Special relativity

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(http://cern.ch/Werner.Herr/CAS2014_Chavannes/rel.pdf)

Why Special Relativity ?

- **Most beams at CERN are relativistic**
- **Strong implications for beam dynamics:**
  - **Transverse dynamics (e.g. momentum compaction, radiation, ...)**
  - **Longitudinal dynamics (e.g. transition, ...)**
  - **Collective effects (e.g. space charge, beam-beam, ...)**
  - **Luminosity in colliders**
  - **Particle lifetime and decay (e.g. μ, π, Z_0, Higgs, ...)**

Small history

- **1678 (Römer, Huygens): Speed of light c is finite ($c \approx 3 \cdot 10^8$ m/s)**
- **1630-1687 (Galilei,Newton): Principles of Relativity**
- **1863 (Maxwell): Electromagnetic theory, light are waves moving through static ether with speed c**
- **1887 (Michelson, Morley): Speed c independent of direction, → no ether**
- **1892 (Lorentz, FitzGerald, Poincaré): Lorentz transformations, Lorentz contraction**
- **1897 (Larmor): Time dilation**
- **1905 (Einstein): Principles of Special Relativity**
- **1907 (Einstein, Minkowski): Concepts of Spacetime**

OUTLINE

- **Principle of Relativity (Newton, Galilei)**
  - **Motivation and Idea**
  - **Formalism, examples**
- **Principle of Special Relativity (Einstein)**
  - **Why ?**
  - **Formalism and consequences**
  - **Four-vectors and applications (accelerators)**

Mathematical derivations and proofs are mostly avoided ...

Reading Material

- A. Einstein, Zur Elektrodynamik bewegter Körper, Ann. Phys. 17, (1905).
- R.P. Feynman, Feynman lectures on Physics, Vol. 1 + 2, (Basic Books, 2011).
- R.P. Feynman, Six not-so-easy pieces, (Basic Books, 2011).
- J. Freund, Special Relativity, (World Scientific, 2008).
- J. Hafele and R. Keating, Science 177, (1972) 166.

Setting the scene ..

- **Where we describe physics and physics laws:**
  - **In space coordinates:** $\vec{x} = (x, y, z)$
  - **In time:** t
- **A "Frame":**
  - **Where we observe physical phenomena and properties as function of their position** $\vec{x}$ **and moment in time** t**.**
  - **In different frames** $\vec{x}$ **and** t **are usually different.**
- **An "Event":**
  - **Something happening at** $\vec{x}$ **at time** t **is an "event", given by four numbers** $(x, y, z),\ t$

Setting the scene ..

Assuming two frames F and F':

- **Event described in F using:** $\vec{x} = (x, y, z)$ **and** t
- **Event described in F' using:** $\vec{x'} = (x', y', z')$ **and** t'

Principles of Relativity (Newton)

We always observe and describe physics in a certain Frame

Laws of physics should be invariant

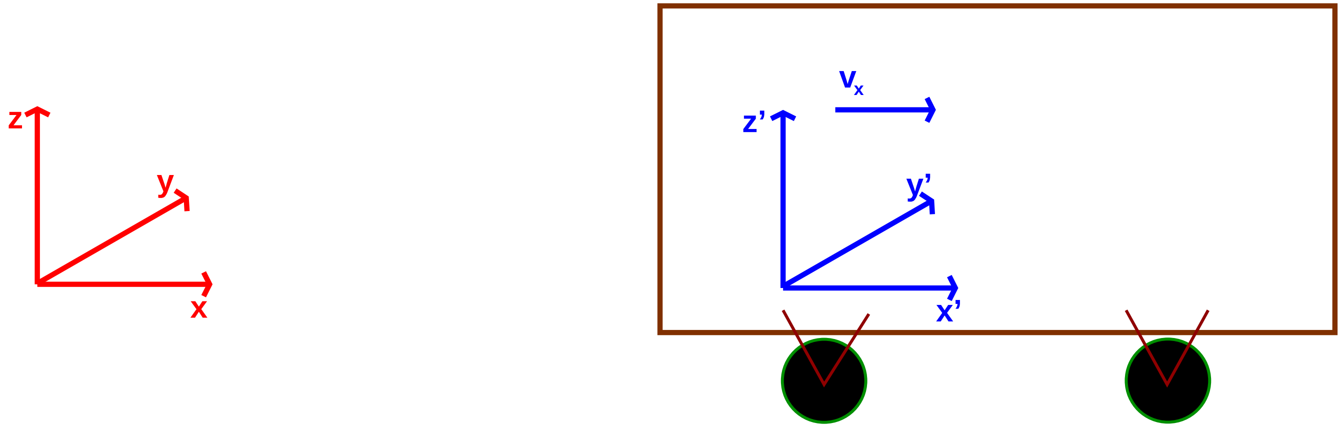
i.e. the same when we change the frame, for example:

- **Frame displaced in Space**
- **Frame displaced in Time**
- **Frame moving at constant speed, an Inertial System**

→ **More formal: "Physical laws have the same form in all inertial systems, they are invariant"**

Principles of Relativity (Galilei, Newton)

Assume a frame at rest (*F*) and another frame (*F'*) moving in x-direction with constant velocity $\vec{v} = (v_x, 0, 0)$

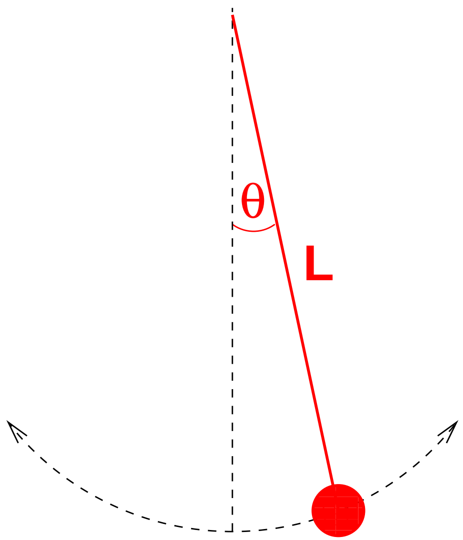


Example: we would like to have

$Force = m \cdot a$ **and** $Force' = m \cdot a'$

(Mass m is the same in all frames)

Example: Pendulum

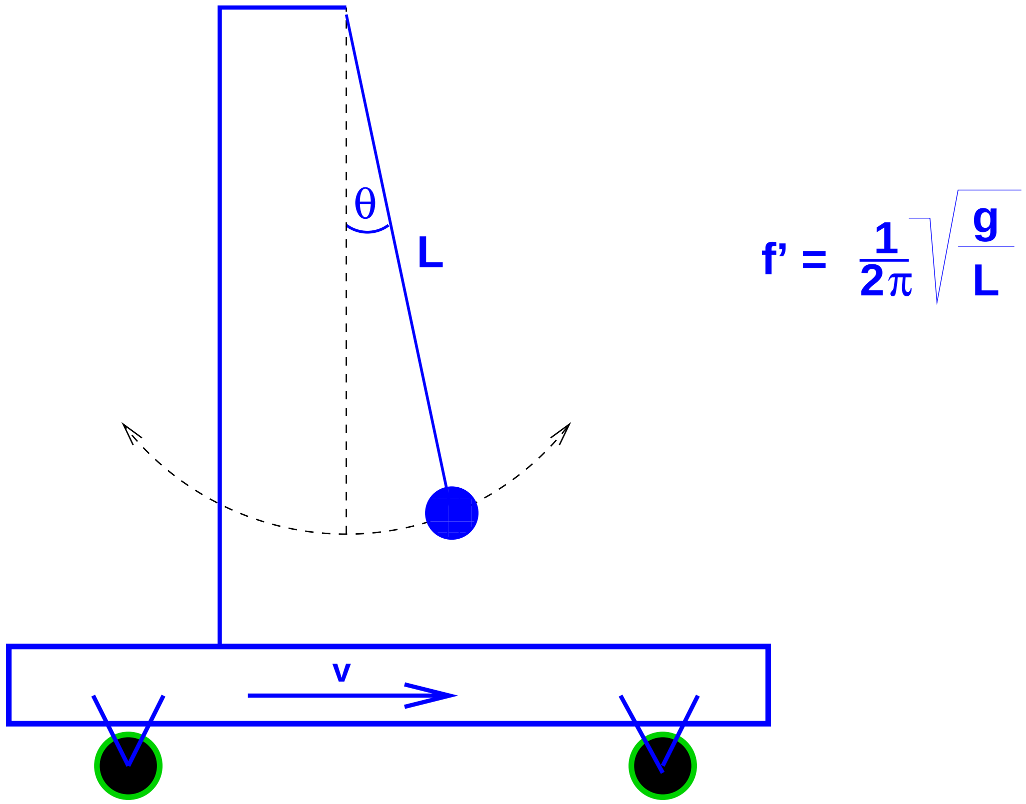


$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

Motion described by coordinate Θ

Frequency of a pendulum is f

Example: Pendulum



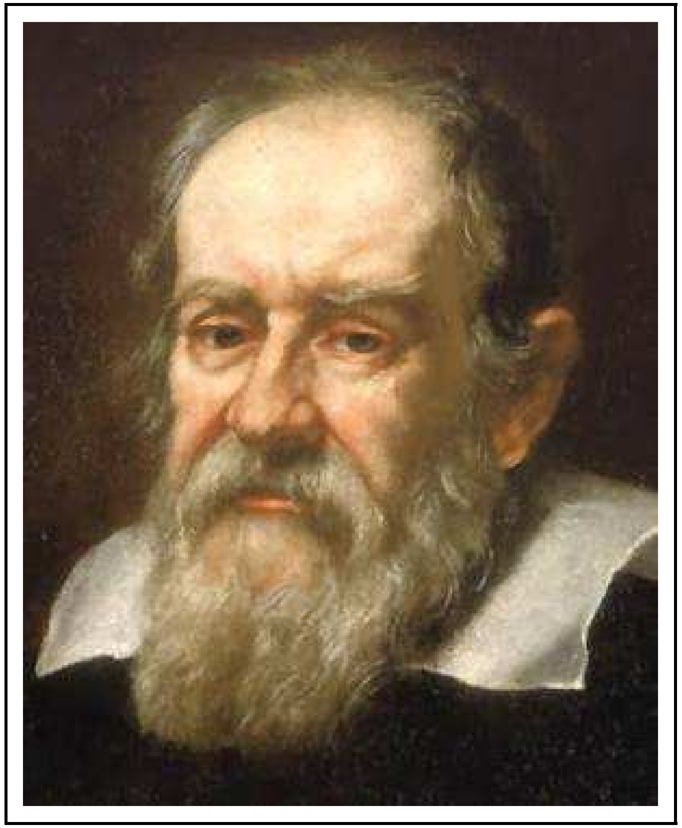
Motion described by coordinate Θ

Frequency of a pendulum the same in all inertial systems: f = f'

How do we describe invariant laws ?

