Nonlinear Dynamics: Part 1

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CAS: Introduction to Accelerator Physics
Budapest, Hungary
October 2016

In these two lectures on nonlinear dynamics, we shall introduce a number of topics:

- Mathematical tools for modelling nonlinear dynamics:
  – *power series (Taylor) maps; symplectic maps.*

- Effects of nonlinear perturbations:
  – *resonances; tune shifts; dynamic aperture.*

- Analysis methods:
  – *normal form analysis; frequency map analysis.*

We shall discuss these aspects of the subject in the context of two types of accelerator system:

1. a bunch compressor (a single-pass system);

2. a storage ring (a multi-turn system).

Our aim is to provide an introduction to some of the key concepts of nonlinear dynamics in particle accelerators.

By the end of the first lecture, you should be able to:

- describe some of the sources of nonlinearities in particle accelerators;
- outline some of the tools used for modelling nonlinear dynamics in accelerators;
- explain the significance and potential impact of nonlinear dynamics in some accelerator systems.

Particle motion through simple components such as drifts, dipoles and quadrupoles can be represented by *linear transfer maps*.

For example, in a drift space:

$$x_1 = x_0 + L p_{x0}, \tag{1}$$

$$p_{x1} = p_{x0}, \tag{2}$$

where x_0 and p_{x0} are the horizontal co-ordinate and (scaled) horizontal momentum at the entrance of the drift space; x_1 and p_{x1} are the horizontal co-ordinate and momentum at the exit of the drift space, and L is the length of the drift space.

Note that:

$$p_x = \frac{\gamma m v_x}{P_0} \approx \frac{dx}{ds}, \tag{3}$$

where γ is the relativistic factor, m is the rest mass of the particle, v_x is the horizontal velocity, and P_0 is the reference momentum.

Linear transfer maps can be written in terms of matrices.

For example, for a drift space of length L:

$$\begin{pmatrix} x_1 \\ p_{x1} \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ p_{x0} \end{pmatrix}. \tag{4}$$

In general, a linear transformation can be written:

$$\vec{x}_1 = R\vec{x}_0 + \vec{A}, \tag{5}$$

where $\vec{x}_0$ and $\vec{x}_1$ are the initial and final phase space vectors, with components (x_0, p_{x0}) and (x_1, p_{x1}), respectively.

R is a matrix (the *transfer matrix*) and $\vec{A}$ is a vector.

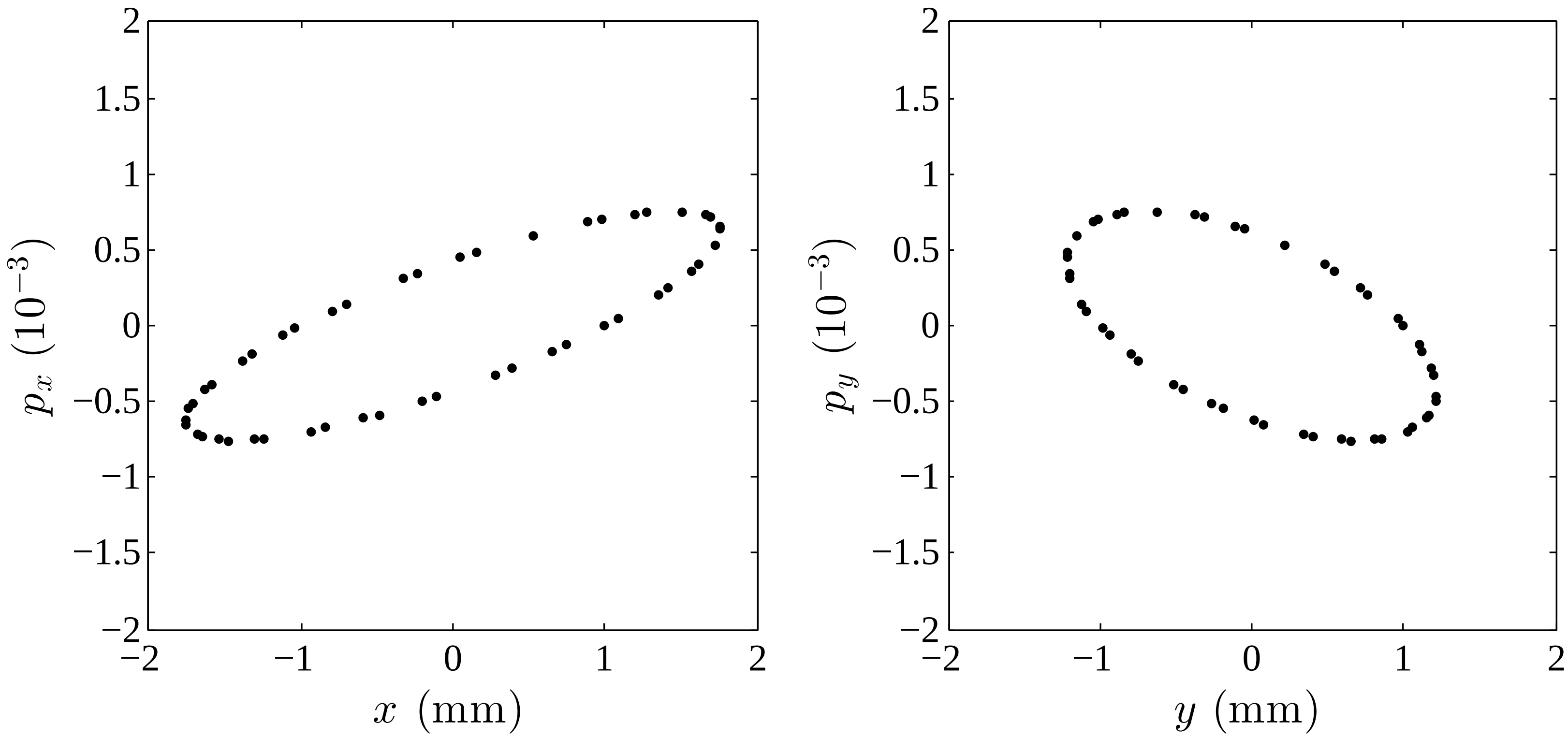
The components of R and $\vec{A}$ are constant, i.e. they do not depend on $\vec{x}_0$.

