Mathematical and Numerical Methods for Non-linear Dynamics (an introduction)

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http://cern.ch/Werner.Herr/CAS2011/lectures/Chios_methods.pdf

For many more details:
http://cern.ch/Werner.Herr/METHODS

Primary purpose of this lectures

- Need to introduce new tools for non-linear dynamics
- Give an overview of the *modern**) tools used in accelerator physics
- Necessarily incomplete
- Avoid mathematical derivations, rather give *"raison d'etre"* and *"mode d'emploi"*

*) *modern*: "contemporary", not "fashionable" !

Recommended Bibliography:

E. Forest, *Beam Dynamics - A New Attitude and Framework*
Harwood Academic Publishers, 1998.

A. Chao, *Lecture Notes on Topics in Accelerator Physics*
SLAC, 2001.

A. Chao and M. Tigner, *Handbook of Accelerator Physics and Engineering*
World Scientific Publishing, 1998.

Outline of this lectures

- Motivation, introduction and classical concepts
- New concepts and modern techniques
  - Maps
  - Hamiltonian theory (for our purpose)
  - Computation: maps, symplectic integration
  - Analysis: Lie transforms, normal forms
  - Analysis: Differential algebra

Treatment of LINEAR dynamics

- **Standard introduction using Hill's equation (for simplicity: show for one dimension first):**

$$\frac{d^2x(s)}{ds^2} + K(s)x(s) = 0$$

- $K(s)$ **periodic, smooth function**
- **Is that true ?**
- **No, is only approximate**

Treatment of LINEAR dynamics

- **Used to "derive" Courant-Snyder ansatz:**

$$x(s) = \sqrt{\beta(s) \cdot \epsilon} \;\cdot\; cos(\mu(s) + \mu_0)$$

$$x'(s) = \sqrt{\frac{\epsilon}{\beta(s)}} \;\cdot\; (sin(\mu(s) + \mu_0) + \alpha \cdot cos(\mu(s) + \mu_0))$$

- **Is the solution to any system that is: confined and periodic !**
- **Do we need Hill's equation ?**

Treatment of DISTORTED dynamics

- **Hill's equation with distortions**
- **For both transverse dimensions:**

$$\frac{d^2x(s)}{ds^2} + K(s)x(s) = \frac{F_x(x,y,s)}{v \cdot p}$$

$$\frac{d^2y(s)}{ds^2} + K(s)y(s) = \frac{F_y(x,y,s)}{v \cdot p}$$

Treatment of DISTORTED dynamics

- **What are F_x, F_y ?**

$$F_x(x,y,s) = -v \cdot B_y(x,y,s)$$

$$F_y(x,y,s) = -v \cdot B_x(x,y,s)$$

- $B_x(x,y,s), B_y(x,y,s)$ **can be any higher order electromagnetic field**

Treatment of DISTORTED dynamics

- We have to re-write:

$$\frac{d^2x(s)}{ds^2} + K(s)x(s) = \frac{F_x(x,y,s)}{v \cdot p}$$

- as (similar for the other plane):

$$\frac{d^2x(s)}{ds^2} + K(s)x(s) = \sum_{i,j,k,l \geq 0} p_{ijkl}(s) x^i x'^j y^k y'^l$$

- Very non-linear differential equation to solve ...

→ Enter the field of non-linear dynamics

Can we deal with that ?

- Under certain circumstances:
  - All $p_{ijkl}(s)$ are perturbations, i.e. (very) small
  - Only a few $p_{ijkl}(s)$ are non-zero
  - You can avoid resonances
  - Perturbations are smooth or possibly periodic
- Would you build a 3 billion Euro machine on these assumptions ?

A disclaimer ...

- Can be useful to understand and demonstrate certain aspects
- See Oliver Brüning's lecture
- For practical work on realistic machine:
  - New tools required
  - It is much easier that you think (.. and other people tell you !)

A better framework

Start with the differential equation:

$$\frac{d^2x(s)}{ds^2} + K(s)x(s) = \sum_{i,j,k,l\geq 0} p_{ijkl}(s)x^i x'^j y^k y'^l$$

- Bad news: We have no global analytical solution
- Good news: An analytical solution is not needed !

A better framework

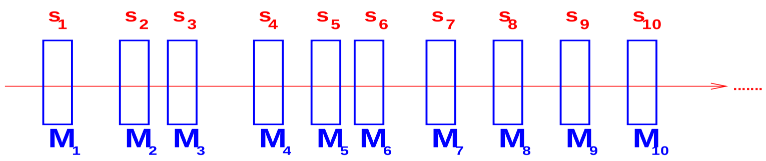
- **Why not ?**
- **We do not want to know:**
  - **The particle's position and momentum at 2h 45min 22.3s ?**
- **We do want to know:**
  - **Is the beams stable for a long time ?**
  - **Is the motion confined ?**
- **Can we get a framework to get that (easily) ?**

A better framework - go back 100 years ...

- **"Old" to "New" classical dynamics:**
  - **Topology and properties of phase space (see Oliver's lecture)**
  - **Chaotic motion, non-integrable systems**
  - **Sensitivity to initial conditions**

How does an accelerator really look like ?

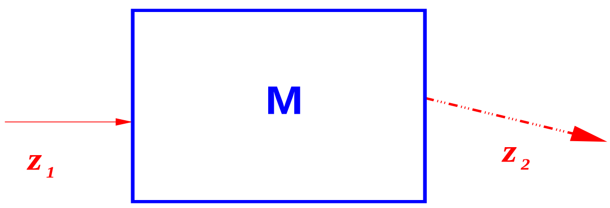
- Magnetic and electric fields are (in general) not a continuous function of s
- A (finite) sequence of machine elements M at longitudinal positions s_1, s_2, s_3,...:



- Cannot be described by a smooth differential equation

How can an element really be described ?

- You need to describe what happens to the particle in M
- Assume each element M (e.g. magnet) acts on the beam locally in a deterministic way, functionally independent of all other elements



- In general: $\vec{z}_2 \neq \vec{z}_1$

How is an element described ?

- Let $\vec{z}_1, \vec{z}_2$ describe a quantity (coordinates, beam sizes ...) before and after the element
- Take an machine element (e.g. magnet) and build a mathematical model $\mathcal{M}$
  - → In general: $\vec{z}_2 \; = \; \mathcal{M} \; (\vec{z}_1)$
  - → $\mathcal{M}$ is a so-called map
  - → Very important: no need to know what happens in the rest of the machine !!
- The complete sequence of MAPS connects the pieces together to make a ring (or beam line)

MAPS transform coordinates through an element

- Use coordinate vector: $\vec{z} \; = \; (x, \quad x' = \frac{\partial x}{\partial s}, \quad y, \quad y' = \frac{\partial y}{\partial s})$ $^{*)}$
- $\mathcal{M}$ transforms the coordinates $\vec{z}_1(s_1)$ at position s_1 to new coordinates $\vec{z}_2(s_2)$ at position s_2:

$$\vec{z}_2(s_2) = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \mathcal{M} \circ \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1} = \mathcal{M} \circ \vec{z}_1(s_1)$$

$^{*)}$ not unique, see later

What can $\mathcal{M}$ be ?

- Any "description" to go from $\vec{z}_1$ to $\vec{z}_2$
- This "description" can be:
  - A simple linear matrix or transformation
  - A non-linear transformation (Taylor series, Lie Transform ...)
  - High order integration algorithm
  - A computer program, subroutine etc.
- Let us look at linear theory first !

Simple examples (one dimensional)

Focusing quadrupole of length L and strength k:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = \begin{pmatrix} \cos(L \cdot k) & \frac{1}{k} \cdot \sin(L \cdot k) \\ k \cdot \sin(L \cdot k) & \cos(L \cdot k) \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

Quadrupole with short length L (i.e.: $1 \gg L^2 \cdot k^2$)

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = \begin{pmatrix} 1 & 0 \\ k^2 \cdot L (= -\frac{1}{f}) & 1 \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

They are $\mathcal{M}$aps, describe the movement in an element (quadrupole)

Interlude: there was already a trap ... !

According to B. Holzer (lectures) or K. Wille (textbook):

$$k = \frac{1}{B\rho}\frac{dB_y}{dx}$$

According to "Handbook for Accelerator Physics" (A. Chao):

$$k^2 = \frac{1}{B\rho}\frac{dB_y}{dx}$$

→ **The lesson: check what people use !!**

(remember Air Canada 143)

Interlude: it can be worse ... !

You also find:

$$K^2 = k = \frac{1}{B\rho}\frac{dB_y}{dx}$$

Often different conventions in simulation programs !

Some programs want fields, not gradients !
Found this construction:

$$B_y = 0.1 \cdot k \cdot x \cdot B \cdot \rho$$

Putting the "pieces" together

Starting from a position s_0 and applying all maps (for N elements) in sequence around a ring with circumference C to get the **One-Turn-Map** (OTM) for the position s_0 (for one dimension only):

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0 + C} = \mathcal{M}_1 \circ \mathcal{M}_2 \circ \ldots \circ \mathcal{M}_N \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$\Longrightarrow \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0 + C} = \mathcal{M}_{ring}(s_0) \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

Why are we interested in $\mathcal{M}_{ring}$?

- In the purely linear case $\mathcal{M}_{ring}$ corresponds to a matrix
  - $\mathcal{M}_{ring}$ allows to derive global quantities (Q, orbit, etc...)
  - Allows the analysis of imperfections (and their correction !)
  - "Straightforward" to formally extend it to complicated (e.g. non-linear) problems

First extension: two dimensions

- **Extend vectors for coordinates**
  $(x, x') \rightarrow (x, x', y, y')$
- **Extend transfer maps/matrices**

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11} & m_{12} & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{43} & m_{44} \end{pmatrix} \circ \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1}$$

Extension to two dimensions (coupling)

- **The horizontal and vertical motion can be coupled:**

$\rightarrow$ **Additional elements in matrix**

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \circ \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1}$$

- **Q: what about the β-functions now ??**

Analysis of the One-Turn-Map

- We have obtained a map for the whole ring
- Have to get now the information we want:
  - Optics parameters (Tune, Twiss function, ..)
  - Closed orbit
  - Stability
  - etc. ...
- How to analyse a MAP (first: a matrix) ???

  (see also B. Holzer lecture, but reality comes here)

Normal forms

- Maps can be transformed into (Jordan) Normal Forms
- Original maps and normal form are equivalent, but ...
- Easily used to analyse the maps:
  - Get parameters (Q, Q', Twiss function, ..)
  - Study invariants, etc.
  - Extract (non-linear) tune shifts
  - For resonance analysis
  - etc. ...
- Idea is to make a transformation to get a simpler form for the map

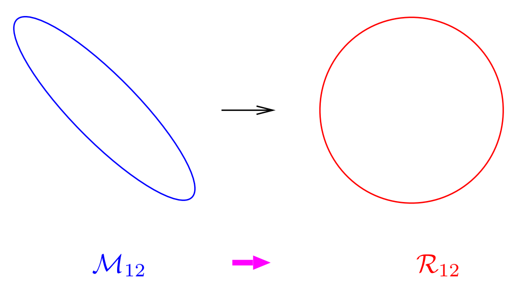
Normal forms

Assume the map $\mathcal{M}_{12}$ propagates the variables from location 1 to location 2, we try to find a transformation $\mathcal{A}$ such that:

$$\mathcal{A}\mathcal{M}_{12}\mathcal{A}^{-1} = \mathcal{R}_{12}$$

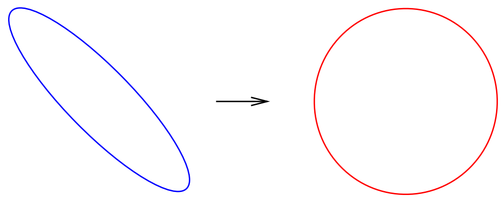
- **The map $\mathcal{R}_{12}$ is:**
  - **A "Jordan Normal Form", (or at least a very simplified form of the map)**
  - **Example: $\mathcal{R}_{12}$ becomes a pure rotation**
- **The map $\mathcal{R}_{12}$ describes the same dynamics as $\mathcal{M}_{12}$, but:**
  - **All coordinates are transformed**
  - **This transformation "analyses" the non-trivial part**

Normal forms - linear case



- **Pictorial form of the transformation**
- **Motion on a complicated ellipse becomes motion on a circle (i.e. a pure rotation)**

Normal forms - linear case



$$M = \mathcal{A} \circ \mathcal{R}(\Delta\mu) \circ \mathcal{A}^{-1} \qquad \text{or :} \qquad \mathcal{R}(\Delta\mu) = \mathcal{A}^{-1} \circ M \circ \mathcal{A}$$

Normal forms - linear case

Assume the one-turn-map $\mathcal{M}(s)$ at the position s is (e.g. lecture on transverse dynamics):

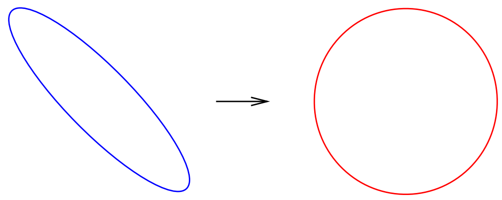
$$\mathcal{M}(s) = \begin{pmatrix} cos(\Delta\mu) + \alpha(s)sin(\Delta\mu) & \beta(s)sin(\Delta\mu) \\ -\gamma(s)sin(\Delta\mu) & cos(\Delta\mu) - \alpha(s)sin(\Delta\mu) \end{pmatrix}$$

- **Describes the motion on a phase space ellipse**
- **Re-write $\mathcal{M}$ such that one part $\mathcal{R}$ becomes a pure rotation (a circle), i.e.:**

$$\mathcal{A}\mathcal{R}\mathcal{A}^{-1} = \mathcal{M}$$

Slide 33

Normal forms - linear case



$$M = \mathcal{A} \circ \mathcal{R}(\Delta\mu) \circ \mathcal{A}^{-1} \quad \text{or :} \quad \mathcal{R}(\Delta\mu) = \mathcal{A}^{-1} \circ M \circ \mathcal{A}$$

with

$$\mathcal{A} = \begin{pmatrix} \sqrt{\beta(s)} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta(s)}} \end{pmatrix} \quad \text{and} \quad \mathcal{R} = \begin{pmatrix} cos(\Delta\mu) & sin(\Delta\mu) \\ -sin(\Delta\mu) & cos(\Delta\mu) \end{pmatrix}$$

Slide 34

Normal forms - linear case

We had:

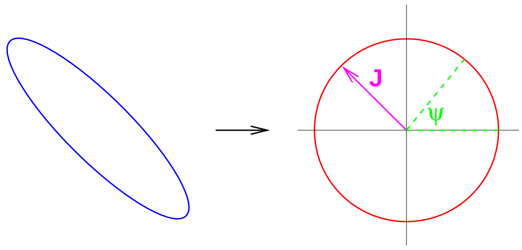
$$M = \mathcal{A} \circ \mathcal{R}(\Delta\mu) \circ \mathcal{A}^{-1} \quad \text{or :} \quad \mathcal{R}(\Delta\mu) = \mathcal{A}^{-1} \circ M \circ \mathcal{A}$$

with

$$\mathcal{A} = \begin{pmatrix} \sqrt{\beta(s)} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta(s)}} \end{pmatrix} \quad \text{and} \quad \mathcal{R} = \begin{pmatrix} cos(\Delta\mu) & sin(\Delta\mu) \\ -sin(\Delta\mu) & cos(\Delta\mu) \end{pmatrix}$$

- **This is just the Courant-Snyder transformation to get $\beta, \alpha, ...$ etc., $\Delta\mu$ is the tune !**
- **That is: the Courant-Snyder analysis is just a normal form transform of the linear one turn matrix**
- **Works in more than one dimension**

Interlude: action - angle variables



- Once the particles "travel" on a circle:
  - Radius (say: $\sqrt{2J}$, with $J = \frac{x^2+x'^2}{2}$) is constant (invariant of motion): action J
  - Phase advances by constant amount: angle Ψ

Another example: coupling

Assume a one-turn-matrix in 2D:

$$T = \begin{pmatrix} M & n \\ m & N \end{pmatrix}$$

M,m,N,n are 2-by-2 matrices. In case of coupling: $m \neq 0, n \neq 0$.
we can try to re-write as:

$$T = \begin{pmatrix} M & n \\ m & N \end{pmatrix} = V R V^{-1}$$

with:

$$R = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} \gamma I & C \\ -C^t & \gamma I \end{pmatrix}$$

What have we obtained ?

