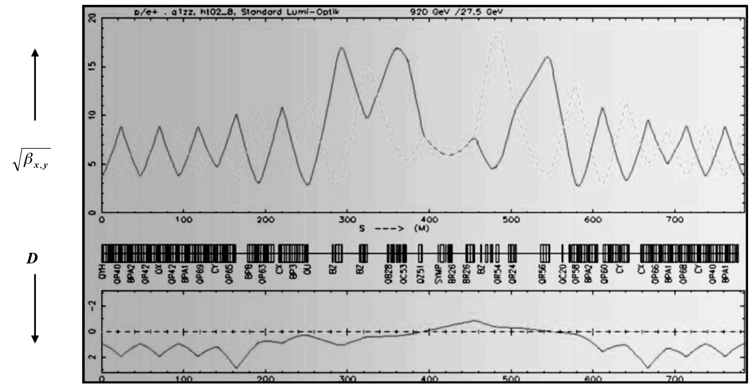
Lattice Design in Particle Accelerators
Bernhard Holzer, CERN



1952: Courant, Livingston, Snyder:
Theory of strong focusing in particle beams

Lattice Design: ***„… how to build a storage ring"***

High energy accelerators → circular machines
somewhere in the lattice we need a number of dipole magnets,
that are bending the design orbit to a closed ring

Geometry of the ring:
centrifugal force = Lorentz force

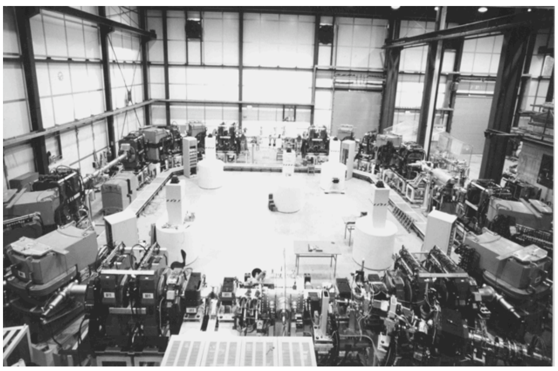
$$e * v * B = \frac{mv^2}{\rho}$$

$$\rightarrow e * B = \frac{mv}{\rho} = p / \rho$$

$$\rightarrow B * \rho = p / e$$

p = momentum of the particle,
ρ = curvature radius

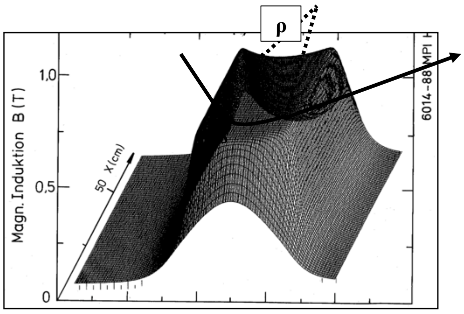
Bρ= beam rigidity



Example: heavy ion storage ring TSR
8 dipole magnets of equal bending strength

1.) Circular Orbit:

„… defining the geometry“



field map of a storage ring dipole magnet

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho}$$

$$\alpha = \frac{B * dl}{B * \rho}$$

The angle swept out in one revolution must be 2π, so

$$\alpha = \frac{\int B dl}{B * \rho} = 2\pi \qquad \rightarrow \qquad \int B dl = 2\pi * \frac{p}{q}$$

… for a full circle

Nota bene: $\frac{\Delta B}{B} \approx 10^{-4}$ ***is usually required !!***

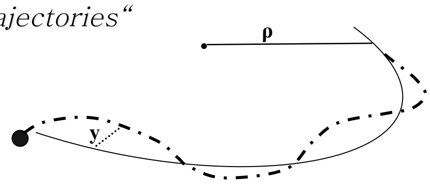
Example LHC:

7000 GeV Proton storage ring
dipole magnets N = 1232
l = 15 m
q = +1 e

$$\int \boldsymbol{B\, dl} \approx \boldsymbol{N\, l\, B} = 2\pi\ \boldsymbol{p / e}$$

$$\boldsymbol{B} \approx \frac{2\pi\ 7000\ 10^{9}\, \boldsymbol{eV}}{1232\ \ 15\, \boldsymbol{m}\ \ 3\ 10^{8}\, \frac{\boldsymbol{m}}{\boldsymbol{s}}\ \ \boldsymbol{e}} = \underline{\underline{8.3\, \boldsymbol{Tesla}}}$$