The transfer matrix for a section of beamline can be found by multiplying the transfer matrices for the accelerator components within that section.

For a periodic beamline (i.e. a beamline constructed from a repeated unit) the transfer matrix for a single period can be parameterised in terms of the Courant–Snyder parameters (α, β, γ) and the phase advance, μ:

$$R = \begin{pmatrix} \cos(\mu) + \alpha \sin(\mu) & \beta \sin(\mu) \\ -\gamma \sin(\mu) & \cos(\mu) - \alpha \sin(\mu) \end{pmatrix}. \tag{6}$$

If the beamline is stable, then the characteristics of the particle motion can be represented by a *phase space portrait* showing the co-ordinates and momenta of a particle after an increasing number of passes through full periods of the beamline.

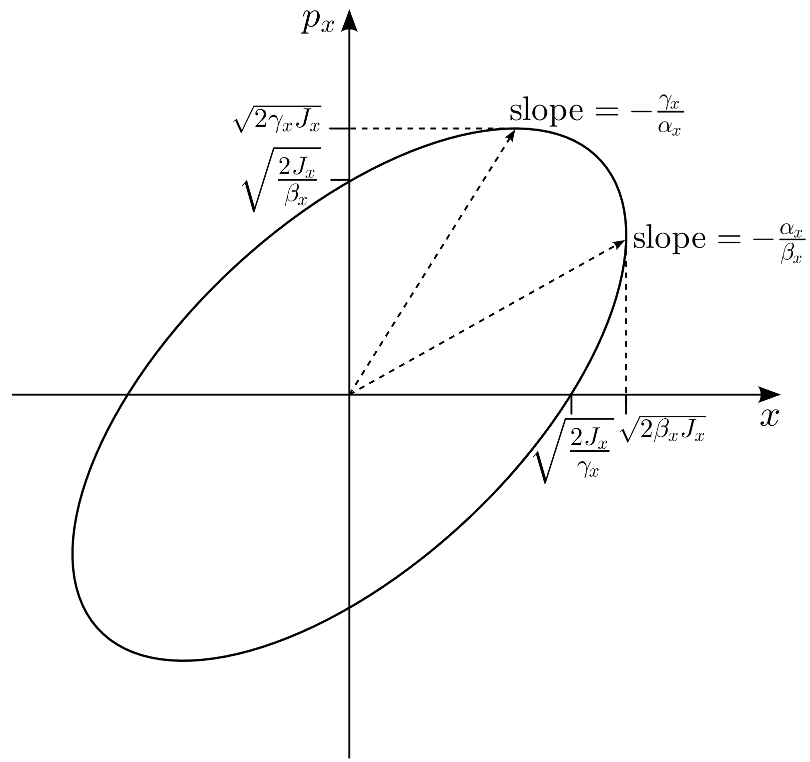


If the transfer map for each period is linear, then the phase space portrait is an ellipse with area πJ_x.

J_x is called the *betatron action*, and characterises the amplitude of the betatron oscillations.

The shape of the ellipse is described by the Courant–Snyder parameters.

The rate at which particles move around the ellipse (phase advance per period) is independent of the betatron action.



Nonlinearities in the particle dynamics can come from a number of different sources, including:

- stray fields in drift spaces;
- higher-order multipole components in dipoles and quadrupoles;
- higher-order multipole magnets (sextupoles, octupoles...) used to control various properties of the beam;
- effects of fields generated by a bunch of particles on individual particles within the bunch (space-charge forces, beam-beam effects...)

The effects of nonlinearities can be varied and quite dramatic. It is important to have some understanding of nonlinear dynamics for optimising the design and operation of many accelerator systems.

As an example, consider (the vertical component of) the field in a sextupole magnet:

$$\frac{B_y}{B\rho} = \frac{1}{2} k_2 x^2, \tag{7}$$

where $B\rho = P_0/q$ is the beam rigidity, and k_2 is the normalised sextupole gradient.

In the "thin lens" approximation, the deflection of a particle on passing through the sextupole is:

$$\Delta p_x = -\frac{1}{B\rho} \int B_y \, ds \approx -\frac{1}{2} k_2 L x^2, \tag{8}$$

where L is the length of the sextupole.

Hence, the transfer map for a sextupole in the thin lens approximation is:

$$x_1 = x_0, \tag{9}$$

$$p_{x1} = p_{x0} - \frac{1}{2} k_2 L x^2. \tag{10}$$

A nonlinear transfer map can be represented as a *power series*:

$$x_1 = A_1 + R_{11}x_0 + R_{12}p_{x0} + T_{111}x_0^2 + T_{112}x_0p_{x0} + T_{122}p_{x0}^2 + \ldots \tag{11}$$

$$p_{x1} = A_2 + R_{21}x_0 + R_{22}p_{x0} + T_{211}x_0^2 + T_{212}x_0p_{x0} + T_{222}p_{x0}^2 + \ldots \tag{12}$$

The coefficients R_{ij} are components of the transfer matrix R.