- **The matrix R is our simple rotation:**
  - A **and** B **are the one-turn-matrices for the "normal modes"**
  - **The matrix** C **contains the "coupling coefficients"**
  - **The matrix** V **transforms from the coordinates** (x, x', y, y') **into the "normal mode" coordinates** (w, w', v, v') **via the expression:**

    $(x, x', y, y') = V(w, w', v, v')$

Normal forms - linear case

This is extremely useful when map is applied k **times (e.g.** k **turns):**

$$M^k(x, x') = AR^k\ A^{-1}(x, x') = AR^k(X, X')$$

- **For multi-turns: study effect of map in normalized coordinates**
- **Easier to apply** k **times using the simple map (e.g. a rotation of** μ **becomes just a rotation** $k \cdot \mu$**)**
- **The** A **just transforms back to physical coordinates at the end (once !)**

The general philosophy (linear systems):

- Describe your elements by a linear map
- Combine all maps into a ring or beam line to get the linear one turn matrix
- Normal form analysis of the linear one turn matrix will give all the information

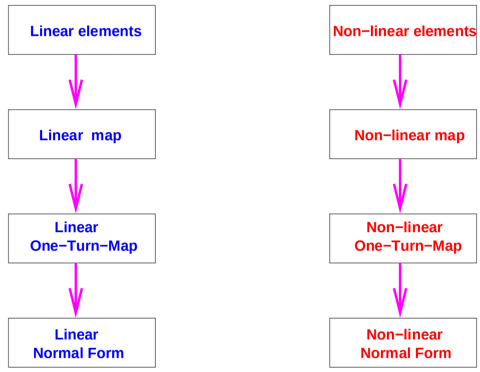
No need for any assumptions !

The general philosophy (non-linear systems):

- Describe your elements by a non-linear map
- Combine all maps into a ring or beam line to get the non-linear one turn map
- Normal form analysis of the non-linear one turn map will give all the information

No need for any assumptions !

The general philosophy



General formalism for all cases !

Various types of non-linear MAPS

- Choice depends on the application
  - Taylor maps
  - Symplectic integration techniques
  - Lie transformations
  - Truncated power series algebra (TPSA), can also generate Taylor map from tracking
  - ...

Slide 43

(A key concept: Symplecticity)

- Not all possible maps are allowed !
- Requires for a matrix $\mathcal{M}$ → $\mathcal{M}^T \cdot S \cdot \mathcal{M} = \mathcal{S}$

  with:

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

- It basically means: $\mathcal{M}$ is area preserving and

$$\lim_{n\to\infty} \mathcal{M}^n = \text{finite} \quad \Longrightarrow \quad \det\ \mathcal{M} = 1$$

Slide 44

Introducing non-linear elements

Effect of a (short) quadrupole depends linearly on amplitude (re-written from the matrix form):

$$\vec{z}(s_2) = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1} + \begin{pmatrix} 0 \\ k_1 \cdot x_{s_1} \\ 0 \\ k_1 \cdot y_{s_1} \end{pmatrix}$$

→ $\vec{z}(s_2) = \mathbf{M} \cdot \vec{z}(s_1)$

→ $\mathbf{M}$ is a matrix

Slide 45

Non-linear elements (e.g. sextupole)

Effect of a (thin) sextupole with strength k_2 is:

$$\vec{z}(s_2) = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_2} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s_1} + \begin{pmatrix} 0 \\ k_2 \cdot (x_{s_1} \cdot y_{s_1}) \\ 0 \\ \frac{1}{2} k_2 \cdot (x_{s_1}^2 \ - \ y_{s_1}^2) \end{pmatrix}$$

→ $\vec{z}(s_2) \ = \ \mathcal{M} \ \circ \ \vec{z}(s_1)$

→ **$\mathcal{M}$ is not a matrix, i.e. cannot be expressed by matrix multiplication**

Slide 46

Non-linear elements

Cannot be written in linear matrix form !
We need something like:

$$\begin{aligned} z_1(s_2) = x(s_2) \quad = \quad & R_{11} \cdot x \quad + R_{12} \cdot x' \quad + R_{13} \cdot y + ... \\ & + T_{111} \cdot x^2 + T_{112} \cdot xx' + T_{122} \cdot x'^2 + \\ & + T_{113} \cdot xy + T_{114} \cdot xy' + ... \\ & + U_{1111} \cdot x^3 + U_{1112} \cdot x^2 x' + ... \end{aligned}$$

and the equivalent for all other variables ...

Higher order (Taylor -) MAPS:

We have (for: j = 1...4):

$$z_j(s_2) \;=\; \sum_{k=1}^{4} R_{jk} z_k(s_1) \;+ \sum_{k=1}^{4}\sum_{l=1}^{4} T_{jkl} z_k(s_1) z_l(s_1)$$

$$\text{Let's call it}: \quad \mathcal{A}_2 \;=\; [R, T] \quad (\text{second order map } \mathcal{A}_2)$$

Higher orders can be defined as needed ...

$$\mathcal{A}_3 \;=\; [R, T, U] \quad \Longrightarrow \quad + \sum_{k=1}^{4}\sum_{l=1}^{4}\sum_{m=1}^{4} U_{jklm} z_k(s_1) z_l(s_1) z_m(s_1)$$

Higher order (Taylor -) MAPS:

Example: complete second order map for a (thick) sextupole with length L and strength K (in 4D):

$$\begin{aligned}
x_2 &= x_1 + L x_1' & -K\left(\tfrac{L^2}{4}(x_1^2 - y_1^2) + \tfrac{L^3}{12}(x_1 x_1' - y_1 y_1') + \tfrac{L^4}{24}(x_1'^2 - y_1'^2)\right) \\
x_2' &= x_1' & -K\left(\tfrac{L}{2}(x_1^2 - y_1^2) + \tfrac{L^2}{4}(x_1 x_1' - y_1 y_1') + \tfrac{L^3}{6}(x_1'^2 - y_1'^2)\right) \\
y_2 &= y_1 + L y_1' & +K\left(\tfrac{L^2}{4} x_1 y_1 + \tfrac{L^3}{12}(x_1 y_1' + y_1 x_1') + \tfrac{L^4}{24}(x_1' y_1')\right) \\
y_2' &= y_1' & +K\left(\tfrac{L}{2} x_1 y_1 + \tfrac{L^2}{4}(x_1 y_1' + y_1 x_1') + \tfrac{L^3}{6}(x_1' y_1')\right)
\end{aligned}$$

⚠ **Definition of K not unique, can differ by some factor !!**

Symplecticity for higher order MAPS

- Truncated Taylor expansions are not matrices !!
- It is the associated Jacobian matrix $\mathcal{J}$ which must fulfil the symplecticity condition:

$$\mathcal{J}_{ik} = \frac{\partial z_2^i}{\partial z_1^k}$$

$$\mathcal{J} \quad \text{must fulfil:} \quad \mathcal{J}^t \cdot \mathcal{S} \cdot \mathcal{J} = \mathcal{S}$$

- In general: $\mathcal{J}_{ik} \neq$ const → for truncated Taylor map can be difficult to fulfil for all z

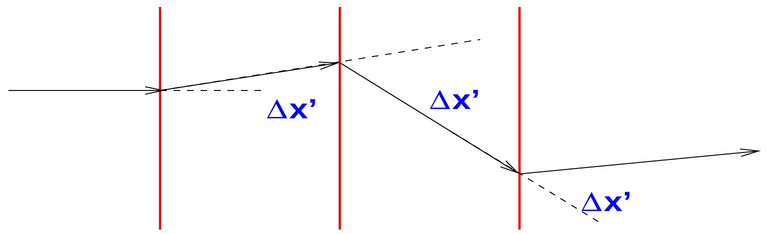
Thick versus thin magnets

- Real magnets have a finite length, i.e. thick magnets
- Thick magnet: field and length used to compute effect, i.e. the map
- Consequence: they are not always linear elements (even not dipoles, quadrupoles)
- For thick, non-linear magnets closed solution for maps often does not exist

Thick versus thin magnets

- Thin "magnet": let the length go to zero, but keep field integral finite (constant)
- Thin dipoles and quadrupoles are linear elements
- Thin elements are much easier to use ...

Moving through thin elements



- No change of amplitudes x and y
- The momenta x' and y' receive an amplitude dependent deflection (kick)

➡ $x' \rightarrow x' + \Delta x'$ and $y' \rightarrow y' + \Delta y'$

Using thin elements

- **Can we approximate a thick element by thin element(s) ?**
  - **Yes, when the length is small or does not matter**
  - **Yes, when we can model the thick magnet correctly**
  - **What about accuracy, symplecticity etc. ??**
  - **Demonstrate with some simple examples**

Thick → thin quadrupole

$$\mathcal{M}_{s \rightarrow s+L} = \begin{pmatrix} \cos(L \cdot K) & \frac{1}{K} \cdot \sin(L \cdot K) \\ -K \cdot \sin(L \cdot K) & \cos(L \cdot K) \end{pmatrix}$$

- **Exact map (matrix) for quadrupole**
- **What happens when we make it thin ?**
  - **Accuracy ?**
  - **Symplecticity ?**
- **(What follows is valid for all elements)**

Accuracy of thin lenses

$$\mathcal{M}_{s\to s+L} = \begin{pmatrix} \cos(L\cdot K) & \frac{1}{K}\cdot\sin(L\cdot K) \\ -K\cdot\sin(L\cdot K) & \cos(L\cdot K) \end{pmatrix}$$

- **Start with exact map**
- **Taylor expansion in "small" length L:**

$$L^0\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + L^1\cdot\begin{pmatrix} 0 & 1 \\ -K^2 & 0 \end{pmatrix} + L^2\cdot\begin{pmatrix} -\frac{K^2}{2} & 0 \\ 0 & -\frac{K^2}{2} \end{pmatrix} + \ldots$$

Accuracy of thin lenses (B)

- **Keep up to first order term in** L

$$\mathcal{M}_{s\to s+L} = L^0\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + L^1\cdot\begin{pmatrix} 0 & 1 \\ -K^2 & 0 \end{pmatrix}$$

$$\mathcal{M}_{s\to s+L} = \begin{pmatrix} 1 & L \\ -K^2\cdot L & 1 \end{pmatrix} + \mathcal{O}(L^2)$$

- **Precise to first order** $\mathcal{O}(L^1)$
- **det** $\mathcal{M} \neq$ **1, non-symplectic**

Accuracy of thin lenses (C)

$$\mathcal{M}_{s\to s+L} = \begin{pmatrix} 1 & L \\ -K^2\cdot L & 1 \end{pmatrix} + \mathcal{O}(L^2)$$

→

$$\mathcal{M}_{s\to s+L} = \begin{pmatrix} 1 & L \\ -K^2\cdot L & 1-K^2L^2 \end{pmatrix}$$

- **Precise to first order $\mathcal{O}(L^1)$**
- **"symplectified" by adding term $-K^2L^2$**
  (it is wrong to $\mathcal{O}(L^2)$ anyway ...)

Accuracy of thin lenses

- **Keep up to <u>second order</u> term in L**

$$\mathcal{M}_{s\to s+L} = \begin{pmatrix} 1-\frac{1}{2}K^2L^2 & L \\ -K^2\cdot L & 1-\frac{1}{2}K^2L^2 \end{pmatrix} + \mathcal{O}(L^3)$$

- **Precise to second order $\mathcal{O}(L^2)$**
- **More accurate than (C), but not symplectic**

Accuracy of thin lenses (D)

- **Symplectification like:**

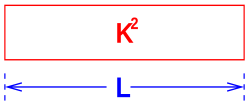
$$\mathcal{M}_{s \to s+L} = \begin{pmatrix} 1 - \frac{1}{2}K^2L^2 & L - \frac{1}{4}K^2L^3 \\ -K^2 \cdot L & 1 - \frac{1}{2}K^2L^2 \end{pmatrix} + \mathcal{O}(L^3)$$

- **Precise to second order $\mathcal{O}(L^2)$**
- **Fully symplectic**

Accuracy of thin lenses

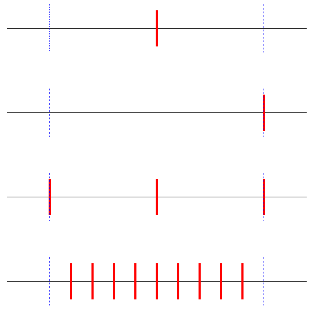
- **Looks like we made some arbitrary changes and called it "symplectification"**
- **Is there a physical picture behind the approximations ?**
- **Yes, geometry of thin lens kicks ...**
- **A thick element is split into thin elements with drifts between them**

Thick → thin quadrupole



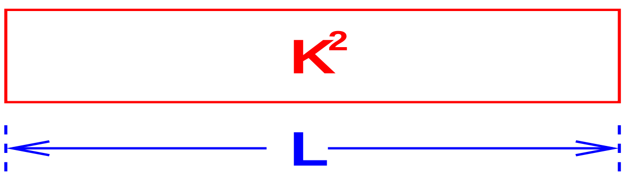
quadrupole of finite length

options:

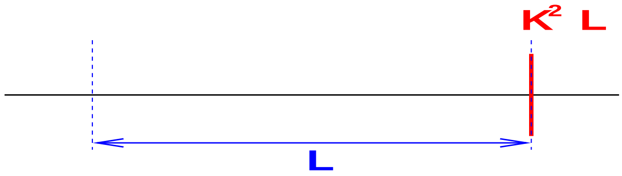


Which is a good strategy ? → accuracy and speed

Thick quadrupole ..