„ Focusing forces … single particle trajectories“



$$y'' + K * y = 0$$

$K = -k + 1/\rho^2$ **hor. plane**

$K = k$ **vert. plane**

dipole magnet $\frac{1}{\rho} = \frac{B}{p/q}$

quadrupole magnet $k = \frac{g}{p/q}$

Example: HERA Ring:
Bending radius: ρ = 580 m
Quadrupol Gradient: g = 110 T/m

k = 33.64*10^-3 /m^2
1/ρ^2 = 2.97 *10^-6 /m^2

***For estimates in large accelerators** the weak focusing term $1/\rho^2$ can **in general** be neglected*

Solution for a focusing magnet

$$y(s) = y_0 * \cos(\sqrt{K} * s) + \frac{y'_0}{\sqrt{K}} * \sin(\sqrt{K} * s)$$

$$y'(s) = -y_0 * \sqrt{K} * \sin(\sqrt{K} * s) + y'_0 * \cos(\sqrt{K} * s)$$

Or written more convenient in matrix form:

$$\begin{pmatrix} y \\ y' \end{pmatrix}_s = M * \begin{pmatrix} y \\ y' \end{pmatrix}_0$$

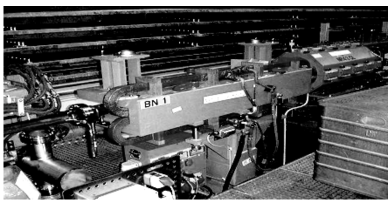
Hor. focusing Quadrupole Magnet

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l) \\ -\sqrt{K} \sin(\sqrt{K} * l) & \cos(\sqrt{K} * l) \end{pmatrix}$$

Hor. defocusing Quadrupole Magnet

$$M_{QD} = \begin{pmatrix} \cosh(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sinh(\sqrt{K} * l) \\ \sqrt{K} \sinh(\sqrt{K} * l) & \cosh(\sqrt{K} * l) \end{pmatrix}$$

Drift space $M_{Drift} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$



$$\boxed{M_{lattice} = M_{QF1} * M_{D1} * M_{QD} * M_{D1} * M_{QF2} \ldots}$$

2.) Reminder: Beam Dynamics Language

Transfer Matrix M

describes the transformation of amplitude x and angle x' through a number of lattice elements

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

... and can be expressed by the optics parameters

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0-\alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

**** we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.***
**** and nothing but the α β γ at these positions.***

**** ... !***

Periodic Lattices

In the case of periodic lattices the transfer matrix can be expressed as a function of a set of periodic parameters α, β, γ

$$M(s) = \begin{pmatrix} \cos\psi_{period} + \alpha_s \sin\psi_{period} & \beta_s \sin\psi_{period} \\ -\gamma_s \sin\psi_{period} & \cos\psi_{period} - \alpha_s \sin\psi_{period} \end{pmatrix} \qquad \psi_{period} = \int_s^{s+L} \frac{ds}{\beta(s)}$$

ψ = phase advance per period:

For stability of the motion in periodic lattice structures it is required that

$$|trace(M)| < 2$$

In terms of these new periodic parameters the solution of the equation of motion is

$$y(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\Phi(s) - \delta)$$

$$y'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta}} * \left\{\sin(\Phi(s) - \delta) + \alpha\cos(\Phi(s) - \delta)\right\}$$

Transformation of α, β, γ

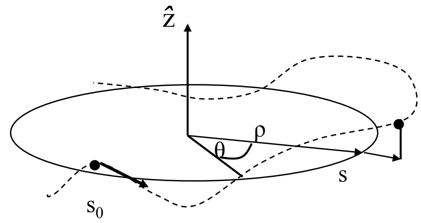
consider two positions in the storage ring: s_0 , s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

since ε = const:

$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$\varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$



express x_0, x'_0 as a function of x, x′.