The coefficients of higher-order (nonlinear) terms are conventionally represented by T_{ijk} (second order), $U_{ijk\ell}$ (third order) and so on.

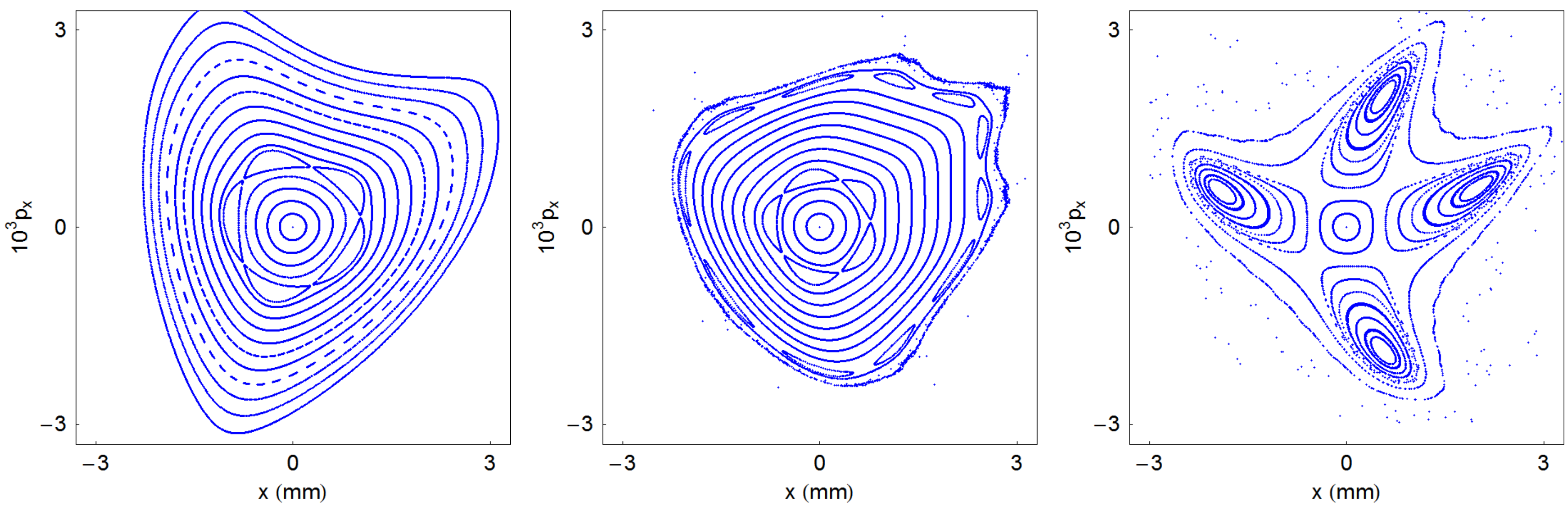
The values of the indices correspond to the components of the phase space vector, thus:

index value	1	2	3	4	5	6
component	x	p_x	y	p_y	z	δ

Nonlinearities in a periodic beamline can have a number of effects:

- the shape of the phase space ellipse can become distorted;
- the phase advance per period can depend on the betatron amplitude (i.e. depends on the action J_x);
- the motion can be stable for small amplitude, but unstable at large amplitude;
- features such as “phase space islands” (closed loops around points away from the origin) can appear in the phase space portrait...

Effects of nonlinearities



We shall discuss the effects of nonlinearities in periodic beamlines in the second lecture.

In the remainder of this lecture, we shall look in more detail at the effects of nonlinearities in a single-pass beamline: a bunch compressor.

We shall see how nonlinear effects can impact the performance of a bunch compressor if they are not properly taken into account in the design of the system.

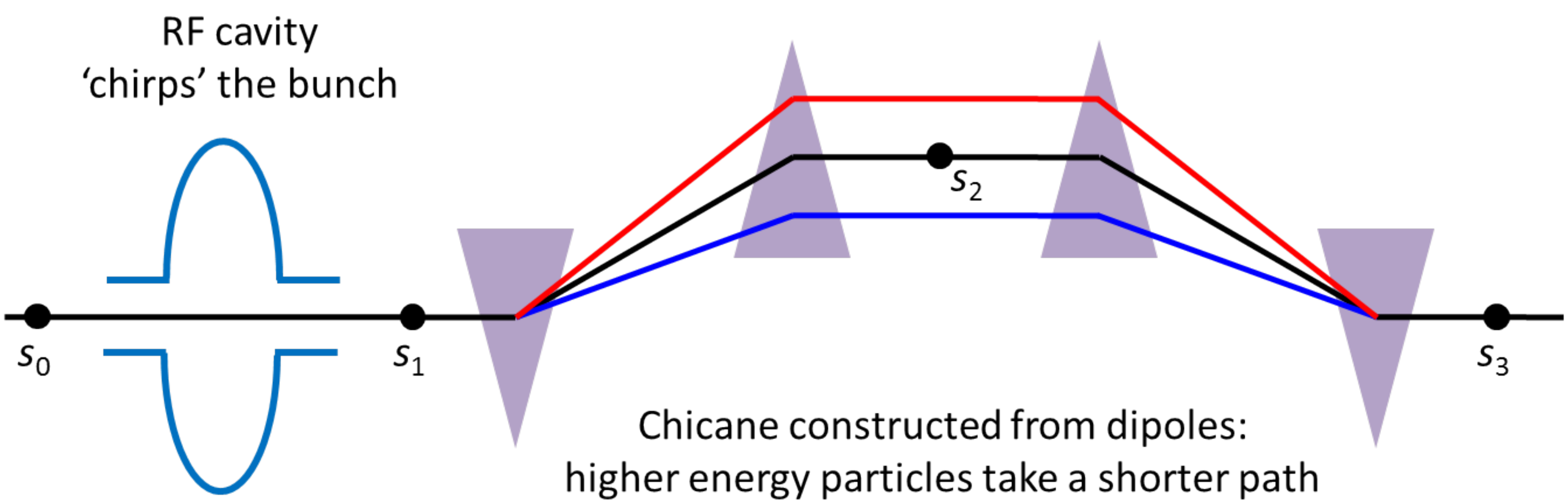
A bunch compressor reduces the length of a bunch, by performing a rotation in longitudinal phase space.