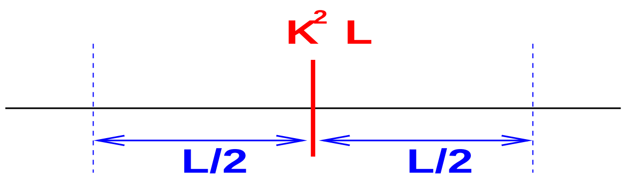
First order ..



- One thin quadrupole "kick" and one drift combined
- Resembles "symplectification" of type (C)

$$\begin{pmatrix} 1 & 0 \\ -K^2 \cdot L & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & L \\ -K^2 \cdot L & 1 - K^2L^2 \end{pmatrix}$$

Second order ..



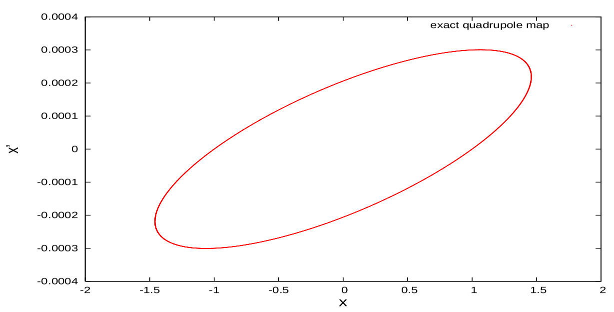
- One thin quadrupole "kick" between two drifts
- Resembles more accurate, symplectic model of type (D)

$$\begin{pmatrix} 1 & \frac{1}{2}L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -K^2 \cdot L & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2}L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}K^2L^2 & L - \frac{1}{4}K^2L^3 \\ -K^2 \cdot L & 1 - \frac{1}{2}K^2L^2 \end{pmatrix}$$

Accuracy of thin lenses

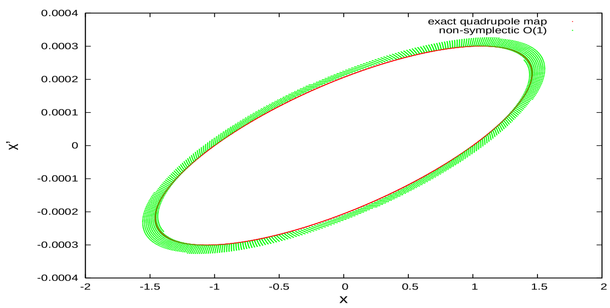
- One kick at the end (or beginning):
  - Error (inaccuracy) of first order $\mathcal{O}(L^1)$
- One kick in the centre:
  - Error (inaccuracy) of second order $\mathcal{O}(L^2)$
- It is very relevant how to apply thin lenses
- Aim should be to be precise and fast (and simple to implement)

What is the point ???



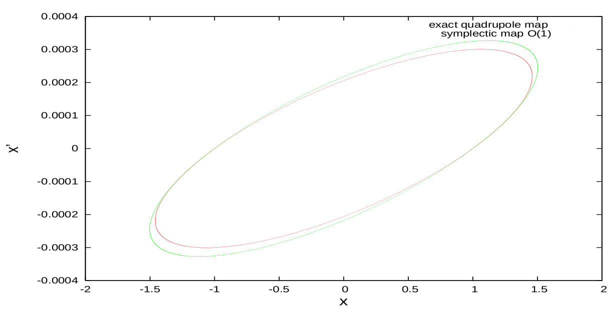
- Phase ellipse - quadrupole exact solution

Quadrupole non-symplectic solution



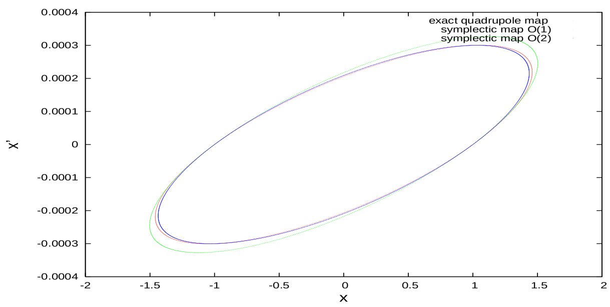
- Non-symplecticity: particles spiral towards outside

Quadrupole symplectic $\mathcal{O}(L^1)$ solution



- symplecticity: but phase space ellipse not accurate

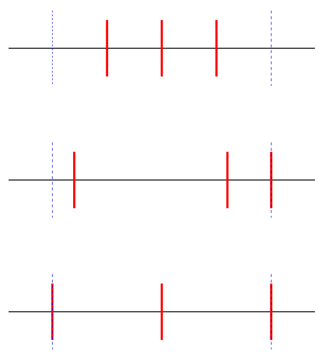
Quadrupole symplectic $\mathcal{O}(L^2)$ solution



- symplecticity: phase space ellipse accurate enough

Slide 69

Can we do better ?



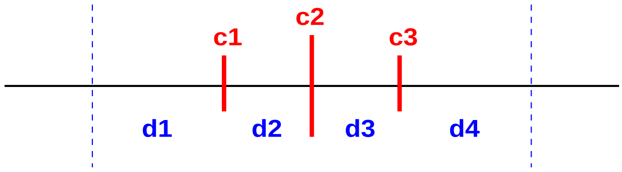
- Try more slices, e.g. 3 kicks:
- How to put them ?
- Hope you are already alerted ...
- Allow that they are at different positions and have different strengths
- Minimize the inaccuracy

Question: is one of the options obviously wrong ? If yes, why ?

Slide 70

Can we do better ?

- Try a model with 3 kicks:

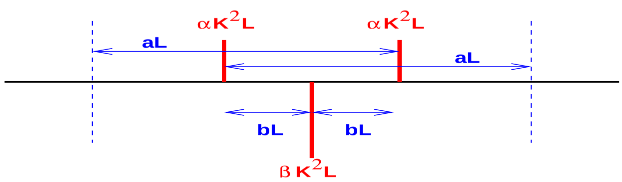


→ To get best accuracy (i.e. deviation from exact solution):

- Optimize kicks c1, c2, c3
- Optimize drifts d1, d2, d3, d4

Can we do better ?

- Try a model with 3 kicks:



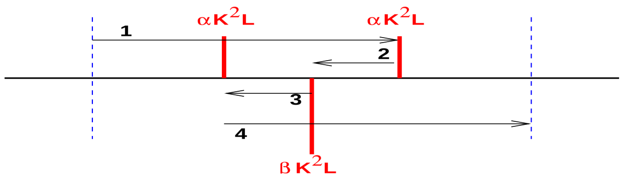
- with:

  a ≈ 0.6756, b ≈ -0.1756, $\alpha \approx$ 1.3512, $\beta \approx$ -1.7024

- we have a $\mathcal{O}(4)$ integrator ...
- (a $\mathcal{O}(6)$ integrator would require 9 kicks (!) ...)

Can we do better ?

- **Try a model with 3 kicks:**



- **Must track backwards ! Change interpretation !**
- **Thin lenses not a new sequence of magnets (a la MAD)**
- **What about space charge calculations ?**

Symplectic integration

- **What we do is** **Symplectic Integration**
- **From a lower order integration scheme (1 kick), construct higher order scheme**
- **Formally (for the formulation of $S_k(t)$ see later):**
  - **From a 2nd order scheme (1 kick) $S_2(t)$ we construct a 4th order scheme (3 kicks = 3 x 1 kick) like:**

    $S_4(t) = S_2(x_1 t) \circ S_2(x_0 t) \circ S_2(x_1 t)$ **with:**

$$x_0 \;=\; \frac{-2^{1/3}}{2-2^{1/3}} \;\approx\; -1.7024 \qquad x_1 \;=\; \frac{1}{2-2^{1/3}} \;\approx\; 1.3512$$

Symplectic integration

- **Can be considered as an iterative scheme (see e.g. H. Yoshida, 1990, E. Forest, 1998[2]):**
  - **If $S_{2k}(t)$ is a symmetric integrator of order $2k$, then:**

    $S_{2k+2}(t) = S_{2k}(x_1 t) \circ S_{2k}(x_0 t) \circ S_{2k}(x_1 t)$ **with:**

$$x_0 = \frac{-\sqrt[2k+1]{2}}{2 - \sqrt[2k+1]{2}} \qquad x_1 = \frac{1}{2 - \sqrt[2k+1]{2}}$$

- **Higher order integrators can be obtained in a similar way**

[2] E. Forest, "Beam Dynamics, A New Attitude and Framework", 1998

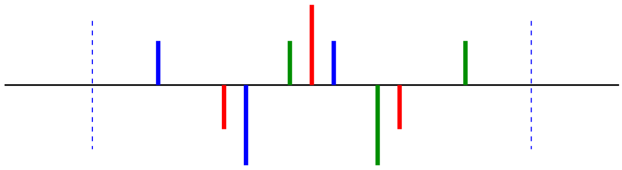
Symplectic integration

- **Example: From a 4th order to 6th order**

  $S_6(t) = S_4(x_1 t) \circ S_4(x_0 t) \circ S_4(x_1 t)$

- **We get 3 times 4th order with 3 kicks each, we have the 9 kick, 6th order integrator mentioned earlier**

Integrator of order 6



- Requires 9 kicks
- We have 3 interleaved 4th order integrators
- Can be used in iterative scheme

Some remarks:

- We have used a linear map (quadrupole) to demonstrate the integration
- Can that be applied for other maps (solenoids, higher order, non-linear maps) ?
  - Yes !!
  - We get the same integrators !
  - Proof and systematic (and easy) extension in the form of Lie operators[2] (see later)

→ Best accuracy for thin lenses !

[2] H. Yoshida, Physics Letters A, Volume 150 (1990) 262.

Accuracy of thin lenses

What about accuracy of non-linear elements ?

assume a general case:

$$x'' = f(x)$$

- **Disadvantage : usually a closed solution through the element does not exist, integration necessary**
- **Advantage : They are usually thin (thinner than dipoles, quadrupoles ...)**
  - **Dipoles: $\approx$ 14.3 m**
  - **Quadrupole: $\approx$ 2 - 5 m**
  - **Sextupoles, Octupoles: $\approx$ 0.30m**
- **Can try our simplest thin lens approximation first ...**

Accuracy of thin lenses - our $\mathcal{O}(2)$ model

$$1.Step \quad \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1+L/2} = \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

$$2.Step \quad \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1+L/2} = \begin{pmatrix} x \\ x' + \Delta x' \end{pmatrix}_{s_1+L/2}$$

$$3.Step \quad \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1+L} = \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1+L/2}$$

Accuracy of thin lenses

Assume the general case:

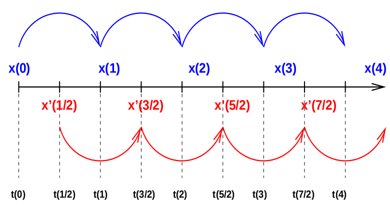
$$x'' = f(x)$$

- **Using this thin lens approximation (type D, $\mathcal{O}(2)$) gives:**

$$x'(L) \approx x'_0 + Lf(x_0 + \frac{L}{2}x'_0)$$
$$x(L) \approx x_0 + \frac{L}{2}(x'_0 + x'(L))$$

- **This is also called "leap frog" algorithm/integration**
- **It is symplectic (... and time reversible) !!**

Interlude ...



For any: $x'' = f(x, x', t)$ we can solve it by:

$$x'_{i+3/2} \approx x'_{i+1/2} + f(x_{i+1})\Delta t$$
$$x_{i+1} \approx x_i + x'_{i+1/2}\Delta t$$

Accuracy of thin lenses

Accuracy of "leap frog" algorithm/integration"

the (exact) Taylor expansion gives:

$$x(L) \quad = \quad x_0 + x_0' L + \frac{1}{2} f(x_0) L^2 + \frac{1}{6} x_0' f'(x_0) L^3 + ...$$

the (approximate) "leap frog" algorithm gives:

$$x(L) \quad = \quad x_0 + x_0' L + \frac{1}{2} f(x_0) L^2 + \frac{1}{4} x_0' f'(x_0) L^3 + ...$$

- **Errors are $\mathcal{O}(L^3)$**
- **For small L acceptable, and symplectic, extend to our symplectic integration**

Accuracy of thin lenses

For bar/coffee discussions:

why did I write:

$$x'' \quad = \quad f(x)$$

and not:

$$x'' \quad = \quad f(x, x')$$

Accuracy of thin lenses

An application, assume a (1D) sextupole with (definition of k not unique !):

$$x'' = k \cdot x^2 = f(x)$$

using the thin lens approximation (type D) gives:

$$x(L) = x_0 + x'_0 L + \frac{1}{2} k x_0^2 L^2 + \frac{1}{2} k x_0 x'_0 L^3 + \frac{1}{8} k x_0'^2 L^4$$

$$x'(L) = x'_0 + k x_0^2 L + k x_0 x'_0 L^2 + \frac{1}{4} k x_0'^2 L^3$$

Map for thick sextupole of length L in thin lens approximation, accurate to $\mathcal{O}(L^2)$

Some comments:

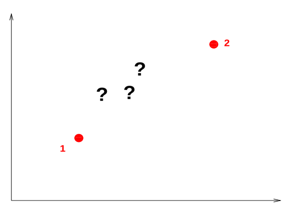
- **We have interleaved kicks with drifts**
- **Is that always necessary ?**
  - **No !**
  - **Can be any map with an exact expression**
  - **In most cases the exact map is a linear map (matrix)**
- **We have derived element maps for tracking from the equation of motion using this technique → can track now**

The plan now ...

- Extend all concepts to non-linear dynamics
  - Lagrangian and Hamiltonian dynamics
  - How to use that → Lie transforms
  - How to analyse that → Non-linear normal forms
  - How to analyse that better → Differential Algebra (DA)
- Avoid abstract definitions and formulation, but:
  - Intuitive (but correct !) treatment
  - Useful formulae and examples
  - Real life examples and demonstration (DA)

Hamilton principle

- Problem: describe the motion of a system (e.g. 1 or more particles) between times t_1 and t_2



- Describe by coordinates q_i $\qquad (i = 1, n)$

  n are degrees of freedom of the system

Hamilton principle

- **Describe motion by a function** L

$$L(q_1, ...q_n, \dot{q_1}, ...\dot{q_n}, t)$$

$(q_1, ...q_n)$ **... generalized coordinates**

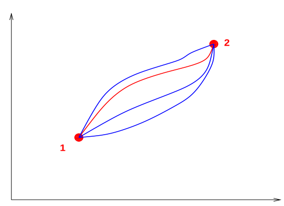
$(\dot{q_1}, ...\dot{q_n})$ **... generalized velocities**

- **The function** L **defines the Lagrange function**
- **The integral** $I = \int L(q_i, \dot{q}_i, t)dt$ **defines the action**

Without proof or derivation:
$L = T - V$ **= kinetic energy - potential energy**

Hamilton principle

$$I = \int_1^2 L(q_i, \dot{q}_i, t)dt = extremum$$



- **Hamiltonian principle: system moves such that the action** I **becomes an extremum**

Extremum principle ?

