Collective Effect I
(Space Charge)

G. Franchetti, GSI

CERN Accelerator – School

Budapest, 2-14 / 10 / 2016

The dynamics of particles follow the Lorenz law

$$\frac{d\vec{p}}{dt} = e\vec{E} + e\vec{v} \times \vec{B}$$

$$\vec{p} = m\gamma\vec{v}$$

E,B can be external field. From magnets and RF systems

But E,B can be field also generated by the beam itself

The beam generate the fields B, E through Maxwell laws

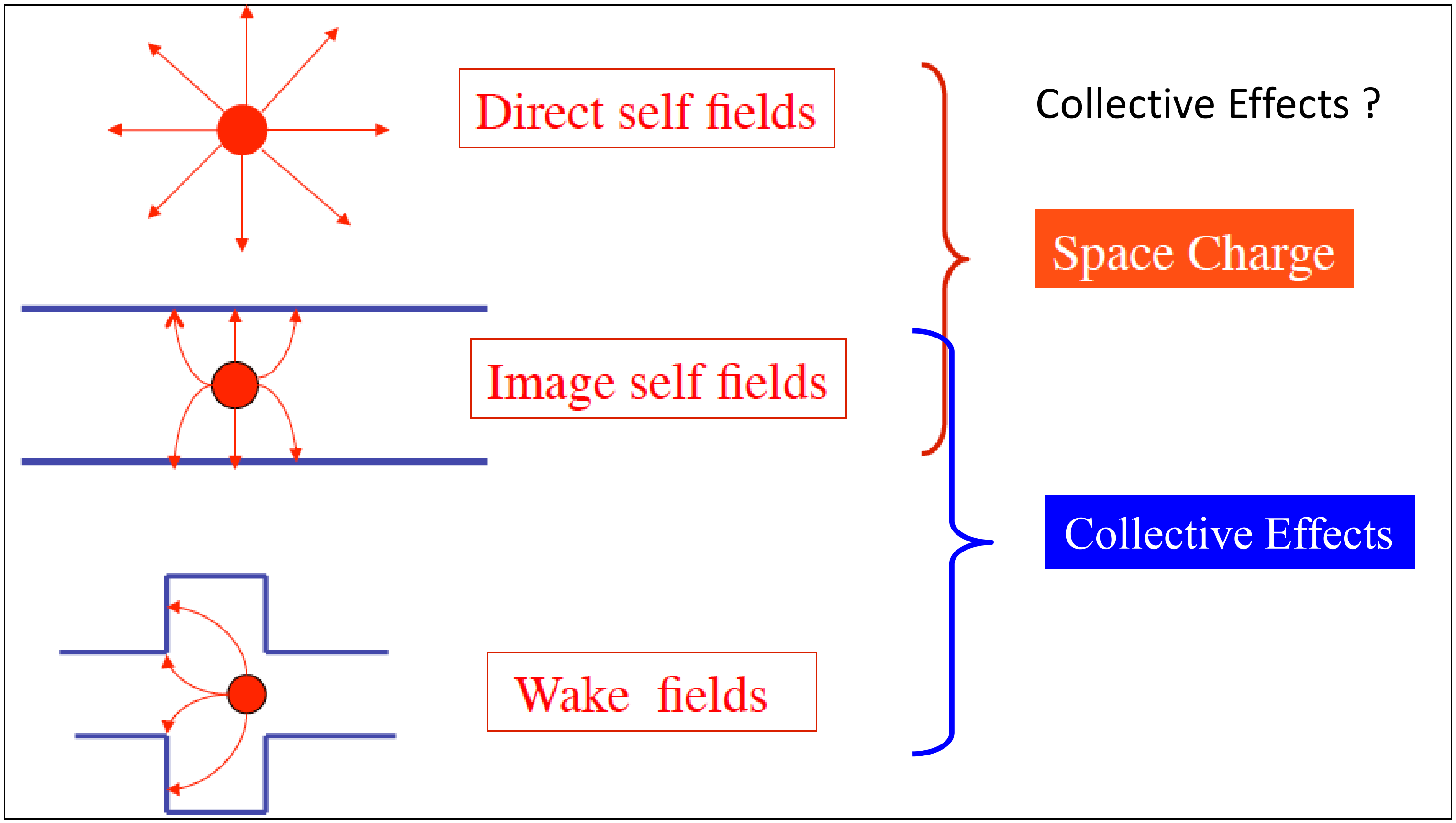
$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Type of fields



