}$, after which it goes "out of resonance", and it loses the phase synchronism with the external driving force.

We can say that at any time t*, only those oscillators with a frequency ω, such that $\boldsymbol{\delta_\omega < \pi/t^*}$ maintain a phase relation with the external force, and keep absorbing energy from the shaking force.

The longer we wait, the narrower the frequency bandwidth $\delta_\omega$ of synchronous oscillators, the less the number of oscillator absorbing energy.

The center mass (CM) of the oscillator's system, initially at rest, will start oscillating with growthing amplitude which, however, will remain bounded. The CM position is given by the average displacement obtained weighting x(t) with the normalized distribution G(ω):

$$\bar{x}_{CM}(t) = -\int_{-\infty}^{\infty} G(\omega) \frac{A}{\Omega^2 - \omega^2} \left( \cos\omega t - \cos\Omega t \right) d\omega$$

Since $\delta_\omega = \Omega - \omega << \omega_0$

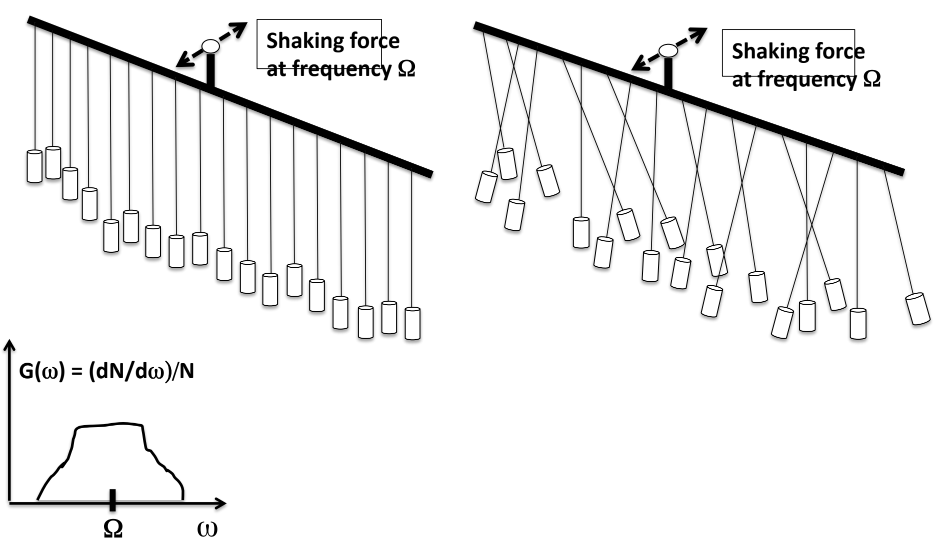
and $\Omega^2 - \omega_0^2 \cong 2\omega_0 \delta_\omega$

We get:

$$\overline{x}_{CM}(t) \cong \frac{A}{2\omega_0}\left[\pi G(\Omega) sin\Omega t - cos\Omega t \ \mathrm{P.V.}\int_{-\infty}^{\infty} \frac{G(\omega)}{\omega - \Omega} d\omega\right]$$

The average oscillation amplitude of the system does not increase with time, il remains limited as time goes to infinity.

The masses oscillate inchoerently, the center of mass motion will be bounded.



$$\bar{x}_{CM}(t) \cong \frac{A}{2\omega_0}\left[\pi G(\Omega)\,sin\,\Omega t + cos\,\Omega t\ \text{P.V.}\int_{-\infty}^{\infty}\frac{G(\omega)}{\omega-\Omega}d\omega\right]$$

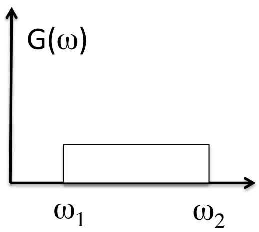
Dispersion relation? Let us assume that the shaking force is proportional to the displacement of the center of mass.