- **We have described an experiment in rest frame F**
- **How can we describe it seen from a moving frame F' ?**
  - **Need to transform coordinates (x, y, z) and time t to describe (translate) results of measurements and observations to the moving system (x', y', z') and t'.**
  - **For Newton's principle of relativity need Galilei transformation for:**
    (x, y, z) **and** $t \longrightarrow (x', y', z')$ **and** t'**.**
- **Then laws should look the same, have the same form**

Galilei transformation



$$
\begin{aligned}
x' &= x - v_x t \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}
$$

Consequences of Galilei transformation

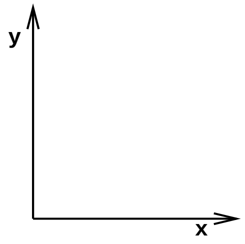
Velocity and transformation in x-coordinates. Only for simplicity, can always rotate the coordinate system.

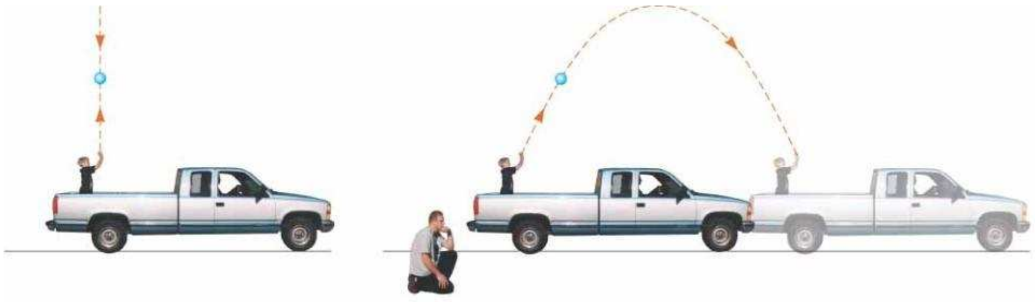
For Galilei transformation:

Space and time are <u>independent</u> quantities

Space coordinates are changed, time is not changed !

Illustration of Galilei transformation:





F'(car) F(observer)

- In car frame F' moving with speed v_x:

  Ball starts with vertical velocity $v'_y = v'_0$

  Ball goes up and down

- In rest frame for observer F: ball describes a curve (parabola ?)

Illustration of Galilei transformation:

Equation of motion in moving frame $x'(t')$ and $y'(t')$:

$$x'(t') \;=\; 0, \quad v'_y(t') \;=\; v'_0 \;-\; g \cdot t'$$

$$y'(t') \;=\; \int v'_y(t')\mathrm{d}t' \;=\; v'_0 \cdot t' \;-\; \frac{1}{2} g t'^2$$

Illustration of Galilei transformation:

From moving frame:

$$y'(t') \;=\; v'_0 \cdot t' \;-\; \frac{1}{2} g t'^2$$

To get equation of motion in rest frame $x(t)$ **and** $y(t)$**:**

Galilei transform: $y(t) \;\equiv\; y'(t')$, $\;t \;\equiv\; t'$, $\;x(t) = x' + v_x \cdot t \;=\; v_x \cdot t$

To get y **as function of** x **we can re-write:**

$$\Rightarrow \quad t' \;=\; t \;=\; \frac{x}{v_x}$$

and get for the trajectory in the rest frame:

$$y(x) \;=\; \frac{v_0}{v_x} \cdot x \;-\; \frac{1}{2} g \frac{x^2}{v_x^2}$$

This is a parabola, observed from the rest frame.

Consequences of Galilei transformation

Velocities can be added

- **From Galilei transformation, take derivative:**

$$x' = x - v_x t$$

$$\dot{x}' = \dot{x} - v_x \quad \Rightarrow \quad v' = v - v_x$$

- **A car moving with speed v' in a frame moving with speed v_x we have in rest frame $v = v' + v_x$**

Problems with Galilei transformation

- **Maxwell's equations are different when Galilei transformations are applied (because they predict the speed of light)**
- **Could exceed it when velocities are added:**

  $0.8 \cdot c + 0.5 \cdot c = 1.3 \cdot c$ **?**

  c = 299792458.000 m/s
- **From experiments: Speed of light is upper limit and the same in all frames and all directions**
- **Enter Einstein: principles of special relativity**

Principle(s) of Special Relativity (Einstein)

All physical laws (e.g. Maxwell's) in inertial frames must have equivalent forms, in particular:

Speed of light c must be the same in all frames

- **Cannot distinguish between inertial frames by measuring speed of light:**
  - **Cannot determine absolute speed of an inertial frame**
  - **No absolute space, no absolute time**
- **Need Transformations (not Galileaen) which make the physics laws (Maxwell !) look the same !**

Coordinates must be transformed differently

- **Transformation must keep speed of light constant**

Constant speed of light requires:

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \;\Rightarrow\; x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$$

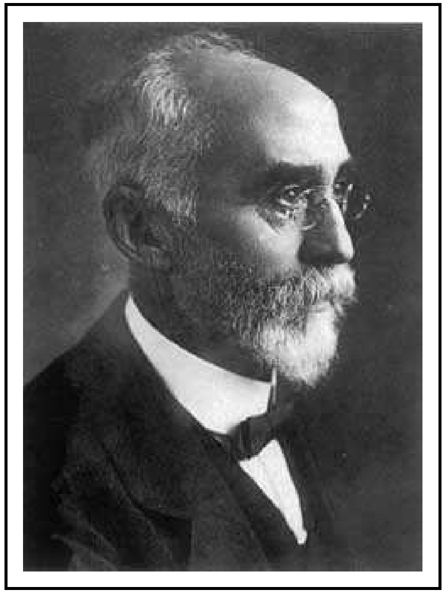
(front of a light wave)

- **To fulfill this condition: Time must be changed by transformation as well as space coordinates**

- **Transform** $(x, y, z),\ t \ \rightarrow\ (x', y', z'),\ t'$

$\Rightarrow$ **Defines the Lorentz transformation**

Lorentz transformation



$$x' = \frac{x - vt}{\sqrt{(1 - \frac{v^2}{c^2})}} = \gamma \cdot (x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{v \cdot x}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}} = \gamma \cdot (t - \frac{v \cdot x}{c^2})$$

Transformation for constant velocity v along x-axis

Time is now also transformed

Definitions: relativistic factors

$$\beta_r = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{(1 - \frac{v^2}{c^2})}} = \frac{1}{\sqrt{(1 - \beta_r^2)}}$$

β_r **relativistic speed:** $\beta_r = [\mathbf{0},\ \mathbf{1}]$

γ **relativistic factor:** $\gamma = [\mathbf{1},\ \infty]$

(unfortunately, you will also see other β **and** γ **... !)**

Consequences of Einstein's interpretation

- **Space and time and <u>NOT</u> independent quantities**
- **Time has no absolute meaning**
- **Relativistic phenomena:**
  - **Velocities cannot exceed speed of light**
  - **(Non-) Simultaneity of events in independent frames**
  - **Lorentz contraction**
  - **Time dilation**
- **Formalism with four-vectors introduced (see later)**

Addition of velocities

Galilei: $v = v_1 + v_2$

With Lorentz transform we have:

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

for $v_1, v_2, v_3, ... = 0.5c$ **we get:**

$\mathbf{0.5c + 0.5c = 0.8c}$

$\mathbf{0.5c + 0.5c + 0.5c = 0.93c}$

$\mathbf{0.5c + 0.5c + 0.5c + 0.5c = 0.976c}$

$\mathbf{0.5c + 0.5c + 0.5c + 0.5c + 0.5c = 0.992c}$

➡ **Speed of light can never be exceeded by adding velocities !**

Special case: 0.5c + 1.0c = 1.0c

- Simultaneity -

Simultaneity between moving frames
(the least intuitive concept)