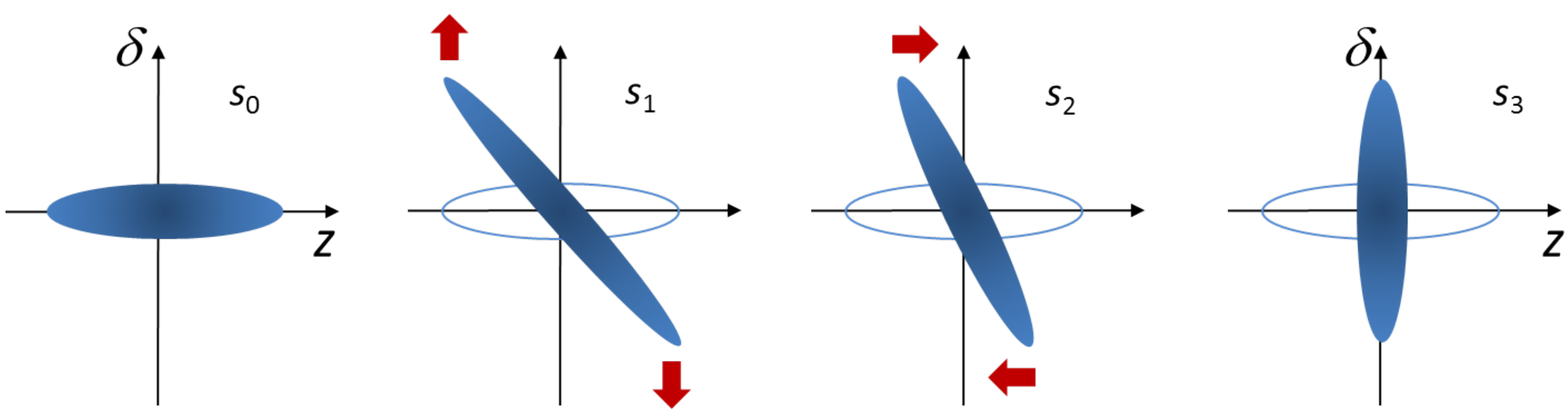
Bunch compressors are used, for example, in free electron lasers to increase the peak current.

We shall follow these steps in our analysis:

1. Outline the structure of the bunch compressor.
2. Specify the parameters based on linear dynamics.
3. Perform an analysis of the linear and nonlinear effects.
4. Adjust the parameters to compensate nonlinear effects.

Bunch compressor: structure and operation





Distribution of particles 'rotates' in longitudinal phase space (area is conserved).

The rf cavity is designed to "chirp" the bunch, i.e. to provide a change in energy deviation as a function of longitudinal position z within the bunch ($z > 0$ at the head of the bunch).

The energy deviation δ of a particle with energy E is defined as:

$$\delta = \frac{E - E_0}{E_0}, \tag{13}$$

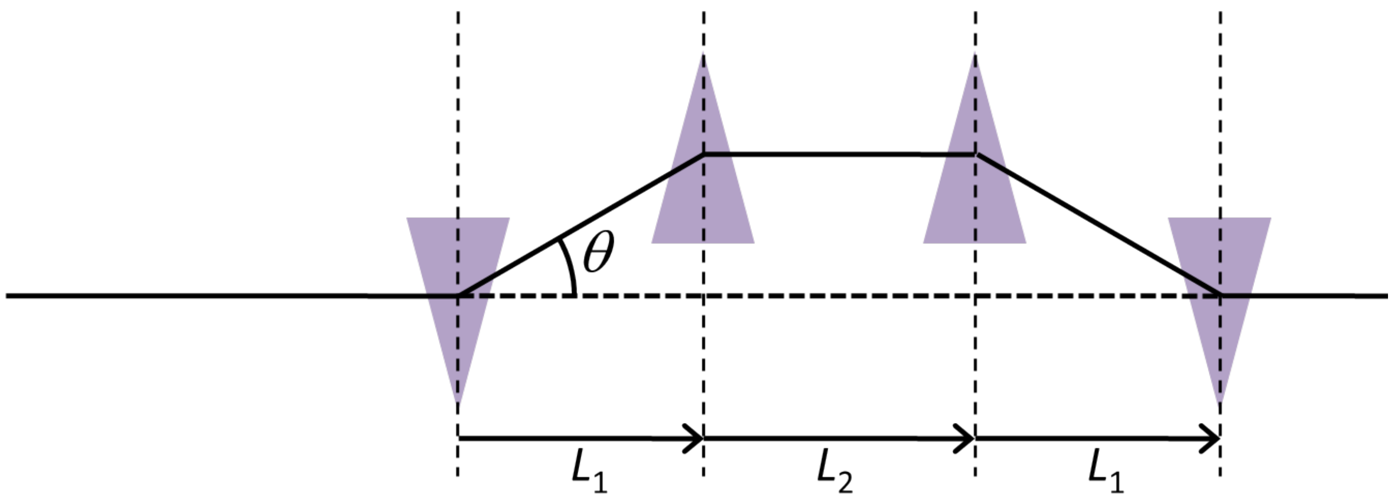
where E_0 is the reference energy for the system.