... remember $W = m_{11} m_{22} - m_{12} m_{21} = 1$

$$\left.\begin{aligned} \begin{pmatrix} x \\ x' \end{pmatrix}_0 &= M^{-1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_s \\ M^{-1} &= \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix} \end{aligned}\right\} \longrightarrow \begin{aligned} x_0 &= m_{22} x - m_{12} x' \\ x_0' &= -m_{21} x + m_{11} x' \end{aligned}$$

inserting into ε $\quad \varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$

$$\varepsilon = \beta_0 (m_{11} x' - m_{21} x)^2 + 2\alpha_0 (m_{22} x - m_{12} x')(m_{11} x' - m_{21} x) + \gamma_0 (m_{22} x - m_{12} x')^2$$

sort via x, x′and compare the coefficients to get ....

The new parameters α, β, γ can be transformed through the lattice via the lattice matrix elements defined above.

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{12}m_{21} + m_{22}m_{11} & -m_{12}m_{22} \\ m_{12}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s1}$$

the optical parameters depend on the focusing properties of the lattice,
... and can be optimised accordingly !!!

... and here starts the lattice design !!!

Most simple example: drift space

$$M_{drift} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

particle coordinates

$$\begin{pmatrix} x \\ x' \end{pmatrix}_l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \end{pmatrix}_0 \quad \rightarrow \quad \boxed{\begin{aligned} x(l) &= x_0 + l * x_0' \\ x'(l) &= x_0' \end{aligned}}$$

transformation of twiss parameters:

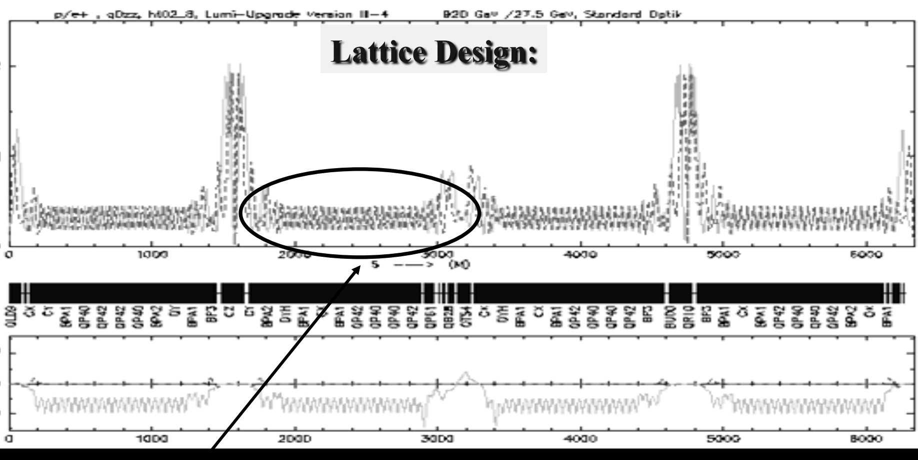
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_l = \begin{pmatrix} 1 & -2l & l^2 \\ 0 & 1 & -l \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0 \qquad \boxed{\beta(s) = \beta_0 - 2l * \alpha_0 + l^2 * \gamma_0}$$

Stability ...?

$$trace(M) = 1 + 1 = 2$$

→A periodic solution doesn't exist in a lattice built exclusively out of drift spaces.

Lattice Design:

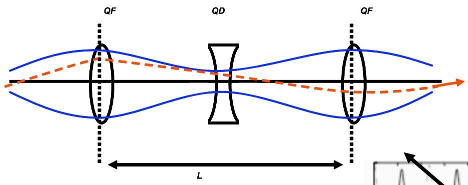


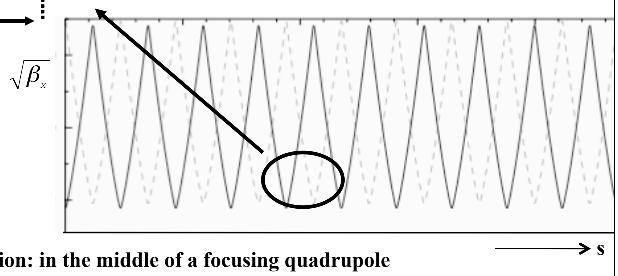
Arc: regular (periodic) magnet structure:
bending magnets → define the energy of the ring
main focusing & tune control, chromaticity correction,
multipoles for higher order corrections

Straight sections: drift spaces for injection, dispersion suppressors,
low beta insertions, RF cavities, etc....
... and the high energy experiments if they cannot be avoided