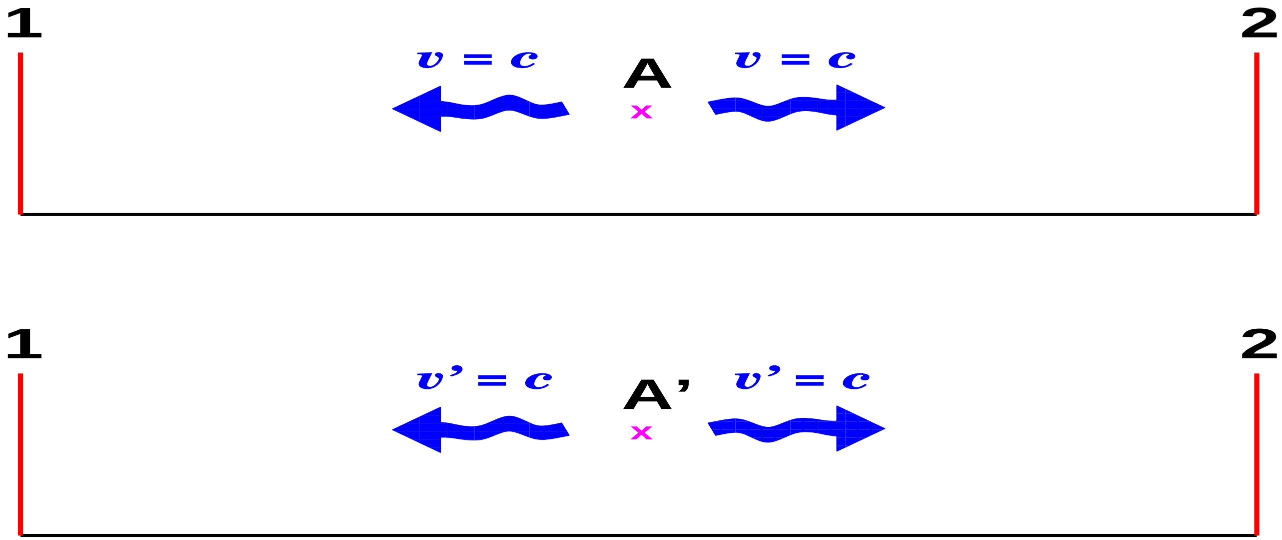
- **Assume two events in frame F at positions x_1 and x_2 happen simultaneously at times $t_1 = t_2$:**

$$t'_1 = \frac{t_1 - \frac{v \cdot x_1}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}} \quad \textbf{and} \quad t'_2 = \frac{t_2 - \frac{v \cdot x_2}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}}$$

$x_1 \neq x_2$ **in** F **implies that** $t'_1 \neq t'_2$ **in frame** F' **!!**

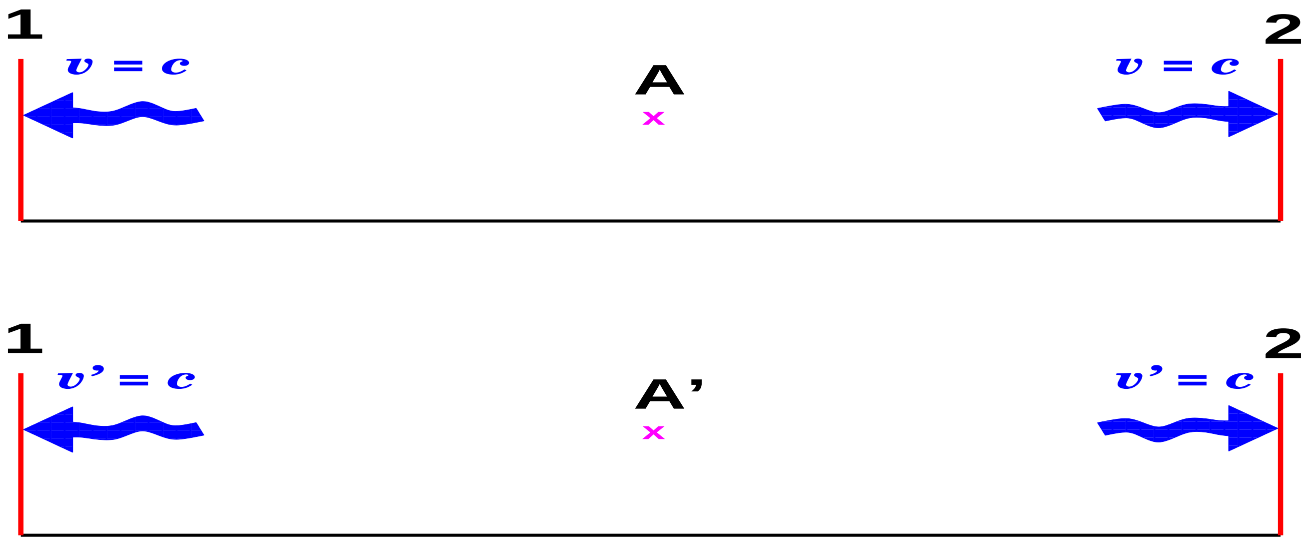
- **Two events simultaneous at positions x_1 and x_2 in F are not simultaneous in F'**

Simultaneity between moving frames



- **System with a light source (x) and detectors (1, 2) and one observer (A) in this frame, another (A') outside**
- **System at rest $\rightarrow$ observation the same in A and A'**

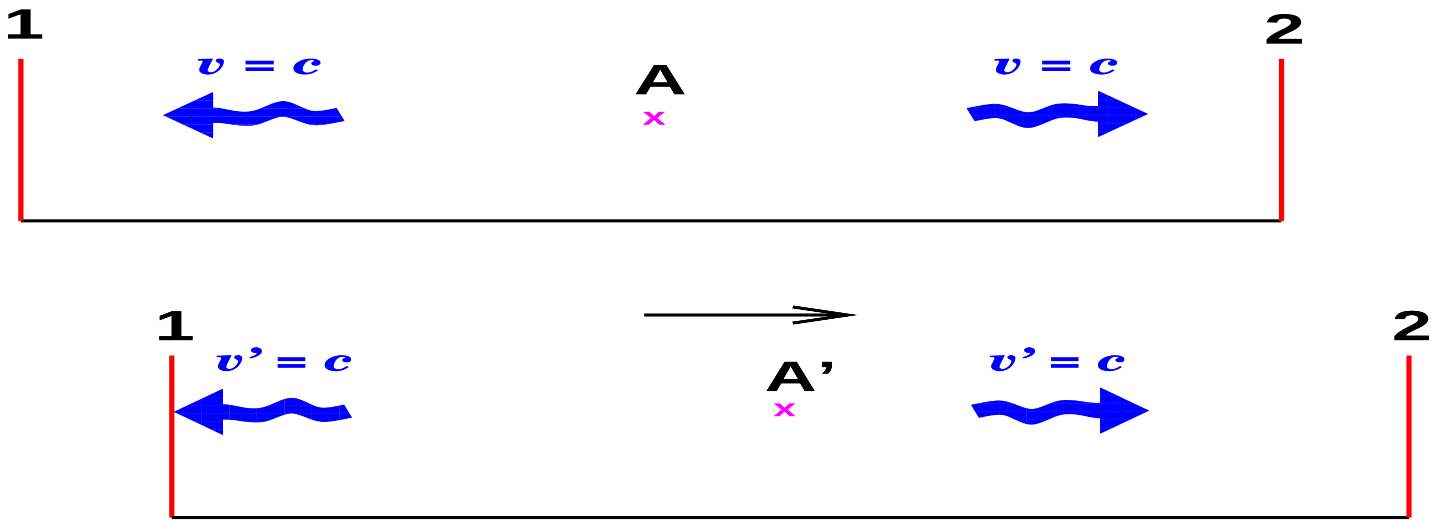
Simultaneity between moving frames



Speed of light is the same in both frames, (no adding of speeds):

> **For A: both flashes arrive simultaneously in 1,2**
>
> **For A': both flashes arrive simultaneously in 1,2**
>
> **What if the frame is moving relative to A' ?**

Simultaneity between moving frames



Speed of light is the same in both frames, (no adding of speeds):

For A: both flashes arrive simultaneously in 1,2

For A': flash arrives first in 1, later in 2

A simultaneous event in F is not simultaneous in F'

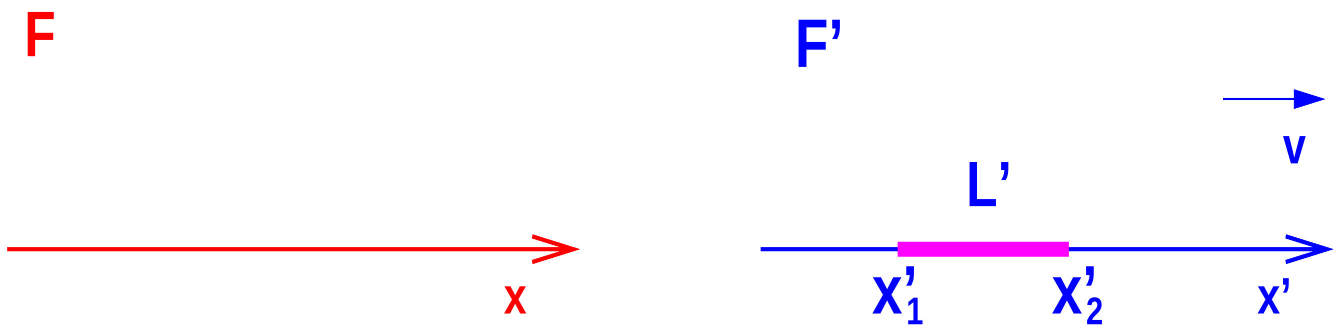
Why do we care ??

Why care about simultaneity ?

