Transverse Beam Dynamics II

II) The State of the Art in High Energy Machines: ♪

The Theory of Synchrotrons:
Linear Beam Optics
The Beam as Particle Ensemble
Emittance and Beta-Function
Colliding Beams & Luminosity
„... how does it work ?"
„...does it ?"

Astronomer Hill:

differential equation for motions with periodic focusing properties „Hill's equation"

Example: particle motion with periodic coefficient

equation of motion: $x''(s) - k(s)x(s) = 0$

restoring force $\neq$ const,
k(s) = depending on the position s
k(s+L) = k(s), periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.

7.) The Beta Function

„it is convenient to see"

... after some beer ... general solution of Mr Hill can be written in the form:

Ansatz:

$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \phi)$$

ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s+L) = \beta(s)$$

ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter, cannot be changed by the foc. properties.
scientifiquely spoken: area covered in transverse x, x´ phase space … and it is constant !!!

Ψ(s) = „phase advance" of the oscillation between point „0" and „s" in the lattice.
For one complete revolution: number of oscillations per turn „Tune"

$$Q_y = \frac{1}{2\pi} \cdot \int \frac{ds}{\beta(s)}$$

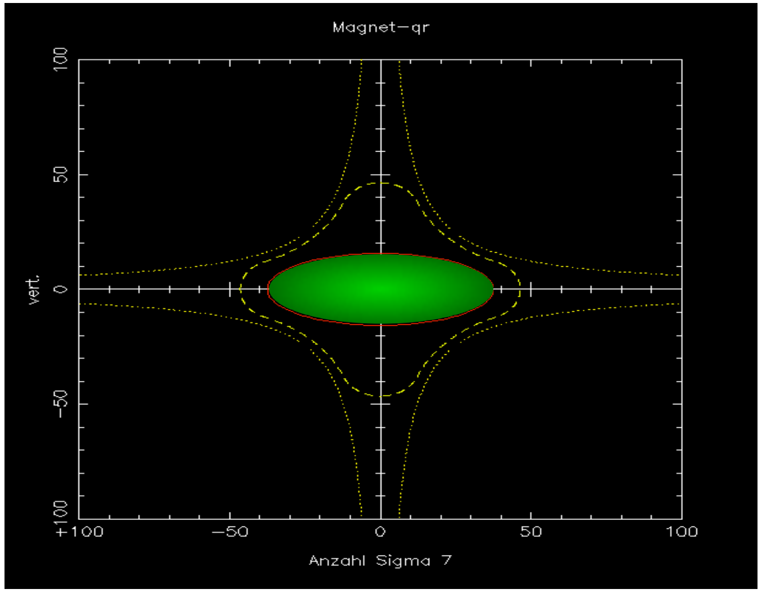
The Beta Function

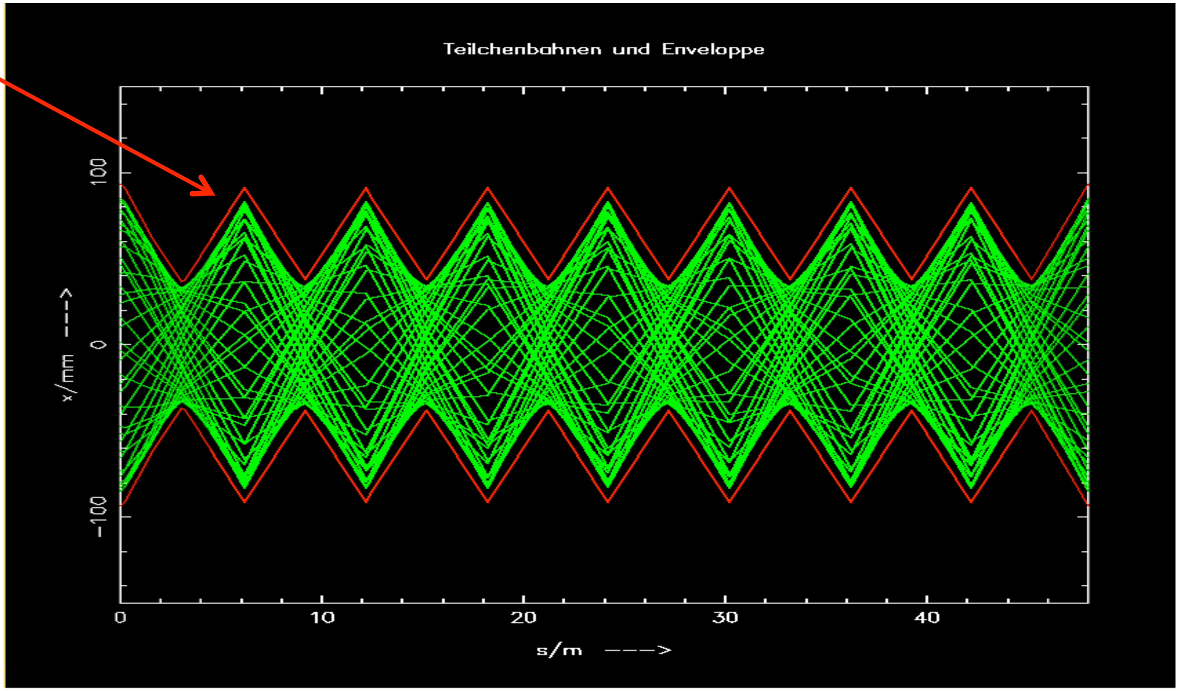
Amplitude of a particle trajectory:

$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \varphi)$$

Maximum size of a particle amplitude

$$\hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$





β determines the beam size
(... the envelope of all particle trajectories at a given position "s" in the storage ring.

It reflects the periodicity of the magnet structure.

8.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

$$\left\{\begin{array}{ll} (1) & x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\phi) \\ (2) & x'(s) = -\dfrac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\,\{\alpha(s)\cos(\psi(s)+\phi)+\sin(\psi(s)+\phi)\} \end{array}\right.$$

from (1) we get

$$\cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\varepsilon}\,\sqrt{\beta(s)}}$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

Insert into (2) and solve for ε

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

**** ε is a constant of the motion … it is independent of „s"***

**** parametric representation of an ellipse in the x x' space***

**** shape and orientation of ellipse are given by α, β, γ***

Phase Space Ellipse

particel trajectory: $x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$

max. Amplitude: $\boxed{\hat{x}(s) = \sqrt{\varepsilon\beta}}$ $\longrightarrow$ ***x´ at that position …?***

… put $\hat{x}(s)$ ***into*** $\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$ ***and solve for x´***

$$\varepsilon = \gamma\cdot\varepsilon\beta + 2\alpha\sqrt{\varepsilon\beta}\cdot x' + \beta x'^2$$

$$\longrightarrow \boxed{x' = -\alpha\cdot\sqrt{\varepsilon/\beta}}$$

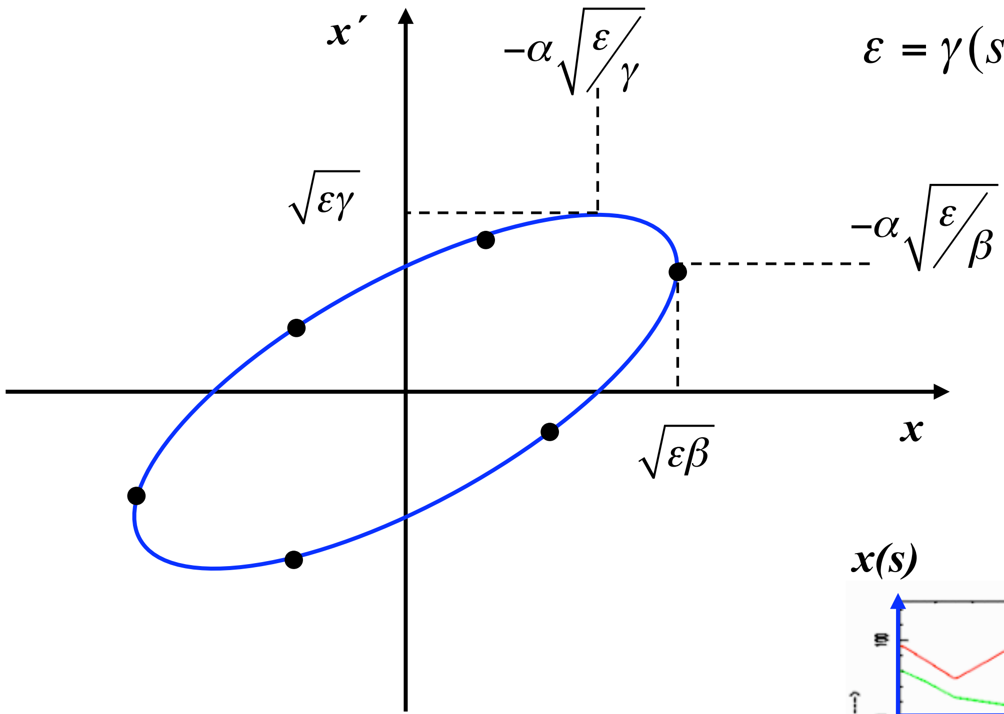
* ***A high β-function means a large beam size and a small beam divergence.*** ***!***
… et vice versa !!!