How does it looks?

The dynamics of each particle follows the equation

$$\frac{d\vec{p}}{dt} = e\vec{E}_{RF} + e\vec{v} \times \vec{B}_M + e\vec{E}_b + e\vec{v} \times \vec{B}_b$$

the origin of the fields is independent on the beam. External fields

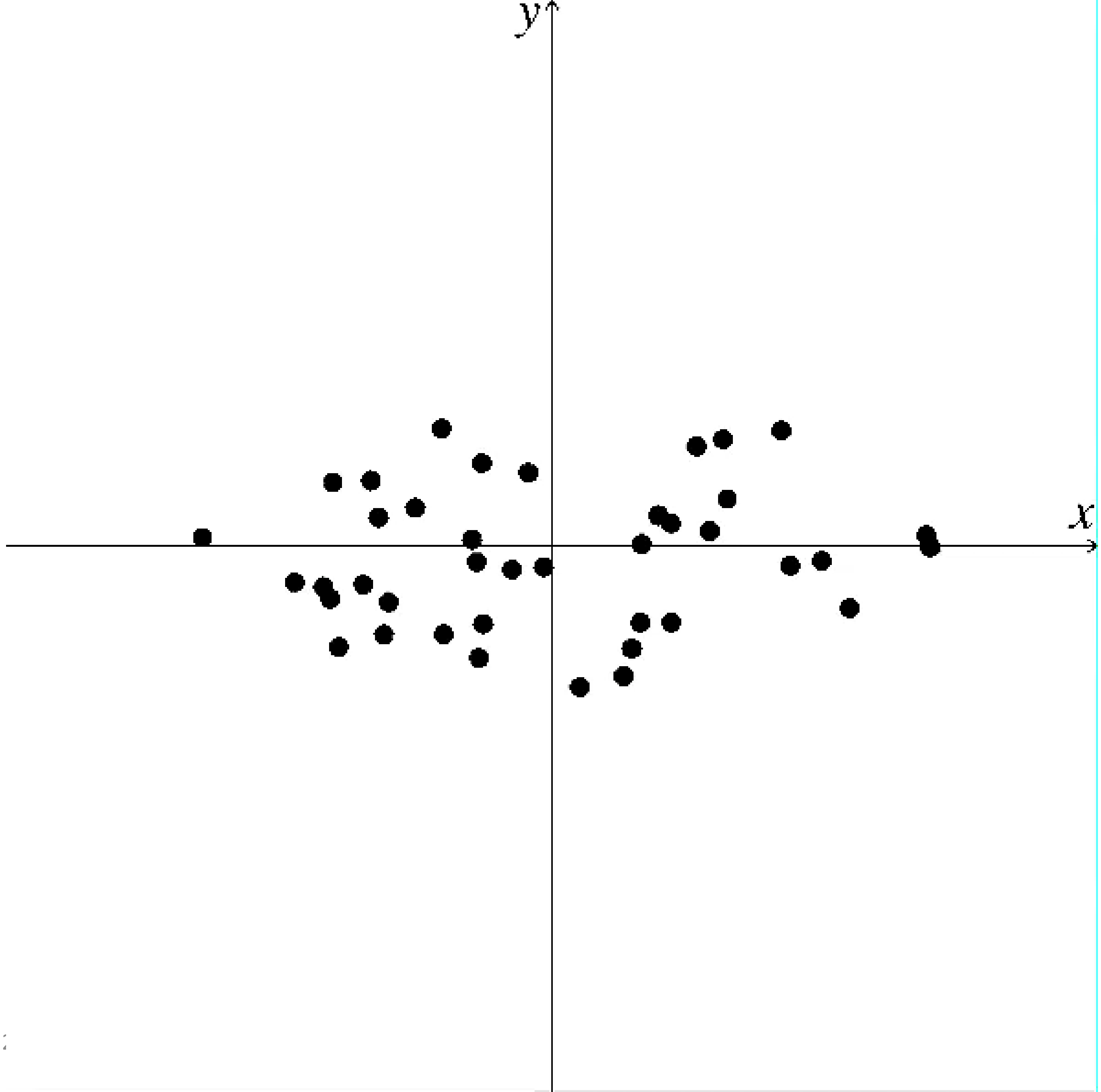
The origin of the fields is dependent on the beam itself