- **Simultaneity is not frame independent**
- **This is a key in special relativity**
- **Most paradoxes are explained by that (although not the twin paradox) !**
- **Different observers see a different reality, in particular the sequence of events can change !**
  - **For $t_1 < t_2$ we may find (not always !) a frame where $t_1 > t_2$ (concept of before and after depends on the observer)**

-   Lorentz contraction   -

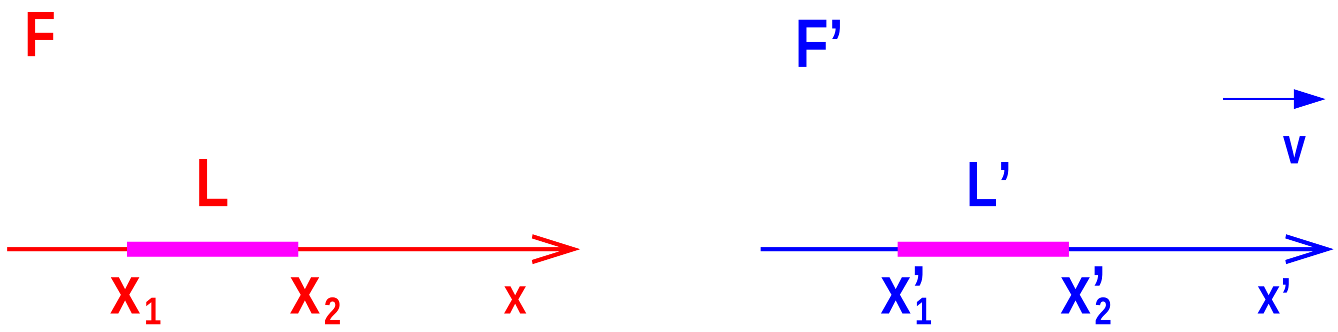
Consequences: length measurement



How to measure the length of an object ?

Have to measure position of <u>both</u> ends <u>simultaneously !</u>
Length of a rod in F' is $L' = x'_2 - x'_1$, measured simultaneously at a fixed time t' in frame F', what is the length L seen in F ??

Consequences: length measurement



We have to measure simultaneously (!) the ends of the rod at a fixed time t in frame F, i.e.: $L = x_2 - x_1$ →

$$x'_1 = \gamma \cdot (x_1 - vt) \quad \text{and} \quad x'_2 = \gamma \cdot (x_2 - vt)$$

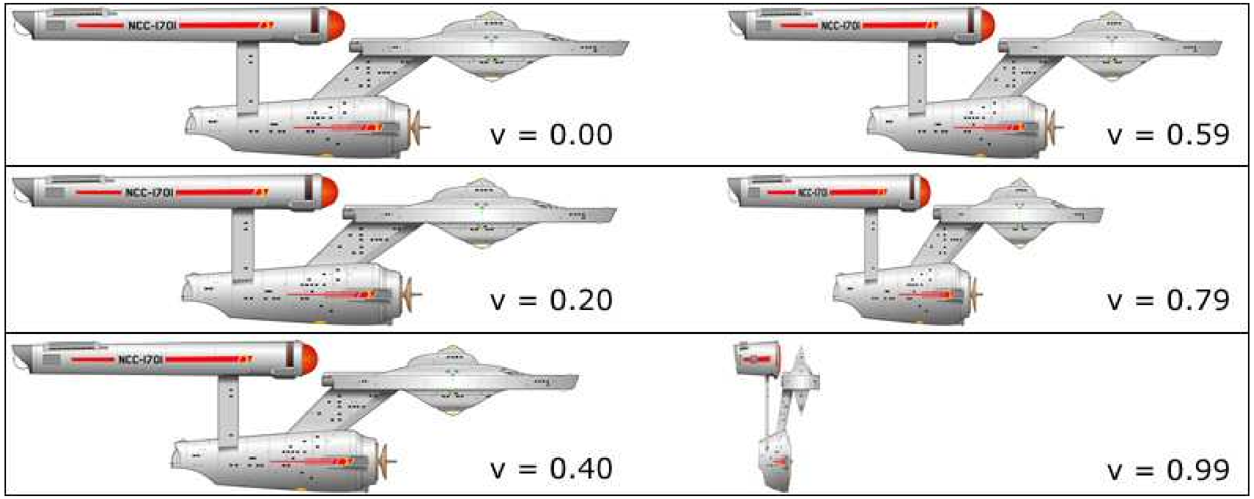
$$L' = x'_2 - x'_1 = \gamma \cdot (x_2 - x_1) = \gamma \cdot L$$

→ $\boxed{L = L'/\gamma}$

Lorentz contraction

- **In moving frame an object has always the same length (it is invariant, our principle !)**
- **From stationary frame moving objects appear contracted by a factor γ (Lorentz contraction)**
- **Why do we care ?**
- **Turn the argument around: assume length of a proton bunch appears always at 0.1 m in laboratory frame (e.g. in the RF bucket), what is the length in its own (moving) frame ?**
  - **At 5 GeV ($\gamma \approx$ 5.3) $\rightarrow$ L' = 0.53 m**
  - **At 450 GeV ($\gamma \approx$ 480) $\rightarrow$ L' = 48.0 m**

Lorentz contraction - schematic



Spaceship seen from earth

→ **Appears shorter at higher velocities**

Lorentz contraction - schematic

Earth seen from spaceship

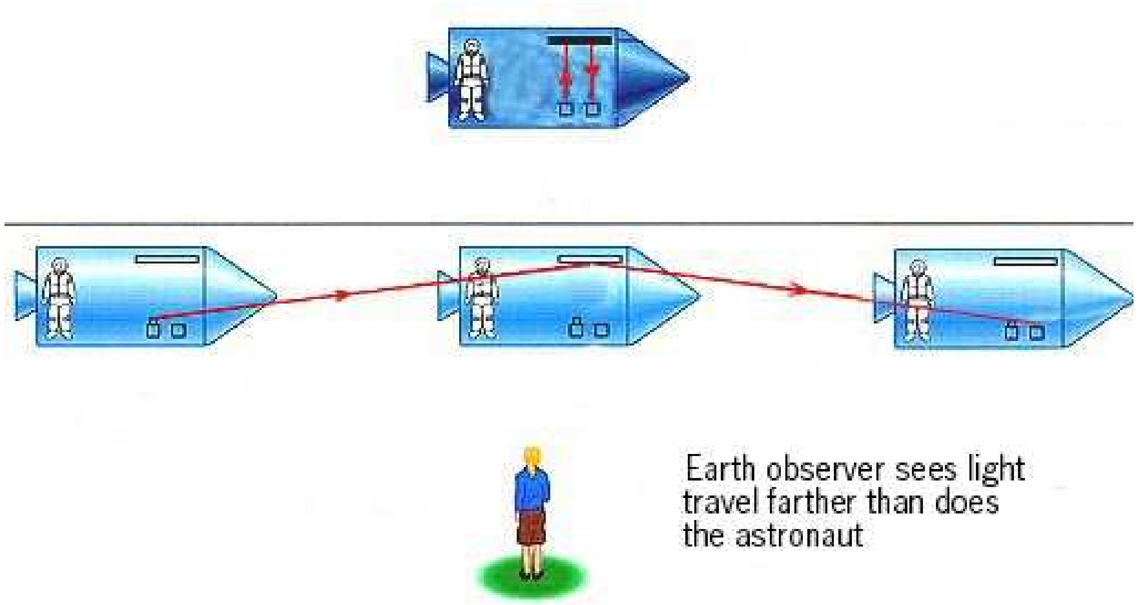
➡ Both observers see the other object contracted

➡ No inertial frame is privileged

- Time dilation -

Time dilation - the dilemma

Reflection of light between 2 mirrors seen in rocket and earth



Does longer travel for the same time mean that c is different ?

Time dilation - derivation

A clock measures time differences:

$\Delta t \;=\; t_2 \;-\; t_1$ **in frame F**

$\Delta t' \;=\; t'_2 \;-\; t'_1$ **in frame F'**

For Lorentz transformation of time in moving frame to rest frame we have:

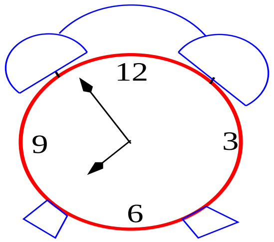
$$t'_1 = \gamma(t_1 - \frac{v \cdot x}{c^2}) \quad \text{and} \quad t'_2 = \gamma(t_2 - \frac{v \cdot x}{c^2})$$

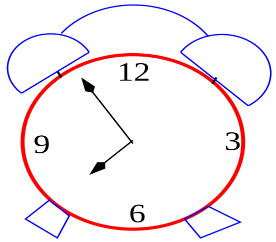
$$\Delta t' = t'_2 - t'_1 = \gamma \cdot (t_2 - t_1) = \gamma \cdot \Delta t$$

$$\Rightarrow \boxed{\Delta t' = \gamma \Delta t}$$

Seen from the rest frame: time in moving frame goes slower ..

