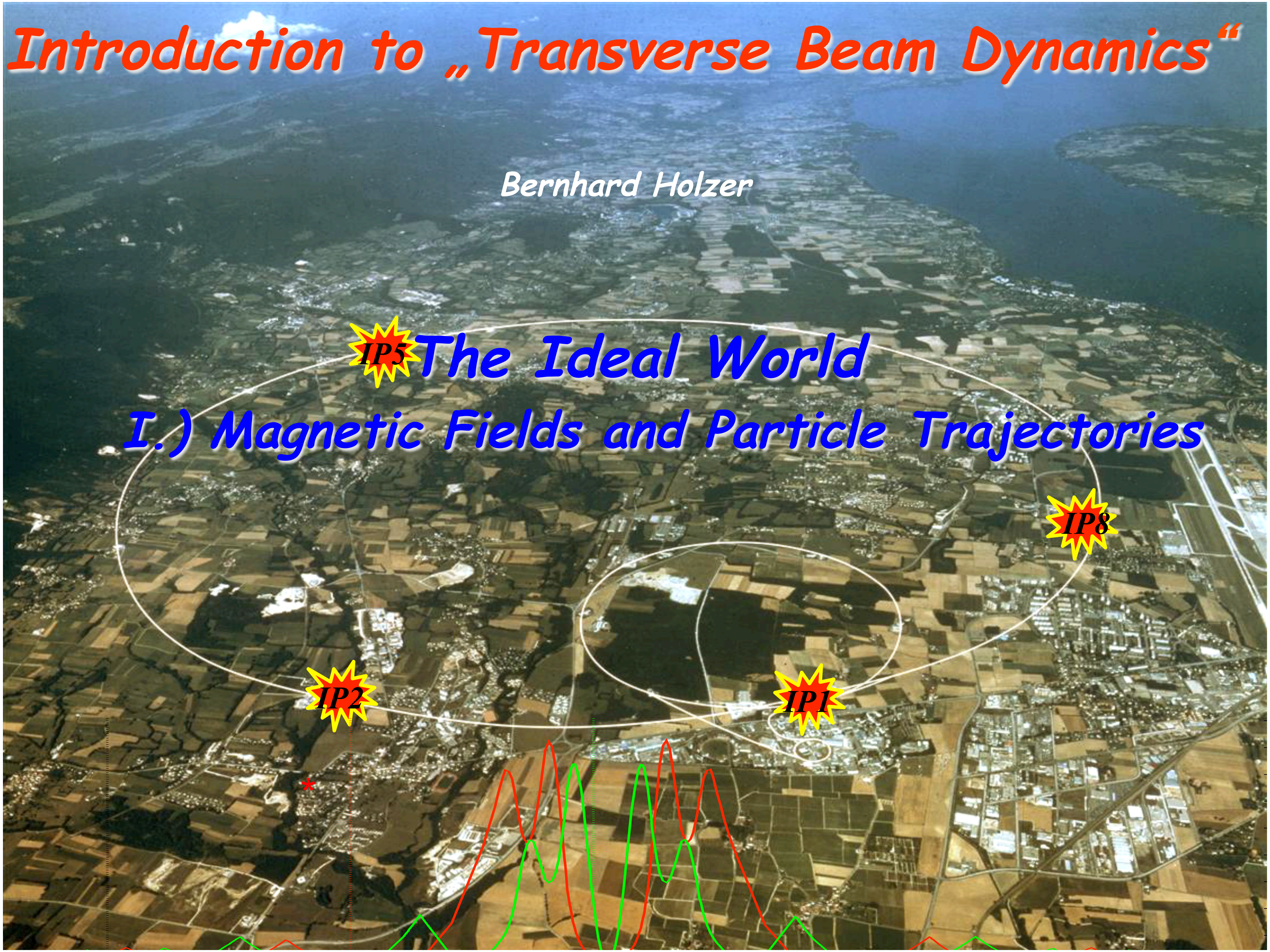
Introduction to „Transverse Beam Dynamics“

Bernhard Holzer

The Ideal World

I.) Magnetic Fields and Particle Trajectories



Focusing forces and particle trajectories:

normalise magnet fields to momentum
(remember: ***B*ρ = p / q*** *)*

Dipole Magnet

$$\frac{B}{p/q} = \frac{B}{B\rho} = \frac{1}{\rho}$$

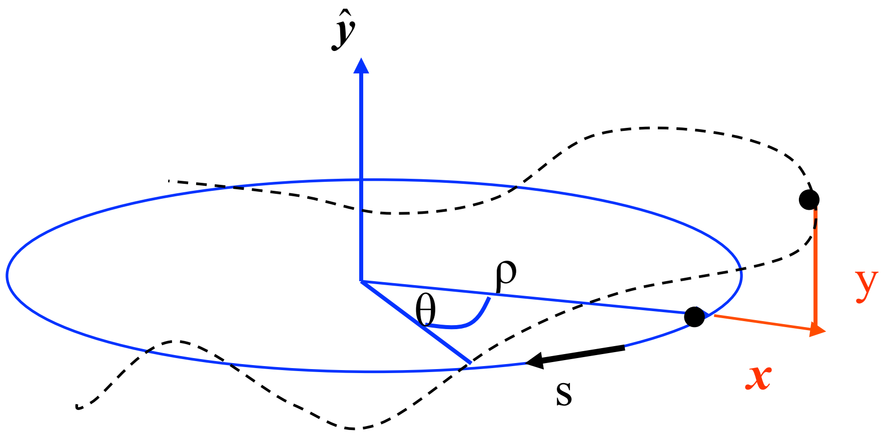
Quadrupole Magnet

$$k := \frac{g}{p/q}$$

The Equation of Motion:

* *Equation for the horizontal motion:*

$$x'' + x\left(\frac{1}{\rho^2} + k\right) = 0$$



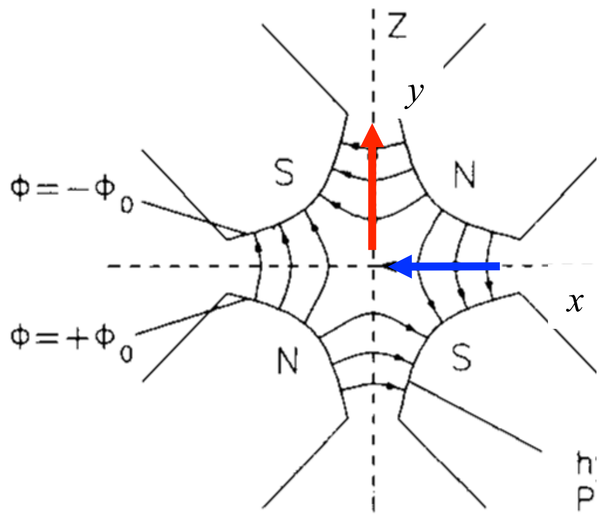
x = particle amplitude
x' = angle of particle trajectory (wrt ideal path line)

* *Equation for the vertical motion:*

$$\frac{1}{\rho^2} = 0$$ *no dipoles … in general …*

$k \leftrightarrow -k$ *quadrupole field changes sign*

$$y'' - k\,y = 0$$



5.) Solution of Trajectory Equations

Define … hor. plane: K= 1/ρ² + k

… vert. Plane: K = - k

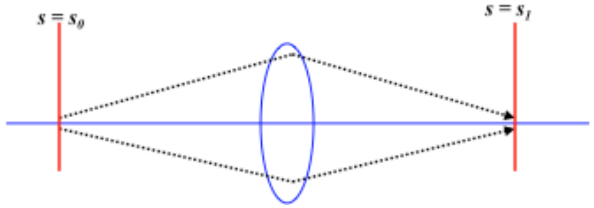
$$x'' + K\,x = 0$$

Differential Equation of harmonic oscillator … with spring constant K

Ansatz: *Hor. Focusing Quadrupole K > 0:*

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x_0' \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x_0' \cdot \cos(\sqrt{|K|}s)$$



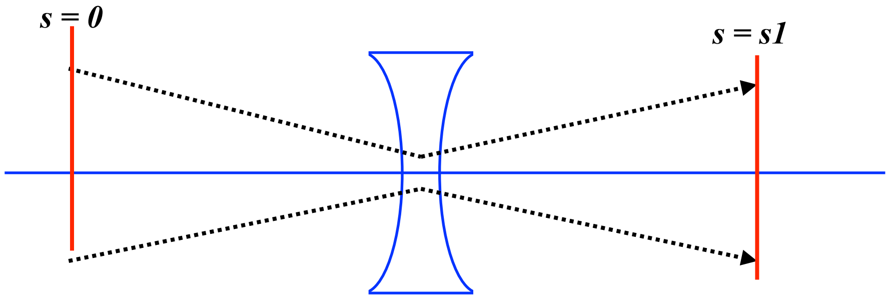
For convenience expressed in matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

$$M_{foc} = \begin{pmatrix} \cos\left(\sqrt{|K|}l\right) & \frac{1}{\sqrt{|K|}} \sin\left(\sqrt{|K|}l\right) \\ -\sqrt{|K|} \sin\left(\sqrt{|K|}l\right) & \cos\left(\sqrt{|K|}l\right) \end{pmatrix}$$

hor. defocusing quadrupole:

$$x'' - K\,x = 0$$



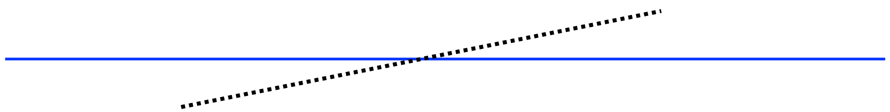
Ansatz: ***Remember from school***

$$x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s)$$

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}l \\ \sqrt{|K|}\sinh\sqrt{|K|}l & \cosh\sqrt{|K|}l \end{pmatrix}$$

drift space:

$$K = 0$$



$$x(s) = x_0' * s$$

$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

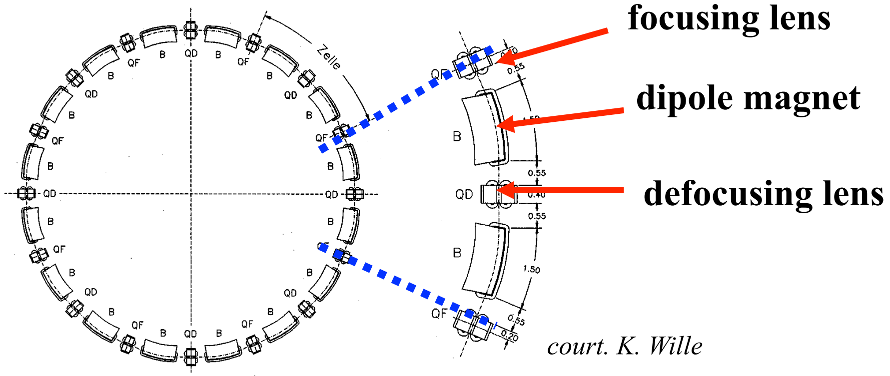
! ***with the assumptions made, the motion in the horizontal and vertical planes are independent „ ... the particle motion in x & y is uncoupled“***

Transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

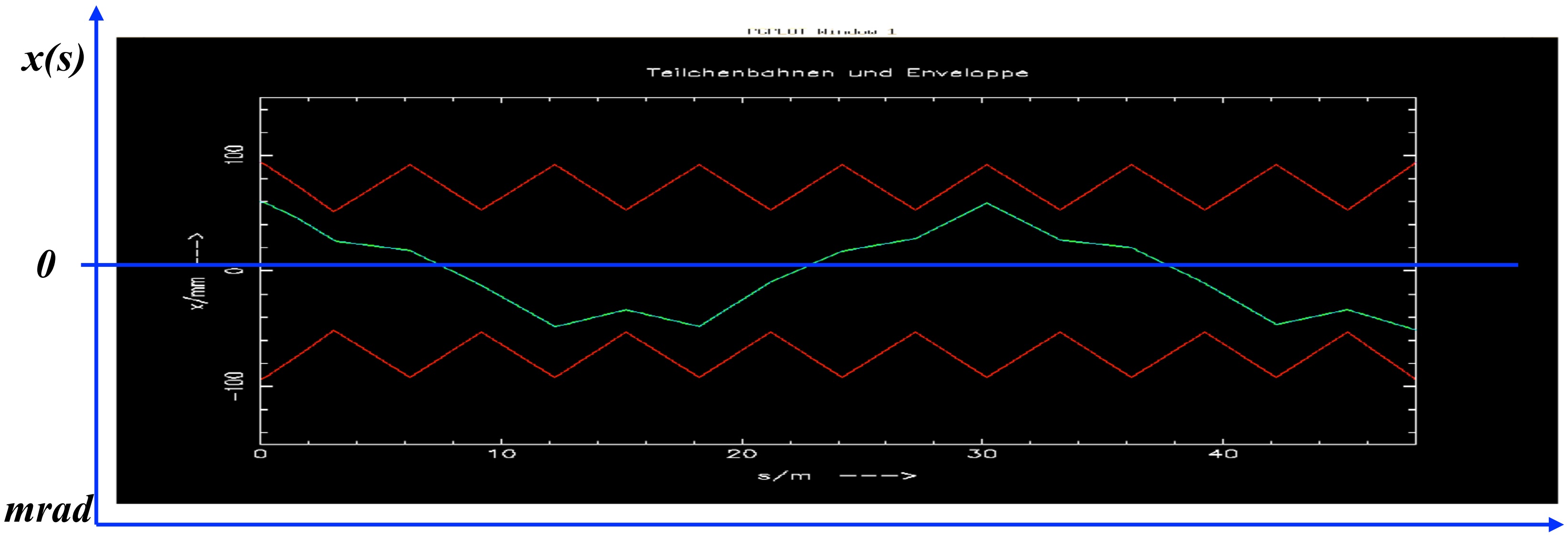
$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D*.....}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



court. K. Wille

in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator ! ! !