$$A\,cos\,\Omega t = \Re\left(\bar{x}_{CM}e^{-i\Omega t}\right)$$

$$A\,sin\,\Omega t = \Im\left(\bar{x}_{CM}e^{-i\Omega t}\right)$$

$$\cancel{\bar{x}_{CM}e^{-i\Omega t}} \cong \frac{\cancel{\bar{x}_{CM}e^{-i\Omega t}}}{2\omega_0}\left[\text{P.V.}\int_{-\infty}^{\infty}\frac{G(\omega)}{\omega-\Omega}d\omega - i\pi G(\Omega)\right]$$

Example - Uniform distribution




$$G(\omega) = \begin{cases} \dfrac{1}{\omega_2-\omega_1}, & \omega_1 < \omega < \omega_2 \\ 0; & elsewhere \end{cases}$$

$$\bar{x}_{CM}(t) \cong \frac{A}{2\omega_0(\omega_2-\omega_1)}\left[\pi\,sin\,\Omega t + ln\left(\frac{\omega_2-\Omega}{\Omega-\omega_1}\right)cos\,\Omega t\right]$$

Looking at the energy absorbed by the system of oscillators:

$$U \propto \frac{A^2}{\omega_0^2 \delta_\omega^2} \sin^2 \frac{\delta_\omega}{2} t$$

$$U_{tot} \propto N \frac{A^2}{\omega_0^2} \int_{-\infty}^{\infty} G(\Omega - \delta_\omega) \frac{\sin^2 \frac{\delta_\omega}{2} t}{\delta_\omega^2} d\delta_\omega$$

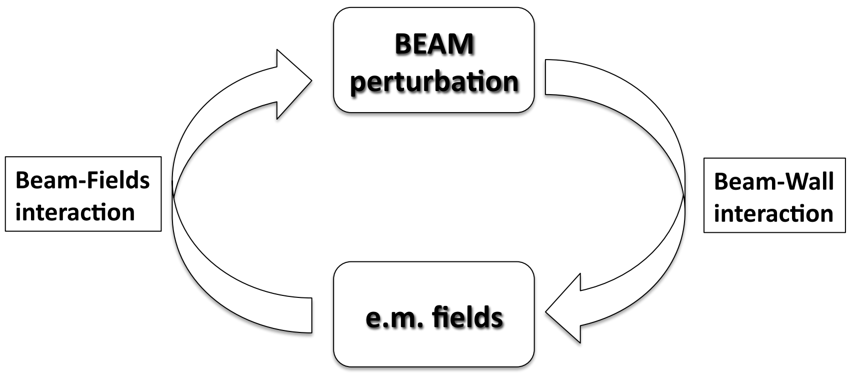
$$U_{tot} \propto N \frac{A^2}{\omega_0^2} \frac{\pi}{2} G(\Omega) t$$

We find that it growths with time !!!!

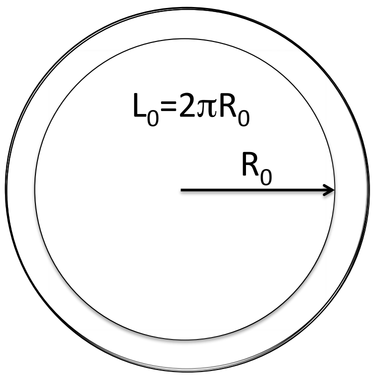
## 4. Beams in particle accelerators

We consider a beam circulating inside an accelerator, and assume that for this system there exists an equilibrium state.

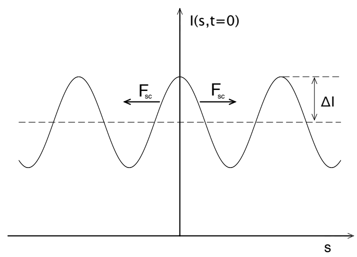
We wander whether a small perturbation around the equilibrium state will grow (instability) or decay (stability).




## Longitudinal Instabilities in coasting Beams




$$I(s,t) = I_0 + \Delta I e^{i(ks-\omega t)}$$




The wavelength of the perturbation is a submultiple of the machine length $L_0$ such that $k=2\pi/\lambda=2\pi n/L_0= n/R_0$
In the LHS picture the number of perturbation wavelengths in the ring is n=4**,** therefore $k=4/R_0$

Coherent instabilities are caused by the electromagnetic interaction of the beam perturbation current with the walls of the vacuum chamber.