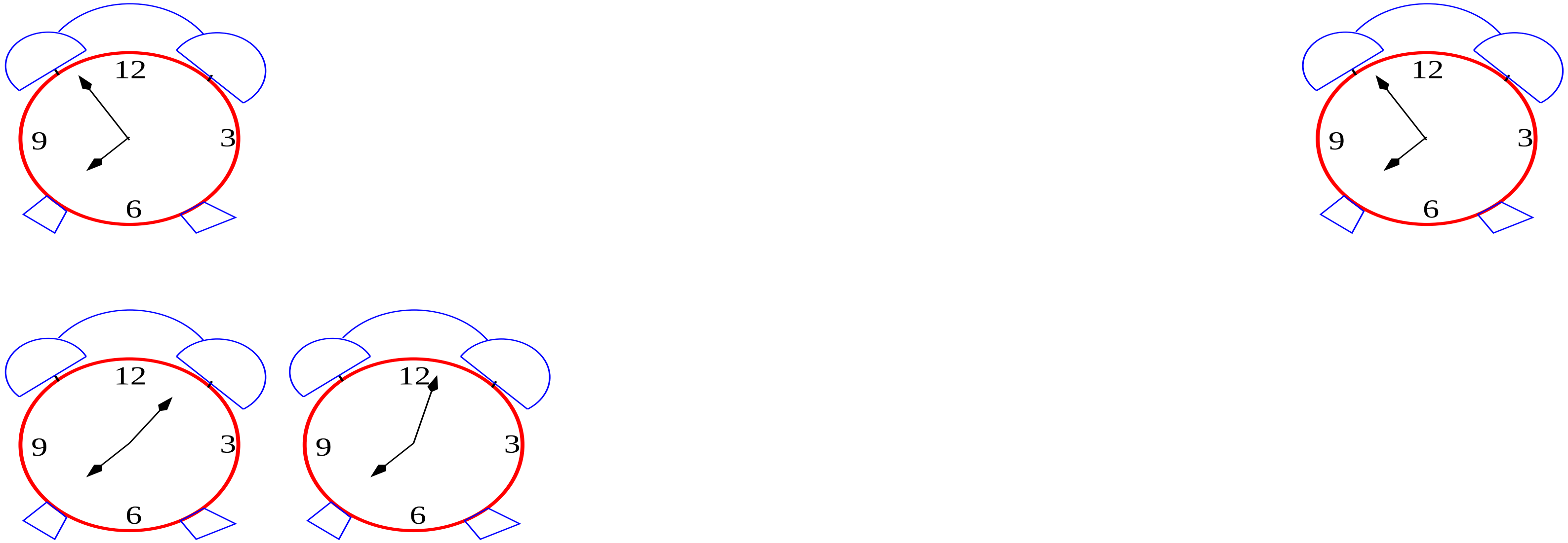
Moving clocks go slower





Moving clocks go slower

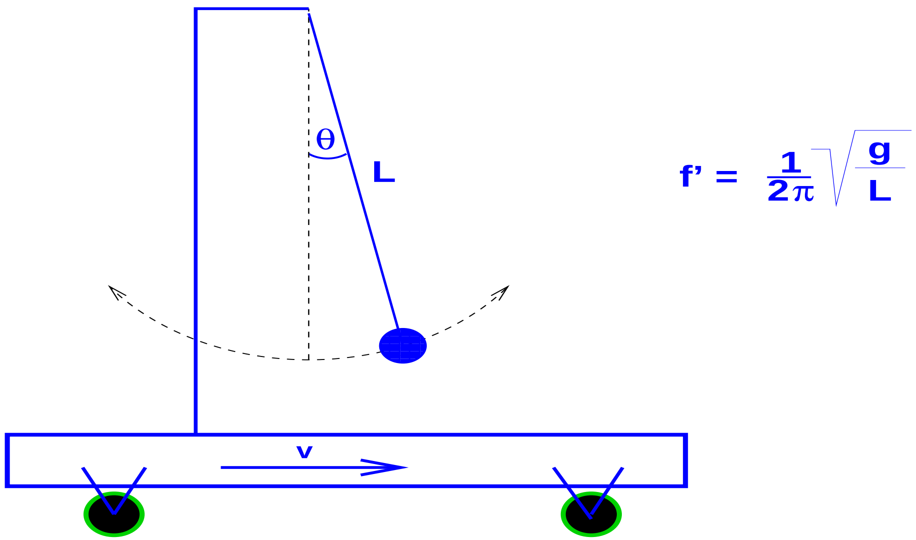
Ten minutes later ...



Travel by airplane:

On a flight from Montreal to Geneva, the time is slower by 25 - 30 ns (considering only special relativity) !

Remember the pendulum ?



From the outside observer:

<u>**Galilei:**</u> **frequency** f **the same in all inertial systems**

<u>**Einstein:**</u> **frequency** f' **smaller by factor** γ **(seen from rest system)**

Time dilation

In moving frame time appears to run slower

Why do we care ?

- μ **have lifetime of 2** μ**s (**$\equiv$ **600 m)**
- **For** $\gamma \geq$ **150, they survive 100 km to reach earth from upper atmosphere**
- **They can survive more than 2** μ**s in a** μ**-collider**
- **Generation of neutrinos from the SPS beams**

Proper Length and Proper Time

Time and distances are relative :

- τ **is a fundamental time: proper time** τ
- **The time measured by an observer in its own frame**
- **From frames moving relative to it, time appears longer**

- $\mathcal{L}$ **is a fundamental length: proper length** $\mathcal{L}$
- **The length measured by an observer in its own frame**
- **From other frames it appears shorter**

The importance of "proper time"

$\Delta\tau$ is the time interval measured <u>inside</u> the moving frame

Back to μ-decay

- **μ lifetime is $\approx$ 2 μs**
- **μ decay in $\approx$ 2 μs in their frame, i.e. using the "proper time"**
- **μ decay in $\approx$ $\gamma\cdot$ 2 μs in the laboratory frame, i.e. earth**
- **μ appear to exist longer than 2 μs in the laboratory frame, i.e. earth**

The meaning of "proper time"

- **How to make neutrinos (e.g. CNGS) ??**
  - **Let pions decay:** $\pi \rightarrow \mu + \nu_\mu$
  - π**-mesons have lifetime of 2.6** $\cdot$ **10**$^{-8}$ **s (i.e. 7.8 m)**
  - **For 40 GeV** π**-mesons:** $\gamma = 288$
  - **In laboratory frame: decay length is 2.25 km (required length of decay tunnel)**

First summary

- **Physics laws the same in different moving frames ...**
- **Speed of light c is maximum possible speed and constant**
- **Constant speed of light requires Lorentz transformation**
- **Moving objects appear shorter**
- **Moving clocks seem to go slower**
- **No absolute space or time: where it happens and when it happens is not independent**

→ **Next: how to calculate something and applications ...**

Introducing four-vectors

Since space and time are not independent, must reformulate physics taking both into account:

Separated time and space (Euclidean space):

$$t, \quad \vec{a} \;=\; (x, y, z)$$

Replace by vector including the time (Minkowski space):

$$A \;=\; (ct, x, y, z)$$

(time t multiplied by c to get the same units)
This is a position four-vector, you also find a^μ instead of A

Definitions of four-vectors

Not a unique definition in literature, one can find:

$$
\begin{aligned}
&= (ct, x, y, z) \\
&= (ct, -x, -y, -z) \\
&= (x, y, z, ct) \\
&= (-x, -y, -z, ct) \\
&= (ict, x, y, z) \\
&= (-ict, x, y, z) \\
&= (....)
\end{aligned}
$$

Always define them when you use them !

With four-vectors, Lorentz transformation can be written in a compact form with matrix multiplication:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Nota bene: this matrix is also used for Lorentz transformation of fields, Lorentz force, derivatives, etc. ...

Scalar products revisited

Define a scalar product for (usual) vectors like: $\vec{a} \cdot \vec{b}$,

$$\vec{a} \;=\; (x_a, y_a, z_a) \qquad\qquad \vec{b} \;=\; (x_b, y_b, z_b)$$

Standard definition (Euclidean geometry):

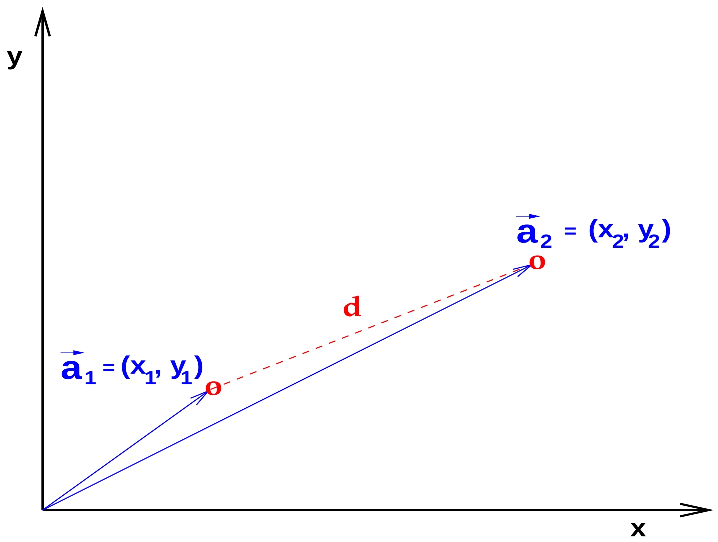
$$\vec{a} \cdot \vec{b} \;=\; (x_a, y_a, z_a) \;\cdot\; (x_b, y_b, z_b) = (x_a \cdot x_b \;+ y_a \cdot y_b \;+\; z_a \cdot z_b)$$

This product of two vectors is a <u>scalar</u> (number) not a vector, and it has a meaning:

$$\vec{a} \cdot \vec{a} \;=\; (x_a, y_a, z_a) \;\cdot\; (x_a, y_a, z_a) = (x_a \cdot x_a \;+\; y_a \cdot y_a \;+\; z_a \cdot z_a) \;=\; d^2$$

d **is the length of the vector** $\vec{a}$ **!**

More: distance between events in space



Distance between two points (here in 2D): d

$$d^2 \;=\; (\vec{a}_2 \;-\; \vec{a}_1)^2$$

$$d^2 \;=\; (x_2 - x_1,\; y_2 - y_1)^2 \;=\; (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Distance d **is always positive !**

Scalar products for four-vectors

Define a scalar product for four-vectors like: $A \odot B$

$$A = (ct_a, x_a, y_a, z_a) \qquad B = (ct_b, x_b, y_b, z_b)$$

$$A \odot B = ct_a \cdot ct_b - \vec{a} \cdot \vec{b} = (ct_a \cdot ct_b \; - \; x_a \cdot x_b \; - \; y_a \cdot y_b \; - \; z_a \cdot z_b)$$

Note the – sign !!

Does it have a meaning ?

Distance between events in space-time

We can describe a distance in the space-time between two points A_1 and A_2:

$$\Delta X = A_2 - A_1 = (ct_2 - ct_1,\ x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$$

Scalar product of the difference is the distance^2 $= D^2$:

$$D^2 \ = \ \Delta A^2 \ = \ c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

D^2 **can be positive (time-like) or negative (space-like)**

Another example: $X = (ct, \vec{x})$, $X' = (ct', \vec{x'})$

$$X \odot X = c^2t^2 - x^2 - y^2 - z^2$$

and

$$X' \odot X' = c^2t'^2 - x'^2 - y'^2 - z'^2$$

we have:

$$X \odot X = X' \odot X'$$

because this is our condition for constant speed of light c **!**

This product is an invariant

Invariant Quantities have the same value in all inertial frames (like c**)**

Why bother about four-vectors ?