3.) The FoDo-Lattice

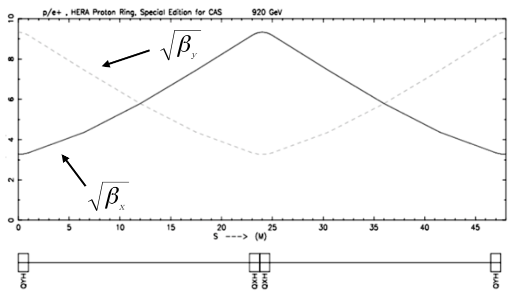
A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in between.
(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)





Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell μ = 45°,
→ calculate the twiss parameters for a periodic solution

Periodic Solution of a FoDo Cell



Output of the optics program:

Nr	*Type*	*Length*	*Strength*	β_x	α_x	φ_x	β_z	α_z	φ_z
		m	*1/m2*	*m*		*1/2π*	*m*		*1/2π*
0	*IP*	*0,000*	*0,000*	*11,611*	*0,000*	*0,000*	*5,295*	*0,000*	*0,000*
1	*QFH*	*0,250*	*-0,541*	*11,228*	*1,514*	*0,004*	*5,488*	*-0,781*	*0,007*
2	*QD*	*3,251*	*0,541*	*5,488*	*-0,781*	*0,070*	*11,228*	*1,514*	*0,066*
3	*QFH*	*6,002*	*-0,541*	*11,611*	*0,000*	*0,125*	*5,295*	*0,000*	*0,125*
4	*IP*	*6,002*	*0,000*	*11,611*	*0,000*	*0,125*	*5,295*	*0,000*	*0,125*

QX= 0,125 QZ= 0,125

$0.125 * 2\pi = 45^0$

Can we understand what the optics code is doing ?

matrices

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l_q) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l_q) \\ -\sqrt{K} \sin(\sqrt{K} * l_q) & \cos(\sqrt{K} * l_q) \end{pmatrix}, \qquad M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1_d \end{pmatrix}$$

strength and length of the FoDo elements

K = +/- 0.54102 m^{-2}
lq = 0.5 m
ld = 2.5 m

The matrix for the complete cell is obtained by multiplication of the element matrices

$$M_{FoDo} = M_{qf\,h} * M_{ld} * M_{qd} * M_{ld} * M_{qf}$$

Putting the numbers in and multiplying out ...

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

The transfer matrix for 1 period gives us all the information that we need !

1.) is the motion stable?

$$trace(M_{FoDo}) = 1.415 \rightarrow \underline{\underline{< 2}}$$

2.) Phase advance per cell

$$M(s) = \begin{pmatrix} \cos\psi_{cell} + \alpha_s \sin\psi_{cell} & \beta_s \sin\psi_{cell} \\ -\gamma_s \sin\psi_{cell} & \cos\psi_{cell} - \alpha_s \sin\psi_{cell} \end{pmatrix}$$

$$\cos\psi_{cell} = \frac{1}{2} trace(M) = 0.707$$

$$\psi_{cell} = \cos^{-1}\left(\frac{1}{2} trace(M)\right) = \underline{\underline{45}}$$

3.) hor β-function

$$\beta = \frac{m_{12}}{\sin\psi_{cell}} = \underline{\underline{11.611\,m}}$$

4.) hor α-function

$$\alpha = \frac{m_{11} - \cos\psi_{cell}}{\sin\psi_{cell}} = \underline{\underline{0}}$$

Can we do a bit easier ?
We can ... in thin lens approximation !

Matrix of a focusing quadrupole magnet:

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K}*l) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}*l) \\ -\sqrt{K}\sin(\sqrt{K}*l) & \cos(\sqrt{K}*l) \end{pmatrix}$$

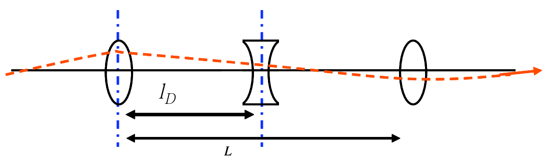
If the focal length *f* is much larger than the length of the quadrupole magnet,

$$f = \frac{1}{kl_Q} >> l_Q$$

the transfer matrix can be aproximated using ♪ $kl_q = const,\ l_q \to 0$

$$M = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

4.) FoDo in thin lens approximation



$$l_D = L/2,$$
$$\tilde{f} = 2f$$

Calculate the matrix for a half cell, starting in the middle of a foc. quadrupole:

$$M_{half\ Cell} = M_{QD/2} * M_{lD} * M_{QF/2}$$

$$M_{half\ Cell} = \begin{pmatrix} 1 & 0 \\ \frac{1}{\tilde{f}} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l_D \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ -\frac{1}{\tilde{f}} & 1 \end{pmatrix}$$

note: $\tilde{f}$ denotes the focusing strength of half a quadrupole, so $\tilde{f} = 2f$

$$M_{half\ Cell} = \begin{pmatrix} 1 - \frac{l_D}{\tilde{f}} & l_D \\ \frac{-l_D}{\tilde{f}^2} & 1 + \frac{l_D}{\tilde{f}} \end{pmatrix}$$

for the second half cell set f → -f

FoDo in thin lens approximation

Matrix for the complete FoDo cell

$$M = \begin{pmatrix} 1+\frac{l_D}{\tilde{f}} & l_D \\ -\frac{l_D}{\tilde{f}^2} & 1-\frac{l_D}{\tilde{f}} \end{pmatrix} * \begin{pmatrix} 1-\frac{l_D}{\tilde{f}} & l_D \\ -\frac{l_D}{\tilde{f}^2} & 1+\frac{l_D}{\tilde{f}} \end{pmatrix}$$

$$M = \begin{pmatrix} 1-\frac{2l_D^2}{\tilde{f}^2} & 2l_D(1+\frac{l_D}{\tilde{f}}) \\ 2(\frac{l_D^2}{\tilde{f}^3}-\frac{l_D}{\tilde{f}^2}) & 1-2\frac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

Now we know, that the phase advance is related to the transfer matrix by

$$\cos\psi_{cell} = \frac{1}{2} trace(M) = \frac{1}{2} * (2 - \frac{4l_d^2}{\tilde{f}^2}) = 1 - \frac{2l_d^2}{\tilde{f}^2}$$

After some beer and with a little bit of trigonometric gymnastics

$$\cos(x) = \cos^2(\frac{x}{2}) - \sin^2(x/2) = 1 - 2\sin^2(\frac{x}{2})$$

we can calculate the phase advance as a function of the FoDo parameter …

$$\cos\psi_{cell} = 1 - 2\sin^2(\psi_{cell}/2) = 1 - \frac{2l_d^2}{\tilde{f}^2}$$

$$\sin(\psi_{cell}/2) = l_d/\tilde{f} = \frac{L_{cell}}{2\tilde{f}}$$

$$\sin(\psi_{cell}/2) = \frac{L_{cell}}{4f}$$

Example:
45-degree Cell

$L_{Cell} = l_{QF} + l_D + l_{QD} + l_D = 0.5m+2.5m+0.5m+2.5m = 6m$

$1/f = k*l_Q = 0.5m*0.541\ m^{-2} = 0.27\ m^{-1}$

$$\sin(\psi_{cell}/2) = \frac{L_{cell}}{4f} = 0.405$$

$\rightarrow \psi_{cell} = 47.8^\circ$

$\rightarrow \beta = 11.4\ m$

Remember:
Exact calculation yields:

$\rightarrow \psi_{cell} = 45^\circ$

$\rightarrow \beta = 11.6\ m$

Stability in a FoDo structure

SPS Lattice

$$M_{FoDo} = \begin{pmatrix} 1-\frac{2l_D^2}{\tilde{f}^2} & 2l_D(1+\frac{l_D}{\tilde{f}}) \\ 2(\frac{l_D^2}{\tilde{f}^3}-\frac{l_D}{\tilde{f}^2}) & 1-2\frac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

Stability requires:

$$|\boldsymbol{Trace(M)}| < 2$$

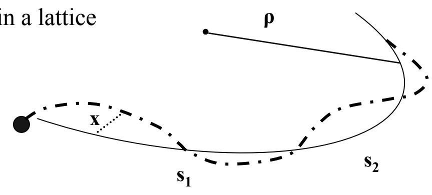
$$|\boldsymbol{Trace(M)}| = \left|2-\frac{4\boldsymbol{l_d^2}}{\tilde{\boldsymbol{f}}^2}\right| < 2$$

$$\rightarrow f > \frac{L_{cell}}{4}$$

For stability the focal length has to be larger than a quarter of the cell length
... don't focus to strong !