* ***In the middle of a quadrupole β = maximum,***
α = zero $\Bigg\}$ $x' = 0$

… and the ellipse is flat

Beam Emittance and Phase Space Ellipse

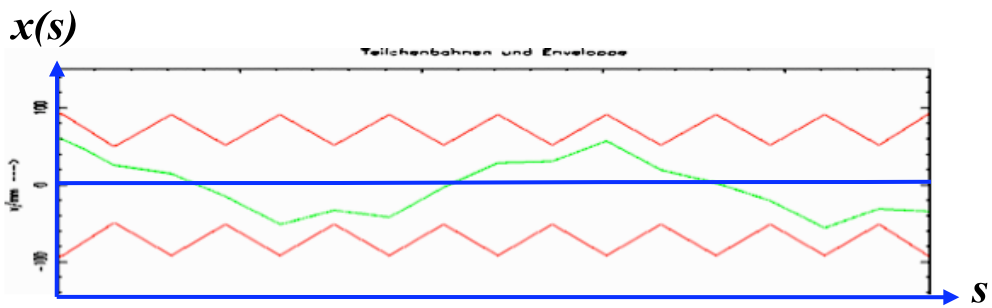
$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \varphi)$$



$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

***Liouville:* in reasonable storage rings**
area in phase space is constant.

$$A = \pi * \varepsilon = const$$

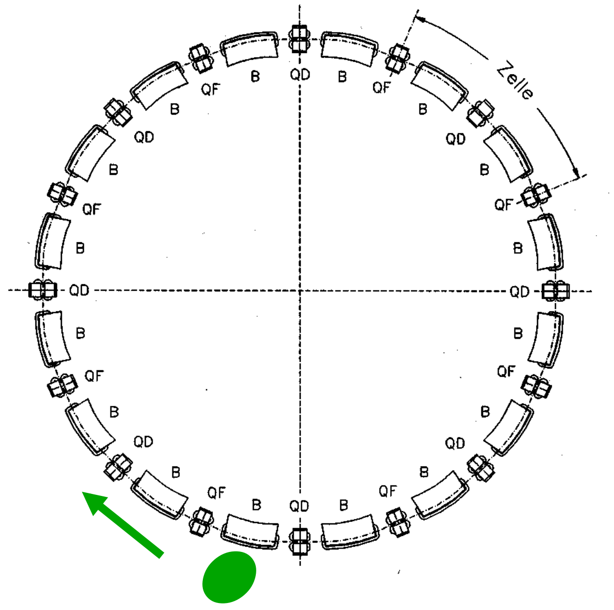


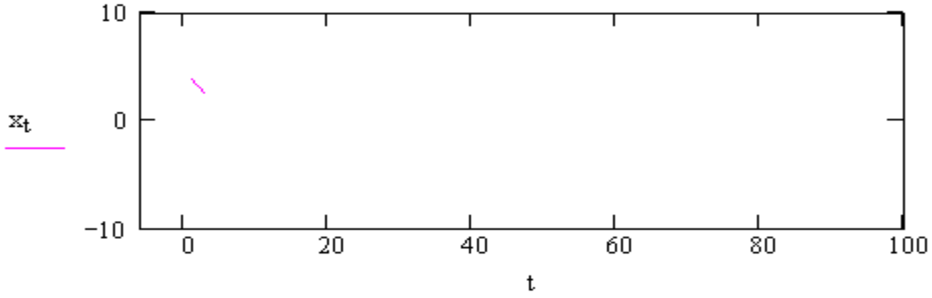
***ε beam emittance** = **woozilycity** of the particle ensemble, **intrinsic beam parameter**,*
cannot be changed by the foc. properties.
***Scientifiquely spoken:** area covered **in transverse x, x´ phase space** … and it is constant !!!*

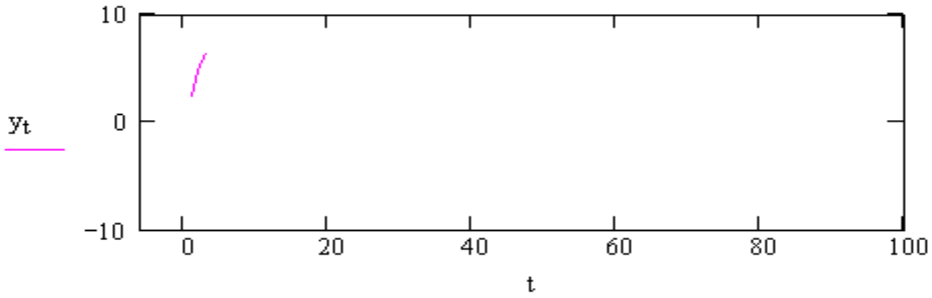
Particle Tracking in a Storage Ring

Calculate x, x´ for each linear accelerator element according to matrix formalism

plot x, x´as a function of „s"