- **Not new:**
  - **Used in optics: Fermat principle**
  - **Quantum mechanics (path integrals)**
  - **General relativity**
  - **...**

Lagrange formalism

Without proof:

$$I = \int_1^2 L(q_i, \dot{q}_i, t)dt = extremum$$

is fulfilled when:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

(Euler - Lagrange equation)

From Lagrangian to Hamiltonian ..

- Lagrangian $L(q_1, ...q_n, \dot{q}_1, ...\dot{q}_n, t)$ in generalized coordinates and velocities
- Provides (n) second order differential equations
- Try to get:
  - Generalized momenta instead of velocities
  - First order differential equations (always solvable)

Corresponding (so-called conjugate) momenta p_j are:

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

Interlude: canonical transformations

- Not treated here in detail (see literature or backup slides)
- Main ideas:
  - Find variables where equations are easier to solve
  - Action-angle variables: lead to invariants of motion (see later)

From Lagrangian to Hamiltonian ..

- **Lagrangian:**
  - n second order equations
  - n-dimensional coordinate space
- **Hamiltonian:**
  - $2n$ first order equations
  - $2n$-dimensional phase space

From Lagrangian to Hamiltonian ..

Once we know what the canonical momenta p_i are: the Hamiltonian is a transformation of the Lagrangian:

$$H(q_j, p_j, t) = \sum_i \dot{q}_i p_i - L(q_j, \dot{q}_j, t)$$

without proof:
$H = T + V$ = kinetic energy + potential energy

we obtain 2 first order equation of motion:

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j = -\frac{dp_j}{dt}, \qquad \frac{\partial H}{\partial p_j} = \dot{q}_j = \frac{dq_j}{dt}$$

Hamiltonian of particle in EM fields

For the Hamiltonian of a (relativistic) particle in a electro-magnetic field we have:

$$\mathcal{H}(\vec{x},\vec{p},t) = c\sqrt{(\vec{p}-e\vec{A}(\vec{x},t))^2 + m_0^2c^2} + e\Phi(\vec{x},t)$$

where $\vec{A}(\vec{x},t)$, $\Phi(\vec{x},t)$ the vector and scalar scalar potential

Using canonical variables and the design path length s as independent variable (bending in x-plane):

$$\mathcal{H} = -(1+\frac{x}{\rho})\cdot\sqrt{(1+\delta)^2 - p_x^2 - p_y^2} + \frac{x}{\rho} + \frac{x^2}{2\rho^2} - \frac{A_s(x,y)}{B_0\rho}$$

where $\delta = (p-p_0)/p$ is relative momentum deviation and $A_s(x,y)$ longitudinal component of the vectorpotential.

Hamiltonian of particle in EM fields

The magnetic fields can be described with the multipole expansion:

$$B_y + iB_x = \sum_{n=1}(b_n + ia_n)(x+iy)^{n-1}$$

and since $\vec{B} = \nabla \times \vec{A}$:

$$A_s = \sum_{n=1}\frac{1}{n}[(b_n + ia_n)(x+iy)^n]$$

- For a large machine ($x \gg \rho$) we expand the root and sort the variables:

Hamiltonian (for large machine) ..

$$\mathcal{H} = \overbrace{\frac{p_x^2 + p_y^2}{2(1+\delta)}}^{kinematic} - \overbrace{\underbrace{\frac{x\delta}{\rho}}_{dispersive} + \underbrace{\frac{x^2}{2\rho^2}}_{focusing}}^{dipole} + \overbrace{\frac{k_1}{2}(x^2 - y^2)}^{quadrupole} + \overbrace{\frac{k_2}{3}(x^3 - 3xy^2)}^{sextupole}$$

- **The Hamiltonian describes <u>exactly</u> the motion of a particle <u>through a magnet</u>**
- **Basis to extend the linear formalism to a non-linear treatment**

But how do we use it ??

Poisson brackets

Introduce Poisson bracket for a differential operator:

$$[f, g] = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

Here the variables q_i, p_i are canonical variables, f and g are functions of q_i and p_i.
We can now write:

$$[q_i, H] = \frac{\partial H}{\partial p_i} = \frac{dq_i}{dt}$$

$$[p_i, H] = -\frac{\partial H}{\partial q_i} = \frac{dp_i}{dt}$$

Poisson brackets

- **Poisson bracket $[g, H]$ describes the time evolution of the system**

It is a special case of:

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$

If H does not explicitely depend on time and:

$$[f, H] = 0$$

implies that f is an invariant of the motion !

Lie transformations

We can define:

$$: f : g = [f, g]$$

where $: f :$ is an operator acting on the function g:

$$: f := [f,]$$

We can define powers as:

$$(: f :)^2 g =: f : (: f : g) = [f, [f, g]] \quad \text{etc.}$$

in particular:

$$e^{:f:} = \sum_{i=0}^{\infty} \frac{1}{i!} (: f :)^i$$

Lie transformations - example

Lie transforms acting on functions like x, p (canonical momentum, instead of x'):

$$:p^2:x = -2p \qquad :p^2:p = 0$$

or more complicated:

$$\begin{aligned} e^{:-Lp^2/2:}x &= x - \frac{1}{2}L:p^2:x + \frac{1}{8}L^2(:p^2:)^2x + .. \\ &= x + Lp \\ e^{:-Lp^2/2:}p &= p - \frac{1}{2}L:p^2:p + ... \\ &= p \end{aligned}$$

This is the transformation of a drift space of length L !!

Lie transformations

Acting on the phase space coordinates:

$$e^{:f:}(x,p)_0 = (x,p)_1$$

that is:

$$e^{:f:}x_0 = x_1$$

$$e^{:f:}p_0 = p_1$$

- **Lie transforms describe how to go from one point to another.**
- **Through a machine element (drift, magnet ...) fully described by f**
- **But what is f ?**

Lie transformations

- **The generator f is the Hamiltonian of the element !**
- **We use the Hamiltonian to descibe the motion through an individual element**
- **Inside a single element the motion is "smooth" (in the full machine it is not !)**

Some formulae for Lie transforms

With a constant, f, g, k arbitrary functions:

$$:a: \; = \; 0 \qquad e^{:a:} \; = \; 1$$

$$:f:a \; = \; 0 \qquad e^{:f:}a \; = \; a$$

$$e^{:f:}g(x) \; = \; g(e^{:f:}x)$$

$$e^{:f:}G(:g:)e^{-:f:} \; = \; G(:e^{:f:}g:)$$

$$e^{:f:}[g,h] = [e^{:f:}g, e^{:f:}h]$$

$$(e^{:f:})^{-1} \; = \; e^{-:f:}$$

and very important:

$$e^{:f:}e^{:g:}e^{-:f:} \; = \; e^{:e^{:f:}g:}$$

Useful formulae for calculations

With x coordinate, p momentum, as usual:

$$:x: \;=\; \frac{\partial}{\partial p} \qquad :p: \;=\; -\frac{\partial}{\partial x}$$

$$:x:^2 \;=\; \frac{\partial^2}{\partial p^2} \qquad :p:^2 \;=\; \frac{\partial^2}{\partial x^2}$$

$$:x^2: \;=\; 2x\frac{\partial}{\partial p} \qquad :p^2: \;=\; -2p\frac{\partial}{\partial x}$$

$$:xp: \;=\; p\frac{\partial}{\partial x} - x\frac{\partial}{\partial p} \qquad :x::p: \;=\; :p::x: \;=\; -\frac{\partial^2}{\partial x \partial p}$$

More useful formulae for calculations

With x coordinate, p momentum, as usual:

$$:p^2:x \;=\; \frac{\partial p^2}{\partial x}\frac{\partial x}{\partial p} - \frac{\partial p^2}{\partial p}\frac{\partial x}{\partial x} \;=\; -2p$$

$$:p^2:p \;=\; \frac{\partial p^2}{\partial x}\frac{\partial p}{\partial p} - \frac{\partial p^2}{\partial p}\frac{\partial p}{\partial x} \;=\; 0$$

$$(:p^2:)^2 x =: p^2 : (:p^2:x) =: p^2 : (-2p) = 0$$

$$(:p^2:)^2 p =: p^2 : (:p^2:p) =: p^2 : (0) = 0$$

More examples (1D):

For:

$$f = -\frac{L}{2}p^2$$

we obtained:

$$\begin{aligned} e^{:f:}x &= x + Lp \\ e^{:f:}p &= p \end{aligned}$$

- **Drift space, seen that already**

More examples (1D):

For:

$$f = -\frac{L}{2}(k^2x^2 + p^2)$$

we would get (try it !):

$$\begin{aligned} e^{:f:}x &= e^{:-\frac{L}{2}(k^2x^2+p^2):}x \\ e^{:f:}p &= e^{:-\frac{L}{2}(k^2x^2+p^2):}p \end{aligned}$$

Remember:

$$e^{:f:}g = \sum_{n=0}^{\infty} \frac{:f:^n}{n!}g$$

More examples (1D):

For:

$$f = -\frac{L}{2}(k^2x^2 + p^2)$$

we would get (try it !):

$$e^{:-\frac{L}{2}(k^2x^2+p^2):}x = \sum_{n=0}^{\infty}\left(\frac{(-k^2L^2)^n}{(2n)!}\cdot x + L\frac{(-k^2L^2)^n}{(2n+1)!}\cdot p\right)$$

$$e^{:-\frac{L}{2}(k^2x^2+p^2):}p = \sum_{n=0}^{\infty}\left(\frac{(-k^2L^2)^n}{(2n)!}\cdot p - k\frac{(-k^2L^2)^n}{(2n+1)!}\cdot x\right)$$

More examples (1D):

For:

$$f = -\frac{L}{2}(k^2x^2 + p^2)$$

we would get (try it !):

$$e^{:f:}x = x\ \cos(kL) + \frac{p}{k}\ \sin(kL)$$

$$e^{:f:}p = -kx\ \sin(kL) + p\ \cos(kL)$$

- **Thick, focusing quadrupole !**

Hamiltonians of some thick machine elements (2D)

dipole:

$$H = -\frac{-x\delta}{\rho} + \frac{x^2}{2\rho^2} + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

quadrupole:

$$H = \frac{1}{2}k_1(x^2 - y^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

sextupole:

$$H = \frac{1}{3}k_2(x^3 - 3xy^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

octupole:

$$H = \frac{1}{4}k_3(x^4 - 6x^2y^2 + y^4) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

Many machine elements

- **We can combine many machine elements by applying one transformation after the other:**

$$e^{:h:} = e^{:f_1:}e^{:f_2:}....e^{:f_N:}$$

- **Not restricted to matrices, i.e. linear elements ...**
- **And arrive at a transformation for the full ring**
  → **a one turn map**
- **The one turn map is the exponential of the effective Hamiltonian:**

$$\mathcal{M}_{ring} = e^{:-C\mathcal{H}_{eff}:}$$

Why all that ???

concatenation very easy:

$$e^{:h:} = e^{:f:}e^{:g:} = e^{:f+g:}$$

when f and g commute (i.e. $[f,g] = [g,f] = 0$)

- **Otherwise formalism exist**
- **The miracles:**
  - **Poisson brackets create symplectic maps**
  - **Exponential form $e^{:h:}$ is always symplectic**
  - **Better: the exponent is directly connected to the invariant of the transfer map !!**

Concatenation

To combine:

$$e^{:h:} = e^{:f:}e^{:g:}$$

We can use the formula (Baker-Campbell-Hausdorff (BCH)):

$$\begin{aligned} h = f \quad & + \; g \; + \; \tfrac{1}{2}[f,g] \; + \; \tfrac{1}{12}[f,[f,g]] \; + \; \tfrac{1}{12}[g,[g,f]] \\ & + \; \tfrac{1}{24}[f,[g,[g,f]]] \; - \; \tfrac{1}{720}[g,[g,[g,[g,f]]]] \\ & - \; \tfrac{1}{720}[f,[f,[f,[f,g]]]] \; + \; \tfrac{1}{360}[g,[f,[f,[f,g]]]] \; + \; ... \end{aligned}$$

or :

$$\begin{aligned} h = f \quad & + \; g \; + \; \tfrac{1}{2} : f : g \; + \; \tfrac{1}{12} : f :^2 g \; + \; \tfrac{1}{12} : g :^2 f \\ & + \; \tfrac{1}{24} : f :: g :^2 f \; - \; \tfrac{1}{720} : g :^4 f \\ & - \; \tfrac{1}{720} : f :^4 g \; + \; \tfrac{1}{360} : g :: f :^3 g \; + \; ... \end{aligned}$$

Concatenation

To combine:

$$e^{:h:} = e^{:f:}e^{:g:}$$

if one of them (f or g) is small, can truncate the series and get a very useful formula. Assume g is small:

$$e^{:f:}\, e^{:g:} \;=\; e^{:h:} \;=\; \exp\left[: f + \left(\frac{:f:}{1-e^{-:f:}}\right) g + \mathcal{O}(g^2) :\right]$$

Non-linear kicks

General thin lens kick $f(x)$:

$$e^{:\int_0^x f(x')dx':}$$

gives for the map:

$$\begin{aligned} x &= x_0 \\ p &= p_0 + f(x) \end{aligned}$$

Example: thin lens multipole of order n ($f(x) = a\cdot x^n$):

$$e^{:a\cdot x^n:}$$

gives for the map:

$$\begin{aligned} x &= x_0 \\ p &= p_0 + anx^{n-1} \end{aligned}$$

From the Hamiltonian to the map

We have seen that given the Hamiltonian f of a machine element is known, the Lie operator becomes:

$$f \quad \rightarrow \quad :f:$$

the corresponding map is than:

$$e^{:f:} \qquad (e^{:-Lf:})$$

This map is always symplectic and we have (in 1D):

$$e^{:f:}x_0 = x_1$$

$$e^{:f:}p_0 = p_1$$

or using $Z \;=\; (x, p_x, y, p_y, ...)$**:**

$$e^{:f:}Z_0 = Z_1$$

From the map to the Hamiltonian

The other question ➡ assuming we have:

$$\mathcal{M} \equiv \begin{pmatrix} \cos(\mu) + \alpha sin(\mu) & \beta \sin(\mu) \\ -\gamma \sin(\mu) & \cos(\mu) - \alpha sin(\mu) \end{pmatrix}$$

i.e.:

$$\mathcal{M}Z_0 = Z_1$$

how do we find the corresponding form for f ?

$$\mathcal{M} \quad \leftrightarrow \quad e^{:f:} \qquad (e^{:-\mu f:})$$

From the map to the Hamiltonian

For the linear matrix we know that f must be a quadratic form in (x, p, ...).
Any quadratic form can be written as:

$$f = -\frac{1}{2} Z^* F Z \qquad [\ = \ -\frac{1}{2}(a \cdot x^2 + b \cdot xp + c \cdot p^2)\ ]$$

where F is a symmetric, positive definite (why ?) matrix.
Then we can write (without proof, see e.g. Dragt):

$$:f: Z = SFZ$$

where S is the "symplecticity" matrix.
Therefore we get for the Lie transformation:

$$e^{:f:} Z \quad \leftrightarrow \quad e^{SF} Z$$

From the map to the Hamiltonian

Since we have $n = 2$, we get (using $Hamilton - Cayley$ theorem):

$$e^{SF} \ = \ \exp\begin{pmatrix} b & c \\ -a & -b \end{pmatrix} = a_0 + a_1 \begin{pmatrix} b & c \\ -a & -b \end{pmatrix}$$

We now have to find a_0 and a_1 !