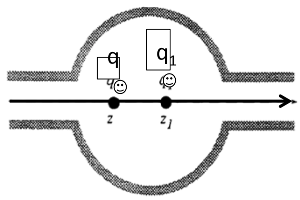
The field generates by the beam perturbation is modified by the walls and causes e.m. forces, proportional to the current, that acts back on the beam. They can lead to a coherent instability.

The average e.m. force over one turn is:

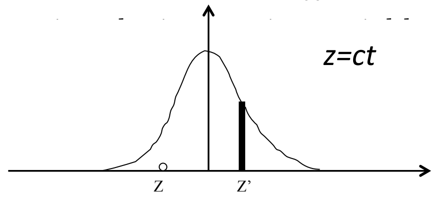
$$\left\langle F_{||}(\Delta z)\right\rangle = \frac{1}{L_0}\int_0^{L_0} F_{||}ds$$

In order to calculate the rate of energy variation of a single particle in one turn due to the beam-wall interaction, we introduce the longitudinal wake function defined as the average energy gain/loss per unit charge.

$$W_{||}(\Delta z) = -\frac{\left\langle F_{||}(\Delta z)\right\rangle L_0}{q q_1}$$




$$\frac{\partial \varepsilon}{\partial t} \simeq \frac{\Delta \varepsilon}{\Delta t} = -\frac{e}{E_0 T_0}\int_{-\infty}^{t} W_{||}\left(ct' - ct\right)\Delta I e^{i(ks-\omega t')}dt'$$



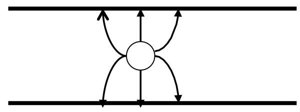

$$\boxed{\frac{\partial \varepsilon}{\partial t} = -\frac{e\Delta I e^{i(ks-\omega t)}}{E_0 T_0} Z_{||}(\omega)}$$

**Impedance**

$$\frac{\partial \varepsilon}{\partial t} = -\frac{e\Delta I e^{i(ks-\omega t)}}{E_0 T_0} Z_{||}(\omega)$$

1) Perfectly conducting circular beal pipe of radius b

$$Z_{||}(\omega) = iZ_0 \frac{R_0 \omega}{c(\beta\gamma)^2} \ln \frac{b}{r}$$

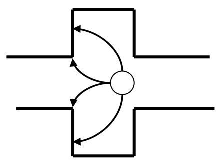


2) Resistive wall circular beal pipe of radius b

$$Z_{||}(\omega) = \frac{R_0}{b} \sqrt{\frac{Z_0|\omega|}{2c\sigma_c}} \left[1 - i\,\mathrm{sign}(\omega)\right]$$

3) RF Resonator mode

$$Z_{||}(\omega) = \frac{R_s}{1 + iQ\left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r}\right)}$$



Consider now a particle with nominal energy $E_0$ which moves in the circular machine with velocity βc on a closed orbit, called the reference orbit, of length $L_0 = 2\pi R_0$.

A particle with a small energy deviation ΔE, with ΔE = βcΔp, travels along a different path with a different speed. The change Δω of its revolution frequency is due to a combination of two effects: the speed and the dispersion in the magnet field.
:

$$\frac{\omega_0 - \bar{\omega}_0}{\bar{\omega}_0} = \frac{\Delta\omega}{\bar{\omega}_0} = -\left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p_0}$$

$$\boxed{\frac{\omega_0-\bar{\omega}_0}{\bar{\omega}_0}=\frac{\Delta\omega}{\bar{\omega}_0}=}-\frac{\eta}{\beta^2}\frac{\Delta E}{E_0}=-\frac{\eta}{\beta^2}\varepsilon$$

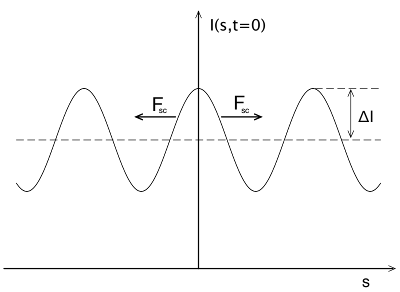
When $\eta > 0$ the machine works above the transition energy, a positive deviation $\varepsilon$ causes a longer trajectory which produces a reduction in the revolution frequency.