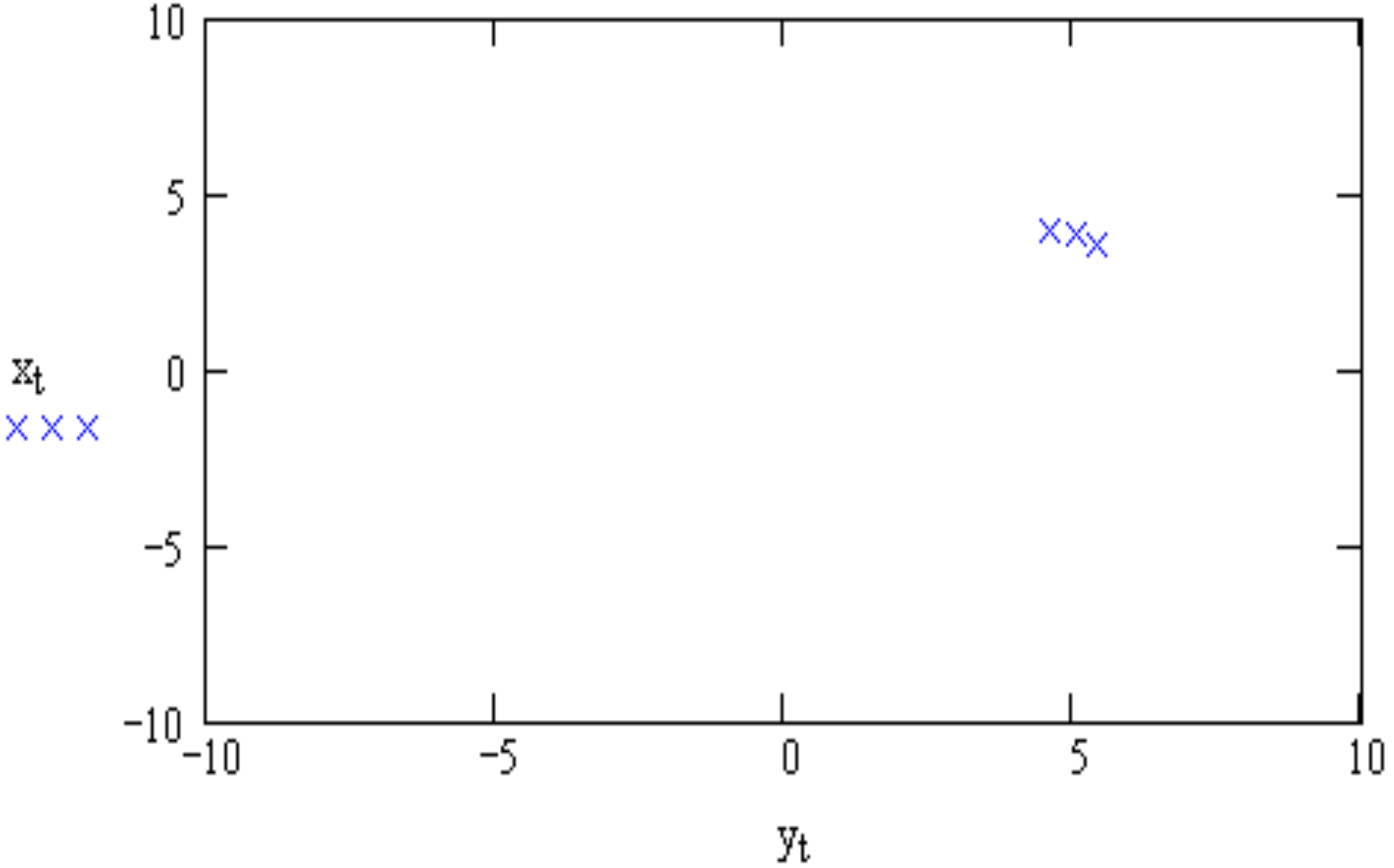




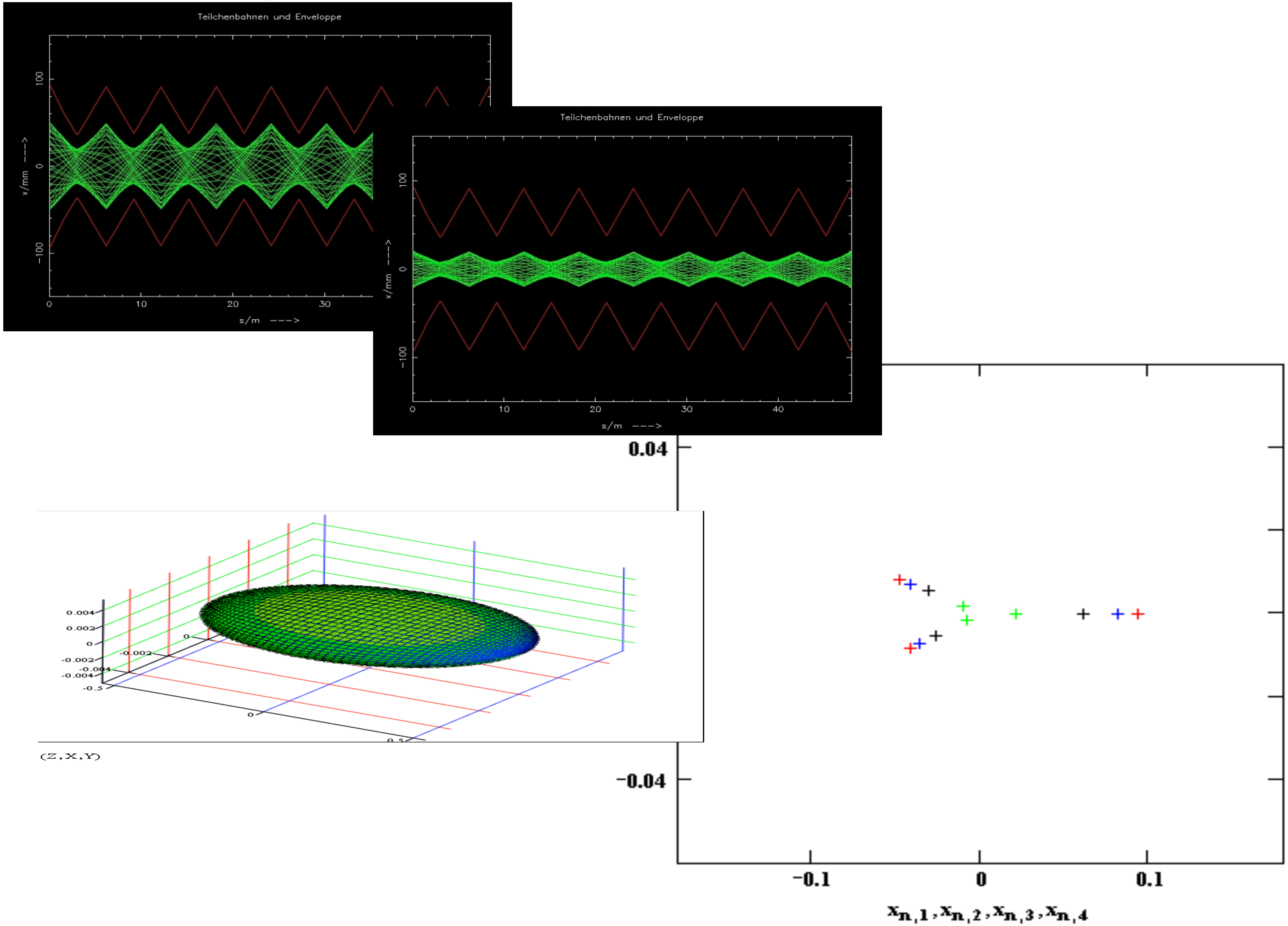


… and now the ellipse:

note for each turn x, x´at a given position „s_1“ and plot in the phase space diagram

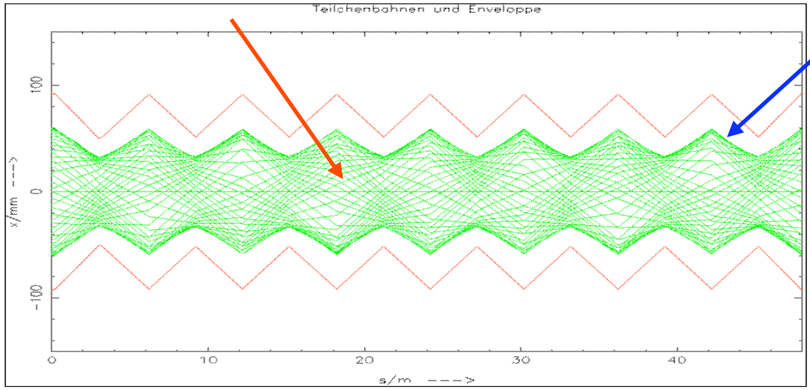


Emittance of the Particle Ensemble:



Emittance of the Particle Ensemble:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\Psi(s)+\phi) \qquad \hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



single particle trajectories, $N \approx 10^{11}$ per bunch

Gauß Particle Distribution:

$$\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$$

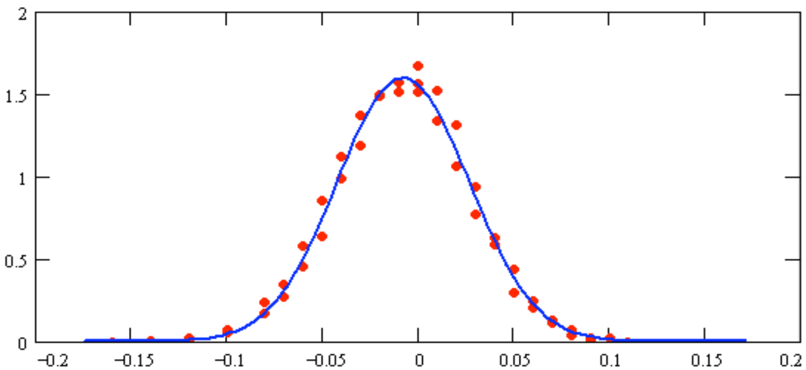
particle at distance 1 σ from centre
↔ 68.3 % of all beam particles

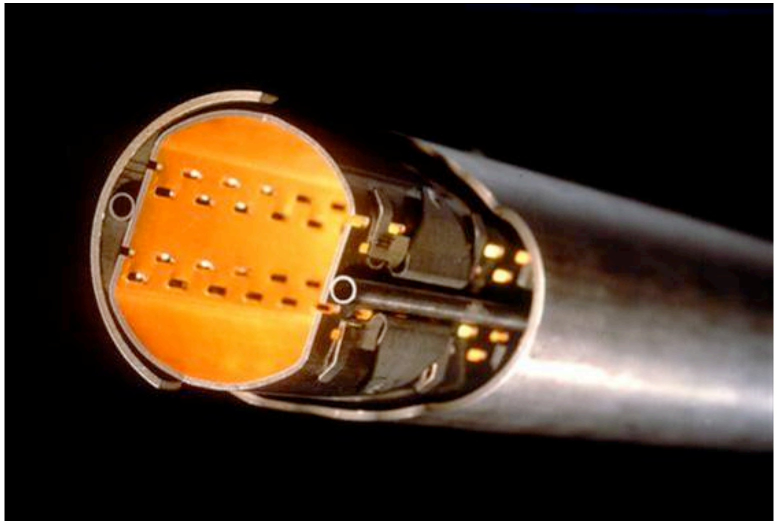
LHC:

$$\beta = 180\, m$$

$$\varepsilon = 5 * 10^{-10}\, m\, rad$$

$$\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5*10^{-10} m * 180\, m} = 0.3\, mm$$