Transformation Matrix in Terms of the Twiss Parameters

Transformation of the coordinate vector (x,x´) in a lattice



$$\begin{pmatrix} \boldsymbol{x(s)} \\ \boldsymbol{x'(s)} \end{pmatrix} = \boldsymbol{M}_{s1,s2} \begin{pmatrix} \boldsymbol{x}_0 \\ \boldsymbol{x'}_0 \end{pmatrix}$$

General solution of the equation of motion

$$\boldsymbol{x(s)} = \sqrt{\varepsilon * \beta(\boldsymbol{s})} * \cos(\psi(\boldsymbol{s}) + \varphi)$$

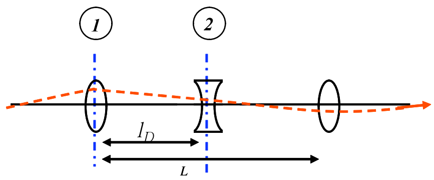
$$x'(s) = \sqrt{\varepsilon / \beta(s)} * \{\alpha(s)\cos(\psi(s)+\varphi) + \sin(\psi(s)+\varphi)\}$$

Transformation of the coordinate vector (x,x´)
expressed as a function of the twiss parameters

$$\boldsymbol{M}_{1\rightarrow 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{12} + \alpha_1 \sin\psi_{12}) & \sqrt{\beta_1\beta_2}\sin\psi_{12} \\ \frac{(\alpha_1-\alpha_2)\cos\psi_{12} - (1+\alpha_1\alpha_2)\sin\psi_{12}}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{12} - \alpha_2\sin\psi_{12}) \end{pmatrix}$$

Transfer Matrix for half a FoDo cell:

$$M_{half\ cell} = \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix}$$



Compare to the twiss parameter form of M

$$M_{1\to2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\psi_{12} + \alpha_1 \sin\psi_{12}) & \sqrt{\beta_1\beta_2}\sin\psi_{12} \\ \frac{(\alpha_1 - \alpha_2)\cos\psi_{12} - (1 + \alpha_1\alpha_2)\sin\psi_{12}}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\psi_{12} - \alpha_2 \sin\psi_{12}) \end{pmatrix}$$

In the middle of a foc (defoc) quadrupole of the FoDo we allways have α = 0, and the half cell will lead us from β_{max} to β_{min}

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\check{\beta}}{\hat{\beta}}}\cos\frac{\psi_{cell}}{2} & \sqrt{\check{\beta}\hat{\beta}}\sin\frac{\psi_{cell}}{2} \\ \frac{-1}{\sqrt{\hat{\beta}\check{\beta}}}\sin\frac{\psi_{cell}}{2} & \sqrt{\frac{\hat{\beta}}{\check{\beta}}}\cos\frac{\psi_{cell}}{2} \end{pmatrix}$$

Solving for β_{max} and β_{min} and remembering that …. $\sin\frac{\psi_{cell}}{2} = \frac{l_d}{f} = \frac{L}{4f}$

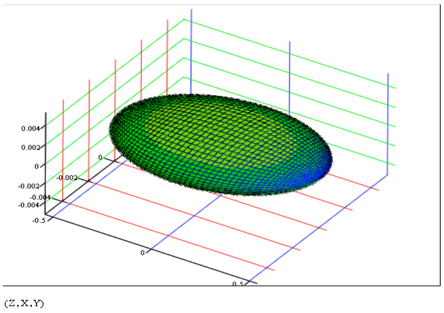
$$\frac{m_{22}}{m_{11}} = \frac{\hat{\beta}}{\bar{\beta}} = \frac{1 + l_d/\tilde{f}}{1 - l_d/\tilde{f}} = \frac{1 + \sin(\psi_{cell}/2)}{1 - \sin(\psi_{cell}/2)}$$

$$\frac{m_{12}}{m_{21}} = \hat{\beta}\bar{\beta} = \tilde{f}^2 = \frac{l_d^2}{\sin^2(\psi_{cell}/2)}$$

$\rightarrow$

$$\hat{\beta} = \frac{(1 + \sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} \; !$$

$$\bar{\beta} = \frac{(1 - \sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} \; !$$



The maximum and minimum values of the β-function are solely determined by the phase advance and the length of the cell.