We have seen the importance of invariants:

- **Ensure equivalence of physics laws in different frames**
- **The solution: write the laws of physics in terms of four vectors**
- **Without proof: any four-vector (scalar) product $F \odot F$ has the same value in all coordinate frames moving at constant velocities with respect to each other:**

$$F \odot F = F' \odot F'$$

Scalar products of four-vectors are invariant !

We have important four-vectors:

$$\begin{array}{ll} \text{Coordinates}: & X = (ct, x, y, z) = (ct, \vec{x}) \\ \text{Velocities}: & V = \frac{dX}{dt} \cdot \frac{dt}{d\tau} \;=\; \gamma(c, \dot{\vec{x}}) = \gamma(c, \vec{v}) \\ \text{Momenta}: & P = mV = m\gamma(c, \vec{v}) = \gamma(mc, \vec{p}) \\ \text{Force}: & F = \frac{dP}{dt} = \gamma\frac{d}{dt}(mc, \vec{p}) \end{array}$$

Any scalar product of two four-vectors:

$X \odot X$**,** $V \odot V$**,** $P \odot P$**,** $P \odot X$**,** $V \odot F$**,... are ALL invariants**

A special invariant

From the velocity four-vector V:

$$V \;=\; \gamma(c, \vec{v})$$

we get the scalar product:

$$V \odot V = \gamma^2(c^2 - \vec{v}^2) = c^2 \;\; !!$$

➡ c **is an invariant, has the same value in all inertial frames**

$$V \;\odot\; V \;\;=\;\; V' \;\odot\; V'$$

➡ **The invariant of the velocity four-vector** V **is the speed of light** c **!**

Transformation of mass

We want the invariance of the formula:

$$\vec{F} \;=\; m \cdot \vec{a} \;=\; m \cdot \frac{d\vec{v}}{dt}$$

Without proof (see e.q. Feynman) momentum conservation in all directions requires that the mass m must also be transformed:

$$m \;=\; m' / \sqrt{1 - \left(\frac{v}{c}\right)^2} \;=\; \gamma \cdot m'$$

In a frame with $v = 0$ we call the mass the <u>rest mass</u> m_0
If the frame moves with a velocity v relative to an observer, she will find the mass increased by the factor γ

Dynamics with four-vectors

Using the expression for the mass m:

$$m \;=\; m_0\sqrt{1-\left(\frac{v}{c}\right)^2} \;=\; \gamma \cdot m_0$$

and expand it for small velocities:

$$m \;\cong\; m_0 \;+\; \frac{1}{2}m_0 v^2\left(\frac{1}{c^2}\right)$$

and multiplied by c^2:

$$mc^2 \;\cong\; m_0c^2 \;+\; \frac{1}{2}m_0 v^2$$

The second term is the kinetic energy

Relativistic energy

Interpretation:

$$E = mc^2 = m_0c^2 + T$$

- **Total energy** E **is** $E = mc^2$
- **Sum of kinetic energy plus rest energy**
- **Energy of particle at rest is** $E_0 = m_0c^2$

$$\boxed{E = m \cdot c^2 = \gamma m_0 \cdot c^2}$$

using the definition of relativistic mass again: $m = \gamma m_0$

Interpretation of relativistic energy

- **For any object, $m \cdot c^2$ is the total energy**
  - **Object can be composite, like proton ..**
  - **m is the mass (energy) of the object "in motion"**
  - **m_0 is the mass (energy) of the object "at rest"**
- **The mass m is not the same in all inertial systems, the rest mass m_0 is ! (To prove it, try $P \odot P'$)**
- **For discussion: what is the mass m of a photon ?**

Practical units

Standard units are not very convenient, easier to use:

$$[\mathbf{E}] = \mathbf{eV} \qquad [\mathbf{p}] = \mathbf{eV}/c \qquad [\mathbf{m}] = \mathbf{eV}/c^2$$

Mass of a proton: $m_p = 1.672 \cdot 10^{-27}$ **Kg**
Energy(at rest): $m_p c^2 = 938$ **MeV = 0.15 nJ**

- **Other example, 1 gram of material equivalent to:**
  - **250000 times the full LHC beam** ($9 \cdot 10^{13}$ **J**)
  - **21.5 kilotons of TNT**

Relativistic mass

The mass of a fast moving particle is increasing like:

$$m = \gamma m_0 = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

- **Why do we care ?**
  - **Particles cannot go faster than c !**
  - **What happens when we accelerate ?**

Relativistic mass

When we accelerate:

- **For v $\ll$ c:**
  - **- E, m, p, v increase ...**
- **For v $\approx$ c:**
  - **- E, m, p increase, but v does not !**

Relativistic energy

We remember that:

$$T \;=\; m_0(\gamma - 1)c^2$$

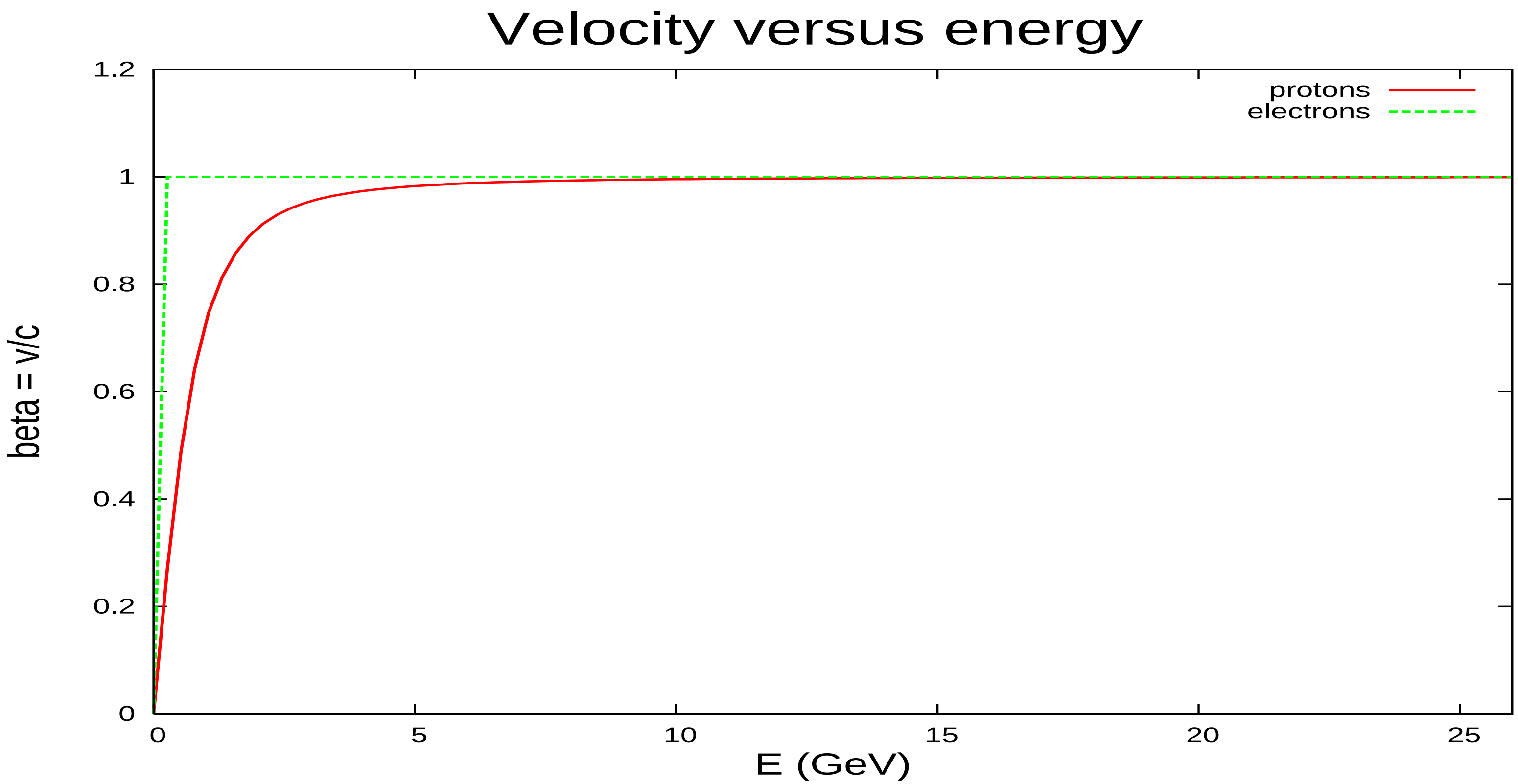
therefore:

$$\gamma \;=\; 1 + \frac{T}{m_0 c^2}$$

we get for the speed v, i.e. β:

$$\beta \;=\; \sqrt{1 - \frac{1}{\gamma^2}}$$

Velocity versus energy (protons)



Why do we care ??