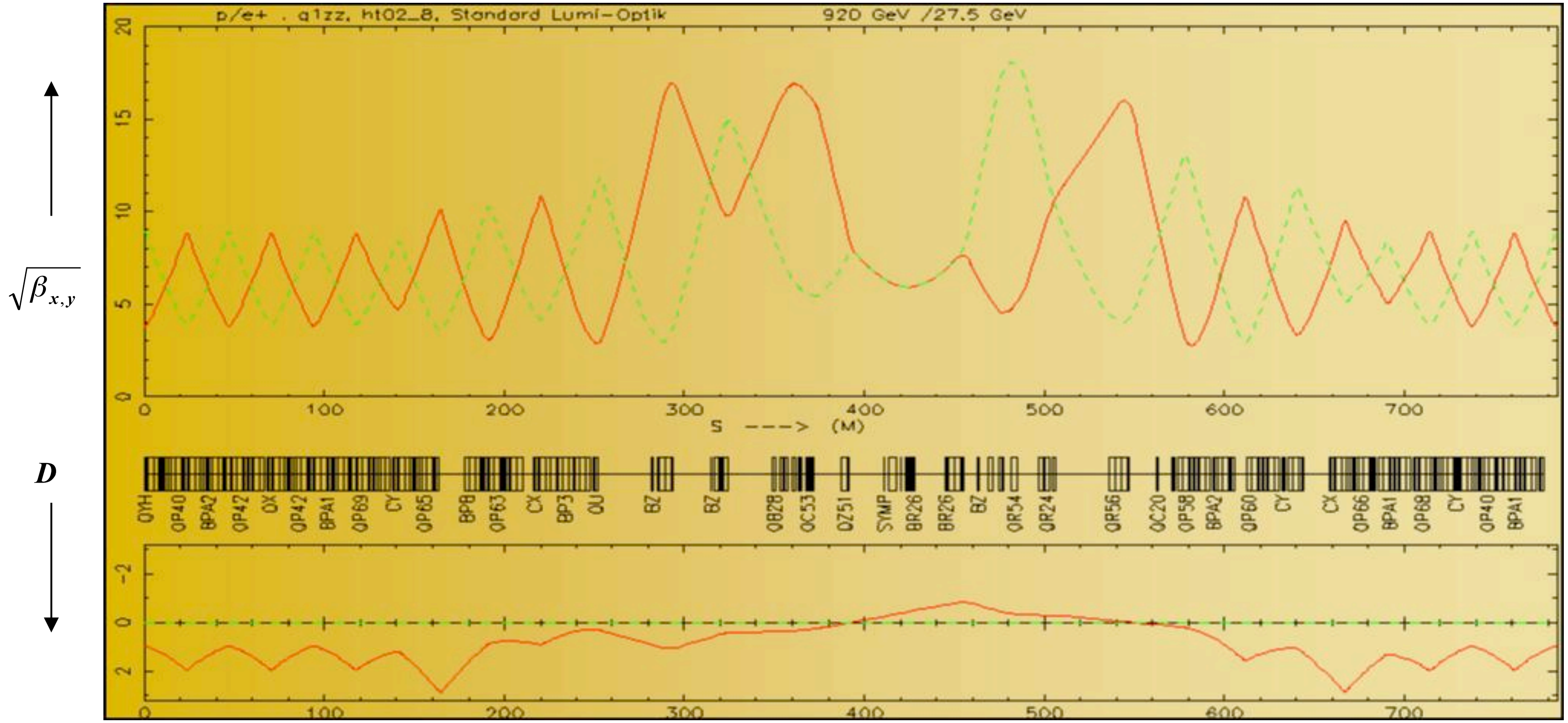




aperture requirements: $r_0 = 12 * \sigma$

The „not so ideal“ World

Lattice Design in Particle Accelerators



1952: Courant, Livingston, Snyder:

Theory of strong focusing in particle beams

Recapitulation: ...the story with the matrices !!!

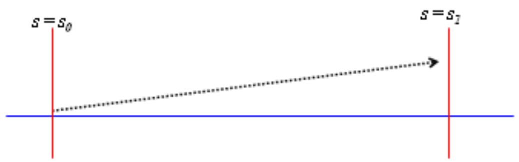
Equation of Motion:

$$x'' + K\,x = 0 \qquad K = 1/\rho^2 - k \;\; \ldots \textit{ hor. plane:}$$

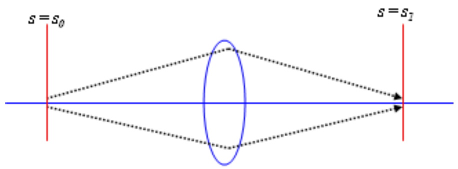
$$K = k \qquad \ldots \textit{ vert. Plane:}$$

Solution of Trajectory Equations

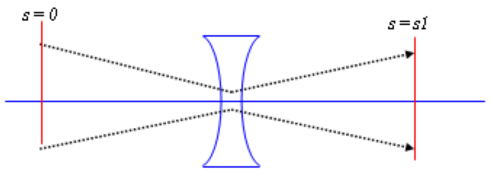
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}l) \\ -\sqrt{|K|}\sin(\sqrt{|K|}l) & \cos(\sqrt{|K|}l) \end{pmatrix}$$



$$M_{defoc} = \begin{pmatrix} \cosh(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}l) \\ \sqrt{|K|}\sinh(\sqrt{|K|}l) & \cosh(\sqrt{|K|}l) \end{pmatrix}$$

$$M_{total} = M_{QF} * M_D * M_B * M_D * M_{QD} * M_D * \quad \ldots$$

9.) Lattice Design: *„… how to build a storage ring“*

Geometry of the ring: $B * \rho = p / e$

p = momentum of the particle,
ρ = curvature radius

Bρ= beam rigidity

Circular Orbit: bending angle of one dipole

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} = \frac{Bdl}{B\rho}$$

The angle run out in one revolution must be 2π, so for a full circle

$$\alpha = \frac{\int Bdl}{B\rho} = 2\pi$$

$$\int Bdl = 2\pi \frac{p}{q}$$

… defines the integrated dipole field around the machine.

Example LHC:

7000 GeV Proton storage ring
dipole magnets N = 1232
l = 15 m
q = +1 e

$$\int B\, dl \approx N\, l\, B = 2\pi\ p / e$$

$$B \approx \frac{2\pi\ 7000\ 10^9\, eV}{1232\ 15\, m\ \ 3\ 10^8 \frac{m}{s}\ \ e} = \underline{\underline{8.3\, Tesla}}$$

10.) Transfer Matrix M … *yes we had the topic already*

general solution of Hill´s equation

$$\begin{cases} x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\} \\ x'(s) = \dfrac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\} + \sin\{\psi(s)+\phi\}\right] \end{cases}$$

remember the trigonometrical gymnastics: sin(a + b) = … etc

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta_s}\left(\cos\psi_s\cos\phi - \sin\psi_s\sin\phi\right)$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}}\left[\alpha_s\cos\psi_s\cos\phi - \alpha_s\sin\psi_s\sin\phi + \sin\psi_s\cos\phi + \cos\psi_s\sin\phi\right]$$

starting at point s(0) = s_0, where we put $\Psi(0) = 0$

$$\left.\begin{aligned} \cos\phi &= \frac{x_0}{\sqrt{\varepsilon\beta_0}} \quad , \\ \sin\phi &= -\frac{1}{\sqrt{\varepsilon}}\left(x'_0\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right) \end{aligned}\right\}$$

inserting above …

$$\underline{x(s)} = \sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\psi_s + \alpha_0 \sin\psi_s\right\} \underline{x_0} + \left\{\sqrt{\beta_s\beta_0}\,\sin\psi_s\right\} \underline{x_0'}$$

$$\underline{x'(s)} = \frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s\right\} \underline{x_0} + \sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\psi_s - \alpha_s \sin\psi_s\right\} \underline{x_0'}$$

which can be expressed ... for convenience ... in matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

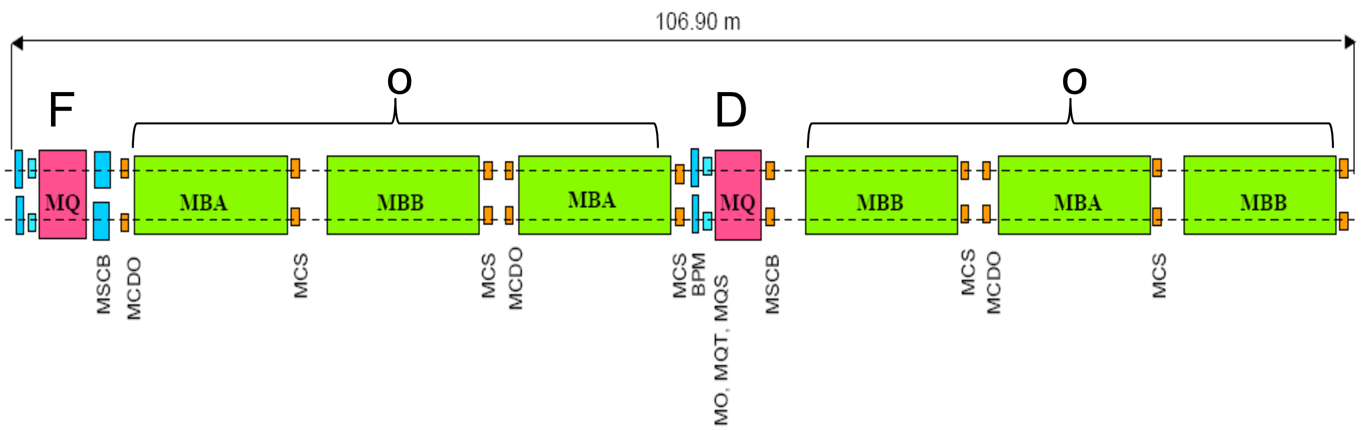
$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