The transfer map for the rf cavity in the bunch compressor is:

$$\begin{aligned} z_1 &= z_0, & (14) \\ \delta_1 &= \delta_0 - \frac{eV}{E_0} \sin\left(\frac{\omega z_0}{c}\right), & (15) \end{aligned}$$

where V is the rf voltage, and $\omega/2\pi$ is the rf frequency.

Neglecting synchrotron radiation, the chicane does not change the energy of the particles. However, the path length L depends on the energy of the particle.



If we assume that the bending angle in a dipole is small, $\theta \ll 1$:

$$L = \frac{2L_1}{\cos\theta} + L_2. \tag{16}$$

The bending angle is a function of the energy of the particle:

$$\theta = \frac{\theta_0}{1+\delta}. \tag{17}$$

The change in the co-ordinate z is the difference between the nominal path length, and the length of the path actually taken by the particle.

Hence, the transfer map for the chicane can be written:

$$
\begin{aligned}
z_2 &= z_1 + 2L_1 \left(\frac{1}{\cos\theta_0} - \frac{1}{\cos(\theta(\delta_1))} \right), && (18) \\
\delta_2 &= \delta_1, && (19)
\end{aligned}
$$

where θ_0 is the nominal bending angle of each dipole in the chicane, and $\theta(\delta)$ is given by (17):

$$
\theta(\delta) = \frac{\theta_0}{1+\delta}.
$$

Clearly, the complete transfer map for the bunch compressor is nonlinear; but how important are the nonlinear terms?

To understand the effects of the nonlinear part of the map, we shall look at a specific example.

First, we will “design” a bunch compressor using only the linear part of the map.

The linear part of a transfer map can be obtained by expanding the map as a Taylor series in the dynamical variables, and keeping only the first-order terms.

After finding appropriate values for the various parameters using the linear transfer map, we shall see how our design has to be modified to take account of the nonlinearities.

To first order in the dynamical variables z and δ, the map for the rf cavity can be written:

$$
\begin{aligned}
z_1 &= z_0, &(20)\\
\delta_1 &= \delta_0 + R_{65} z_0, &(21)
\end{aligned}
$$

where:

$$R_{65} = -\frac{eV}{E_0}\frac{\omega}{c}. \tag{22}$$

The map for the chicane can be written:

$$
\begin{aligned}
z_2 &= z_1 + R_{56}\delta_1, &(23)\\
\delta_2 &= \delta_1, &(24)
\end{aligned}
$$

where:

$$R_{56} = 2L_1 \frac{\theta_0 \sin\theta_0}{\cos^2\theta_0}. \tag{25}$$

As a specific example, consider a bunch compressor for the International Linear Collider:

Initial rms bunch length	$\sqrt{\langle z_0^2 \rangle}$	6 mm
Initial rms energy spread	$\sqrt{\langle \delta_0^2 \rangle}$	0.15%
Final rms bunch length	$\sqrt{\langle z_2^2 \rangle}$	0.3 mm

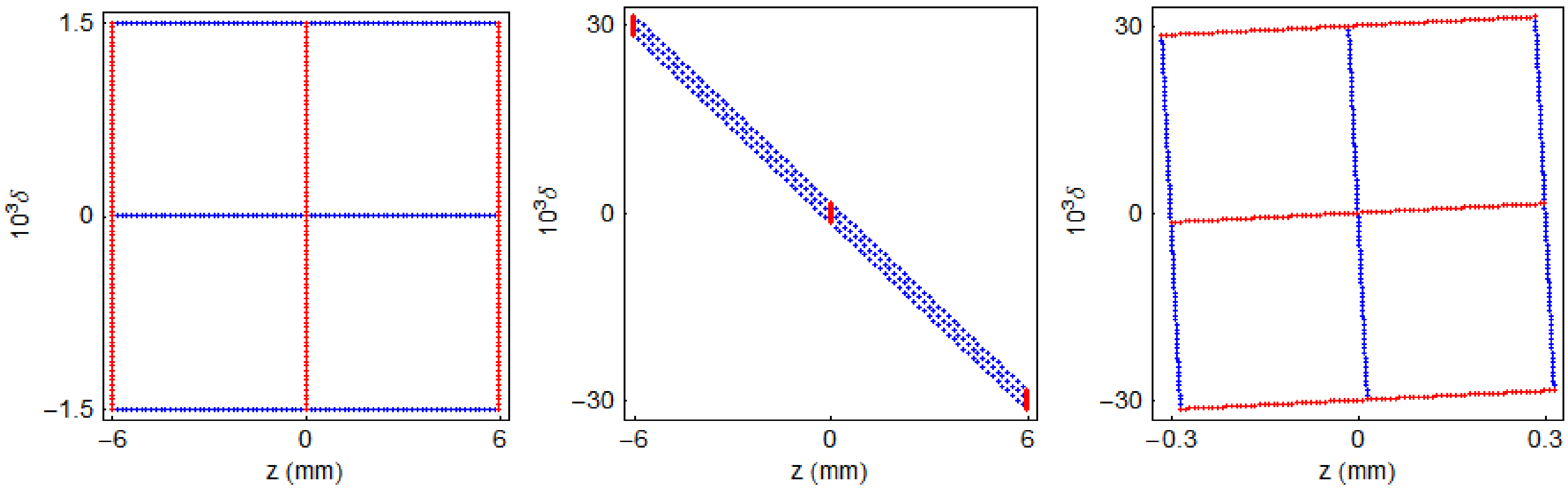
Two constraints determine the values of R_{65} and R_{56}:

- The bunch length should be reduced by a factor 20.
- There should be no "chirp" on the bunch at the exit of the bunch compressor, i.e. $\langle z_2 \delta_2 \rangle = 0$.