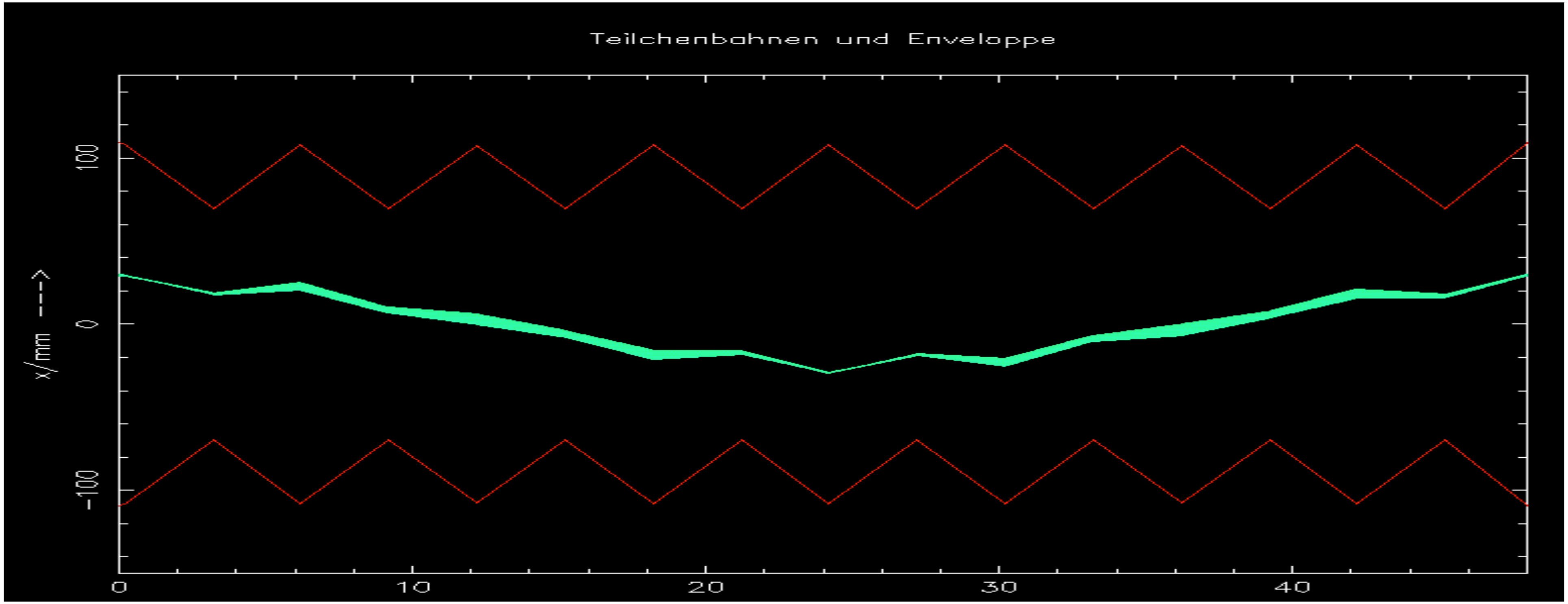
typical values in a strong foc. machine:

$x \approx mm$, $x' \leq mrad$

Resonance Problem:

Why do we have so stupid non-integer tunes ?
"Q = 64.0" sounds much better

Qualitatively spoken: ***Integer tunes lead to a resonant increase of the closed orbit amplitude in presence of the smallest dipole field error.***



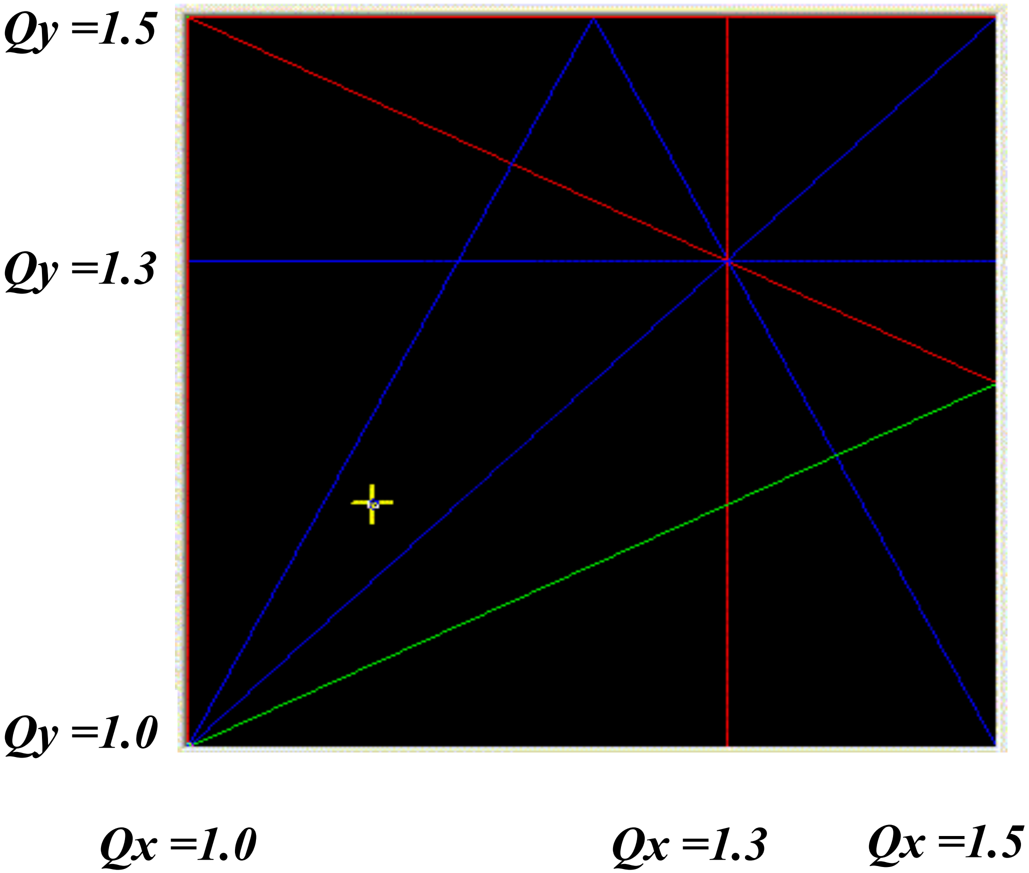
Orbit in case of a small dipole error:

$$x_{co}(s) = \frac{\sqrt{\beta(s)} * \int \frac{1}{\rho_{s1}} \sqrt{\beta_{s1}} * \cos(\psi_{s1} - \psi_s - \pi Q)\, ds}{\underbrace{2\sin \pi Q}}$$

Assume: Tune = integer $Q = 1 \quad \rightarrow \quad 0$

Tune and Resonances

To avoid resonance conditions the frequency of the transverse motion must not be equal to (or a integer multiple of) the revolution frequency



$1*Q_x = 1 \quad -> \quad Q_x = 1$

$2*Q_x = 1 \quad -> \quad Q_x = 0.5$

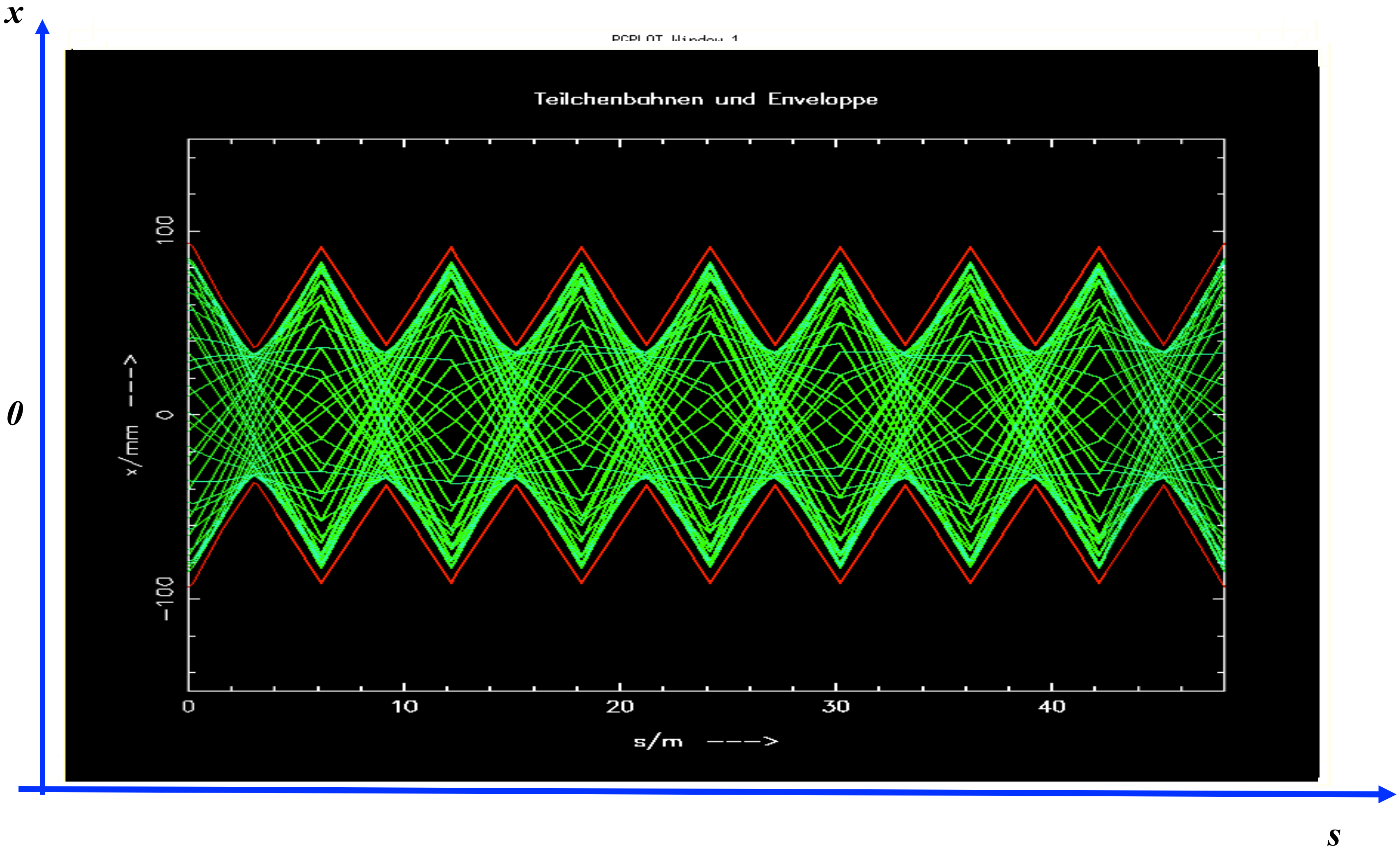
in general:

$$m*Q_x+n*Q_y+l*Q_s = integer$$

Tune diagram up to 3rd order

Question: ***what will happen, if the particle performs a second turn ?***

... or a third one or ... 10^{10} turns



Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill 's equation“

Example: particle motion with periodic coefficient

equation of motion: $x''(s) - k(s)x(s) = 0$

restoring force ≠ const,
k(s) = depending on the position s
k(s+L) = k(s), periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.