* ***we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.***
* ***and nothing but the α β γ at these positions.***
* **... !**

* *Äquivalenz der Matrizen*

LHC: Lattice Design

the ARC 90° FoDo in both planes

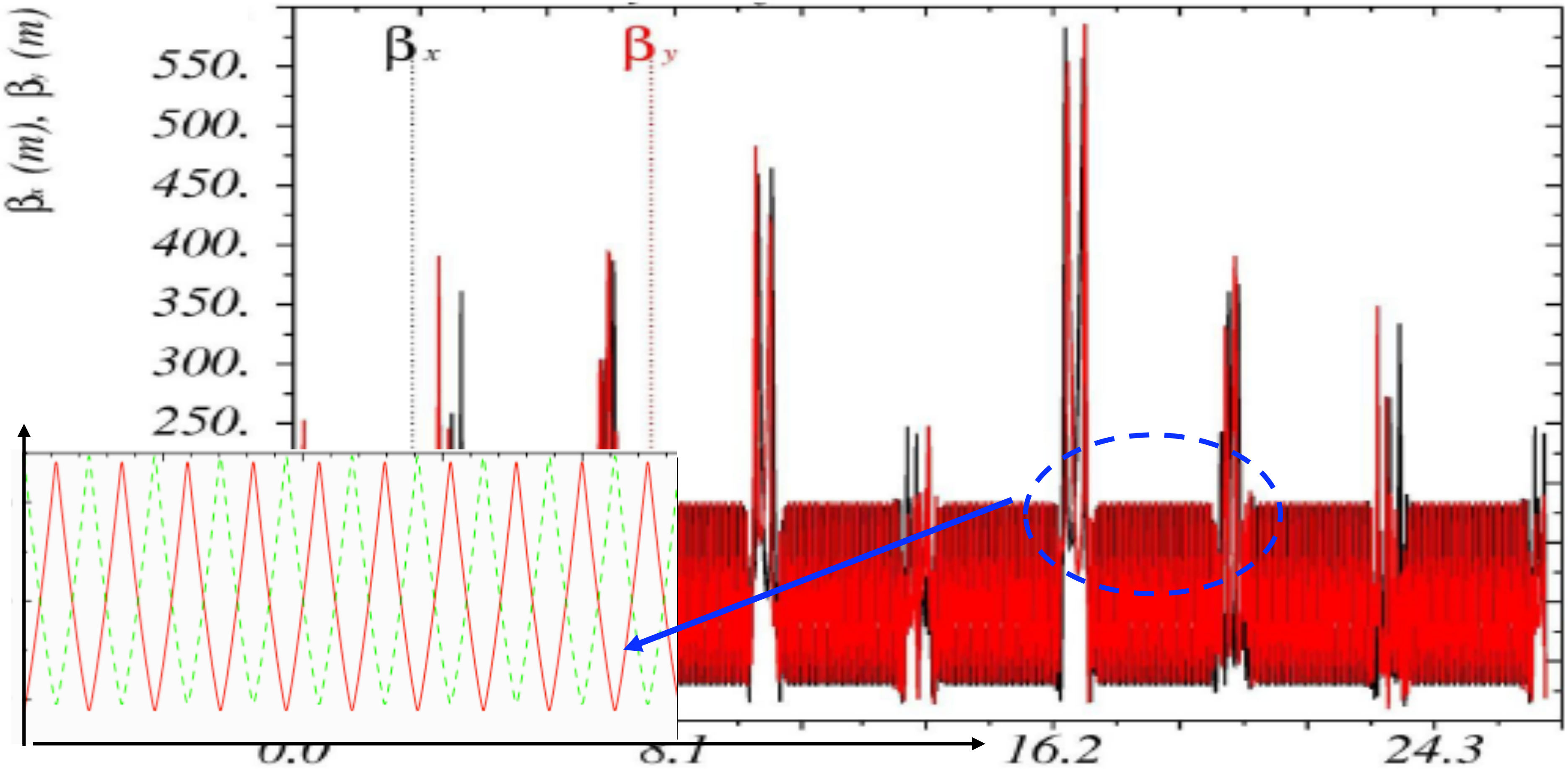


equipped with additional corrector coils

MB: main dipole
MQ: main quadrupole
MQT: Trim quadrupole
MQS: Skew trim quadrupole
MO: Lattice octupole (Landau damping)
MSCB: Skew sextupole
Orbit corrector dipoles
MCS: Spool piece sextupole
MCDO: Spool piece 8 / 10 pole
BPM: Beam position monitor + diagnostics

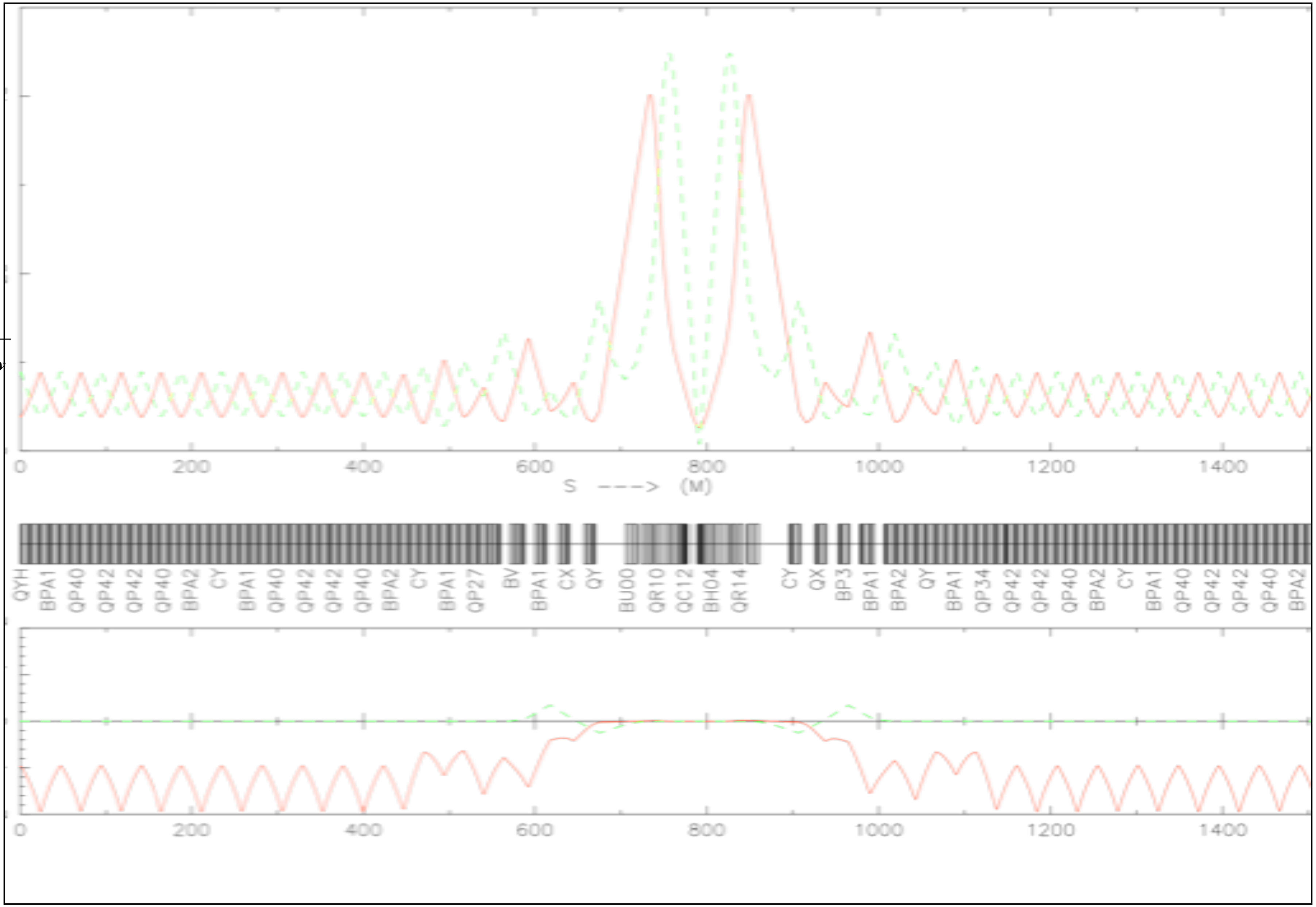
FoDo-Lattice

A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in .
(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)



Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell $\mu = 45°$,
→ calculate the twiss parameters for a periodic solution

11.) Insertions

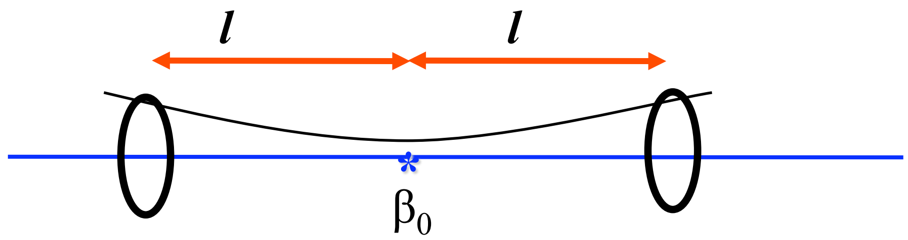


$\sqrt{\beta_{x,y}}$

D

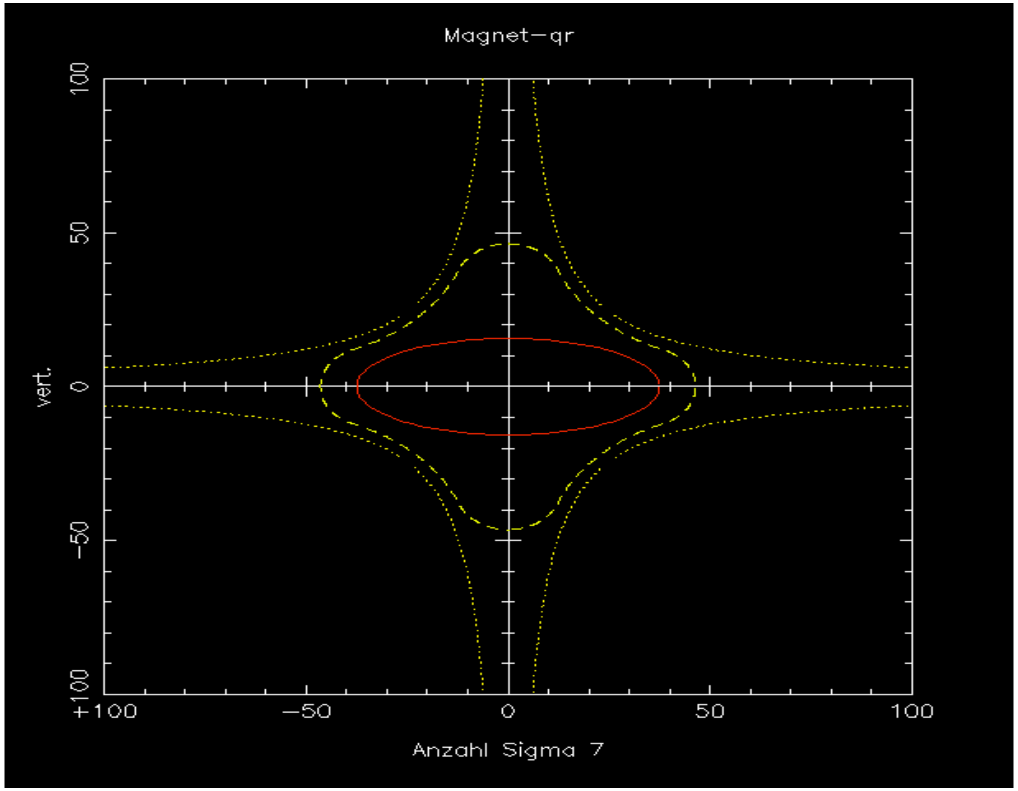
β-Function in a Drift:

$$\beta(\ell) = \beta_0 + \frac{\ell^2}{\beta_0}$$



At the end of a long symmetric drift space the beta function reaches its maximum value in the complete lattice.
-> here we get the largest beam dimension.

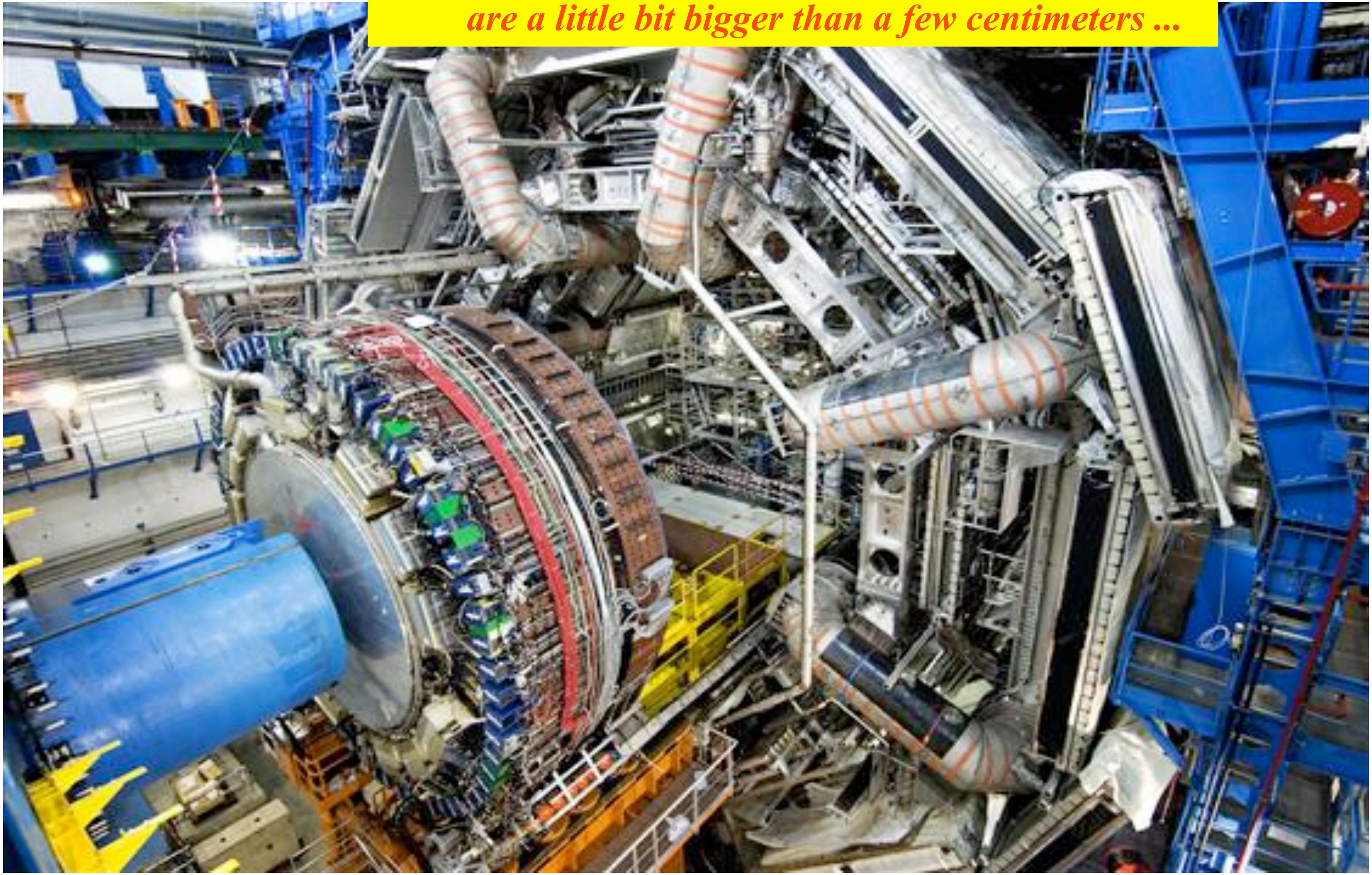
-> keep l as small as possible



7 sigma beam size inside a mini beta quadrupole

... clearly there is an

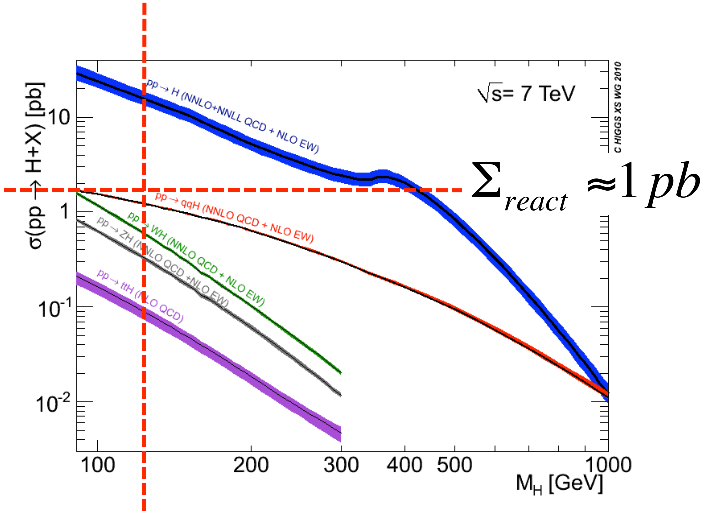
... unfortunately ... in general
high energy detectors that are
installed in that drift spaces
are a little bit bigger than a few centimeters ...