Final form of the transverse equation of motion with space charge

$$\frac{d^2x}{ds^2} + \boxed{k_x x} = \left(\frac{e}{m\gamma v_0^2}\vec{E}_b + \frac{e}{m\gamma v_0^2}\vec{v} \times \vec{B}_b \right)_x$$

$$\frac{d^2y}{ds^2} + \boxed{k_y y} = \left(\frac{e}{m\gamma v_0^2}\vec{E}_b + \frac{e}{m\gamma v_0^2}\vec{v} \times \vec{B}_b \right)_y$$

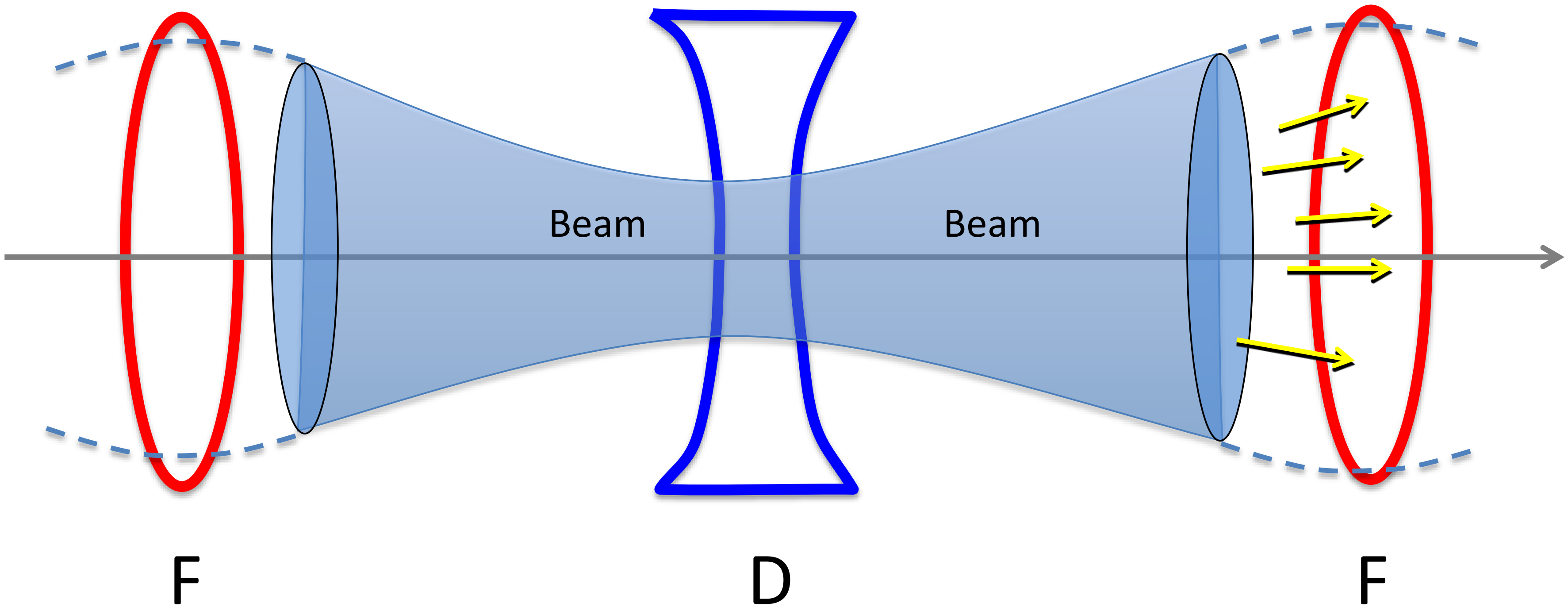
K_x, K_y govern the linear optics


y
x

Model of beam

We neglect the longitudinal forces.

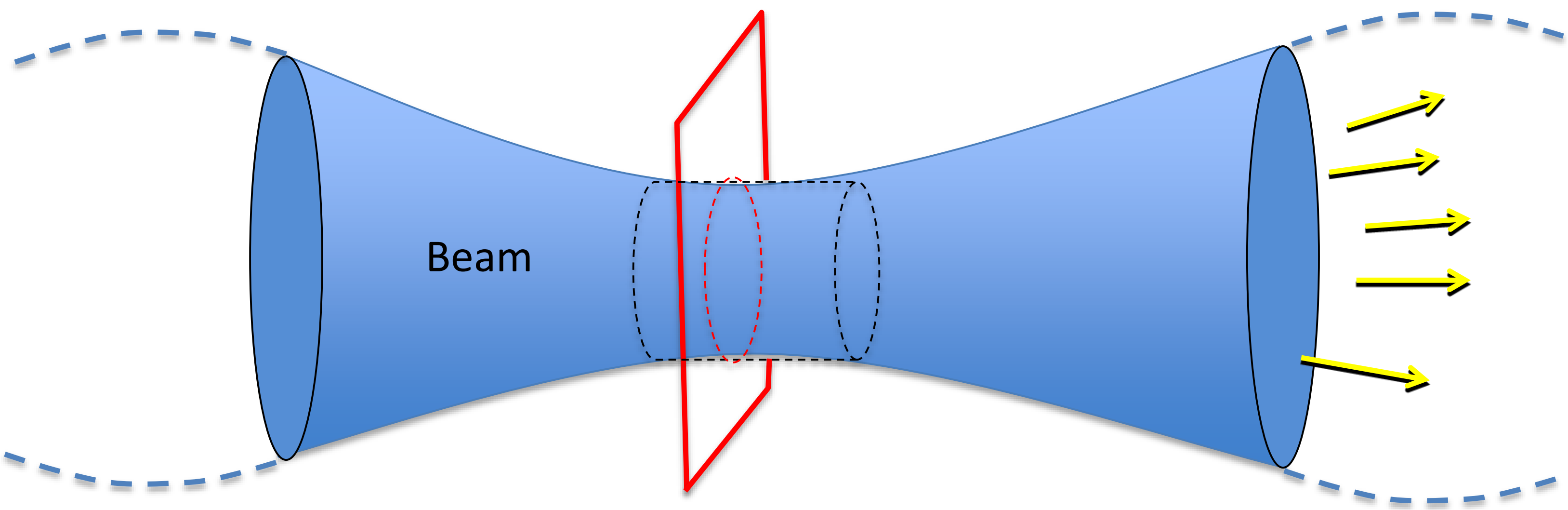
Locally the beam can be seen as a "piece" of a coasting beam