With these constraints, we find (see Appendix A):

$$R_{65} = -4.9937\,\mathrm{m}^{-1}, \quad \text{and} \quad R_{56} = 0.19975\,\mathrm{m}. \tag{26}$$

We can illustrate the effect of the linearised bunch compressor map on phase space using an artificial "window frame" distribution:



The rms bunch length is reduced by a factor of 20 as required, but the rms energy spread is *increased* by the same factor. This is because the transfer map is *symplectic*, so phase space areas are conserved under the transformation.

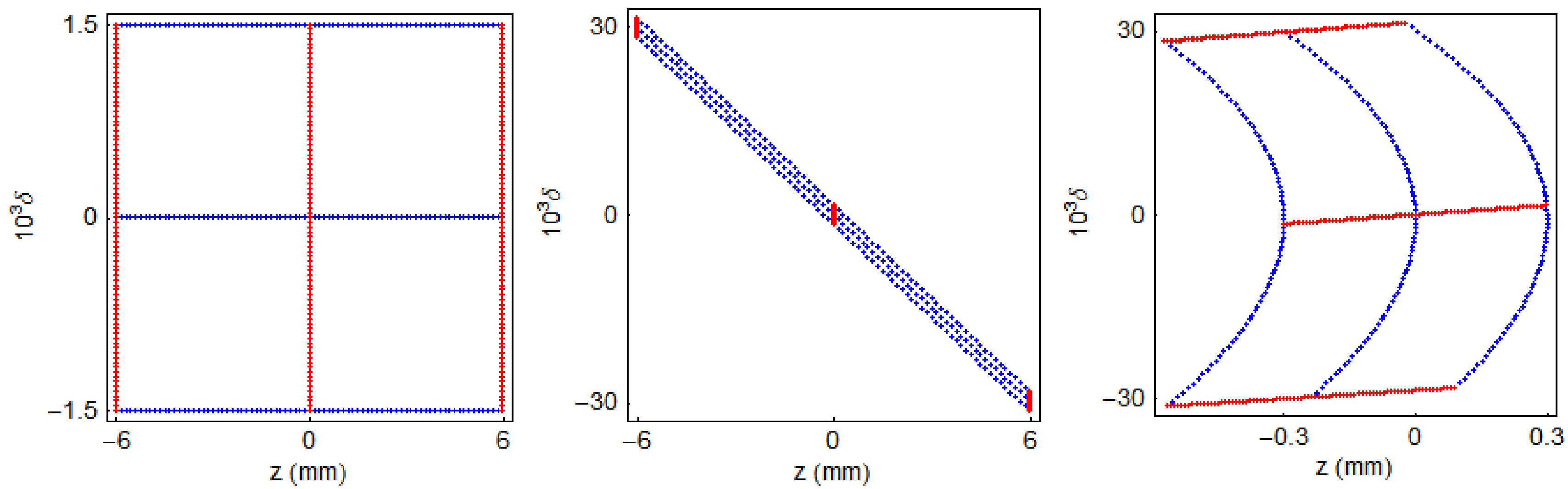
Now let us see what happens when we apply the full nonlinear map for the bunch compressor.

The full map cannot simply be represented by the two coefficients R_{65} and R_{56}: we need to make some assumptions for the rf voltage and frequency, and the dipole bending angle and chicane length.

We have to choose all these parameters so that the "linear" parameters have the appropriate values.

Beam (reference) energy	E_0	5 GeV
RF frequency	f_{rf}	1.3 GHz
RF voltage	V_{rf}	916 MV
Dipole bending angle	θ_0	3°
Dipole spacing	L_1	36.3 m

As before, we illustrate the effect of the bunch compressor map on phase space using a "window frame" distribution:



Although the bunch length has been reduced, there is significant distortion of the distribution: the rms bunch length will be significantly longer than we are aiming for.

To reduce the distortion, we first need to understand where it comes from, which means looking at the map more closely.

Consider a particle entering the bunch compressor with initial phase space co-ordinates z_0 and δ_0. We can write the co-ordinates z_1 and δ_1 of the particle after the rf cavity *to second order* in z_0 and δ_0:

$$z_1 = z_0, \tag{27}$$

$$\delta_1 = \delta_0 + R_{65} z_0 + T_{655} z_0^2. \tag{28}$$

Recall the notation for the coefficients in the map: the first subscript indicates the variable on the left hand side of the equation, and subsequent subscripts indicate the variables in the relevant term.

The co-ordinates of the particle after the chicane are then (to second order):

$$z_2 = z_1 + R_{56}\delta_1 + T_{566}\delta_1^2, \tag{29}$$

$$\delta_2 = \delta_1. \tag{30}$$

If we combine the maps for the rf and the chicane, we get:

$$\begin{aligned} z_2 = {} & (1 + R_{56}R_{65})z_0 + R_{56}\delta_0 \\ & + (R_{56}T_{655} + R_{65}^2 T_{566})z_0^2 \\ & + 2R_{65}T_{566}z_0\delta_0 \\ & + T_{566}\delta_0^2, \end{aligned} \tag{31}$$

$$\delta_2 = \delta_0 + R_{65}z_0 + T_{655}z_0^2. \tag{32}$$

The term that gives the strong nonlinear distortion is the term in z_0^2 in (31). If we can design a system such that the appropriate coefficients satisfy:

$$R_{56}T_{655} + R_{65}^2 T_{566} = 0, \tag{33}$$

then we should be able to reduce the distortion.