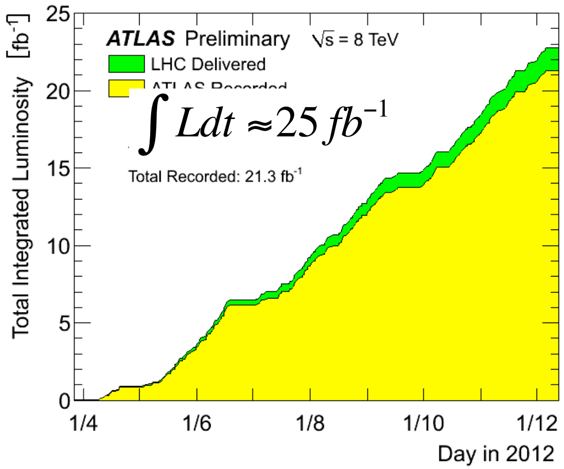
The Mini-β Insertion & Luminosity:

production rate of events is determined by the cross section Σ_{react} and a parameter L that is given by the design of the accelerator:
… the luminosity

$$R = L * \Sigma_{react} \qquad \approx 10^{-12} b \cdot 25 \frac{1}{10^{-15} b} = some\,1000\,H$$



$$\Sigma_{react} \approx 1\,pb$$

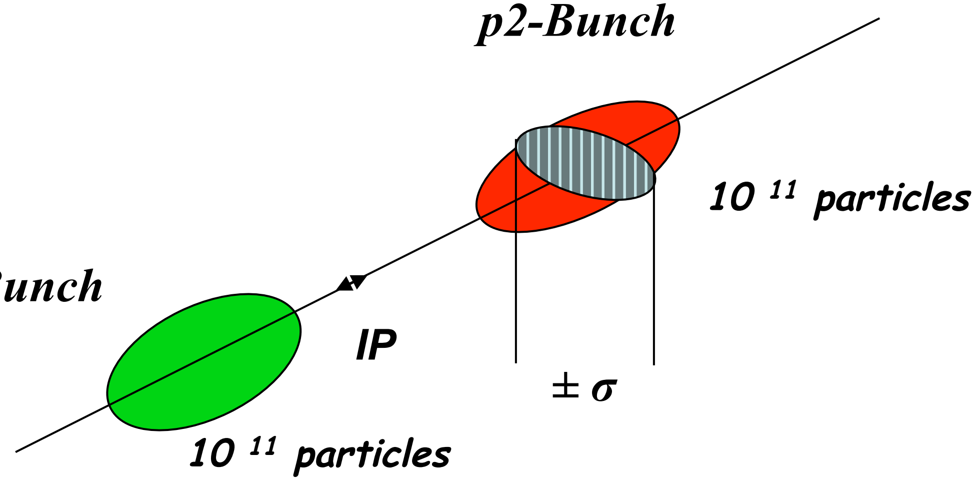


remember:
$1b=10^{-24}\,cm^2$

The luminosity is a storage ring quality parameter and depends on beam size (β !!) and stored current

$$L = \frac{1}{4\pi e^2 f_0\,\mathrm{b}} * \frac{I_1 * I_2}{\sigma^*_x * \sigma^*_y}$$

11.) Luminosity



Example: Luminosity run at LHC

$$\beta_{x,y} = 0.55\ m \qquad f_0 = 11.245\ kHz$$

$$\varepsilon_{x,y} = 5 * 10^{-10}\ rad\ m \qquad n_b = 2808$$

$$\sigma_{x,y} = 17\ \mu m$$

$$L = \frac{1}{4\pi e^2 f_0 n_b} * \frac{I_{p1} I_{p2}}{\sigma_x \sigma_y}$$

$$I_p = 584\ mA$$

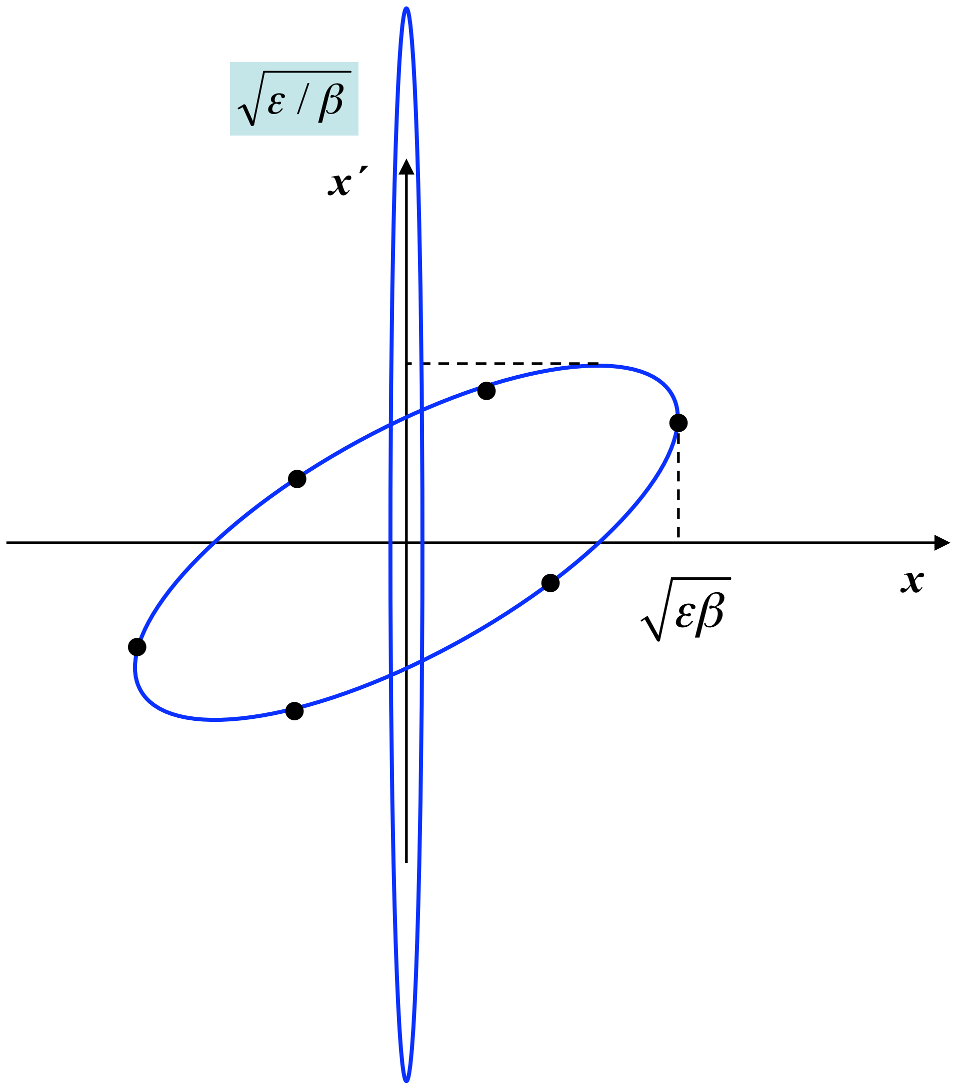
$$L = 1.0 * 10^{34}\ \frac{1}{cm^2 s}$$

Mini-β Insertions: Betafunctions

A mini-β insertion is always a kind of special symmetric drift space.