Model of beam

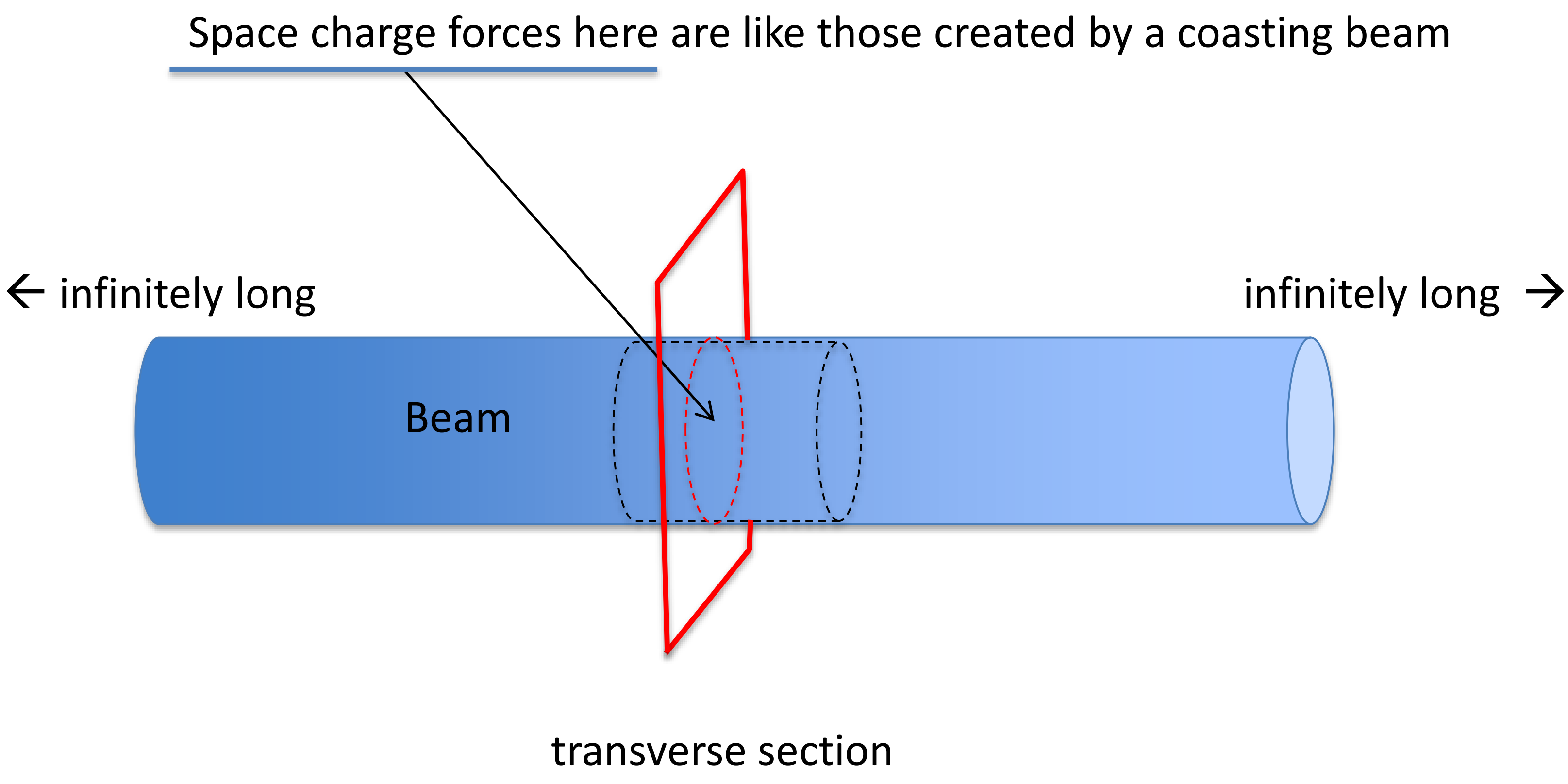
We neglect the longitudinal forces.