The eigenvalues of SF are:

$$\lambda_\pm \ = \ \pm i\sqrt{ac - b^2}$$

From the map to the Hamiltonian

This tells us for the coefficients the conditions:

$$e^{\lambda_+} = a_0 + a_1 \cdot \lambda_+$$

$$e^{\lambda_-} = a_0 + a_1 \cdot \lambda_-$$

and therefore:

$$a_0 = \cos(\sqrt{ac - b^2})$$

$$a_1 = \frac{\sin(\sqrt{ac - b^2})}{\sqrt{ac - b^2}}$$

and

$$e^{SF} \;=\; \cos(\sqrt{ac - b^2}) + \frac{\sin(\sqrt{ac - b^2})}{\sqrt{ac - b^2}} \begin{pmatrix} b & c \\ -a & -b \end{pmatrix}$$

From the map to the Hamiltonian

For a general 2×2 matrix:

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

we get by comparison:

$$\cos(\sqrt{ac - b^2}) = \frac{1}{2} tr(M)$$

and

$$\frac{a}{-m_{21}} = \frac{2b}{m_{11} - m_{22}} = \frac{c}{m_{12}} = \frac{\sqrt{ac - b^2}}{\sin(\sqrt{ac - b^2})}$$

for the quadratic form of f:

$$f = -\frac{1}{2}(a \cdot x^2 + b \cdot xp + c \cdot p^2)$$

From the map to the Hamiltonian

For the example of a drift:

$$\mathcal{M} \equiv \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

we find:

$$a = 0, \quad b = 0, \quad c = L$$

and for the generator:

$$f = -\frac{1}{2}(Lp^2)$$

From the map to the Hamiltonian

For the example of a thin quadrupole:

$$\mathcal{M} \equiv \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

we find:

$$a = \frac{1}{f}, \quad b = 0, \quad c = 0$$

and for the generator:

$$f = -\frac{1}{2f}(x^2)$$

A very important example ...

$$\mathcal{M} \equiv \begin{pmatrix} \cos\mu + \alpha sin(\mu) & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha sin(\mu) \end{pmatrix}$$

corresponds to:

$$e^{:h:} = e^{:f_2:} = e^{:-\mu\frac{1}{2}(\gamma x^2 + 2\alpha x p + \beta p^2):}$$

In this form f is: $-\mu \cdot$ **(Courant-Snyder invariant)**

$$e^{:h:} = e^{:f_2:} = e^{:-\mu\epsilon:}$$

- **We have standard ($e^{:f_2:}$) for the linear one-turn-matrix (a rotation)...**

Normal forms non-linear case

Normal form transformations can be generalized for non-linear maps (i.e. not matrices). If $\mathcal{M}$ is our usual one-turn-map, we try to find a transformation:

$$\mathcal{N} = \mathcal{A}\mathcal{M}\mathcal{A}^{-1}$$

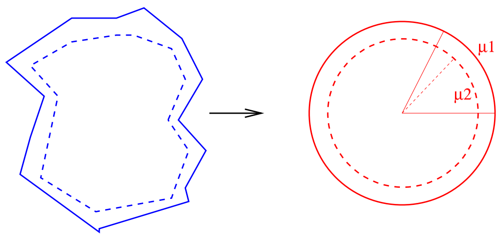
- **where $\mathcal{N}$ is a simple form (like the rotation we had before)**

Of course we now do not have matrices, but find a Lie transform F for:

$$\mathcal{N} = e^{-:h:} = e^{:F:}\mathcal{M}e^{-:F:}$$

Slide 129

Normal forms



- More complicated transformation required
- Transform to coordinates where map is just a rotation
- Rotation angle now amplitude dependent !

Slide 130

Normal forms - non-linear case

The transformation A:

$$A = e^{-:F:}$$

is the canonical transformation which produces our simple form

- Works for any kind of local perturbation
- Formalism and software tools exist to find F (see e.g. Chao[1] or E.Forest, M. Berz, J. Irwin, SSC-166)
- Formalism to find F is valid for any n-th order polynomial
- Once we know $h_{eff}(J_x, J_y, \delta)$ we can derive everything !

[1] A. Chao, Lecture Notes on Topics in Accelerator Physics, 2001

Normal forms - non-linear case

Once we can write the map as (now example in 3D):

$$\mathcal{N} = e^{:h_{eff}(J_x, J_y, \delta):}$$

where h_{eff} depends only on J_x, J_x, and δ, then we have the tunes:

$$Q_x(J_x, J_y, \delta) = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial J_x}$$

$$Q_y(J_x, J_y, \delta) = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial J_y}$$

and the change of path length:

$$\Delta s = -\frac{\partial h_{eff}}{\partial \delta} = \alpha_c \delta$$

➡ **Dependence on J is amplitude detuning, dependence on δ are the chromaticities !**

Example: sextupole (1D)

A linear map followed by a single (weak) sextupole:

$$\mathcal{M} = e^{-\frac{\mu}{2}:x^2+p^2:}\ e^{:f_3:} = e^{:-\mu J_x:}\ e^{:kx^3:}$$

using the BCH formula we get immediately for h and h_{eff}:

$$h = -\mu J + \left(\frac{:-\mu J_x:}{1 - e^{:\mu J_x:}}\right)\cdot kx^3$$

$$h = -\mu J - \frac{3}{8}\mu k(2\beta J)^{3/2}\cdot\left(\frac{sin(3\Psi + \frac{3\mu}{2})}{sin\frac{3\mu}{2}} - \frac{sin(\Psi + \frac{\mu}{2})}{sin\frac{\mu}{2}}\right)$$

and

$$h_{eff} = -h/C$$

Example: sextupole (3D)

When we have h_{eff} in 3D we can write (see e.g. Chao):

$$Q_x(J_x, J_y, \delta) = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial J_x} = \frac{1}{2\pi}(\mu_x - 3k\beta_x D\delta)$$

$$Q_y(J_x, J_y, \delta) = \frac{1}{2\pi}\frac{\partial h_{eff}}{\partial J_y} = \frac{1}{2\pi}(\mu_y + 3k\beta_y D\delta)$$

and the change of path length:

$$\Delta s = -\frac{\partial h_{eff}}{\partial \delta} = \alpha_c\delta - 3kD^3\delta^2 - 3kD(\beta_x J_x - \beta_y J_y)$$

Example: sextupole (1D)

Given the Hamiltonian h:

$$h = -\mu J - \frac{3}{8}\mu k(2\beta J)^{3/2}\cdot\left(\frac{sin(3\Psi + \frac{3\mu}{2})}{sin\frac{3\mu}{2}} - \frac{sin(\Psi + \frac{\mu}{2})}{sin\frac{\mu}{2}}\right)$$

particles move in phase space along constant h.

Back to Cartesian coordinates we get for h:

$$h = -\frac{\mu}{2}(x^2 + x'^2)\frac{3}{8}\mu\beta^{3/2}x[(3x'^2 - x^2)cot\frac{3\mu}{2} - (x^2 + x'^2)cot\frac{\mu}{2} - 4xx']$$

Constant h defines the trajectory in phase space !

What about close to resonance ?

If we have $Q = \frac{\mu}{2\pi} \approx \frac{m}{3}$ (3rd order resonance). Using a "distance to resonance d" as:

$$Q = \frac{m+d}{3} \qquad \text{where}: \ d \ll 1$$

The trick is to observe the motion every 3 turns:

$$\mathcal{M}^3 = (e^{:-\mu J:}e^{:kx^3:})^3 = e^{:3h:}$$

We get a factor:

$$e^{:-3\mu J:} \ = \ e^{:-2\pi dJ:} \quad (\text{because}: \ e^{:-2\pi mJ:} \equiv 1)$$

$$d = \frac{3\mu}{2\pi}$$

What about close to resonance ?

Without proof (but like before, see e.g. Chao), we get:

$$h = -\frac{2\pi}{3}dJ - \frac{\pi}{12}dk(2J)^{3/2} \cdot \left(\frac{sin(3\Psi + \frac{3\mu}{2})}{sin\frac{3\mu}{2}} - \frac{sin(\Psi + \frac{\mu}{2})}{sin\frac{\mu}{2}}\right)$$

For small d ($sin\frac{3\mu}{2} \approx -\pi d$) we can simplify:

$$h \approx -\frac{2\pi}{3}dJ - \frac{1}{\sqrt{2}}k(\beta J)^{3/2}sin(3\Psi)$$

Analysis give fixed points, i.e. (back in Cartesian again):

$$\frac{\partial h}{\partial x} = -\frac{2\pi}{3}dx - \frac{1}{4}\beta^{3/2}(3x'^2 - 3x^2) = 0$$

$$\frac{\partial h}{\partial x'} = -\frac{2\pi}{3}dx' - \frac{1}{4}\beta^{3/2}3xx' = 0$$

Normal forms - non-linear case

Assume a linear rotation (as always) followed by an octupole, the Hamiltonian is:

$$\mathcal{H} = \frac{\mu}{2}(x^2 + p^2) + \frac{x^4}{4} \qquad (p = p_x)$$

- **With the Hamilton's equation leading to:**

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \mu p$$

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -\mu x - x^3$$

Normal forms - non-linear case

The map, written in Lie representation is:

$$\mathcal{M} = e^{-\frac{\mu}{2}:x^2+p^2:} e^{:\frac{x^4}{4}:} = R e^{:\frac{x^4}{4}:}$$

we transform by applying:

$$\mathcal{N} = \mathcal{A}\mathcal{M}\mathcal{A}^{-1} = e^{:F:} \, R e^{:\frac{x^4}{4}:} \, e^{-:F:} = RR^{-1} \, e^{:F:} \, R e^{:\frac{x^4}{4}:} \, e^{-:F:}$$

$$= R e^{:R^{-1}F + \frac{x^4}{4} - F: + O(\epsilon^2)} = R e^{:(R^{-1}-1)F + \frac{x^4}{4}: + O(\epsilon^2)}$$

we have now to choose F to simplify the expression:

$$= (R^{-1} - 1)F + \frac{x^4}{4}$$

Normal forms - non-linear case

We go back to x and p coordinates and with:

$$J = (x^2 + p^2)/2$$

we can write the map:

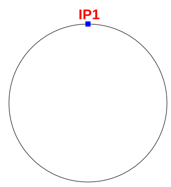
$$M = e^{-:F:} \quad e^{:-\mu J + \frac{3}{8}J^2:} \quad e^{:F:}$$

the term $\frac{3}{8}J^2$ is the tune shift with amplitude we know for an octupole

we have for F in x and p coordinates:

$$F = -\frac{1}{64}\{-5x^4+3p^4+6x^2p^2+x^3p\cdot(8cot(\mu)+4cot(2\mu))+xp^3(8cot(\mu)-4cot(2\mu))\}$$

A real life example: beam-beam interaction



- **Localized distortion, very strong non-linearity**
- **Standard perturbation theory not appropriate**

Effect on invariants - start with single IP

- **Look for invariant h**
- **Linear transfer $e^{:f_2:}$ and beam-beam interaction $e^{:F:}$, i.e.:**

$$e^{:f_2:} \cdot e^{:F:} = e^{:h:}$$

with (see before):

$$f_2 = -\frac{\mu}{2}(\frac{x^2}{\beta} + \beta p_x^2)$$

and (see before):

$$F = \int_0^x dx' f(x')$$

Effect on invariants

using for a Gaussian beam f(x) (see lecture on beam-beam effects):

$$f(x) = \frac{2}{x}(1 - e^{\frac{-x^2}{2\sigma^2}})$$

as usual go to action angle variables Ψ, J:

$$x = \sqrt{2J\beta}\sin\Psi, \qquad p = \sqrt{\frac{2J}{\beta}}\cos\Psi$$

and write F(x) as Fourier series:

$$F(x) = \sum_{n=-\infty}^{\infty} c_n(J) e^{in\Psi}$$

We need:

REMEMBER: with this transform:

$$f_2 = -\mu J$$

and useful properties of Lie operators (any textbook[2]):

$$: f_2 : g(J) = 0, \qquad : f_2 : e^{in\Psi} = in\mu e^{in\Psi}, \qquad g(: f_2 :)e^{in\Psi} = g(in\mu)e^{in\Psi}$$

and the formula (any textbook[2]):

$$e^{:f_2:}\, e^{:F:} \;=\; e^{:h:} \;=\; \exp\left[: f_2 + \left(\frac{: f_2 :}{1 - e^{-:f_2:}}\right) F + \mathcal{O}(F^2) :\right]$$

[2] E. Forest, "Beam Dynamics, A New Attitude and Framework", 1998

Single IP

gives immediately for h:

$$h = -\mu J + \sum_n c_n(J) \frac{in\mu}{1 - e^{-in\mu}} e^{in\Psi}$$

$$h = -\mu J + \sum_n c_n(J) \frac{n\mu}{2\sin(\frac{n\mu}{2})} e^{(in\Psi + i\frac{n\mu}{2})}$$

away from resonance normal form transformation gives:

$$h_n \;=\; -\mu J + c_0(J) \;= const.$$

$$\left[homework : \qquad \frac{dc_0(J)}{dJ}\right]$$

Single IP - analysis of h

$$h = -\mu J + \sum_n c_n(J) \frac{n\mu}{2\sin(\frac{n\mu}{2})} e^{(in\Psi + i\frac{n\mu}{2})}$$

On resonance:

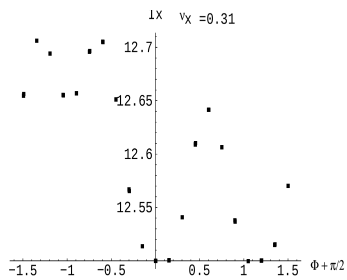
$$Q = \frac{p}{n} = \frac{\mu}{2\pi}$$

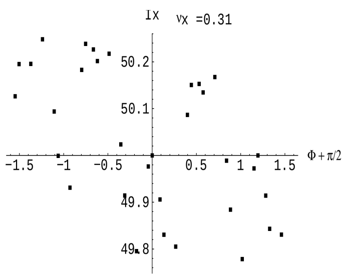
with $c_n \neq 0$:

$$\sin(\frac{n\pi p}{n}) = \sin(p\pi) \equiv 0 \quad \forall \quad integer \quad p$$

and h diverges, find automatically all resonance conditions

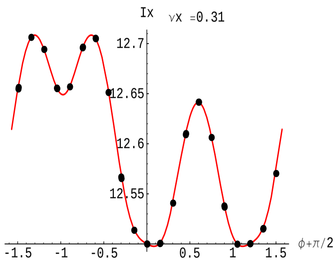
Invariant from tracking: one IP

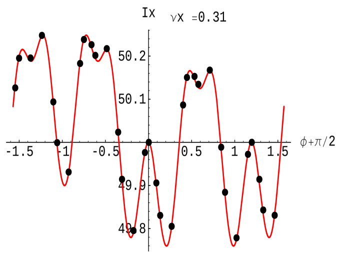




➡ **Shown for $5\sigma_x$ and $10\sigma_x$**

Invariant versus tracking: one IP

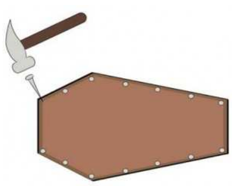




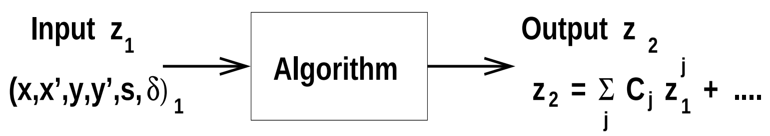
- Shown for $5\sigma_x$ and $10\sigma_x$

Differential Algebra (DA)
a.k.a.
Truncated Power Series Algebra (TPSA)

- Some people like analytic methods
- Not just tracking numbers around the machine !
- Now we put the final nail into the coffin of any other approach ...