→greetings from Liouville

the smaller the beam size
the larger the bam divergence



Mini-β Insertions: some guide lines

** calculate the periodic solution in the arc*

** introduce the drift space needed for the insertion device (detector ...)*

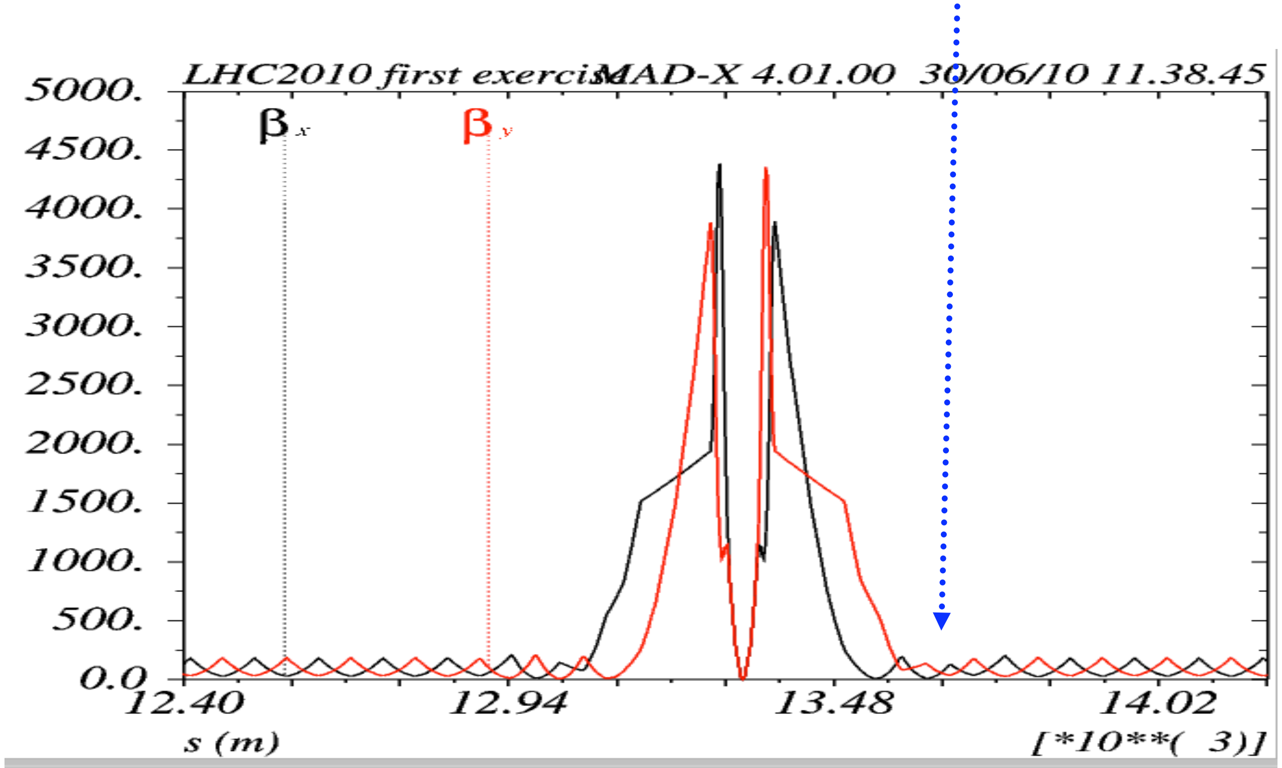
** put a quadrupole doublet (triplet ?) as close as possible*

** introduce additional quadrupole lenses to match the beam parameters to the values at the beginning of the arc structure*

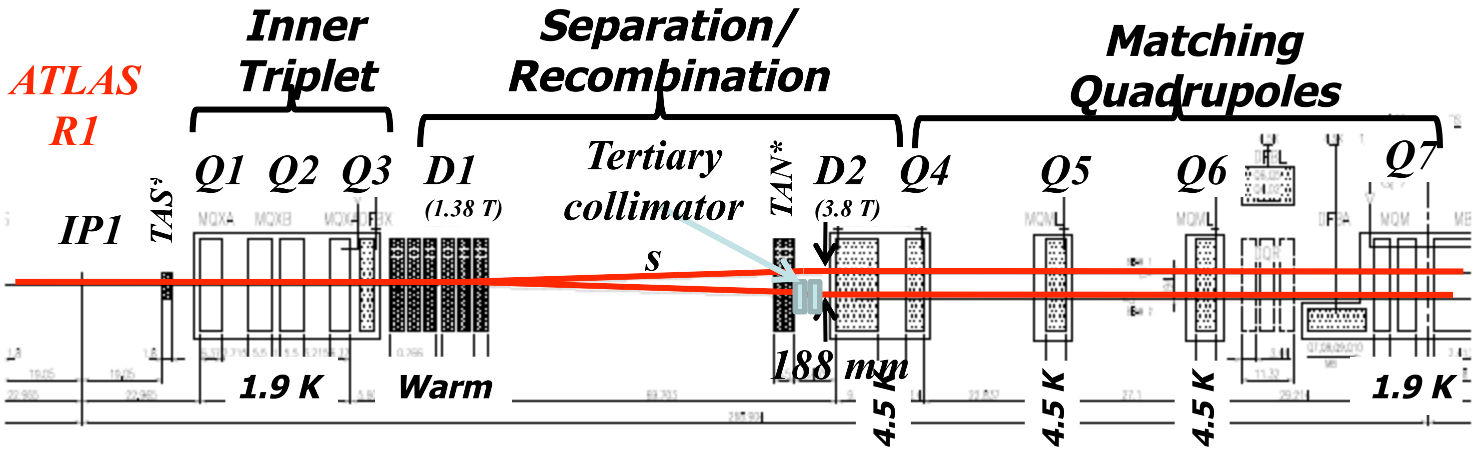
parameters to be optimised & matched to the periodic solution:

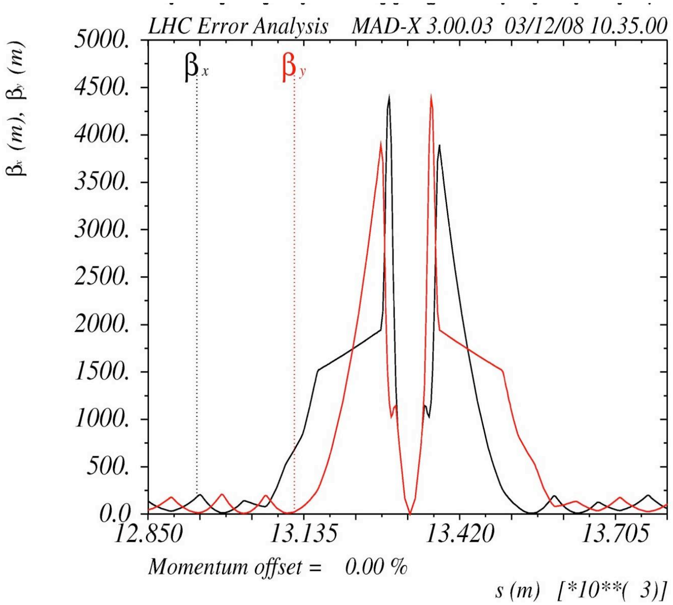
$$\alpha_x,\ \beta_x \qquad D_x,\ D_x'$$
$$\alpha_y,\ \beta_y \qquad Q_x,\ Q_y$$

8 individually powered quad magnets are needed to match the insertion
(... at least)

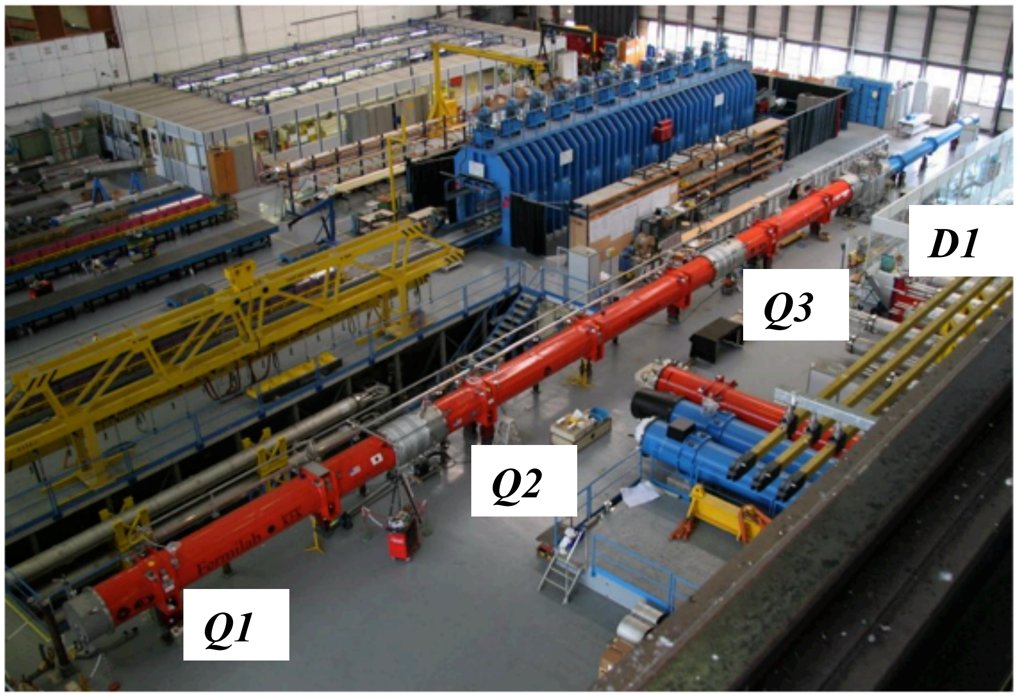


The LHC Insertions





mini β optics



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