The change in the revolution frequency influences the longitudinal position of a particle. We use the quantity z to define the longitudinal coordinate of a particle with respect to the reference one, which has a nominal energy $E_0$.

We observe that a revolution frequency different from $\omega_0$ produces a change in the longitudinal position z in one turn given by the relation:

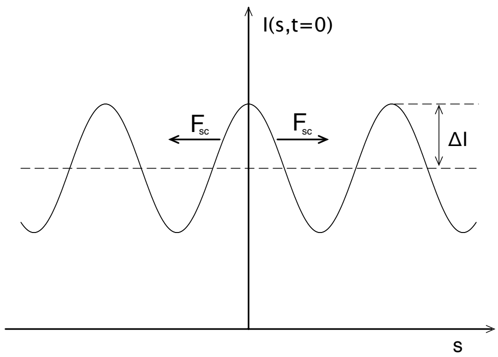
$$\frac{\Delta z}{L_0}=\frac{\Delta\omega}{\bar{\omega}_0} \qquad \frac{\Delta z}{T_0}=\Delta\omega R_0$$

For example, the longitudinal effect of the space charge in a perfectly conducting pipe is a force proportional to $-\partial I/\partial s$.



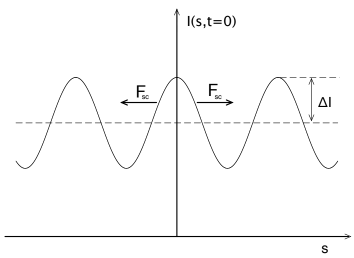
Particles that on the front slope experience a positive force, and, in one turn, their energy will increase.

The contrary will happen to the rear slope of the perturbation.




Above transition, an increase of energy implies a decrease of the revolution frequency. Therefore the particle in the front slope will delay and those in the back crest will anticipate, giving, as a net result, an increase of the height of the crest.

The initial perturbation is thus increased leading to instability, known as negative mass instability.



## Dispersion Relation for coasting beams

The dynamics of a coasting (unbunched) beam can be formalized by means of the Vlasov equation. The formalism is very similar to that we have used for the waves in a perturbed plasma. Here we use *f(z,ε;t)* for the beam distribution function such that:

**BEAM**

$$\int\int f(z,\varepsilon;t)dzd\varepsilon = N$$

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial f}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} = 0$$

**PLASMA**

$$\int f(x,v_x,t)\,dx\,dv_x = N$$

$$\boxed{\frac{\partial f}{\partial t} + v_x\frac{\partial f}{\partial x} + \frac{F_x}{m_e}\frac{\partial f}{\partial v_x} = 0}$$

The beam current can be obtained from the beam distribution function as:

$$I(z;t) = ec\int f(z,\varepsilon;t)d\varepsilon$$

$$f(z,\varepsilon;t) = f_0(\varepsilon) + f_1(\varepsilon)e^{i[kz-(\omega-n\bar{\omega}_0)t]}$$

$$I(z,t) = I_0 + \Delta I e^{i[kz-(\omega-n\bar{\omega}_0)t]}$$

$$\boxed{\frac{\partial f}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial f}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial t} = 0}$$

$$\frac{\partial z}{\partial t} \simeq \frac{\Delta z}{T_0} = \Delta\omega R_0$$

$$I(z;t) = \frac{ecN}{L_0} + ece^{i[kz-(\omega-n\bar{\omega}_0)t]}\int f_1(\varepsilon)d\varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} = -\frac{e\Delta I e^{i(ks-\omega t)}}{E_0T_0}Z_{||}(\omega) = -\frac{ce^2e^{i[kz-(\omega-n\bar{\omega}_0)t]}}{E_0T_0}Z_{||}(\omega)\int f_1(\varepsilon)d\varepsilon$$