Truncated Power Series Algebra

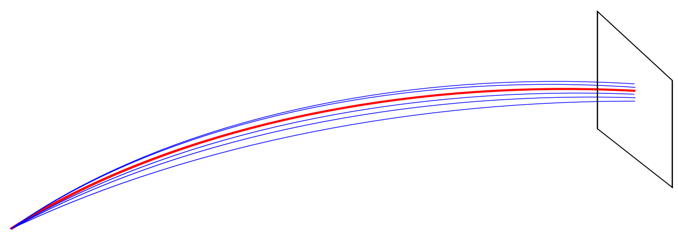


- The tracking of a complicated system relates the output numerically to the input
- Could we imagine something that relates the output algebraically to the input ?
- For example a Taylor series ?

$$z_2 = \sum C_j z_1^j = \sum f^{(n)} z_1^j$$

Why are Taylor series useful ?

Let us study the paraxial behaviour:



- Red line is the ideal orbit
- Blue lines are small deviations
- If we understand how small deviations behave, we understand the system much better

Why are Taylor series useful ?

Now remember the definition of the Taylor series:

$$f(x+\Delta x) = f(x) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x)}{n!} \Delta x^n$$

$$f(x+\Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x^1 + \frac{f''(x)}{2!}\Delta x^2 + \frac{f'''(x)}{3!}\Delta x^3 + ...$$

- **The coefficients determine the behaviour of small deviations Δx from the ideal orbit x**
- **The Taylor expansion does a paraxial analysis of the system**
- **How to get the coefficients without extra work ?**

Numerical differentiation

The problem getting the derivatives $f^{(n)}(a)$ of $f(x)$ at a:

$$f'(a) = \frac{f(a+\epsilon) - f(a)}{\epsilon}$$

- **Need to subtract almost equal numbers and divide by small number.**
- **For higher orders f'', f'''.., accuracy hopeless !**
- **We can use Differential Algebra (DA) (Berz, 1988)**

Differential Algebra

1. **Define a pair (q_0, q_1), with q_0, q_1 real numbers**

Differential Algebra

1. **Define a pair (q_0, q_1), with q_0, q_1 real numbers**
2. **Define operations on a pair like:**

$$(q_0, q_1) + (r_0, r_1) = (q_0 + r_0, q_1 + r_1)$$

$$c \cdot (q_0, q_1) = (c \cdot q_0, c \cdot q_1)$$

$$(q_0, q_1) \cdot (r_0, r_1) = (q_0 \cdot r_0,\ q_0 \cdot r_1 + q_1 \cdot r_0)$$

Differential Algebra

1. Define a pair (q_0, q_1), with q_0, q_1 real numbers

2. Define operations on a pair like:

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$$c \cdot (q_0, q_1) = (c \cdot q_0, c \cdot q_1)$$

$$(q_0, q_1) \cdot (r_0, r_1) = (q_0 \cdot r_0,\ q_0 \cdot r_1 + q_1 \cdot r_0)$$

3. And some ordering:

$$(q_0, q_1) < (r_0, r_1) \quad \text{if} \quad q_0 < r_0 \quad \text{or} \quad (q_0 = r_0 \quad \text{and} \quad q_1 < r_1)$$

$$(q_0, q_1) > (r_0, r_1) \quad \text{if} \quad q_0 > r_0 \quad \text{or} \quad (q_0 = r_0 \quad \text{and} \quad q_1 > r_1)$$

Differential Algebra

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$$(q_0, q_1) + (r_0, r_1) = (q_0 + r_0, q_1 + r_1)$$

$$c \cdot (q_0, q_1) = (c \cdot q_0, c \cdot q_1)$$

$$(q_0, q_1) \cdot (r_0, r_1) = (q_0 \cdot r_0,\ q_0 \cdot r_1 + q_1 \cdot r_0)$$

3. And some ordering:

$$(q_0, q_1) < (r_0, r_1) \quad \text{if} \quad q_0 < r_0 \quad \text{or} \quad (q_0 = r_0 \quad \text{and} \quad q_1 < r_1)$$

$$(q_0, q_1) > (r_0, r_1) \quad \text{if} \quad q_0 > r_0 \quad \text{or} \quad (q_0 = r_0 \quad \text{and} \quad q_1 > r_1)$$

4. This implies something strange:

$$(0,0) < (0,1) < (r,0) \text{ (for any pos. } r\text{)}$$

$$(0,1) \cdot (0,1) = (0,0) \rightarrow (0,1) = \sqrt{(0,0)}!!$$

Differential Algebra

This means that (0,1) is between 0 and ANY real number
→ infinitely small !!!
We call this therefore "differential unit" $d = (0,1) = \delta$.

Of course $(q,0)$ is just the real number q and we define "real part" and "differential part" (a bit like complex numbers..):

$$q_0 = \mathcal{R}(q_0, q_1) \quad \text{and} \quad q_1 = \mathcal{D}(q_0, q_1)$$

With our rules we can further see that:

$$(1,0) \cdot (q_0, q_1) = (q_0, q_1)$$

$$(q_0, q_1)^{-1} = \left(\frac{1}{q_0}, -\frac{q_1}{q_0^2}\right)$$

Differential Algebra

Of course can let a function f act on the pair (or vector) using our rules.

For example:

$$f(x) \to f(x,0)$$

acts like the function f on the real variable x:

$$f(x) = \mathcal{R}[f(x,0)]$$

What about the differential part $\mathcal{D}$?

Differential Algebra

For a function $f(x)$ without proof:

$$\mathcal{D}[f(x+d)] = \mathcal{D}[f((x,0)+(0,1))] = \mathcal{D}[f(x,1)] = f'(x)$$

An example instead:

$$f(x) = x^2 + \frac{1}{x}$$

then using school calculus:

$$f'(x) = 2x - \frac{1}{x^2}$$

For $x = 2$ we get then:

$$f(2) = \frac{9}{2}, f'(2) = \frac{15}{4}$$

Differential Algebra

For x in:

$$f(x) = x^2 + \frac{1}{x}$$

we substitute: x $\rightarrow$ (x, 1) = (2, 1) and use our rules:

$$\begin{aligned} f[(2,1)] &= (2,1)^2 + (2,1)^{-1} \\ &= (4,4) + (\frac{1}{2}, -\frac{1}{4}) \\ &= (\frac{9}{2}, \ \frac{15}{4}) = (f(2), \ f'(2)) \quad !!! \end{aligned}$$

The computation of derivatives becomes an algebraic problem, no need for small numbers, exact !

Differential Algebra - higher orders

1. **The pair $(q_0, 1)$, becomes a vector of order N:**

   $(q_0, 1) \rightarrow (q_0, 1, \mathbf{0, 0, ..., 0})$ $\qquad \delta = (\mathbf{0, 1, 0, 0, 0, ...})$

2. $(q_0, q_1, q_2, ...q_N) + (r_0, r_1, r_2, ...r_N) = (s_0, s_1, s_2, ...s_N)$

   with: $s_i = q_i + r_i$

3. $c \cdot (q_0, q_1, q_2, ...q_N) = (c \cdot q_0, c \cdot q_1, c \cdot q_2, ...c \cdot q_N)$

4. $(q_0, q_1, q_2, ...q_N) \cdot (r_0, r_1, r_2, ...r_N) = (s_0, s_1, s_2, ...s_N)$

   with:

$$s_i = \sum_{k=0}^{i} \frac{i!}{k!(i-k)!} q_k r_{i-k}$$

Differential Algebra

If we had started with:

$$x = (a, 1, 0, 0, 0...)$$

we would get:

$$f(x) = (\ f(a),\ f'(a),\ f''(a),\ f'''(a), ...\ f^{(n)}(a)\ )$$

can be extended to more variables x, y:

$$x = (a, 1, 0, 0, 0...) \qquad dx = (0, 1, 0, 0, 0, ..)$$

$$y = (b, 0, 1, 0, 0...) \qquad dy = (0, 0, 1, 0, 0, ..)$$

and get (with more complicated multiplication rules):

$$f((x + dx), y + dy)) = \left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, ... \right) (x, y)$$

What is the use of that:

Can extract a truncated Taylor map of a beam line or ring by pushing the identity map $f(x) = (a, 1, 0, 0, 0...)$ through the algorithm as if it is a vector in phase space !
The maps are provided with the desired accuracy and to any order.

What is the use of that:

- "Algorithm" can be a mathematical function
- "Algorithm" can be a complex computer code
- Easy using polymorphism of modern languages (see example)
- Normal form analysis on Taylor series is much easier !!

→ We get a Taylor map for a computer code !!!

What is the use of that:

- Demonstrate with simple examples (FORTRAN 95):
  - First show the concept
  - Simple FODO cell
  - Normal form analysis of the FODO cell with sextupoles
- All examples and all source code in:

  /afs/cern.ch/user/z/zwe/public/DA

  Small DA package provided by E. Forest

Look at this small example:

```
PROGRAM DATEST1
use my_own_da
real(8) x,z, dx
my_order=3
dx=0.0
x=3.141592653_8/6.0_8+dx
call track(x, z)
call print(z,6)
END PROGRAM DATEST1

SUBROUTINE TRACK(a, b)
use my_own_da
real(8) a,b
b = sin(a)
END SUBROUTINE TRACK
```

```
PROGRAM DATEST2
use my_own_da
type(my_taylor) x,z, dx
my_order=3
dx=1.0d0.mono.1    ! this is our (0,1)
x=3.141592653_8/6.0_8+dx
call track(x, z)
call print(z,6)
END PROGRAM DATEST2

SUBROUTINE TRACK(a, b)
use my_own_da
type(my_taylor) a,b
b = sin(a)
END SUBROUTINE TRACK
```

[2] Courtesy E. Forest for the small DA package used here ...

Look at the results:

(0,0) 0.5000000000000E+00

(0,0) 0.5000000000000E+00
(1,0) 0.8660254037844E+00
(0,1) 0.0000000000000E+00
(2,0) -0.2500000000000E+00
(0,2) 0.0000000000000E+00
(1,1) 0.0000000000000E+00
(3,0) -0.1443375672974E+00
(0,3) 0.0000000000000E+00
(2,1) 0.0000000000000E+00
(1,2) 0.0000000000000E+00

We have $sin(\frac{\pi}{6}) \; = \; 0.5$ **all right, but what is the rest ??**

Look at the results:

(0,0) 0.5000000000000E+00

(0,0) 0.5000000000000E+00
(1,0) 0.8660254037844E+00
(0,1) 0.0000000000000E+00
(2,0) -0.2500000000000E+00
(0,2) 0.0000000000000E+00
(1,1) 0.0000000000000E+00
(3,0) -0.1443375672974E+00
(0,3) 0.0000000000000E+00
(2,1) 0.0000000000000E+00
(1,2) 0.0000000000000E+00

$$sin(\frac{\pi}{6} + \Delta x) = sin(\frac{\pi}{6}) + cos(\frac{\pi}{6})\Delta x^1 - \frac{1}{2} sin(\frac{\pi}{6})\Delta x^2 - \frac{1}{6} cos(\frac{\pi}{6})\Delta x^3$$

What is the use of that:

- We have used a simple algorithm here (sin) but it can be anything very complex
- We can compute nonlinear maps as a Taylor expansion of anything the program computes
- Simply by:
  - Replacing regular (e.g. REAL) types by TPSA types (my_taylor) i.e. variables x, p are automatically replaced by $(x, 1, 0, ..)$ and $(p, 0, 1, 0, ..)$ etc.
  - Operators and functions $(+, -, *, =, ..., exp, sin,$ ...) automatically overloaded, i.e. behave according to new type

What is the use of that:

Assume the *Algorithm* describes one turn, then:

- Normal tracking:
  - $X_n = (x, p_x, y, p_y, s, \delta)_n \rightarrow X_{n+1} = (x, p_x, y, p_y, s, \delta)_{n+1}$
  - Coordinates after one completed turn
- TPSA tracking:
  - $X_n = (x, p_x, y, p_y, s, \delta)_n \rightarrow X_{n+1} = \sum C_j X_n^j$
  - Taylor expansion after one completed turn
  - Automatically all X_{n+1} where it converges
  - The C_j contain useful information about behaviour
  - Taylor map directly used for normal form analysis

Another example:

- Track through a FODO lattice:

  QF - DRIFT - QD

  Integrate 100 steps in the quadrupoles

  Now we use two variables: x, p = (z(1), z(2))

Another example:

```
program fodo1
use my_own_da
use my_analysis
type(my_taylor) z(2)
type(normalform) NORMAL
type(my_map) M,id

real(dp) L,DL,k1,fix(2),ma(2,2)
integer i,nstep


my_order=3
nstep=100
fix=0.0_dp ! fixed point
id=1
z=fix+id

L=1.d0
DL=L/nstep
k1=1.d0
do i=1,nstep
z(1)=z(1)+DL/2.d0*z(2)
z(2)=z(2)-k1*dl*z(1)
z(1)=z(1)+DL/2.d0*z(2)
enddo


z(1)=z(1)+L*z(2)

do i=1,nstep
z(1)=z(1)+DL/2.d0*z(2)
z(2)=z(2)+k1*dl*z(1)
z(1)=z(1)+DL/2.d0*z(2)
enddo


call print(z(1),6)
call print(z(2),6)
M=z
NORMAL=M
write(6,*) normal%tune, normal%dtune_da
end program fodo1
```

Result:

```
0.166310856124535, ( 0.0000000000, 0.0000000000     )
```

The result is:

```
(0,0) 0.00000000000000
(1,0) -1.45367086074474
(0,1) 2.76712693076176
(2,0) 0.00000000000000
(0,2) 0.00000000000000
(1,1) 0.00000000000000
(3,0) 0.00000000000000
(0,3) 0.00000000000000
(2,1) 0.00000000000000
(1,2) 0.00000000000000

(0,0) 0.00000000000000
(1,0) -1.65241972385291
(0,1) 2.45754057212448
(2,0) 0.00000000000000
(0,2) 0.00000000000000
(1,1) 0.00000000000000
(3,0) 0.00000000000000
(0,3) 0.00000000000000
(2,1) 0.00000000000000
(1,2) 0.00000000000000
```

Only linear elements in the Taylor expansion, the result is the matrix:

$$\Delta x_f = -1.45367 \Delta x_i + 2.76712 \Delta p_i$$

$$\Delta p_f = -1.65241 \Delta x_i + 2.45754 \Delta p_i$$

The output from the normal form analysis is the tune: 0.1663108 !