7.) The Beta Function

„it is convenient to see" ... *after some beer* ... *we make two statements:*

1.) There exists a mathematical function, that defines the envelope of all particle trajectorie
and so can act as measure for the beam size. We call it the β – function.

2.) Whow !!

A particle oscillation can then be written in the form

$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \phi)$$

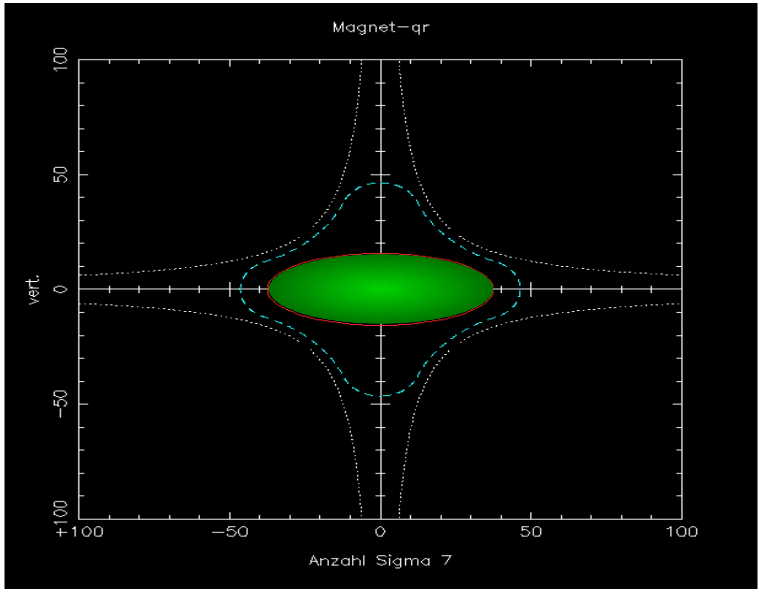
ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s+L) = \beta(s)$$

ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter,
cannot be changed by the foc. properties.
scientifiquely spoken: area covered in transverse x, x´ phase space
… and it is constant !!!

The Beta Function

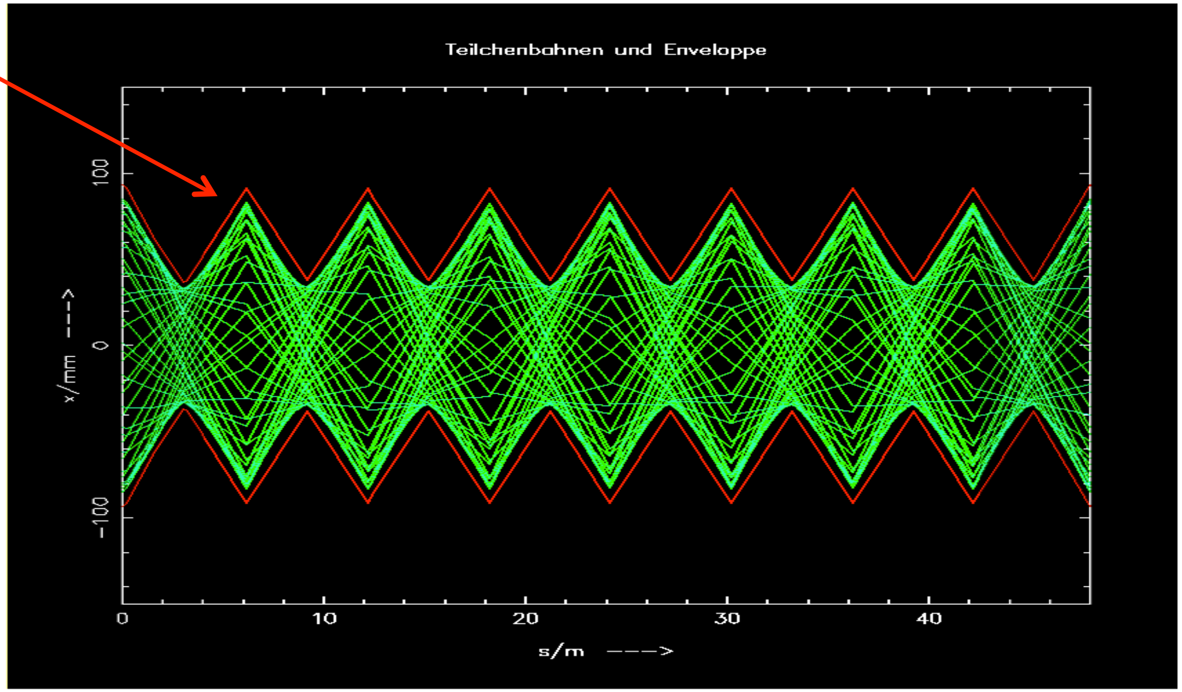


If we obtain the x, x' coordinates of a particle trajectory via

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$

The maximum size of any particle amplitude at a position "s" is given by

$$\hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



β determines the beam size
(... the envelope of all particle trajectories at a given position "s" in the storage ring.

It reflects the periodicity of the magnet structure.

8.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

$$(1)\qquad x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\phi)$$

$$(2)\qquad x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left\{\alpha(s)\cos(\psi(s)+\phi)+\sin(\psi(s)+\phi)\right\}$$

from (1) we get

$$\cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\varepsilon}\,\sqrt{\beta(s)}}$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

Insert into (2) and solve for ε

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

***** *ε* **is a** ***constant*** ***of the motion … it is independent of „s"***
***** ***parametric representation of an ellipse in the x x ' space***
***** ***shape and orientation of ellipse are given by α, β, γ***

Phase Space Ellipse

particel trajectory: $x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$

max. Amplitude: $\boxed{\hat{x}(s) = \sqrt{\varepsilon\beta}}$ $\longrightarrow$ *x´ at that position …?*

… put $\hat{x}(s)$ *into* $\boldsymbol{\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)}$ *and solve for x´*

$$\varepsilon = \gamma \cdot \varepsilon\beta + 2\alpha\sqrt{\varepsilon\beta}\cdot x' + \beta x'^2$$

$$\longrightarrow \boxed{x' = -\alpha \cdot \sqrt{\varepsilon / \beta}}$$

* *A high β-function means a large beam size and a small beam divergence.* *!*
… et vice versa !!!

* *In the middle of a quadrupole β = maximum,*
α = zero

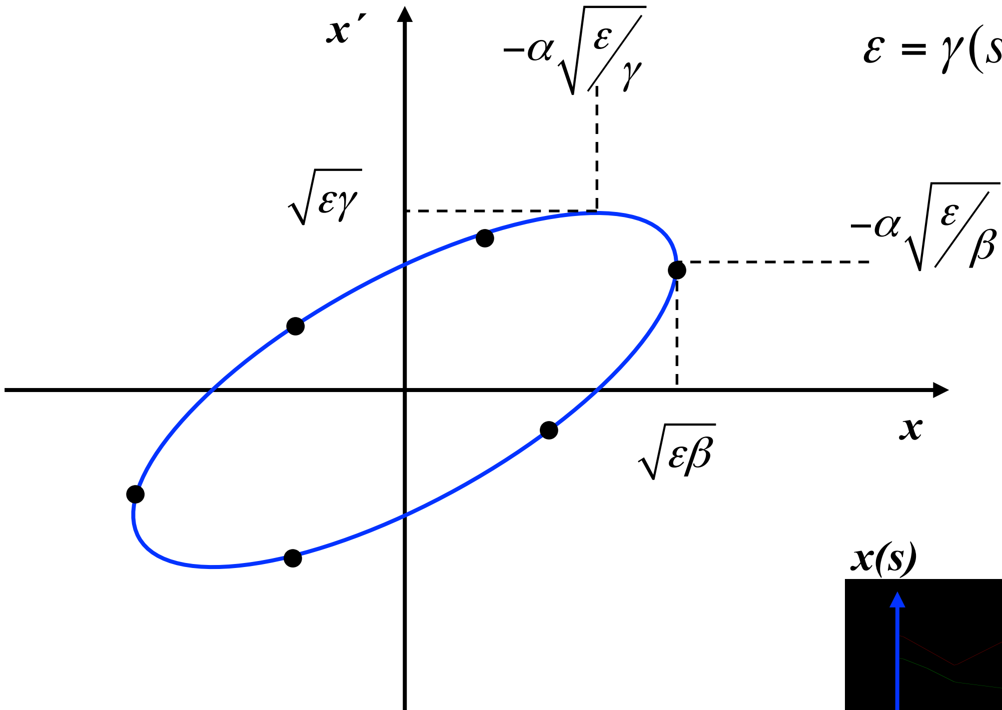
$\left.\begin{matrix}\beta = \text{maximum} \\ \alpha = \text{zero}\end{matrix}\right\} x' = 0$

… and the ellipse is flat

Beam Emittance and Phase Space Ellipse

***In phase space x, x' a particle** oscillation, observed at a given position "s" in the ring*
***is running on an ellipse** ... making Q revolutions per turn.*

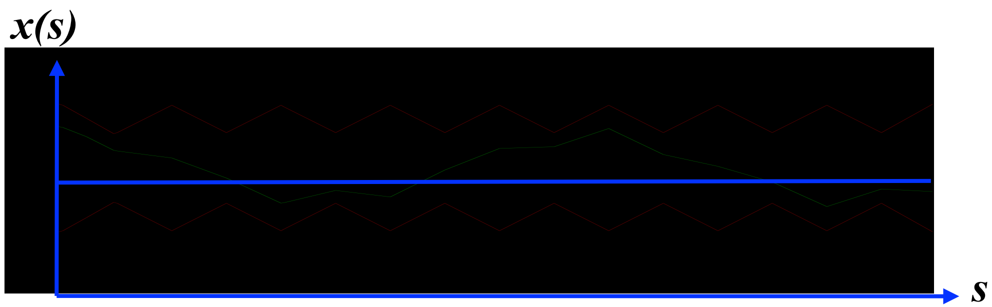
$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \varphi)$$