$$-i\left(\omega - n\bar{\omega}_0 - n\Delta\omega\right) f_1 e^{i[kz-(\omega-n\bar{\omega}_0)t]} =$$
$$= \frac{\partial f_0}{\partial \varepsilon}\frac{e^2cZ_{||}(n\bar{\omega}_0)}{E_0T_0}e^{i[kz-(\omega-n\bar{\omega}_0)t]}\int f_1 d\varepsilon$$

$$f_1 = i\frac{\partial f_0/\partial\varepsilon}{\omega - n\bar{\omega}_0 + n\bar{\omega}_0\eta\varepsilon}\frac{e^2c^2Z_{||}(n\bar{\omega}_0)}{E_0L_0}\int f_1 d\varepsilon$$

$$1 = i\frac{(Z_{||}/n)I_0L_0}{2\pi N(E_0/e)\eta}\int\frac{\partial f_0/\partial\varepsilon}{\frac{(\omega-n\bar{\omega}_0)}{n\bar{\omega}_0\eta}+\varepsilon}d\varepsilon$$

$$\boxed{1 + \frac{e^2}{\varepsilon_0 m_e k}\int\frac{\partial f_0/\partial v_x}{\omega - kv_x}dv_x = 0}$$

Coasting Beams

$$1 = i\frac{(Z_{||}/n)I_0 L_0}{2\pi N(E_0/e)\eta}\left[P.V.\int\frac{\partial f_0/\partial\varepsilon}{\frac{(\omega-n\bar{\omega}_0)}{n\bar{\omega}_0\eta}+\varepsilon}d\varepsilon - i\pi\left(\frac{\partial f_0}{\partial\varepsilon}\right)_{\varepsilon=\frac{(n\bar{\omega}_0-\omega)}{n\bar{\omega}_0\eta}}\right]$$

Plasma waves

$$1+\frac{e^2}{\varepsilon_0 m_e k}\left[P.V.\int\frac{\partial f_0/\partial v_x}{\omega - k v_x}dv_x - \frac{i\pi}{k}\left(\frac{\partial f_0}{\partial v_x}\right)_{v_x=\omega/k}\right]=0$$

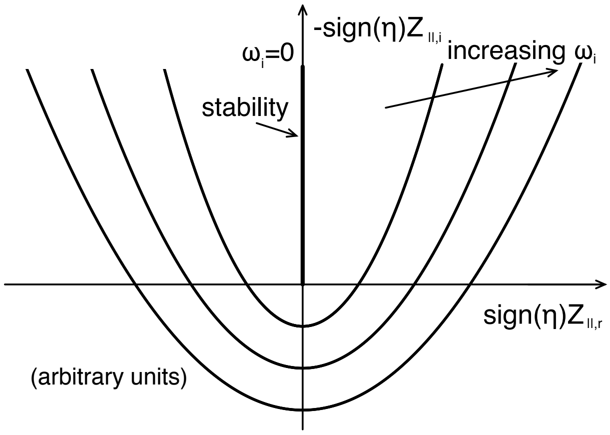
## Monochromatic beam

$$f_0(\varepsilon) = N\frac{\delta(\varepsilon)}{L_0}$$

$$1 = i\frac{(Z_{||}/n)I_0}{2\pi(E_0/e)\eta}\int\frac{\delta'(\varepsilon)}{\frac{(\omega-n\bar{\omega}_0)}{n\bar{\omega}_0\eta}+\varepsilon}d\varepsilon = -i\frac{(Z_{||}/n)I_0}{2\pi(E_0/e)\eta}\frac{\partial}{\partial\varepsilon}\left(\frac{1}{\frac{(\omega-n\bar{\omega}_0)}{n\bar{\omega}_0\eta}+\varepsilon}\right)\Bigg|_{\varepsilon=0}$$

$$1 = i\frac{\eta(Z_{||}/n)I_0}{2\pi(E_0/e)}\left(\frac{n\bar{\omega}_0}{\omega - n\bar{\omega}_0}\right)^2$$

$$\omega = n\bar{\omega}_0 \pm n\bar{\omega}_0 \sqrt{i\frac{\eta(Z_{||}/n)I_0}{2\pi(E_0/e)}}$$




If the machine coupling impedance has a real part, that is a resistive component, ω will always have an imaginary part and therefore the beam will be unstable.