[2] Courtesy E. Forest for the small DA package used here ...

Modified previous example (with sextupole):

```
program fodo3
use my_own_da
use my_analysis
type(my_taylor) z(2)
type(normalform) NORMAL
type(my_map) M,id

real(dp) L,DL,k1,fix(2),ma(2,2)
integer i,nstep


my_order=3
nstep=100
fix=0.0_dp ! fixed point
id=1
z=fix+id

L=1.d0
DL=L/nstep
k1=1.d0

do i=1,nstep
z(1)=z(1)+DL/2.d0*z(2)
z(2)=z(2)-k1*dl*z(1)
z(1)=z(1)+DL/2.d0*z(2)
enddo

z(2)=z(2)-z(1)**2
z(1)=z(1)+L*z(2)

do i=1,nstep
z(1)=z(1)+DL/2.d0*z(2)
z(2)=z(2)+k1*dl*z(1)
z(1)=z(1)+DL/2.d0*z(2)
enddo
z(2)=z(2)-z(1)**2

call print(z(1),6)
call print(z(2),6)
M=z
NORMAL=M
write(6,*) normal%tune, normal%dtune_da
end program fodo3
```

Result:

```
0.166310856124535, ( 0.000000000, -12.5243894659508     )
```

[2] Courtesy E. Forest for the small DA package used here ...

The result is:

```
(0,0) 0.00000000000000
(1,0) -1.45367086074474
(0,1) 2.76712693076176
(2,0) -0.793529405369330
(0,2) -1.92470766801451
(1,1) -2.47168940711356
(3,0) 0.00000000000000
(0,3) 0.00000000000000
(2,1) 0.00000000000000
(1,2) 0.00000000000000

(0,0) 0.00000000000000
(1,0) -1.65241972385291
(0,1) 2.45754057212448
(2,0) -2.90667980020143
(0,2) -9.58167831652595
(1,1) 5.57332088161683
(3,0) -2.30706114745899
(0,3) 10.6518208440132
(2,1) -2.79445255996715
(1,2) 8.08317374091570
```

Now non-linear elements in the Taylor expansion,

The output from the normal form analysis is the tune: 0.166310 and the detuning with amplitude -12.524389 !

[2] Courtesy E. Forest for the small DA package used here ...

Modified previous example (with sextupole):

Remember the normal form transformation:

$$\mathcal{A}\mathcal{M}\mathcal{A}^{-1} = \mathcal{R}$$

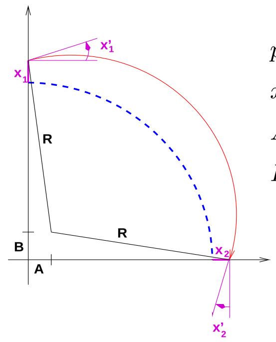
The type normalform in the demonstration package also contains the maps $\mathcal{A}$ and $\mathcal{R}$!

Question: what would be the result of something like that ?

```
r2=(x**2+p**2)*normal%at**(-1)
print sqrt(r2.sub.beta)
print -0.5*(r2.sub.twoalpha),r2.sub.gamma
```

[2] Courtesy E. Forest for the small DA package used here ...

This was trivial - now a (normally) hard one



The exact map:

$$p_2 = \sin(x_2') = -\tfrac{B}{R}$$
$$x_2 = A - R(1-\cos(x_2')) = A - R(1-\sqrt{1-p_2^2})$$
$$A = R \cdot p_1 = R \cdot \sin(x_1')$$
$$B = R(1-\cos(x_1')) + x_1 = R(1-\sqrt{1-p_1^2}) + x_1$$

A 90^0 bending magnet ..

How to apply Differential Algebra here ...

- **Start with initial coordinates in DA style:**

  $x_1 = (0,1,0,...)$

  $p_1 = (0,0,1,...)$ **and have:**

  $A = (0,0,R,0,...)$

  $B = (0,1,0,0,0,R,0,...)$

- **After pushing them through the algorithm:**

  → $x_2 = (0,0,R,-\frac{1}{R},0,0,0...) = (0, \frac{\partial x_2}{\partial x_1}, \frac{\partial x_2}{\partial p_1}, \frac{\partial^2 x_2}{\partial x_1^2}, \frac{\partial^2 x_2}{\partial x_1 \partial p_1}, ...)$

  → $p_2 = (0,-\frac{1}{R},0,0,0,-1,0...) = (0, \frac{\partial p_2}{\partial x_1}, \frac{\partial p_2}{\partial p_1}, \frac{\partial^2 p_2}{\partial x_1^2}, \frac{\partial^2 p_2}{\partial x_1 \partial p_1}, ...)$

- **Automatically evaluates all non-linearities to any desired order ..**

How to apply Differential Algebra here ...

- **Start with initial coordinates in DA style:**

$$x_1 = (0, 1, 0, ...)$$

$$p_1 = (0, 0, 1, ...) \quad \textbf{and have:}$$

$$A = (0, 0, R, 0, ...)$$

$$B = (0, 1, 0, 0, 0, R, 0, ...)$$

- **After pushing them through the algorithm:**

  → $x_2 = (0, 0, R, -\frac{1}{R}, 0, 0, 0...) = (0, \frac{\partial x_2}{\partial x_1}, \frac{\partial x_2}{\partial p_1}, \frac{\partial^2 x_2}{\partial x_1^2}, \frac{\partial^2 x_2}{\partial x_1 \partial p_1}, ...)$

  → $p_2 = (0, -\frac{1}{R}, 0, 0, 0, -1, 0...) = (0, \frac{\partial p_2}{\partial x_1}, \frac{\partial p_2}{\partial p_1}, \frac{\partial^2 p_2}{\partial x_1^2}, \frac{\partial^2 p_2}{\partial x_1 \partial p_1}, ...)$

- **Automatically evaluates all non-linearities to any desired order ..**

Some we know ...

Transfer matrix of a dipole:

$$M_{dipole} = \begin{pmatrix} cos(\frac{L}{R}) & Rsin(\frac{L}{R}) \\ -\frac{1}{R}sin(\frac{L}{R}) & cos(\frac{L}{R}) \end{pmatrix} = \begin{pmatrix} \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial p_1} \\ \frac{\partial p_2}{\partial x_1} & \frac{\partial p_2}{\partial p_1} \end{pmatrix}$$

For a 90^0 bending angle we get:

$$M_{dipole} = \begin{pmatrix} 0 & R \\ -\frac{1}{R} & 0 \end{pmatrix}$$

as computed, but we also have all derivatives and non-linear effects !

What is the use of that:

- Although not strictly an analytic method in the traditional sense:
  - TPSA provide analytic expression (Taylor series) for the one turn map
  - Can be used for tracking
  - Can be analysed for dynamic behaviour of the system
  - Typical use: Normal Form Analysis discussed earlier, rather straightforward from a Taylor expansion

Is there a summary ?

m = z

NORMAL = m

- Get the map m somehow (no matter how)
- Analyse this map (Normal form)

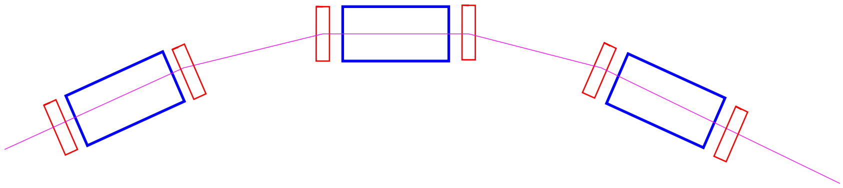
And another summary

- Perturbation treatment limited to:
  - Small perturbations (not in real machines)
  - Pedagogical purpose
- For realistic machines symplectic, iterative mapping is appropriate, using:
  - Symplectic integration
  - Lie transformations and normal form analysis
  - Differential algebra

Back up

- backup slides -

Where to put the elements in an accelerator ?



$$\frac{d^2x}{ds^2} + K(s)\,x = 0$$

- Usually use **s** (pathlength) along "reference path"
- "Reference path" defined geometrically by straight sections and bending magnets

Second order MAPS concatenation

Assume now 2 maps of second order:
$\mathcal{A}_2 = [R^A, T^A]$ and $\mathcal{B}_2 = [R^B, T^B]$

the combined second order map
$\mathcal{C}_2 = \mathcal{A}_2 \circ \mathcal{B}_2$ is $\mathcal{C}_2 = [R^C, T^C]$ with:

$$R^C = R^A \cdot R^B$$

and (after truncation of higher order terms !!):

$$T^C_{ijk} = \sum_{l=1}^{4} R^B_{il} T^A_{ljk} + \sum_{l=1}^{4}\sum_{m=1}^{4} T^B_{ilm} R^A_{lj} R^A_{mk}$$

Symplecticity for higher order MAPS

try truncated Taylor map in 2D, second order:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} R_{11}x_0 + R_{12}x'_0 + T_{111}x_0^2 + T_{112}x_0x'_0 + T_{122}x_0'^2 \\ R_{21}x_0 + R_{22}x'_0 + T_{211}x_0^2 + T_{212}x_0x'_0 + T_{222}x_0'^2 \end{pmatrix}$$

The Jacobian becomes:

$$\mathcal{J} = \begin{bmatrix} R_{11} + 2T_{111}x_0 + T_{112}x'_0 & R_{12} + T_{112}x_0 + 2T_{122}x'_0 \\ R_{21} + 2T_{211}x_0 + T_{212}x'_0 & R_{22} + T_{212}x_0 + 2T_{222}x'_0 \end{bmatrix}$$

symplecticity condition requires that:
det $\mathcal{J}$ = 1 for all x_0 and all x'_0

Symplecticity for higher order MAPS

This is only possible for the conditions:

$$\begin{pmatrix} R_{11}R_{22} - R_{12}R_{21} = 1 \\ R_{11}T_{212} + 2R_{22}T_{111} - 2R_{12}T_{211} - R_{21}T_{112} = 0 \\ 2R_{11}T_{222} + R_{22}T_{112} - R_{12}T_{212} - 2R_{21}T_{122} = 0 \end{pmatrix}$$

- **10 coefficients, but 3 conditions**
- **number of independent coefficients only 7 !**
- **Taylor map requires more coefficients than necessary**
- **e.g. 4D, order 4: coefficients 276 instead of 121**

Canonical transformations

- **With Hamiltonian's equations, still have to solve (2n) differential equations**
- **Not necessarily easy, but:**
  - **More freedom to choose the variables q and p (because they have now "equal" status)**
  - **Try to find variables where they are easy to solve**
- **Change of variables through "canonical transformations"**

Why canonical transformations ?

- **Hamiltonian have one advantage over Lagrangians:**
  - **If the system has a symmetry, i.e. a coordinate q_i does not occur in H (i.e. $\frac{\partial H}{\partial q_i} = 0 \quad \rightarrow \quad \frac{dp_i}{dt} = 0$) → the corresponding momentum p_i is conserved (and the coordinate q_i can be ignored in the other equations of the set).**
  - **Comes also from Lagrangian, but the velocities still occur in $\mathcal{L}$!**

Canonical transformations

Starting with $H(q,p,t)$ get new coordinates:

$$Q_i = Q_i(q,p,t)$$

$$P_i = P_i(q,p,t)$$

and new Hamiltonian $K(Q,P,t)$ with:

$$\frac{\partial K}{\partial Q_j} = -\dot{P}_j = -\frac{dP_j}{dt}, \qquad \frac{\partial K}{\partial P_j} = \dot{Q}_j = \frac{dQ_j}{dt}$$

- **We can two types of canonical transformations**

Canonical transformations - type 1

- **Ideally one would like a Hamiltonian H and coordinates with:**

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j = -\frac{dp_j}{dt} = 0$$

- **Coordinate q_j not explicit in H**
- **p_j is a constant of the motion (!) and:**

$$\frac{dq_j}{dt} = \frac{\partial H(p_1,p_2,..p_n)}{\partial p_j} = F_j(p_1,p_2,..p_n)$$

which can be directly integrated to get $q_j(t)$

Canonical transformations - type 1, example

Harmonic oscillator:

$$H = T + V \quad = \quad \frac{1}{2}mv^2 + \frac{m\omega^2}{2}x^2 \quad = \quad \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2$$

try: $x = \sqrt{\frac{2P}{m\omega}} \cdot sin(X)$ **and** $p = \sqrt{2m\omega P} \cdot cos(X)$ **and we get:**

$$K = \omega P cos^2(X) + \omega P sin^2(X) = \omega P$$

then:

$$\frac{dX}{dt} = \frac{\partial K}{\partial P} = \omega \quad \rightarrow \quad X = \omega t + \alpha$$

back transformation to x,p:

$$x = \sqrt{\frac{2E}{m\omega^2}} sin(\omega t + \alpha)$$

Canonical transformations - type 2

- **Find a transformation of q, p at time t to values q_0, p_0 at time $t = 0$.**

$$q = q(q_0, p_0, t)$$

$$p = p(q_0, p_0, t)$$

- **The transformations ARE the solution of the problem !**

For both types: how to find the transformation ?