$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

***Liouville:** in reasonable storage rings*
area in phase space is constant.

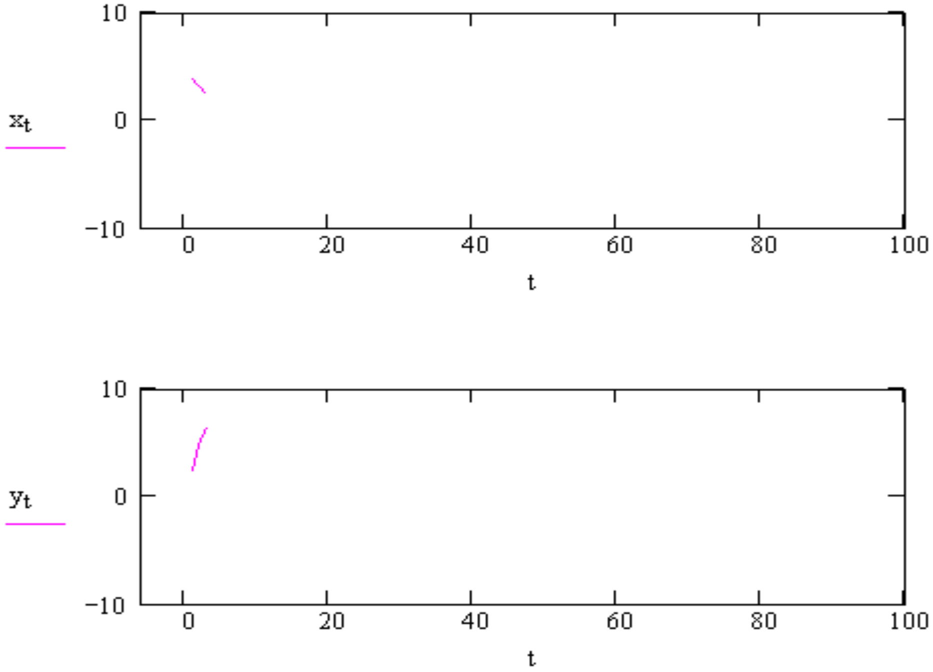
$$A = \pi * \varepsilon = const$$

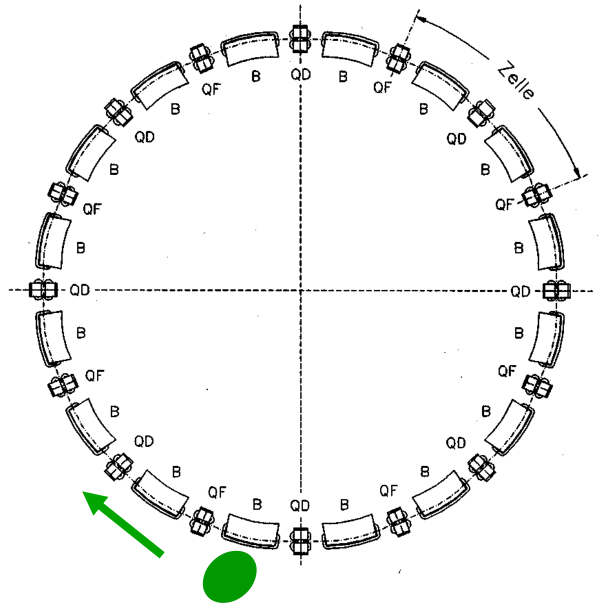


Particle Tracking in a Storage Ring

Calculate x, x´ for each linear accelerator element according to matrix formalism

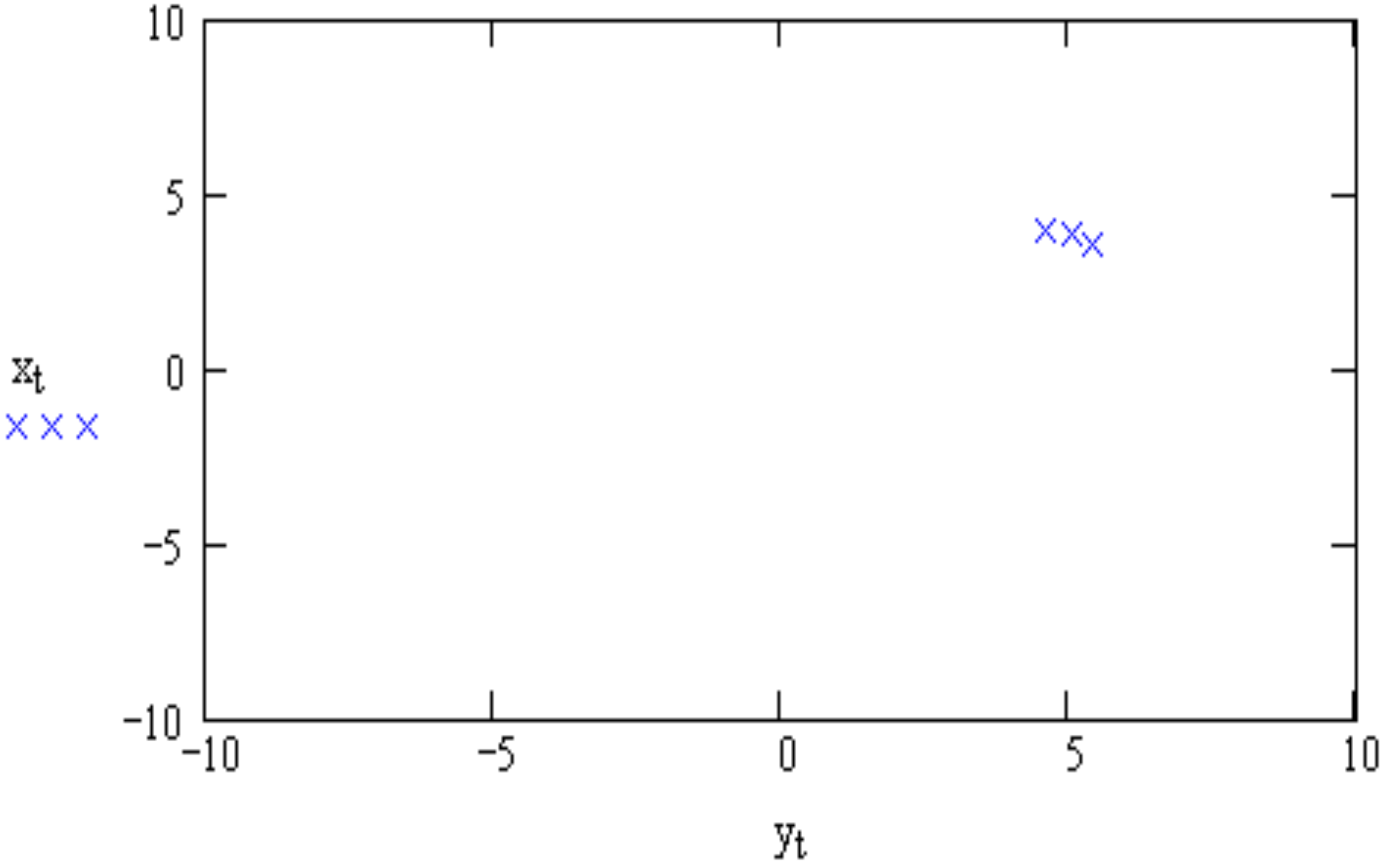
plot x, x´as a function of „s"



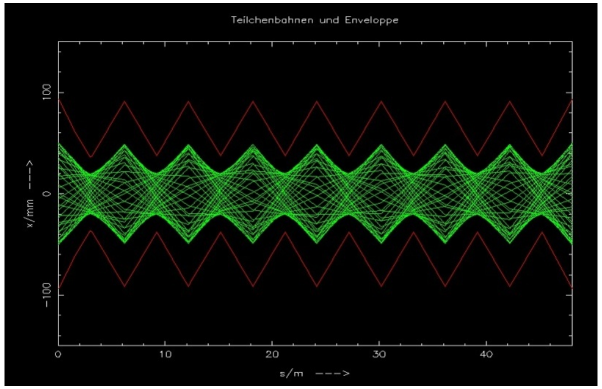


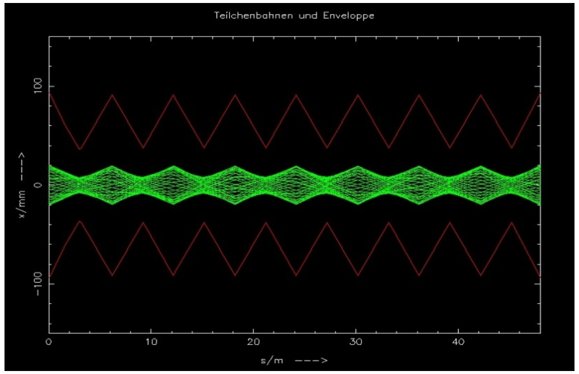
… and now the ellipse:

note for each turn x, x´ at a given position „s_1“ and plot in the phase space diagram



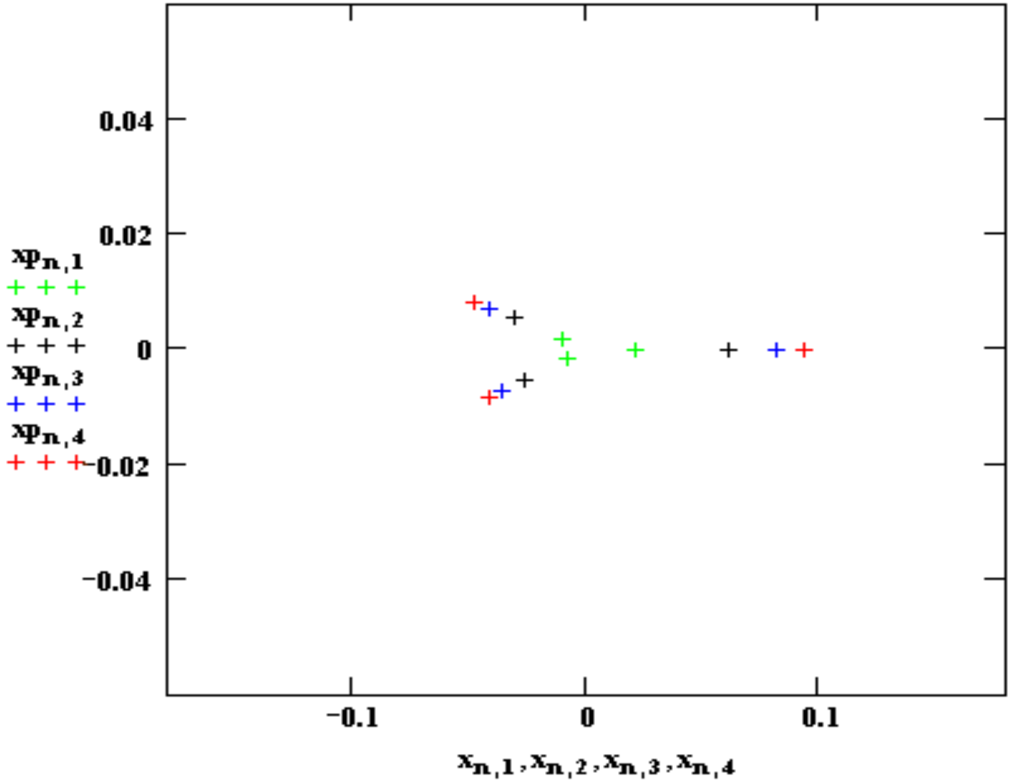
Emittance of the Particle Ensemble:





… to be very clear:

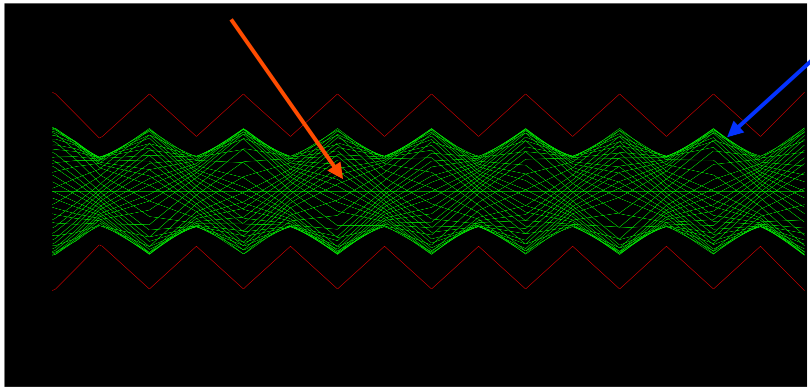
as long as our particle is running on an ellipse in x, x' space everything is alright, the beam is stable and we can sleep well at nights.