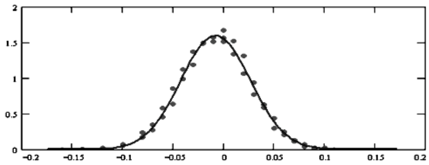
Longer cells lead to larger β

typical shape of a proton bunch in a FoDo Cell

5.) Beam dimension:
Optimisation of the FoDo Phase advance:

In both planes a gaussian particle distribution is assumed, given by the beam emittance ε and the β-function

$$\sigma = \sqrt{\varepsilon\beta}$$



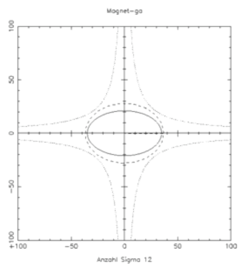
HERA beam size

In general proton beams are „*round*" in the sense that

$$\varepsilon_x \approx \varepsilon_y$$

So for highest aperture we have to minimise the β-function in both planes:

$$r^2 = \varepsilon_x\beta_x + \varepsilon_y\beta_y$$



typical beam envelope, vacuum chamber and pole shape in a foc. Quadrupole lens in HERA

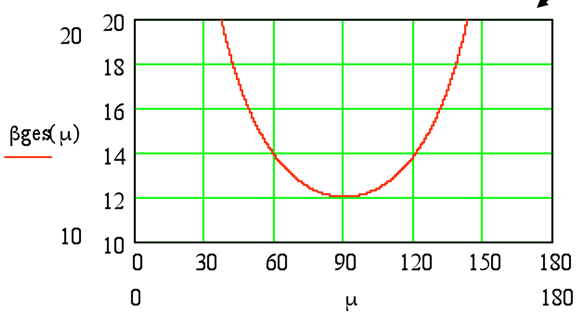
Optimising the FoDo phase advance

$$r^2 = \varepsilon_x\beta_x + \varepsilon_y\beta_y$$

search for the phase advance μ that results in a minimum of the sum of the beta's

$$\hat{\beta} + \breve{\beta} = \frac{(1+\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}} + \frac{(1-\sin\frac{\psi_{cell}}{2})L}{\sin\psi_{cell}}$$

$$\hat{\beta} + \breve{\beta} = \frac{2L}{\sin\psi_{cell}} \qquad \frac{d}{d\psi_{cell}}\left(\frac{2L}{\sin\psi_{cell}}\right) = 0$$



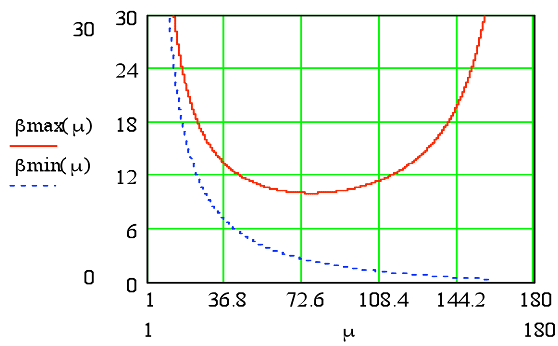
$$\frac{L}{\sin^2\psi_{cell}} * \cos\psi_{cell} = 0 \quad \rightarrow \quad \underline{\underline{\psi_{cell} = 90^\circ}}$$

Electrons are different

electron beams are usually flat, $\varepsilon_y \approx 2 - 10\,\%\,\varepsilon_x$

→ *optimise only* β_{hor}

$$\frac{d}{d\psi_{cell}}(\hat{\beta}) = \frac{d}{d\psi_{cell}} \frac{L(1+\sin\frac{\psi_{cell}}{2})}{\sin\psi_{cell}} = 0 \quad \rightarrow \quad \psi_{cell} = 76^\circ$$