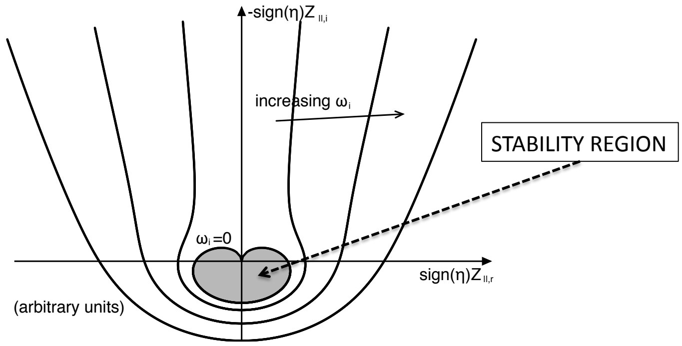
For a pure imaginary impedance stability or instability depends on the sign of η and $Z_i$.

Above transition energy (η>0), the beam is unstable if $Z_i>0$ (capacitive impedance) and stable if $Z_i<0$ (inductive impedance).

## Parabolic energy distribution

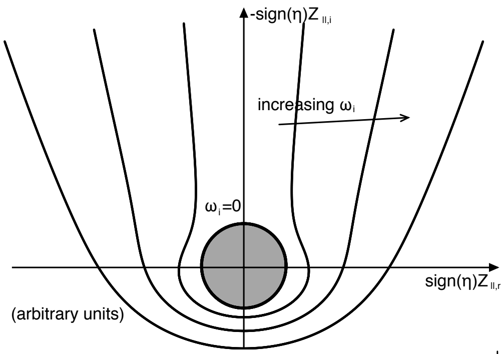
The relative energy deviation ranges between $-\varepsilon_m$ and $\varepsilon_m$

$$f_0(\varepsilon) = \frac{3N}{4L_0\varepsilon_m}\left[1-\left(\frac{\varepsilon}{\varepsilon_m}\right)^2\right]$$




A.G. Ruggero, V.G. Vaccaro, CERN Report ISR-TH/68-33 (1968)

## tri-elliptical energy distribution




$$f_0(\varepsilon) = \frac{8N}{3\pi L_0\varepsilon_m}\left[1-\left(\frac{\varepsilon}{\varepsilon_m}\right)^2\right]^{3/2}$$

Kheil-Shnell Stability Criterion (F=0.68)

$$\left|\frac{Z_{||}}{n}\right| \le F\frac{(E_0/e)|\eta|\varepsilon_{1/2}^2}{I_0\beta^2}$$

## Bunched Beam (Longitudinal)

The effect of Landau damping on bunched beam dynamics is a complex problem. However, a simplified and approximated expression, similar to the Keil-Schnell stability criterion for the coasting beam, has been proposed by D. Boussard in case of short range wake fields (acting on the sigle bunch) at high frequencies.

The idea is that for high frequency fields generated by the beam, a bunched beam can be considered as a coasting one, provided we use the bunched beam peak current in the threshold criterion.

Thus, considering a Gaussian energy distribution, we end up with the Boussard criterion

$$\left|\frac{Z_{||}}{n}\right| < \frac{2\pi(E_0/e)|\eta|\sigma_\varepsilon^2}{\hat{I}}$$

The Boussard criterion can be used to get a first evaluation of the threshold current of a ring for a given impedance Z/n. It depends on the particle energy, the energy spread, and on the factor $\eta$.

The threshold corersponds to the maximum single bunch current one can store in a storage ring, keeping the beam stable. Above the threshold current, we enter in the regime of the "microwave instability", however the beam is not lost.

The microwave instabilities will heat the beam, increasing the energy spread such to restore the threshold condition.

$$\left|\frac{Z_{||}}{n}\right| = \frac{2\pi(E_0/e)|\eta|\sigma_\varepsilon^2}{\hat{I}}$$

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