If however we have scattering at the rest gas, or non-linear fields, or beam collisions (!) the particle will perform a jump in x' and ε will increase

Emittance of the Particle Ensemble:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi) \qquad \hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



single particle trajectories, N ≈ 10 [11] per bunch

Gauß Particle Distribution: $$\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$$

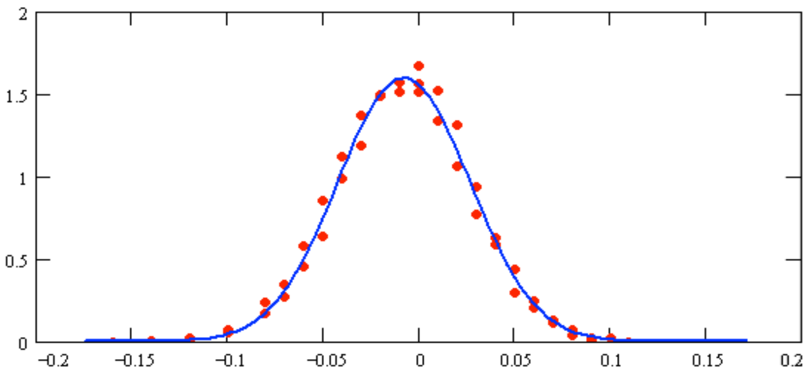
particle at distance 1 σ from centre
↔ 68.3 % of all beam particles

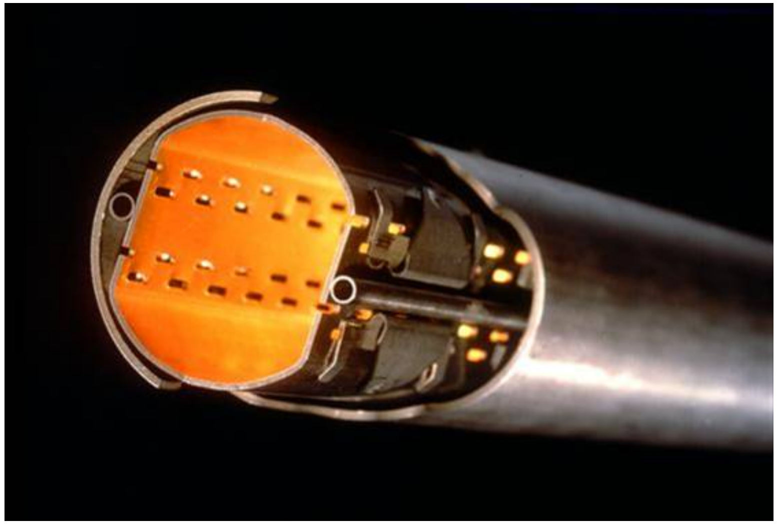
LHC:

$$\beta = 180\,m$$

$$\varepsilon = 5 * 10^{-10}\,m\,rad$$

$$\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5 * 10^{-10}\,m * 180\,m} = 0.3\,mm$$

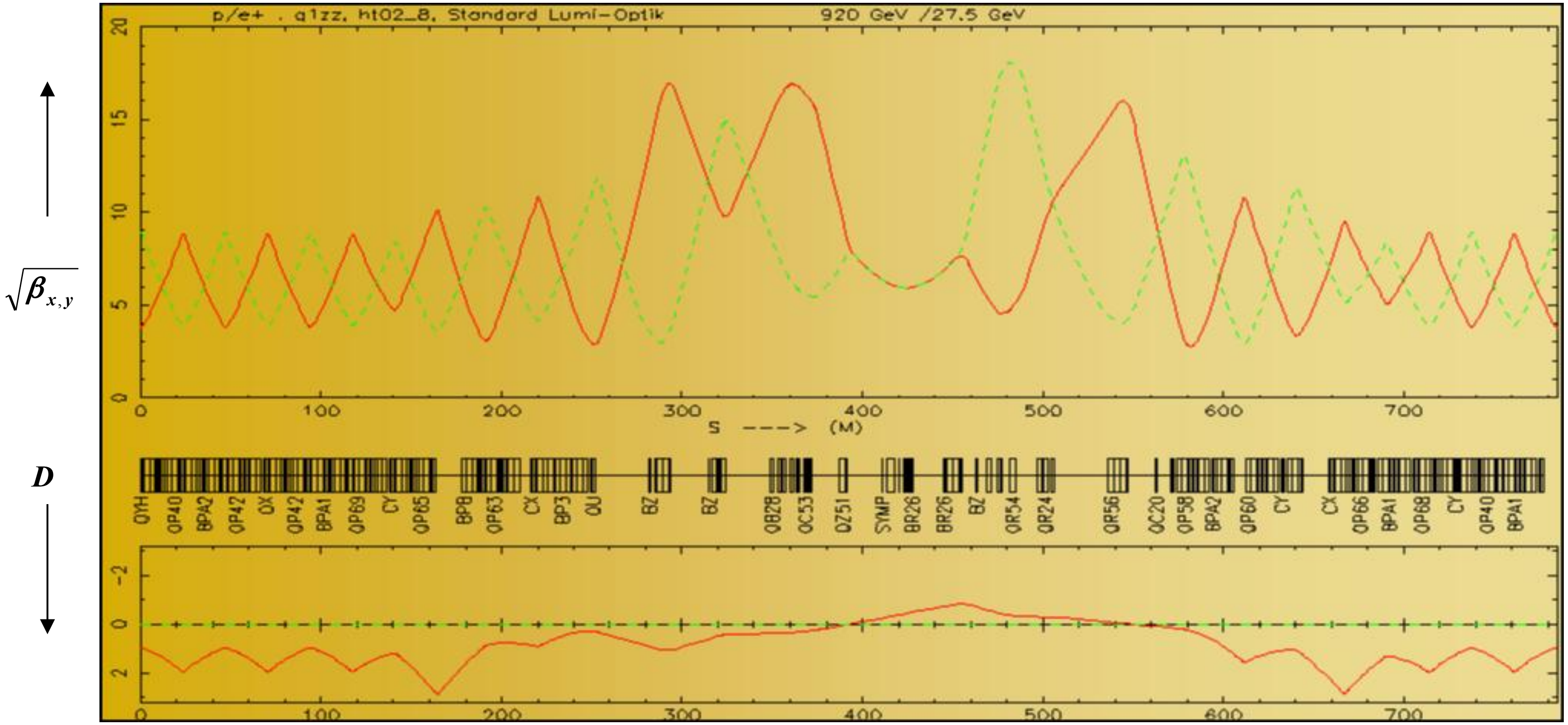




aperture requirements: $r_0 = 12 * \sigma$

The „not so ideal“ World

Lattice Design in Particle Accelerators



1952: Courant, Livingston, Snyder:

Theory of strong focusing in particle beams

Recapitulation: ...the story with the matrices !!!

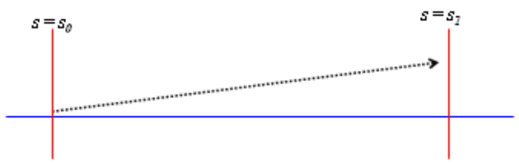
Equation of Motion:

$$x'' + K\,x = 0 \qquad K = 1/\rho^2 - k \;\ldots\; \textit{hor. plane:}$$

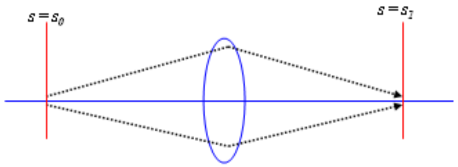
$$K = k \qquad \ldots\; \textit{vert. Plane:}$$

Solution of Trajectory Equations

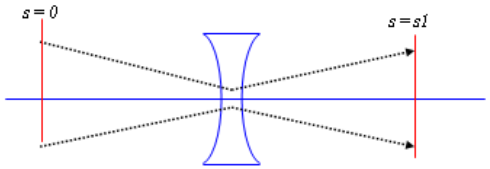
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}\,l) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}\,l) \\ -\sqrt{|K|}\sin(\sqrt{|K|}\,l) & \cos(\sqrt{|K|}\,l) \end{pmatrix}$$



$$M_{defoc} = \begin{pmatrix} \cosh(\sqrt{|K|}\,l) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}\,l) \\ \sqrt{|K|}\sinh(\sqrt{|K|}\,l) & \cosh(\sqrt{|K|}\,l) \end{pmatrix}$$

$$M_{total} = M_{QF} * M_D * M_B * M_D * M_{QD} * M_D * \quad \ldots$$

9.) Lattice Design: *„… how to build a storage ring"*

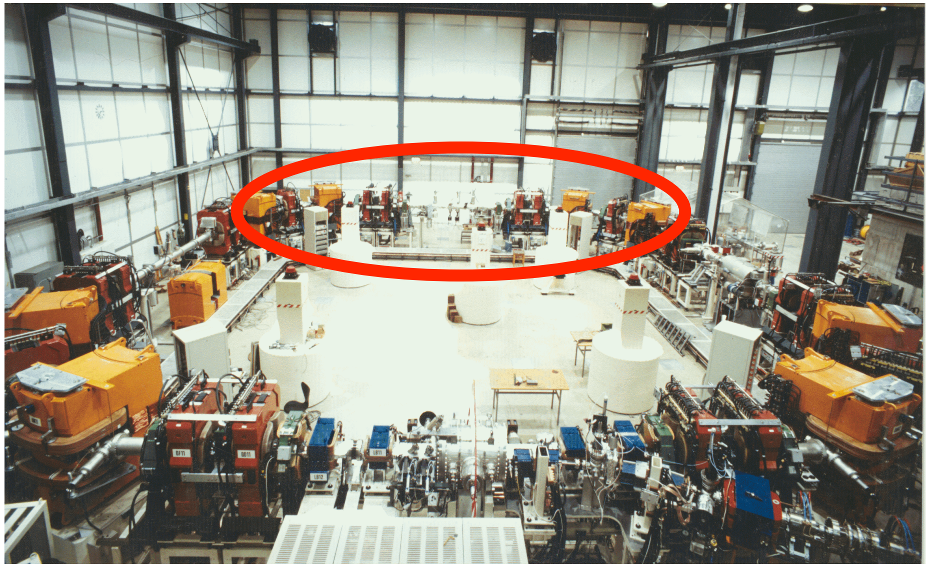
Geometry of the ring: $B * \rho = p / e$

p = momentum of the particle,
ρ = curvature radius

Bρ= beam rigidity

Circular Orbit: bending angle of one dipole

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} = \frac{Bdl}{B\rho}$$



The angle run out in one revolution must be 2π, so for a full circle

$$\alpha = \frac{\int Bdl}{B\,\rho} = 2\pi$$

$$\int Bdl = 2\pi \frac{p}{q}$$

… defines the integrated dipole field around the machine.

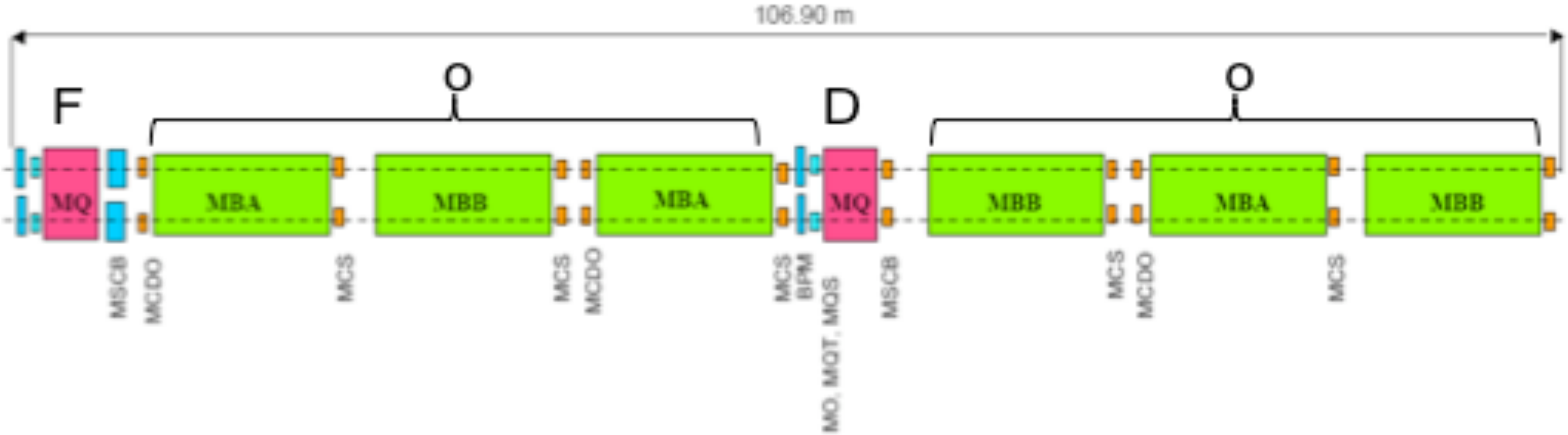
Example LHC:

7000 GeV Proton storage ring
dipole magnets N = 1232
l = 15 m
q = +1 e

$$\int B\,dl \approx N\,l\,B = 2\pi\;p/e$$

$$B \approx \frac{2\pi\;\;7000\;\;10^{9}\,eV}{1232\;\;15\,m\;\;3\;10^{8}\,\frac{m}{s}\;\;\;e} = 8.3\;Tesla$$

LHC: Lattice Design
the ARC 90° FoDo in both planes

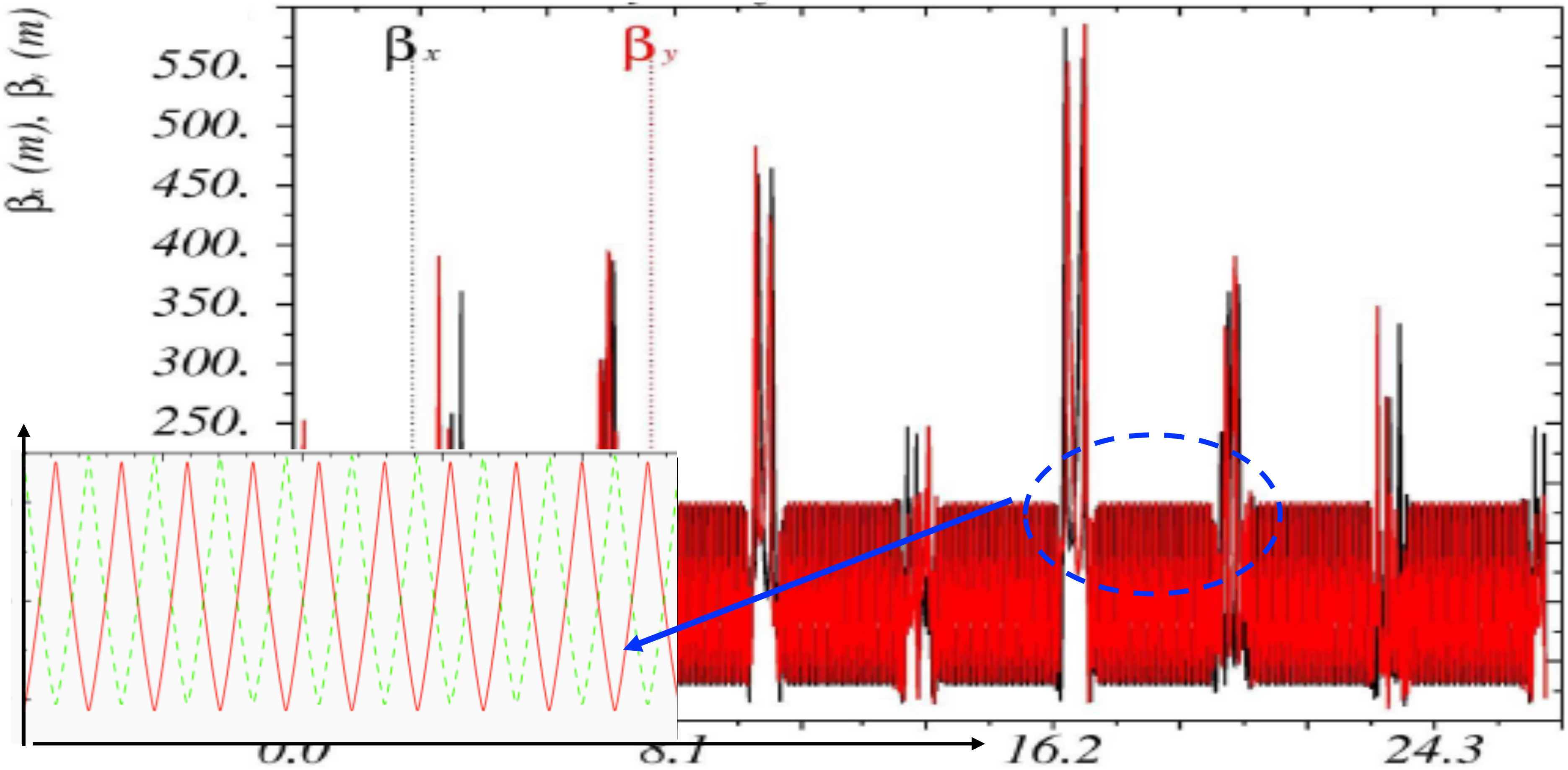


equipped with additional corrector coils

MB: main dipole
MQ: main quadrupole
MQT: Trim quadrupole
MQS: Skew trim quadrupole
MO: Lattice octupole (Landau damping)
MSCB: Skew sextupole
Orbit corrector dipoles
MCS: Spool piece sextupole
MCDO: Spool piece 8 / 10 pole
BPM: Beam position monitor + diagnostics

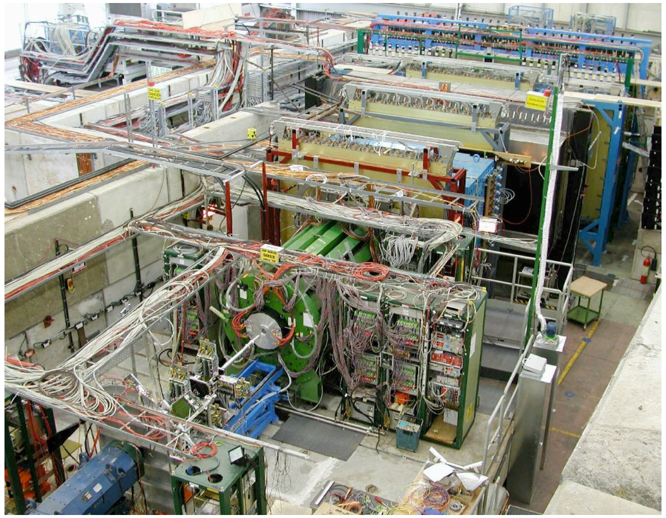
FoDo-Lattice

A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in between.
(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)



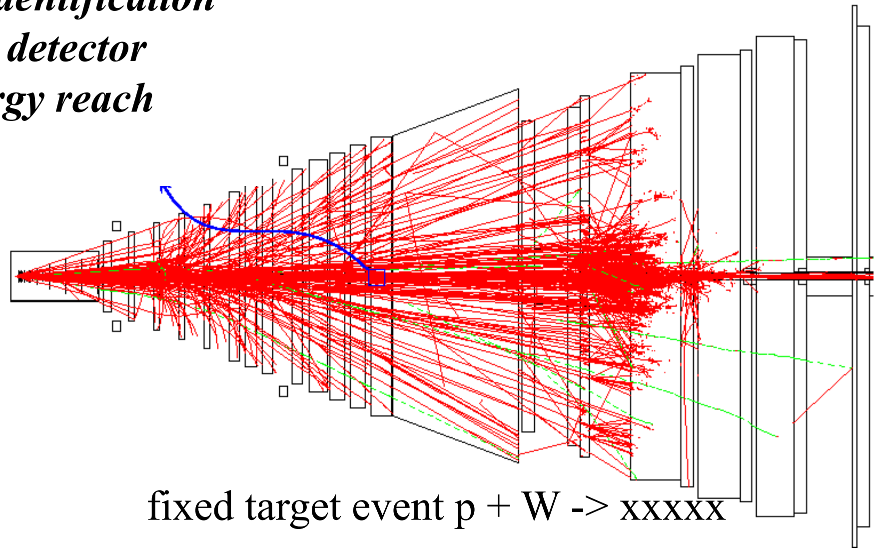
Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell $\mu = 45°$,
→ calculate the twiss parameters for a periodic solution

Fixed target experiments:



HARP Detector, CERN

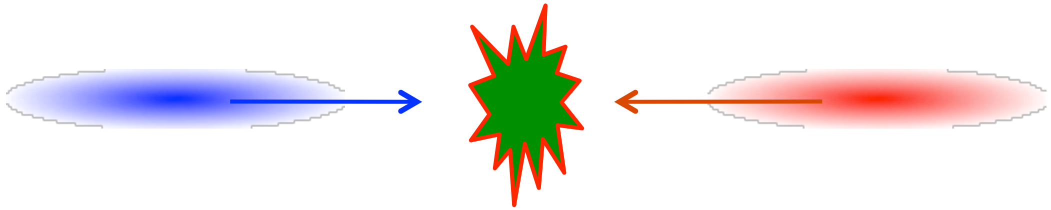
high event rate
easy track identification
asymmetric detector
limited energy reach

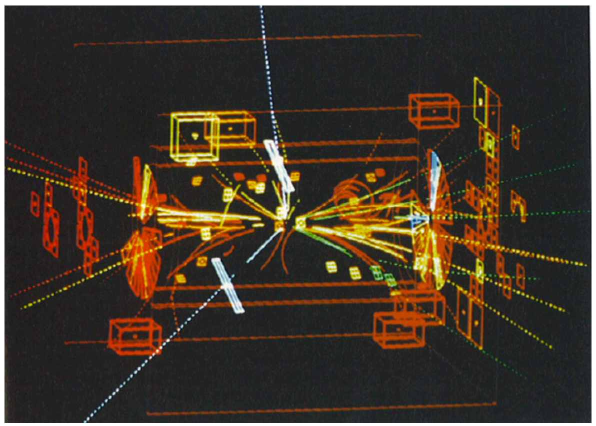


fixed target event p + W -> xxxxx

Collider experiments:

$E=mc^2$



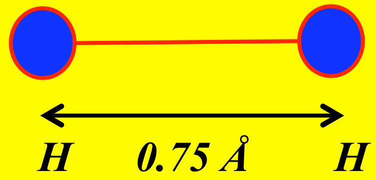


low event rate (luminosity)
challenging track identification
symmetric detector
$E_{lab} = E_{cm}$

Z_0 boson discovery at the UA2 experiment (CERN).
The Z_0 boson decays
into a e+e− pair, shown as white dashed lines.

Particle Density in matter

Hydrogen molecule

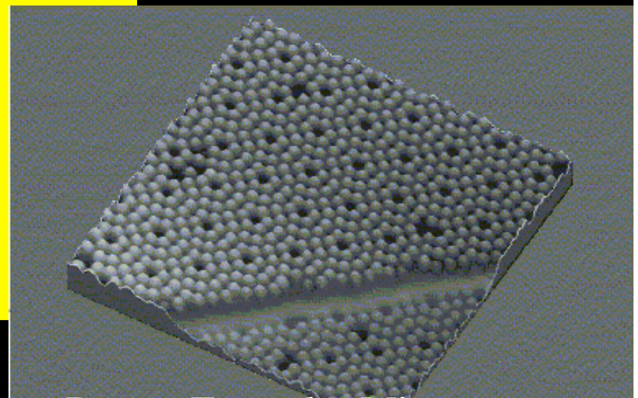


Atomic Distance in Hydrogen Molecule

$R_B \approx 0.5$ Å

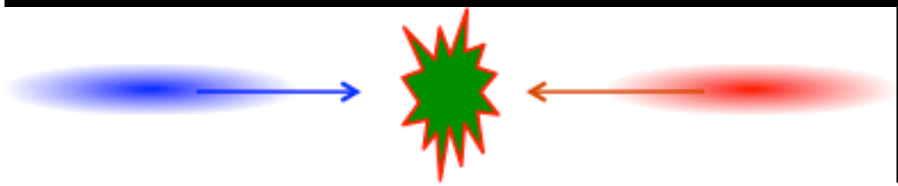
in solids / fluids $\lambda \approx 2.1 \; \ldots \; 2.9$ Å

in gases $\lambda \approx 33$ Å $= 3.3$ nm



Raster Tunnel e-Microscope

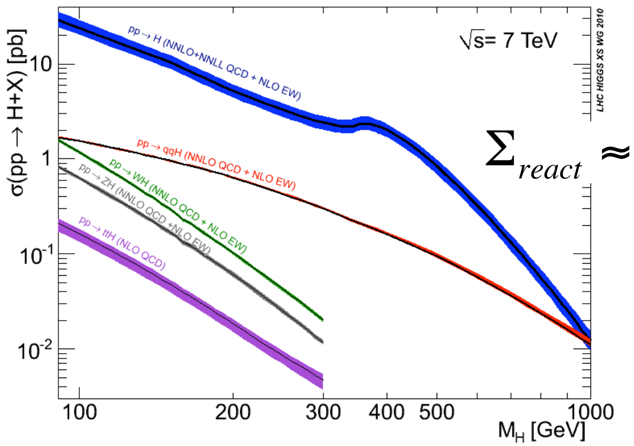
Particle Distance in Accelerators: $\lambda \approx = 6000$ Å $= 600$ nm (Arc LHC)



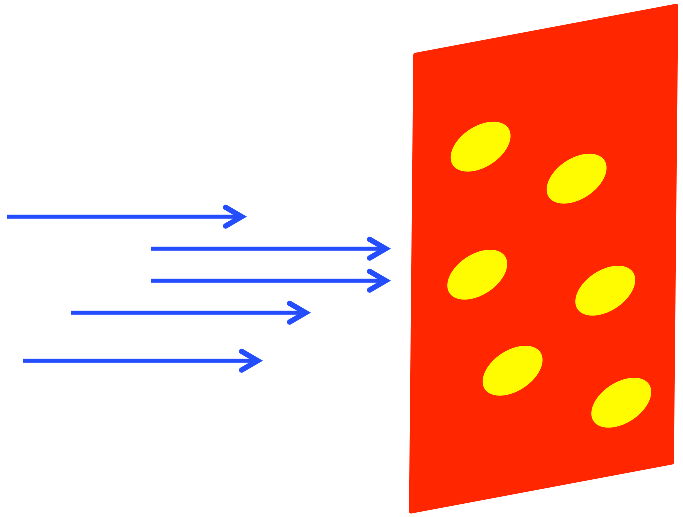
court. Jens Molge

Problem: Our particles are VERY small !!