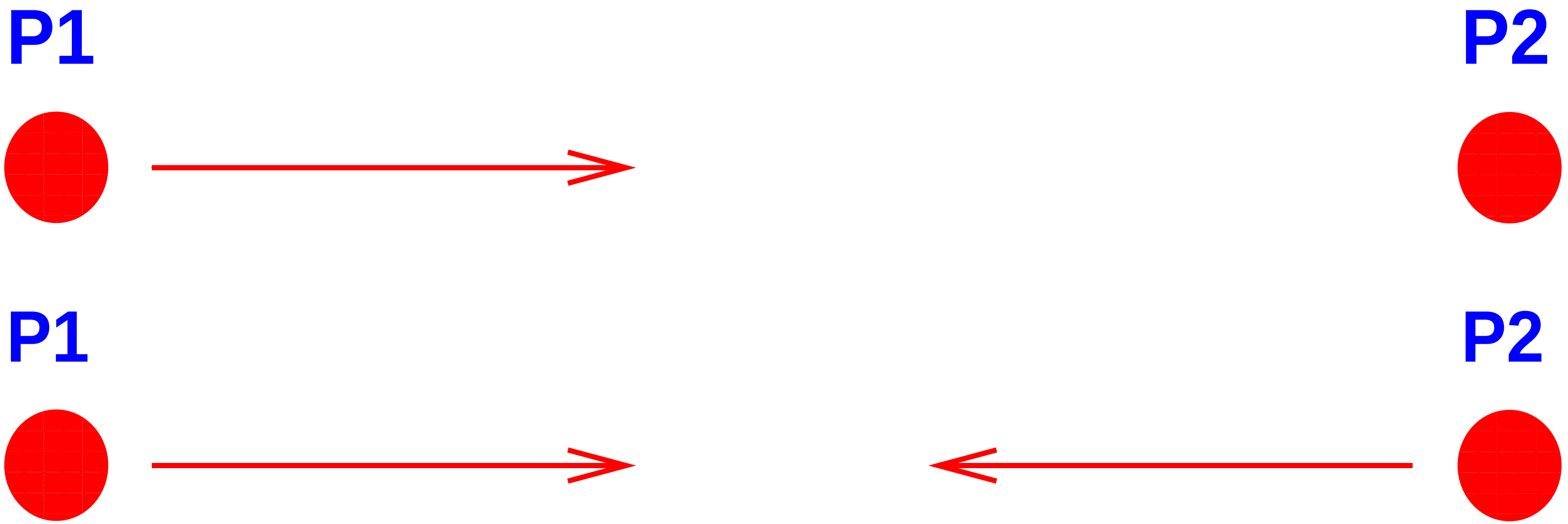
E (GeV)	v (km/s)	γ	β	T (LHC)
450	299791.82	479.74	0.99999787	88.92465 μs
7000	299792.455	7462.7	0.99999999	88.92446 μs

- **For identical circumference very small change in revolution time**
- **If path for faster particle slightly longer, the faster particle arrives later !**

Four vectors

- Use of four-vectors simplify calculations significantly
- Follow the rules and look for invariants
- In particular kinematic relationships, e.g.
  - Particle decay (find mass of parent particle)
  - Particle collisions

Particle collisions



What is the available collision energy ?

Particle collisions - collider

Assume identical particles and beam energies, colliding head-on

The four momentum vectors are:

$$P1 = (E, \vec{p}) \qquad P2 = (E, -\vec{p})$$

The four momentum vector in centre of mass system is:

$$P^* = P1 + P2 = (E + E, \vec{p} - \vec{p}) = (2E, \vec{0})$$

Particle collisions - collider

The four momentum vector in centre of mass system is:

$$P^* = P1 + P2 = (E + E, \vec{p} - \vec{p}) = (2E, \vec{0})$$

The square of the total available energy s in the centre of mass system is the momentum invariant:

$$s = P^* \odot P^* = 4E^2$$

$$E_{cm} = \sqrt{P^* \odot P^*} = 2E$$

i.e. in a (symmetric) collider the total energy is twice the beam energy

Particle collisions - fixed target

The four momentum vectors are:

$$P1 = (E, \vec{p}) \qquad P2 = (m_0, \vec{0})$$

The four momentum vector in centre of mass system is:

$$P^* = P1 + P2 = (E + m_0, \vec{p})$$

Particle collisions - fixed target

With the above it follows:

$$P^* \odot P^* = E^2 + 2m_0E + m_0^2 - \vec{p}^2$$

since $E^2 - \vec{p}^2 = m_0^2$ **we get:**

$$s = 2m_0E + m_0^2 + m_0^2$$

if E **much larger than** m_0 **we find:**

$$E_{cm} = \sqrt{s} = \sqrt{2m_0E}$$

Particle collisions - fixed target

Homework: try for $E1 \neq E2$ and $m1 \neq m2$

Examples:

collision	**beam energy**	$\sqrt{s}$ **(collider)**	$\sqrt{s}$ **(fixed target)**
pp	**315 (GeV)**	**630 (GeV)**	**24.3 (GeV)**
pp	**7000 (GeV)**	**14000 (GeV)**	**114.6 (GeV)**
e+e−	**100 (GeV)**	**200 (GeV)**	**0.320 (GeV)**

Kinematic invariant

We need to make cross sections (and therefore luminosity) invariant !
This is done by a calibration factor which is (without derivation):

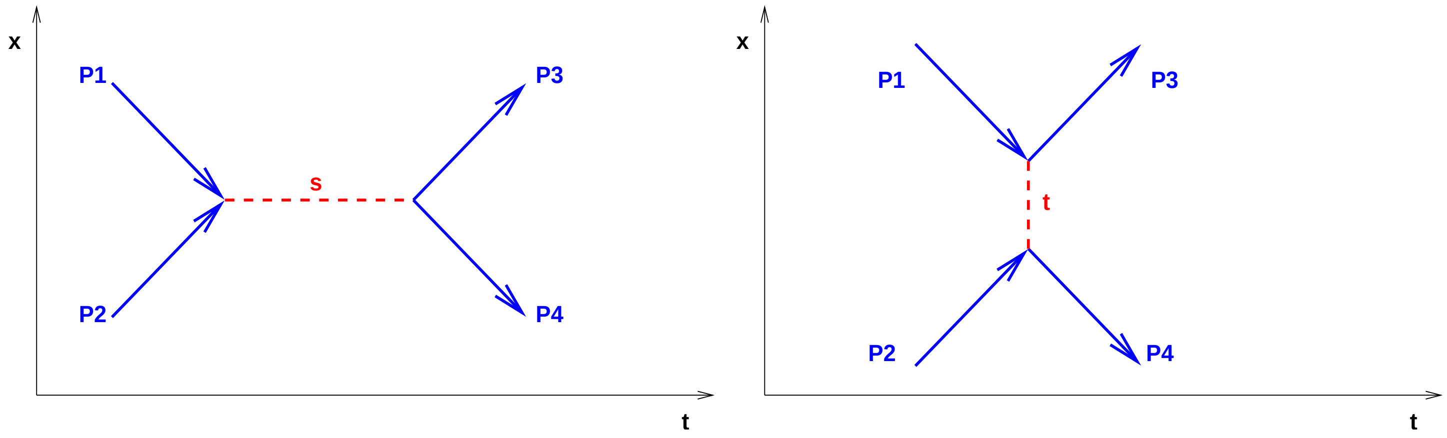
$$K = \sqrt{(\vec{v_1} - \vec{v_2})^2 - (\vec{v_1} \times \vec{v_2})^2/c^2}$$

Here $\vec{v_1}$ and $\vec{v_2}$ are the velocities of the two (relativistic) beams.
For a (symmetric) collider, e.g. LHC, we have:

$$\vec{v_1} = -\vec{v_2}, \qquad \vec{v_1} \times \vec{v_2} \ = \ 0 \quad \mathrm{head-on!}$$

$\rightarrow$ $K \ = \ 2 \ \cdot \ c$ **!**

For completeness ...



Squared centre of mass energy:

$$s = (P1 + P2)^2 = (P3 + P4)^2$$

Squared momentum transfer in particle scattering (small t - small angle, see again lecture on Luminosity):

$$t = (P1 - P3)^2 = (P2 - P4)^2$$

Kinematic relations

We have already seen a few, e.g.:

- $T = E - E_0 = (\gamma - 1)E_0$
- $E = \gamma \cdot E_0$
- $E_0 = \sqrt{E^2 - c^2p^2}$
- **etc. ...**

Very useful for everyday calculations ➡

Kinematic relations

	cp	**T**	**E**	γ
$\beta =$	$\frac{1}{\sqrt{(\frac{E_0}{cp})^2+1}}$	$\sqrt{1-\frac{1}{(1+\frac{T}{E_0})^2}}$	$\sqrt{1-(\frac{E_0}{E})^2}$	$\sqrt{1-\gamma^{-2}}$
cp =	cp	$\sqrt{T(2E_0+T)}$	$\sqrt{E^2-E_0^2}$	$E_0\sqrt{\gamma^2-1}$
$E_0 =$	$\frac{cp}{\sqrt{\gamma^2-1}}$	$T/(\gamma-1)$	$\sqrt{E^2-c^2p^2}$	E/γ
T =	$cp\sqrt{\frac{\gamma-1}{\gamma+1}}$	**T**	$E-E_0$	$E_0(\gamma-1)$
$\gamma =$	$cp/E_0\beta$	$1+T/E_0$	E/E_0	γ