- **Without details: Hamilton-Jacobi equation ...**

Extension: general monomials

Monomials in x and p of orders n and m $(x^n p^m)$

$$e^{:ax^n p^m:}$$

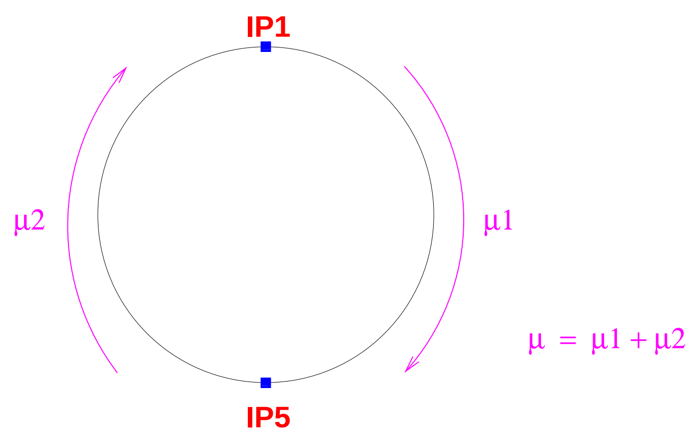
gives for the map (for $n \neq m$):

$$e^{:ax^n p^m:}x = x \cdot [1 + a(n-m)x^{n-1}p^{m-1}]^{m/(m-n)}$$
$$e^{:ax^n p^m:}p = p \cdot [1 + a(n-m)x^{n-1}p^{m-1}]^{n/(n-m)}$$

gives for the map (for $n = m$):

$$e^{:ax^n p^n:}x = x \cdot e^{-anx^{n-1}p^{n-1}}$$
$$e^{:ax^n p^n:}p = p \cdot e^{anx^{n-1}p^{n-1}}$$

Collision scheme - two IPs



Two IPs

➡ **two transfers f_2^1, f_2^2 and two beam-beam kicks F^1, F^2, first IP at μ_1, second IP at μ:**

$$
\begin{aligned}
&= e^{:f_2^1:}\, e^{:F^1:}\, e^{:f_2^2:}\, e^{:F^2:} \;=\; e^{:h_2:} \\
&= e^{:f_2^1:}\, e^{:F^1:}\, e^{-:f_2^1:}\, e^{:f_2^1:}\, e^{:f_2^2:}\, e^{:F^2:} \;=\; e^{:h_2:} \\
&= e^{:f_2^1:}\, e^{:F^1:}\, e^{-:f_2^1:}\, e^{:f_2:}\, e^{:F^2:}\, e^{-:f_2:}\, e^{:f_2:} = e^{:h_2:} \\
&= e^{:e^{-:f_2^1:}F^1:}\, e^{:e^{-:f_2:}F^2:}\, e^{:f_2:} = e^{:h_2:}
\end{aligned}
$$

$$f_2 = -\mu A, \quad f_2^1 = -\mu_1 A, \quad and \quad f_2^2 = -\mu_2 A$$

Two IPs

here a miracle occurs (remember $g(:f_2:)e^{in\Psi} = g(in\mu)e^{in\Psi}$):

$$e^{:f_2^1:}e^{in\Psi} = e^{in\mu_1}e^{in\Psi} = e^{in(\mu_1+\Psi)}$$

i.e. the Lie transforms of the perturbations are phase shifted[2]. Therefore:

$$e^{:e^{-:f_2^1:}F^1:}\, e^{:e^{-:f_2:}F^2:}\, e^{:f_2:} = e^{:h_2:}$$

becomes simpler with substitutions of $\Psi_1 = \Psi + \mu_1$ and $\Psi = \Psi + \mu$ in F^1 and F:

$$e^{:F^1(\Psi_1):}e^{:F(\Psi):}e^{:f_2:} \quad \Rightarrow \quad e^{:F^1(\Psi_1)+F(\Psi):}e^{:f_2:}$$

[2] E. Forest, "Beam Dynamics, A New Attitude and Framework", 1998

Two IPs

gives for h_2:

$$h_2 = -\mu A + \sum_{n=-\infty}^{\infty} \frac{n\mu c_n(A)}{2\sin(n\frac{\mu}{2})} e^{-in(\Psi+\mu/2+\mu_1)} + e^{-in(\Psi+\mu/2)}$$

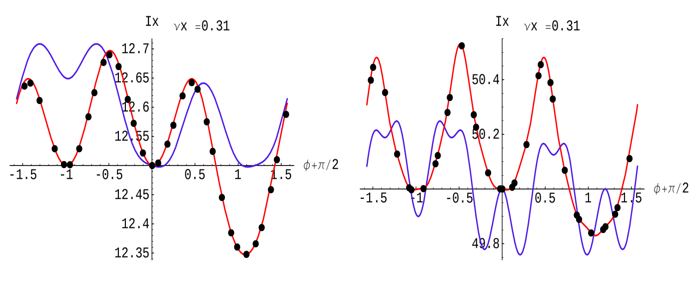
$$h_2 = -\mu A + 2c_0(A) + \underbrace{\sum_{n=1}^{\infty} \frac{2n\mu c_n(A)}{2\sin(n\frac{\mu}{2})} \cos(n(\Psi + \frac{\mu}{2} + \frac{\mu_1}{2}))\cos(n\frac{\mu_1}{2})}_{\text{interesting part}}$$

Nota bene, because of:

$$e^{:F(\Psi):}e^{:f_2:} \rightarrow e^{:F^1(\Psi_1)+F(\Psi):}e^{:f_2:}$$

can be generalized to more interaction points ...

Invariant versus tracking: two IPs



→ **Shown for $5\sigma_x$ and $10\sigma_x$**

Recap: Hamiltonian for a finite length element

We have from the Hamiltonian equations for the motion through an element with the Hamiltonian H for the element of length L:

$$\frac{dq}{dt} = [q, H] =: -H : q \quad \text{(from lecture 5)}$$

$$\rightarrow \frac{d^k q}{dt^k} = (: -H :)^k q$$

$$\rightarrow q(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\frac{d^k q}{dt^k} \right) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (- : H :)^k = e^{:-tH:}$$

with independent variable s instead of t (nota bene: $s_0 = 0, t_0 = 0$):

$$\rightarrow q(s) = e^{:-LH:}$$

Lie transformations on moments:

We have used Lie transformations mainly to propagate coordinates and momenta, i.e. like:

$$e^{:f:} x_0 = x_1$$

$$e^{:f:} p_0 = p_1$$

or using $Z = (x, p_x, y, p_y, ...)$:

$$e^{:f:} Z_0 = Z_1$$

- **Remember: can be applied to any function of x and p !!**
- **In particular to moments like $x^2, xp, p^2, ...$**

Lie transformations on moments

Assume a matrix M of the type:

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

described by a generator f, we have for the Lie transformation on the moment:

$$e^{:f:}x^2 = (e^{:f:}x)^2 \qquad \text{(see lecture 5)}$$

therefore:

$$(e^{:f:}x)^2 = (m_{11}x + m_{12}p)^2$$

$$(e^{:f:}x)^2 = m_{11}^2 x^2 \ + \ 2\ m_{11}m_{12}xp \ + \ m_{12}^2 p^2$$

More on moments

To summarize the moments:

$$\begin{pmatrix} x^2 \\ xp \\ p^2 \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11}^2 & 2m_{11}m_{12} & m_{12}^2 \\ m_{11}m_{21} & m_{11}m_{22} + m_{12}m_{21} & m_{12}m_{22} \\ m_{21}^2 & 2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \circ \begin{pmatrix} x^2 \\ xp \\ p^2 \end{pmatrix}_{s_1}$$

This is the well known transfer matrix for optical parameters

A real life example: beam-beam interaction*)

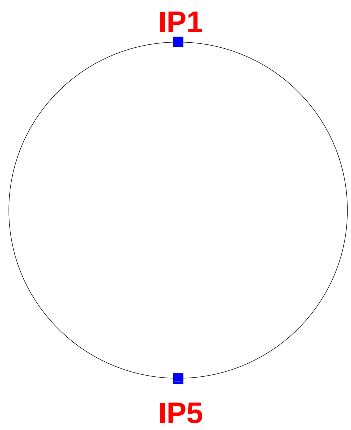
- Beam-beam interaction very non-linear
- Important to understand stability
- Non-linear effects such as amplitude detuning very important

Our questions ?

- How does the particles behave in phase space ?
- Do we have an invariant ?
- Can we calculate the invariant ?

*) From: W. Herr, D. Kaltchev, LHC Project Report 1082, (2008).

Collision scheme - two IPs



Start with single IP

"Classic" (B.C.) approach:

- Interaction point at beginning (end) of the ring (very local interactions, δ-functions)
- s-dependent Hamiltonian and perturbation theory:

$$\mathcal{H} \;=\; .... + \delta(s)\epsilon V$$

- Disadvantages:
  - for several IPs endless mathematics
  - conceptually and computationally easier method

Effect on invariants - start with single IP

Look for invariants h, (see e.g. Dragt[1]), and evaluate for different number of interactions and phase advance.
Very well suited for local distortions (e.g. beam-beam kick)
Linear transfer $e^{:f_2:}$ and beam-beam interaction $e^{:F:}$, i.e.:

$$e^{:f_2:} \;\cdot\; e^{:F:} \;=\; e^{:h:}$$

with

$$f_2 \;=\; -\frac{\mu}{2}(\frac{x^2}{\beta} + \beta p_x^2)$$

and

$$F = \int_0^x dx' f(x')$$

[1] A. Dragt, AIP Conference proceedings, Number 57 (1979)

Effect on invariants

using for a Gaussian beam f(x):

$$f(x) = \frac{2}{x}(1 - e^{\frac{-x^2}{2\sigma^2}})$$

as usual go to action angle variables Ψ, A:

$$x = \sqrt{2A\beta}\sin\Psi, \qquad p = \sqrt{\frac{2A}{\beta}}\cos\Psi$$

and write F(x) as Fourier series:

$$F(x) = \sum_{n=-\infty}^{\infty} c_n(A)e^{in\Psi} \qquad \text{with :} \qquad c_n(A) \; = \; \frac{1}{2\pi}\int_0^{2\pi} e^{-in\Psi}F(x)d\Psi$$

Slide 209

We need:

REMEMBER: with this transform:

$$f_2 = -\mu A$$

and useful properties of Lie operators (any textbook[2]):

$$: f_2 : g(A) = 0, \qquad : f_2 : e^{in\Psi} = in\mu e^{in\Psi}, \qquad g(: f_2 :)e^{in\Psi} = g(in\mu)e^{in\Psi}$$

and the formula (because the beam-beam perturbation is small !):

$$e^{:f_2:}\, e^{:F:} \; = \; e^{:h:} \; = \; \exp\left[: f_2 + \left(\frac{: f_2 :}{1 - e^{-:f_2:}}\right) F + \mathcal{O}(F^2) :\right]$$

[2] E. Forest, "Beam Dynamics, A New Attitude and Framework", 1998

Slide 210

Single IP

gives immediately for h:

$$h = -\mu A + \sum_n c_n(A) \frac{in\mu}{1 - e^{-in\mu}} e^{in\Psi}$$

$$h = -\mu A + \sum_n c_n(A) \frac{n\mu}{2\sin(\frac{n\mu}{2})} e^{(in\Psi + i\frac{n\mu}{2})}$$

away from resonance, a normal form transformation takes away the pure oscillatory part and we have only:

$$h \;=\; -\mu A + c_0(A) \;= const.$$

$$\left[homework : \qquad \frac{dc_0(A)}{dA} \right]$$

Single IP

If you are too lazy or too busy:

$$\Delta Q = \frac{-1}{2\pi} \frac{dc_0(A)}{dA}$$

is the detuning with amplitude, i.e. the amplitude dependent frequency change of the transformation we had before ...

We get:

$$\Delta Q = \frac{-1}{2\pi} \frac{N r_0}{\gamma A} [1 - e^{-A\beta/2\sigma^2} I_0(A\beta/2\sigma^2)]$$

Single IP - analysis of h

$$h = -\mu A + \sum_n c_n(A) \frac{n\mu}{2\sin(\frac{n\mu}{2})} e^{(in\Psi + i\frac{n\mu}{2})}$$

On resonance:

$$Q = \frac{p}{n} = \frac{\mu}{2\pi}$$

with $c_n \neq 0$:

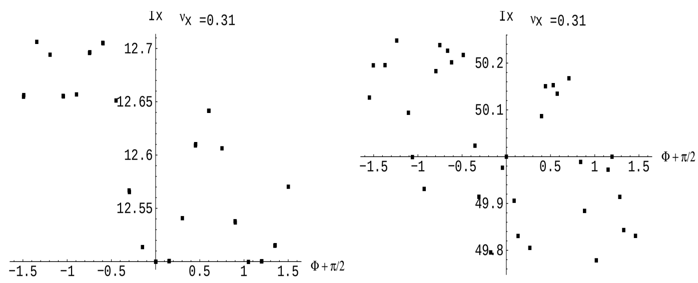
$$\sin(\frac{n\pi p}{n}) = \sin(p\pi) \equiv 0 \quad \forall \quad integer \quad p$$

and h diverges

Invariant versus tracking

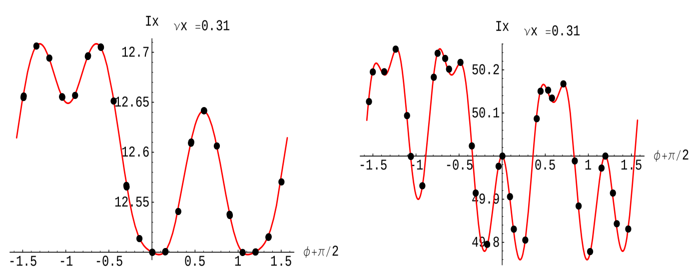
- **Is it useful what we obtained ?**
  - **Debug and compare ("benchmark")**
- **Compare to very simple tracking program:**
  - **linear transfer between interactions**
  - **beam-beam kick for round beam**
  - **compute action $I = \frac{\beta^*}{2\sigma^2}(\frac{x^2}{\beta^*} + p_x^2\beta^*)$**
  - **and phase $\Psi = \arctan(\frac{p_x}{x})$**
  - **compare I with h**

Invariant from tracking: one IP



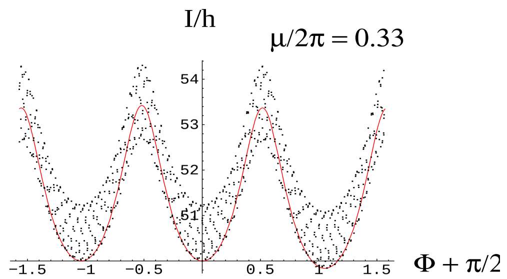
➡ **Shown for $5\sigma_x$ and $10\sigma_x$**

Invariant versus tracking: one IP



➡ **Shown for $5\sigma_x$ and $10\sigma_x$**

Invariant versus tracking:



- Behaviour near a resonances: no more invariant possible
- Envelope of tracking well described