Overall cross section of the Higgs:



$$\Sigma_{react} \approx 1\, pb$$

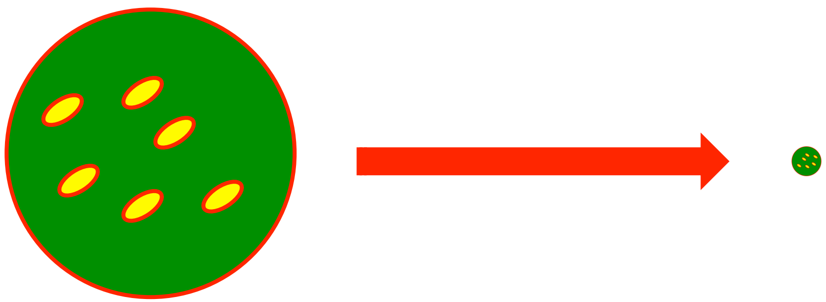


$$1b \;= 10^{-24}\, cm^2$$

$$1pb = \quad 10^{-12} * 10^{-24}\, cm^2 = 1/mio * 1/mio * 1/mio * 1/mio * 1/mio * 1/10000\ mm^2$$

The particles are "very small"

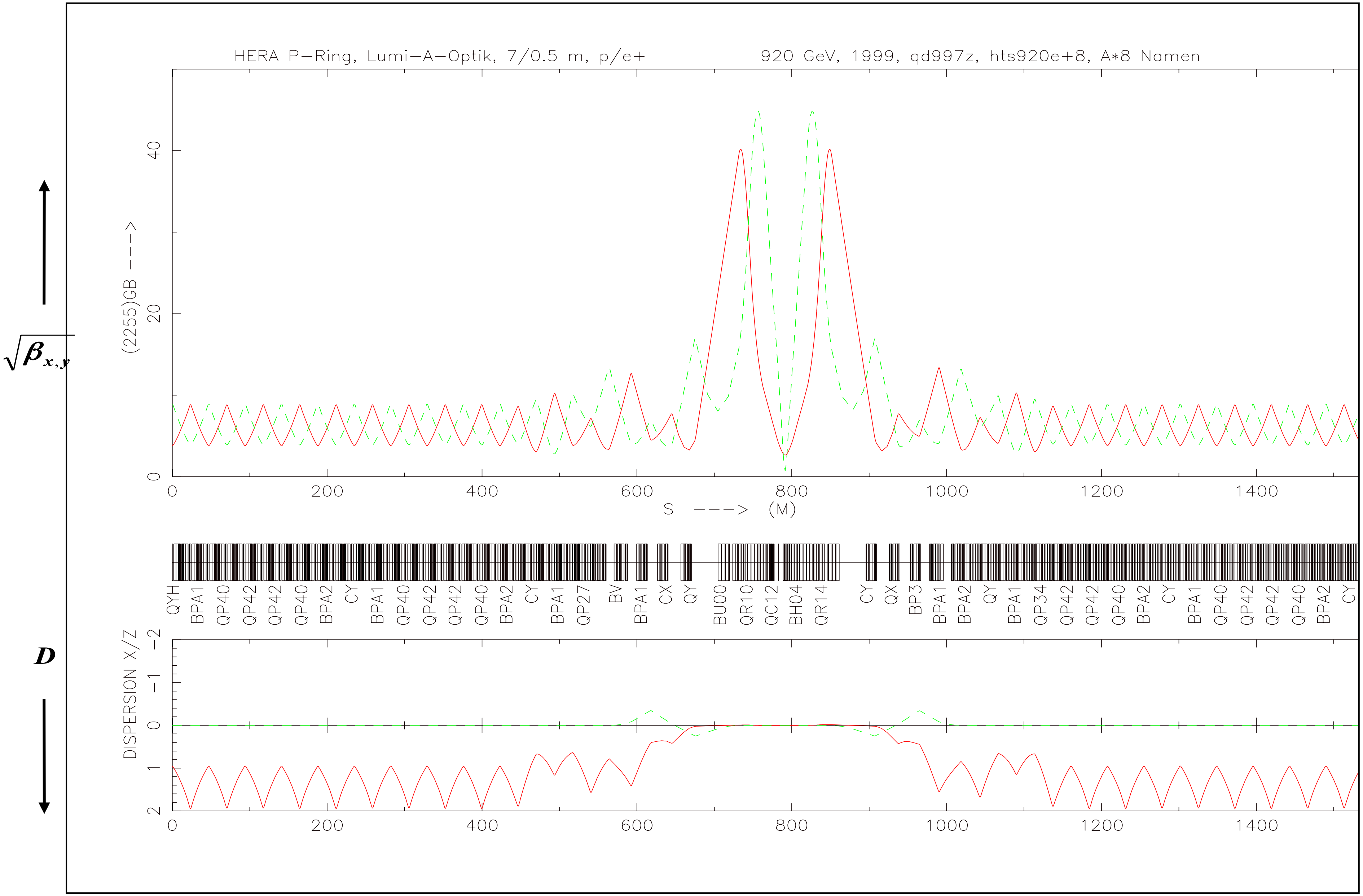
The only chance we have:
compress the transverse beam size … at the IP



LHC typical:

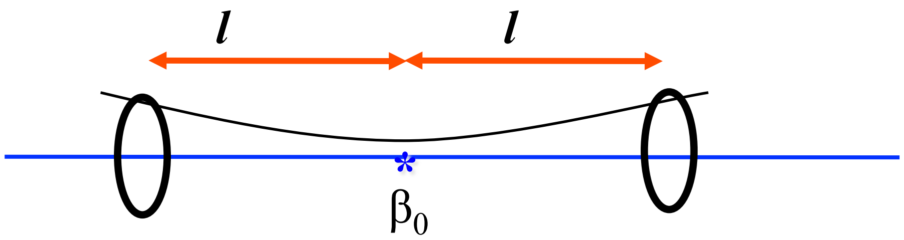
σ = 0.1 mm → 16 µm

11.) Insertions



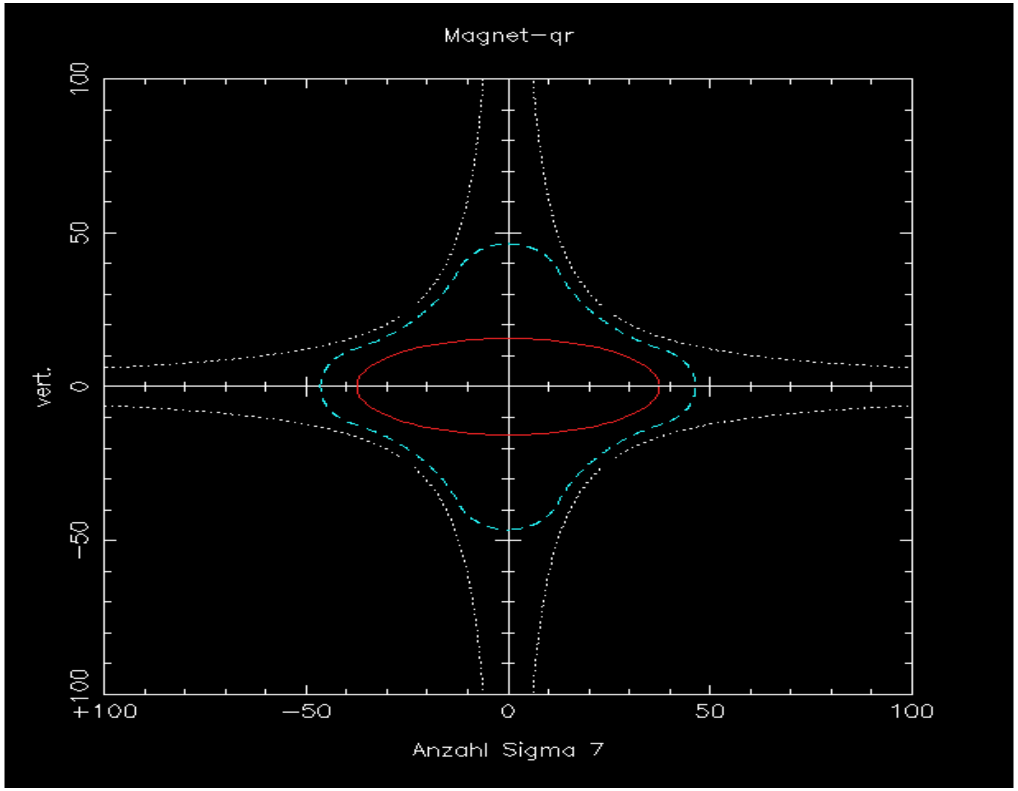
β-Function in a Drift:

$$\beta(l) = \beta_0 + \frac{l^2}{\beta_0}$$



At the end of a long symmetric drift space the beta function reaches its maximum value in the complete lattice.
-> here we get the largest beam dimension.