The values of R_{56} and R_{65} are determined by the required compression factor.

The value of T_{566} is determined by the chicane; by expanding (18) as a Taylor series in δ, we find for $\theta_0 \ll 1$:

$$T_{566} \approx -3L_1\theta_0^2. \tag{34}$$

That leaves us with T_{655}. This is the second-order dependence of the energy deviation on longitudinal position for a particle passing through the rf cavity. But if we inspect the full rf map (15), we find it contains only odd-order terms, unless...

...we operate the rf cavity off-phase. In other words, we have to modify the rf transfer map to:

$$z_1 = z_0, \tag{35}$$

$$\delta_1 = \delta_0 - \frac{eV}{E_0} \sin\left(\frac{\omega z_0}{c} + \phi_0 \right). \tag{36}$$

The first-order coefficient in the transfer map for δ is then:

$$R_{65} = -\frac{eV}{E_0} \frac{\omega}{c} \cos\phi_0. \tag{37}$$

The second-order coefficient is:

$$T_{655} = \frac{1}{2} \frac{eV}{E_0} \left(\frac{\omega}{c} \right)^2 \sin\phi_0. \tag{38}$$

Note that there is also a zeroth-order term, so the bunch ends up with a non-zero mean energy deviation $\langle\delta\rangle$ after the rf cavity; but we can take this into account simply by an appropriate scaling of the field in the chicane.

The linear coefficients are determined by the required compression factor, and the requirement to have zero final correlation $\langle z\delta \rangle$. For the ILC bunch compressor:

$$R_{65} = -4.9937\,\mathrm{m}^{-1}, \quad \text{and} \quad R_{56} = 0.19975\,\mathrm{m}. \tag{39}$$

The value of T_{566} is determined by the parameters of the chicane:

$$T_{566} \approx -3L_1\theta_0^2 = -0.29963\,\mathrm{m}. \tag{40}$$

And the value of T_{655} is determined by the need to correct the second-order distortion of the phase space:

$$R_{56}T_{655} + R_{65}^2 T_{566} = 0 \quad \therefore \quad T_{655} = 37.406\,\mathrm{m}^{-2}. \tag{41}$$

Now, given:

$$R_{65} = -\frac{eV}{E_0}\frac{\omega}{c}\cos\phi_0 = -4.9937\,\mathrm{m}^{-1}, \tag{42}$$

and:

$$T_{655} = \frac{1}{2}\frac{eV}{E_0}\left(\frac{\omega}{c}\right)^2 \sin\phi_0 = 37.406\,\mathrm{m}^{-2}, \tag{43}$$

we find, for $E_0 = 5\,\mathrm{GeV}$ and $\omega = 1.3\,\mathrm{GHz}$:

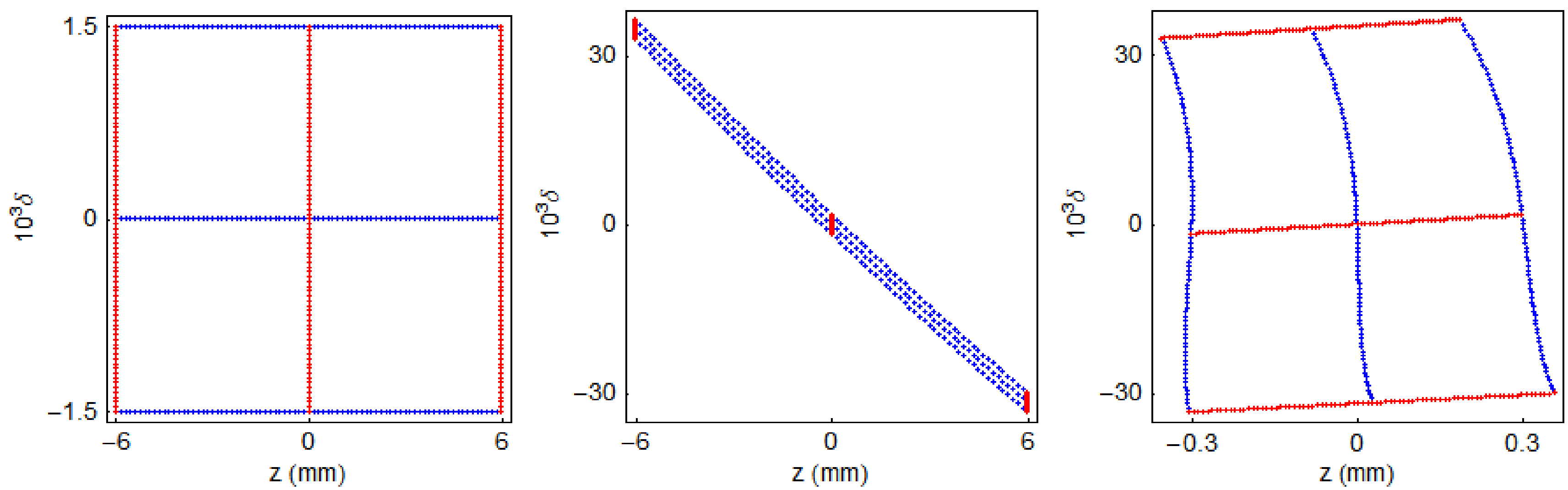
$$V = 1{,}046\,\mathrm{MV}, \quad \text{and} \quad \phi_0 = 28.8^\circ. \tag{44}$$

Operating with this phase, we are providing over a gigavolt of rf to *decelerate* the beam by more than 500 MV.