Kinematic relations

Example: CERN Booster

At injection: T = 50 MeV

➡ $\mathbf{E = 0.988\ GeV,\ p = 0.311\ GeV/c}$

➡ $\gamma = \mathbf{1.0533}$, $\beta = \mathbf{0.314}$

At extraction: T = 1.4 GeV

➡ $\mathbf{E = 2.338\ GeV,\ p = 2.141\ GeV/c}$

➡ $\gamma = \mathbf{2.4925}$, $\beta = \mathbf{0.916}$

Kinematic relations - logarithmic derivatives

	$\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{dE}{E} = \frac{d\gamma}{\gamma}$
$\frac{d\beta}{\beta} =$	$\frac{d\beta}{\beta}$	$\frac{1}{\gamma^2}\frac{dp}{p}$	$\frac{1}{\gamma(\gamma+1)}\frac{dT}{T}$	$\frac{1}{(\beta\gamma)^2}\frac{d\gamma}{\gamma}$
$\frac{dp}{p} =$	$\gamma^2\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$[\gamma/(\gamma+1)]\frac{dT}{T}$	$\frac{1}{\beta^2}\frac{d\gamma}{\gamma}$
$\frac{dT}{T} =$	$\gamma(\gamma+1)\frac{d\beta}{\beta}$	$\left(1+\frac{1}{\gamma}\right)\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{\gamma}{(\gamma-1)}\frac{d\gamma}{\gamma}$
$\frac{dE}{E} =$	$(\beta\gamma)^2\frac{d\beta}{\beta}$	$\beta^2\frac{dp}{p}$	$\left(1-\frac{1}{\gamma}\right)\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$
$\frac{d\gamma}{\gamma} =$	$(\gamma^2-1)\frac{d\beta}{\beta}$	$\frac{dp}{p} - \frac{d\beta}{\beta}$	$\left(1-\frac{1}{\gamma}\right)\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$

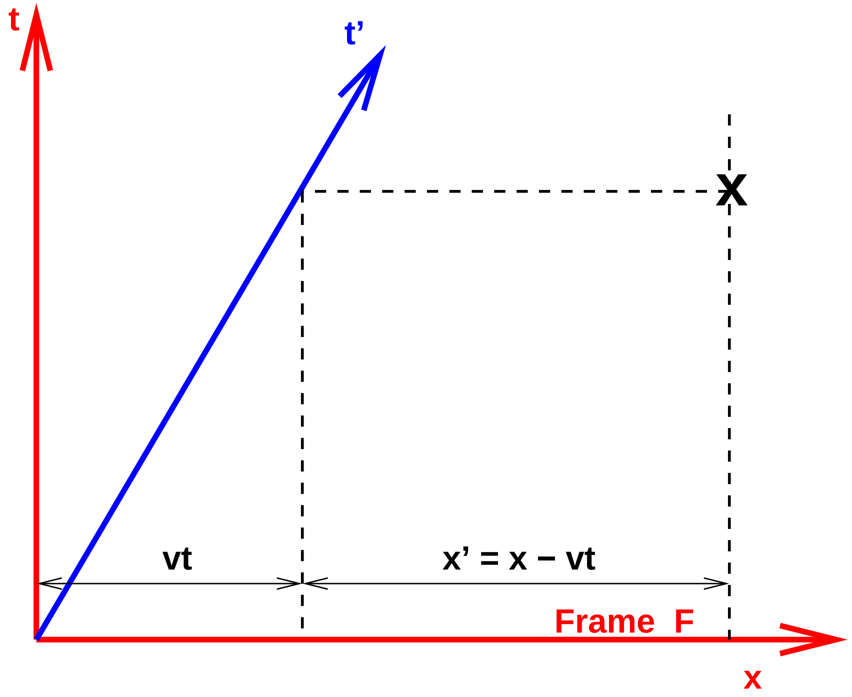
Example LHC (7 TeV): $\frac{\Delta p}{p} \approx 10^{-4} \Rightarrow \frac{\Delta\beta}{\beta} = \frac{\Delta v}{v} \approx 2 \cdot 10^{-12}$

Summary

- **Special Relativity is very simple, derived from basic principles**
- **Relativistic effects vital in accelerators:**
  - **Lorentz contraction and Time dilation**
  - **Invariants !**
  - **Relativistic mass effects**
  - **Modification of electromagnetic field**
- **Find back in later lectures ...**

- BACKUP SLIDES -

Galilei transformation - schematic



➤ **Rest frame and Galilei transformation ...**

Forces and fields

Motion of charged particles in electromagnetic fields $\vec{E}, \vec{B}$ determined by Lorentz force

$$\vec{f} = \frac{d}{dt}(m_0 \gamma \vec{v}) = q \cdot (\vec{E} + \vec{v} \times \vec{B})$$

or as four-vector:

$$F = \frac{dP}{d\tau} = \gamma \left(\frac{\vec{v} \cdot \vec{f}}{c}, \vec{f} \right) = \gamma \left(\frac{1}{c} \frac{dE}{dt}, \frac{d\vec{p}}{dt} \right)$$

Field tensor

Electromagnetic field described by field-tensor $F^{\mu\nu}$:

$$F^{\mu\nu} = \begin{pmatrix} 0 & \frac{-E_x}{c} & \frac{-E_y}{c} & \frac{-E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

derived from four-vector $A_\mu = (\Phi, \vec{A})$ like:

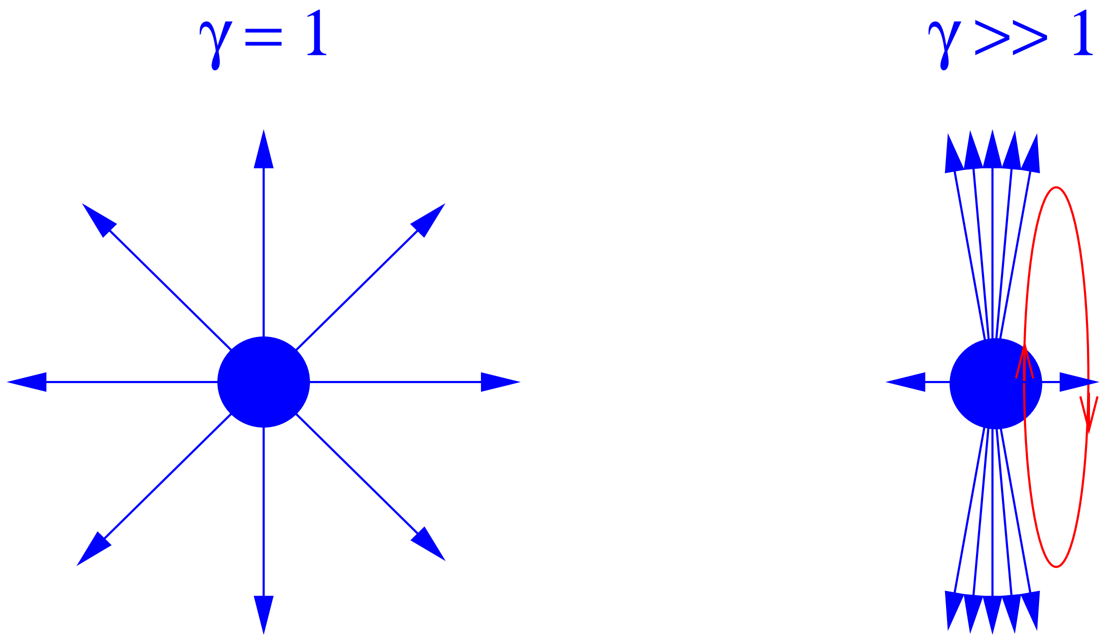
$$F^{\mu\nu} = \delta^\mu A^\nu - \delta^\nu A^\mu$$

Lorentz transformation of fields

$$
\begin{aligned}
\vec{E}'_{\perp} &= \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B}) \\
\vec{B}'_{\perp} &= \gamma\left(\vec{B}_{\perp} - \frac{\vec{v}\times\vec{E}}{c^2}\right) \\
\vec{E}'_{\parallel} &= \vec{E}_{\parallel} \\
\vec{B}'_{\parallel} &= \vec{B}_{\parallel}
\end{aligned}
$$

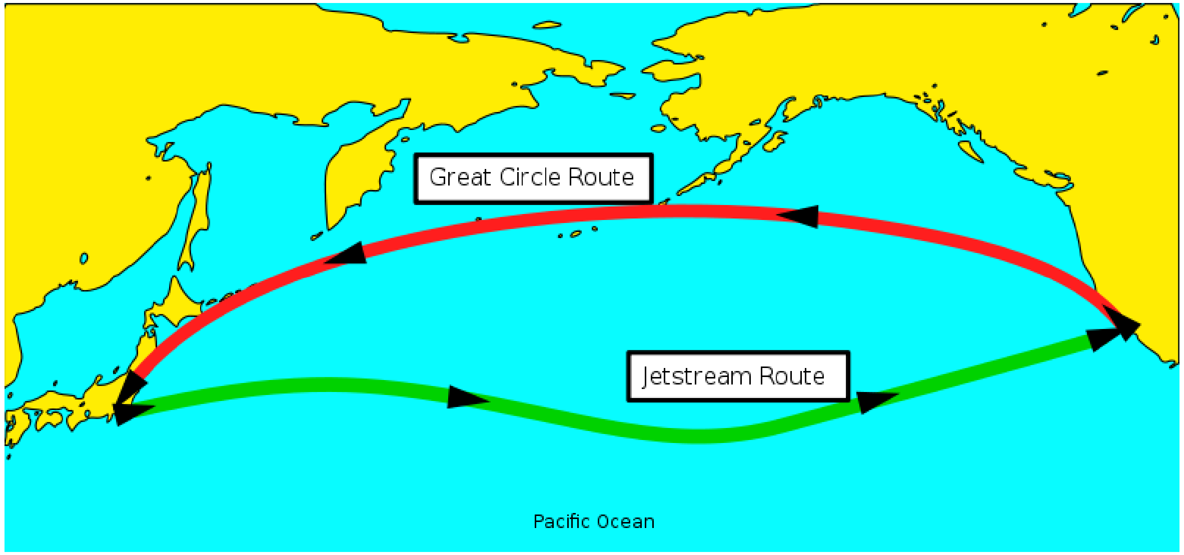
- **Field perpendicular to movement transform**

Lorentz transformation of fields



- **In rest frame purely electrostatic forces**
- **In moving frame $\vec{E}$ transformed and $\vec{B}$ appears**

Addition of velocities (Galilei)
(an everyday example ...)



Jetstream up to 350 - 400 km/hour !

Can save one 1 hour or more on an eastbound flight !