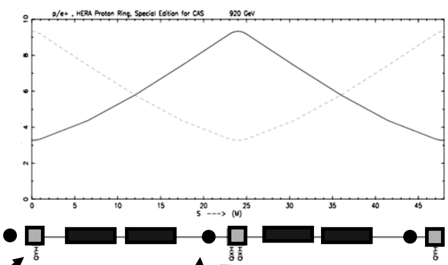


red curve: β_{max}

blue curve: β_{min}

as a function of the phase advance ψ

Orbit distortions in a periodic lattice



field error of a dipole/distorted quadrupole

$$\rightarrow \delta(mrad) = \frac{ds}{\rho} = \frac{\int Bds}{p/e}$$

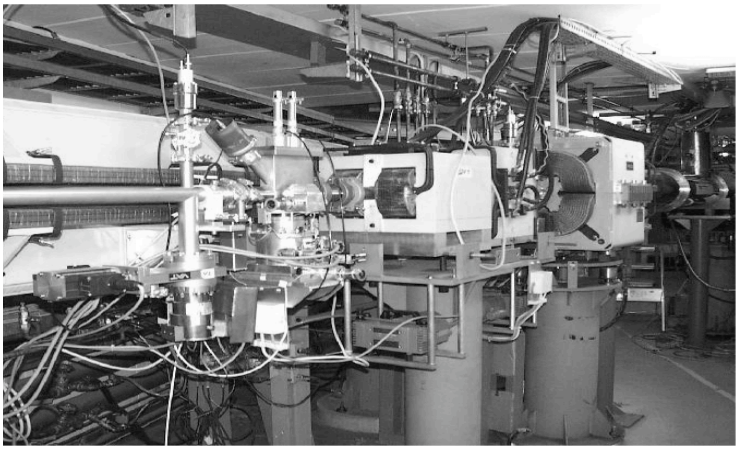
the particle will follow a new closed trajectory, the distorted orbit:

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi Q)} * \oint \sqrt{\beta(\tilde{s})} \frac{1}{\rho(\tilde{s})} \cos(|\psi(\tilde{s}) - \psi(s)| - \pi Q)\, d\tilde{s}$$

* **the orbit amplitude will be large if the β function at the location of the kick is large**
$\beta(\tilde{s})$ **indicates the sensitivity of the beam → here orbit correctors should be placed in the lattice**

* **the orbit amplitude will be large at places where in the lattice β(s) is large → here beam position monitors should be installed**

Orbit Correction and Beam Instrumentation in a storage ring



Elsa ring, Bonn

*

Resumé:

1.) Dipole strength $\int B ds = N * B_0 * l_{eff} = 2\pi \frac{p}{q}$

l_{eff} effective magnet length, N number of magnets

2.) Stability condition $Trace(M) < 2$

for periodic structures within the lattice / at least for the transfer matrix of the complete circular machine

3.) Transfer matrix for periodic cell $M(s) = \begin{pmatrix} \cos\psi_{cell} + \alpha_s \sin\psi_{cell} & \beta_s \sin\psi_{cell} \\ -\gamma_s \sin\psi_{cell} & \cos\psi_{cell} - \alpha_s \sin\psi_{cell} \end{pmatrix}$

α,β,γ depend on the position s in the ring, μ (phase advance) is independent of s

4.) Thin lens approximation $M_{QF} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_Q} & 1 \end{pmatrix}$ $\quad f_Q = \frac{1}{k_Q l_Q}$

focal length of the quadrupole magnet $f_Q = 1/(k_Q l_{Q)} >> l_Q$

5.) Tune (rough estimate)

$$\psi_{period} = \int_s^{s+L} \frac{ds}{\beta(s)}$$

Tune = phase advance in units of 2π

$$Q = N * \frac{\psi_{period}}{2\pi} = \frac{1}{2\pi} * \oint \frac{ds}{\beta(s)} \approx \frac{1}{2\pi} * \frac{2\pi \bar{R}}{\bar{\beta}} = \frac{\bar{R}}{\bar{\beta}}$$

$\bar{R}, \bar{\beta}$ *average radius and β-function*

$$Q \approx \frac{\bar{R}}{\bar{\beta}}$$

6.) Phase advance per FoDo cell

(thin lens appro

$$\sin\frac{\psi_{cell}}{2} = \frac{l_d}{f} = \frac{L_{cell}}{4 f_Q}$$

L_{Cell} length of the complete FoDo cell, f_Q focal length of the quadrupole, μ phase advance per cell

7.) Stability in a FoDo cell

(thin lens approx)

$$f_Q > \frac{L_{Cell}}{4}$$

8.) Beta functions in a FoDo cell

(thin lens approx)

$$\hat{\beta} = \frac{(1+\sin\frac{\psi_{cell}}{2})L_{cell}}{\sin\psi_{cell}} \qquad \breve{\beta} = \frac{(1-\sin\frac{\psi_{cell}}{2})L_{cell}}{\sin\psi_{cell}}$$

L_{Cell} length of the complete FoDo cell, μ phase advance per cell