Because of adiabatic (anti)damping, we will need to reduce the R_{56} of the chicane by a factor E_1/E_0, where E_0 and E_1 are the mean bunch energy before and after the rf, respectively.

Also, the phase space area occupied by the distribution will be increased by a factor E_0/E_1.

As before, we illustrate the effect of the bunch compressor on phase space using a "window frame" distribution. But now we use the parameters determined above, to try to compress by a factor 20, while minimising the second-order distortion:



This looks much better: the dominant distortion now appears to be third-order, and looks small enough that it should not significantly affect the performance of the machine.

We have already learned some useful lessons from this example:

- Nonlinear effects can limit the performance of an accelerator system. Sometimes the effects are small enough that they can be ignored; however, in many cases, a system designed without taking account of nonlinearities will not achieve the specified performance.

- If we take the trouble to analyse and understand the nonlinear behaviour of a system, then, if we are fortunate enough and clever enough, we may be able to devise a means of compensating any adverse effects.

- Nonlinear effects can arise from a number of sources in accelerators, including stray fields, higher-order multipole components in magnets, space-charge...

- The transfer map for a nonlinear element (such as a sextupole) may be represented as a power series in the initial values of the phase space variables.

- The effects of nonlinearities in accelerator system vary widely, depending on the type of system in which they occur (e.g. single-pass, or periodic).

- In some cases, nonlinear effects can limit the performance of an accelerator system. In such cases, it is important to take nonlinearities into account in the design of the system.

Appendix

In a linear approximation, the maps for the rf cavity and the chicane in a bunch compressor may be represented as matrices:

$$M_{\mathrm{rf}} = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}, \qquad M_{\mathrm{ch}} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \tag{45}$$

where:

$$a = \frac{eV}{E_0}\frac{\omega}{c}, \quad \text{and} \quad b = 2L_1 \frac{\theta_0 \sin\theta_0}{\cos^2\theta_0}. \tag{46}$$

The matrix representing the total map for the bunch compressor, M_{bc}, is then:

$$M_{\mathrm{bc}} = M_{\mathrm{ch}} M_{\mathrm{rf}} = \begin{pmatrix} 1-ab & b \\ -a & 1 \end{pmatrix} = \begin{pmatrix} R_{55} & R_{56} \\ R_{65} & R_{66} \end{pmatrix}. \tag{47}$$

The effect of the map is written:

$$\vec{z} \mapsto M_{\mathrm{bc}}\vec{z}, \quad \text{where} \quad \vec{z} = \begin{pmatrix} z \\ \delta \end{pmatrix}. \tag{48}$$

Now we proceed to derive expressions for the required values of the parameters a and b, in terms of the desired initial and final bunch length and energy spread.

We construct the beam distribution *sigma* matrix by taking the outer product of the phase space vector for each particle, then averaging over all particles in the bunch:

$$\Sigma = \langle \vec{z}\,\vec{z}^{\mathsf{T}} \rangle = \begin{pmatrix} \langle z^2 \rangle & \langle z\delta \rangle \\ \langle z\delta \rangle & \langle \delta^2 \rangle \end{pmatrix}. \tag{49}$$

The transformation of Σ under a linear map represented by a matrix M is given by:

$$\Sigma \mapsto M \cdot \Sigma \cdot M^{\mathsf{T}}. \tag{50}$$

Usually, a bunch compressor is designed so that the correlation $\langle z\delta \rangle = 0$ at the start and end of the compressor. In that case, using (47) for the linear map M, and (50) for the transformation of the sigma matrix, we find that the parameters a and b must satisfy:

$$(1 - ab)\frac{a}{b} = \frac{\langle \delta_0^2 \rangle}{\langle z_0^2 \rangle} \tag{51}$$

where the subscript 0 indicates that the average is taken over the *initial* values of the dynamical variables.

Given the constraint (51), the compression factor r is given by:

$$r^2 \equiv \frac{\langle z_1^2 \rangle}{\langle z_0^2 \rangle} = 1 - ab, \tag{52}$$

where the subscript 1 indicates that the average is taken over the final values of the dynamical variables.

We note in passing that the linear part of the map is *symplectic*. A linear map is symplectic if the matrix M representing the map is symplectic, i.e. satisfies:

$$M^{\mathsf{T}} \cdot S \cdot M = S, \tag{53}$$

where, in one degree of freedom (i.e. two dynamical variables), S is the matrix:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{54}$$

In more degrees of freedom, S is constructed by repeating the 2×2 matrix above on the block diagonal, as often as necessary.

In one degree of freedom, it is a necessary and sufficient condition for a matrix to be symplectic, that it has unit determinant: but this condition does *not* generalise to more degrees of freedom.

As a specific example, consider a bunch compressor for the International Linear Collider:

Initial rms bunch length	$\sqrt{\langle z_0^2 \rangle}$	6 mm
Initial rms energy spread	$\sqrt{\langle \delta_0^2 \rangle}$	0.15%
Final rms bunch length	$\sqrt{\langle z_1^2 \rangle}$	0.3 mm

Solving equations (51) and (52) with the above values for rms bunch lengths and energy spread, we find:

$$a = 4.9937\,\mathrm{m}^{-1}, \quad \text{and} \quad b = 0.19975\,\mathrm{m}. \tag{55}$$