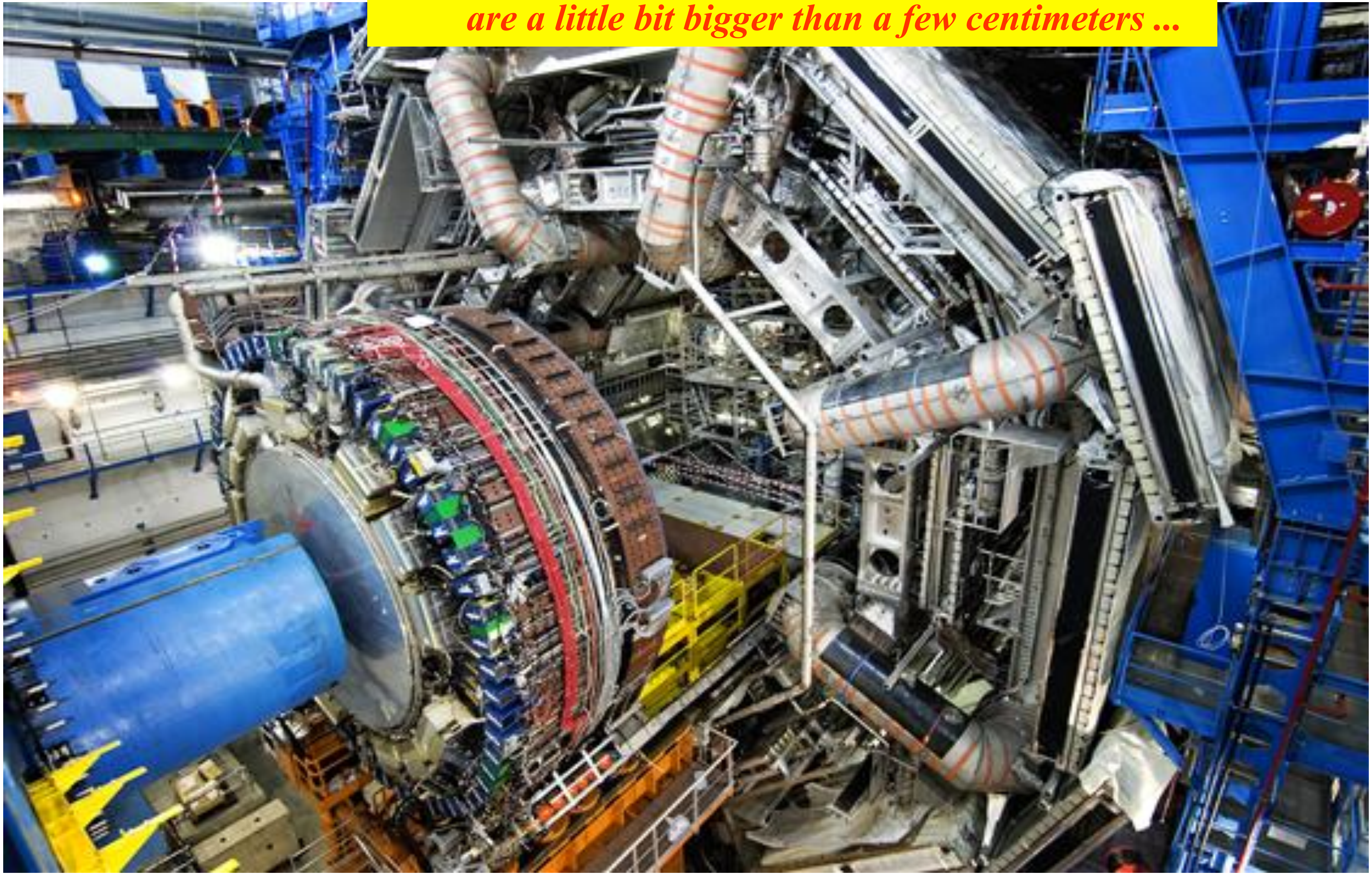
-> keep l as small as possible



7 sigma beam size inside a mini beta quadrupole

... clearly there is an

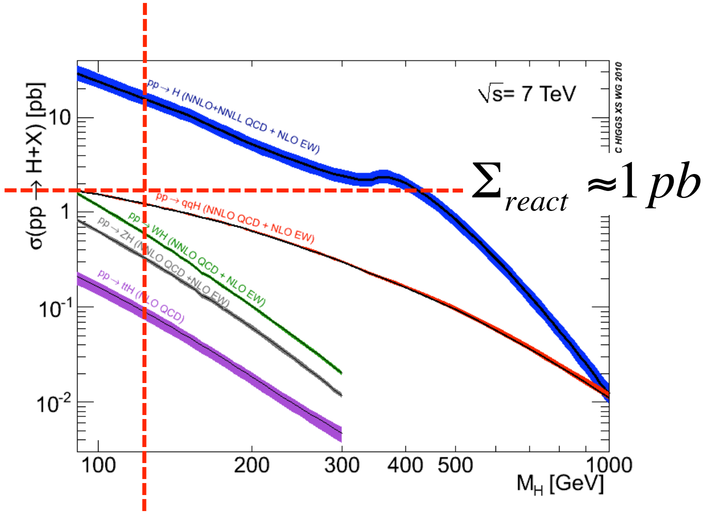
... unfortunately ... in general
high energy detectors that are
installed in that drift spaces
are a little bit bigger than a few centimeters ...

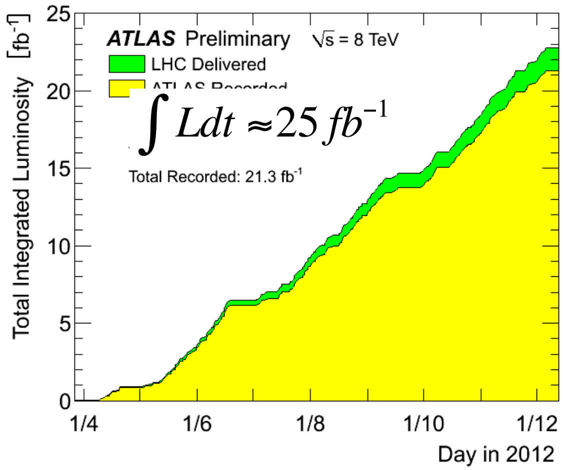


11.) The Mini-β Insertion & Luminosity:

production rate of events is determined by the cross section Σ_{react}
and a parameter L that is given by the design of the accelerator:
… the luminosity

$$R = L * \Sigma_{react} \qquad \approx 10^{-12} b \cdot 25 \frac{1}{10^{-15} b} = some\,1000\,H$$





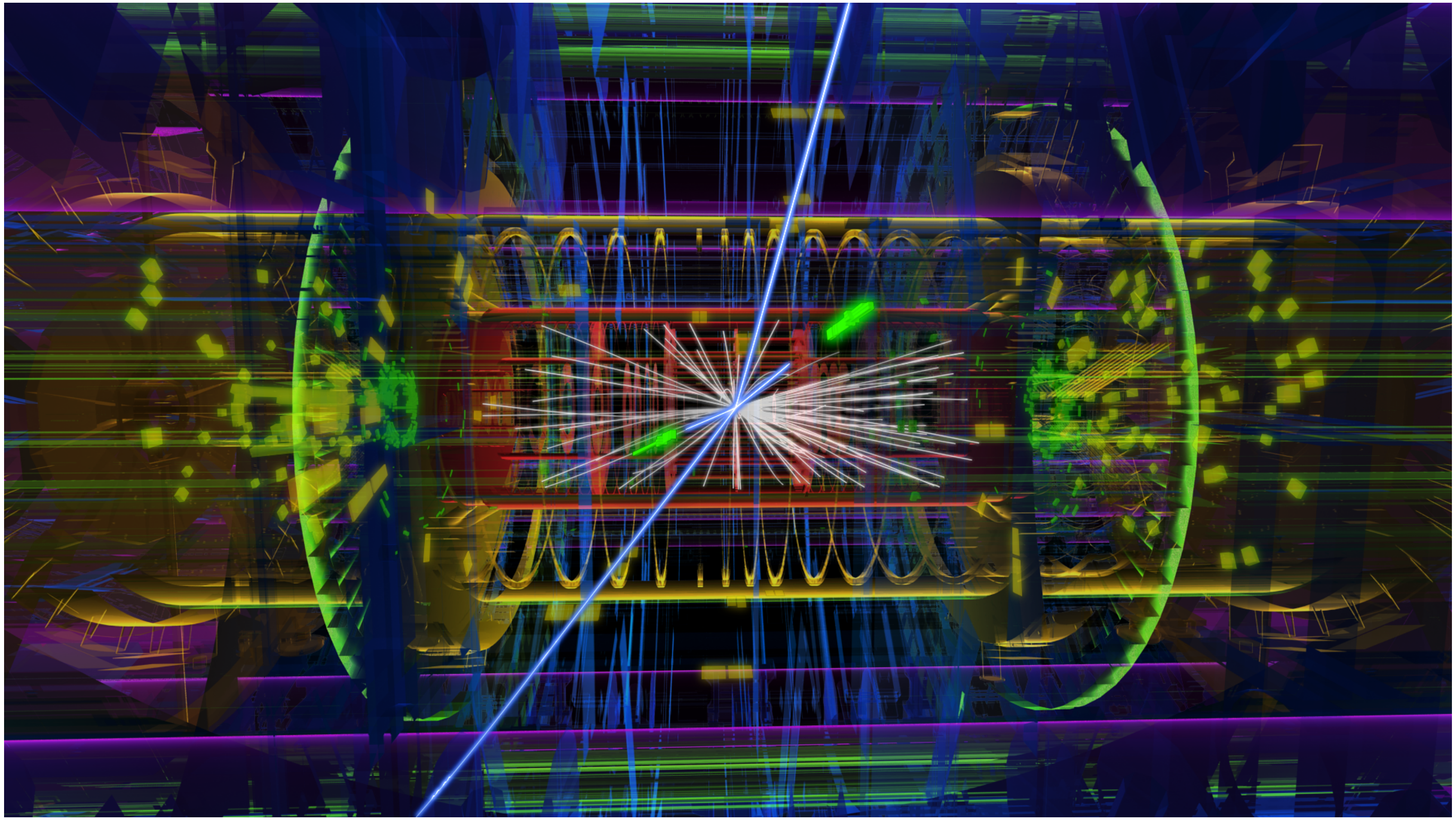
remember:
$1b = 10^{-24}\,cm^2$

The luminosity is a storage ring quality parameter and depends on beam size (β !!) and stored current

$$L = \frac{1}{4\pi e^2 f_0\, b} * \frac{I_1 * I_2}{\sigma^*_x * \sigma^*_y}$$

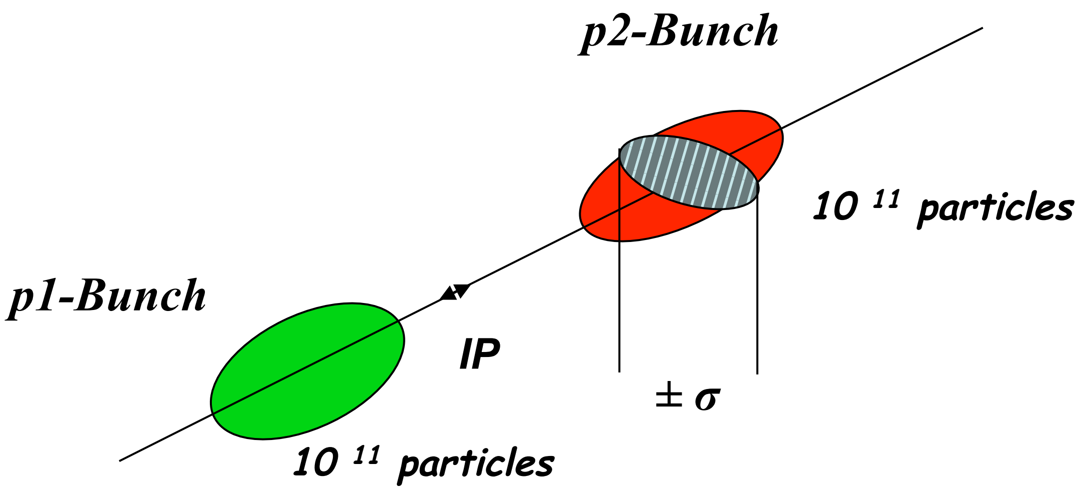
yes ... yes ... there is NO talk without it ...

The Higgs



ATLAS event display: Higgs => two electrons & two muons

Luminosity



Example: Luminosity run at LHC

$$\beta_{x,y} = 0.55\ m \qquad f_0 = 11.245\ kHz$$

$$\varepsilon_{x,y} = 5 * 10^{-10}\ rad\ m \qquad n_b = 2808$$

$$\sigma_{x,y} = 17\ \mu m$$

$$L = \frac{1}{4\pi e^2 f_0 n_b} * \frac{I_{p1} I_{p2}}{\sigma_x \sigma_y}$$

$$I_p = 584\ mA$$

$$L = 1.0 * 10^{34}\ \frac{1}{cm^2 s}$$

The Tune ...

...is the number of these transverse oscillations per turn and corresponds to the „Eigenfrequency" or sound of the particle oscilations. As in any oscillating system (e.g. pendulum) we have toavoid resonance conditions between the eigenfrequemcy of the system (= partcicle) and any external frequency that might act on the beam. Most prominent external frequency is the revolution frequemcy itself ‼ -> avoid integer tunes.

The Beta function shows the overall effect of all focusing fields; it has a certain value (m) that depends on the actual position in the ring.

The beam emittance describes the quality of the particle ensemble. It measure the area in phase space and can be considered like the temperature of a gas. The lower the emittacne the better the beam quality. Together with the beta function it defines the beam dimension.

The lattice cell is the special magnet arrangement of the principle building block in an accelerator. Moist appropriate for high energy accelerators is the FoDo.

The Higgs particle is very small, 10^{-36} cm^2, and so it is difficult to produce.