Locally the beam can be seen as a “piece” of a coasting beam



transverse section

From the point of view of space charge



The lattice strength is adjusted to have the prescribed optics in absence of space charge. That is the functional shape of $k_x(s)$, $k_y(s)$ is independent on the beam energy

However the space charge forces are **not under our control** !

Analysis in the case the beam energy is small

For non moving particles

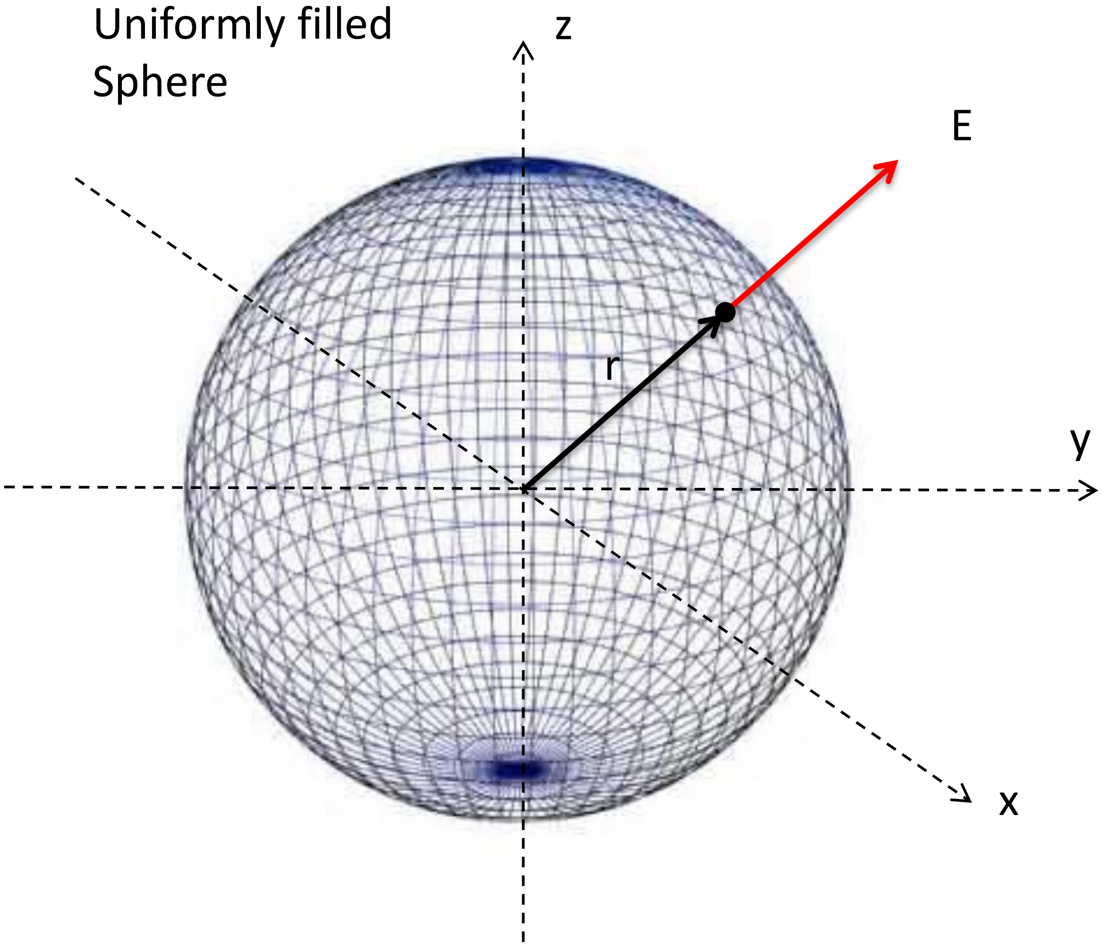
Coulomb electric field

$$\vec{E}(r) = \frac{e}{4\pi\epsilon_0}\sum_i \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3}$$

Much easier

Coulomb Forces

Uniformly filled Sphere



Inside the sphere

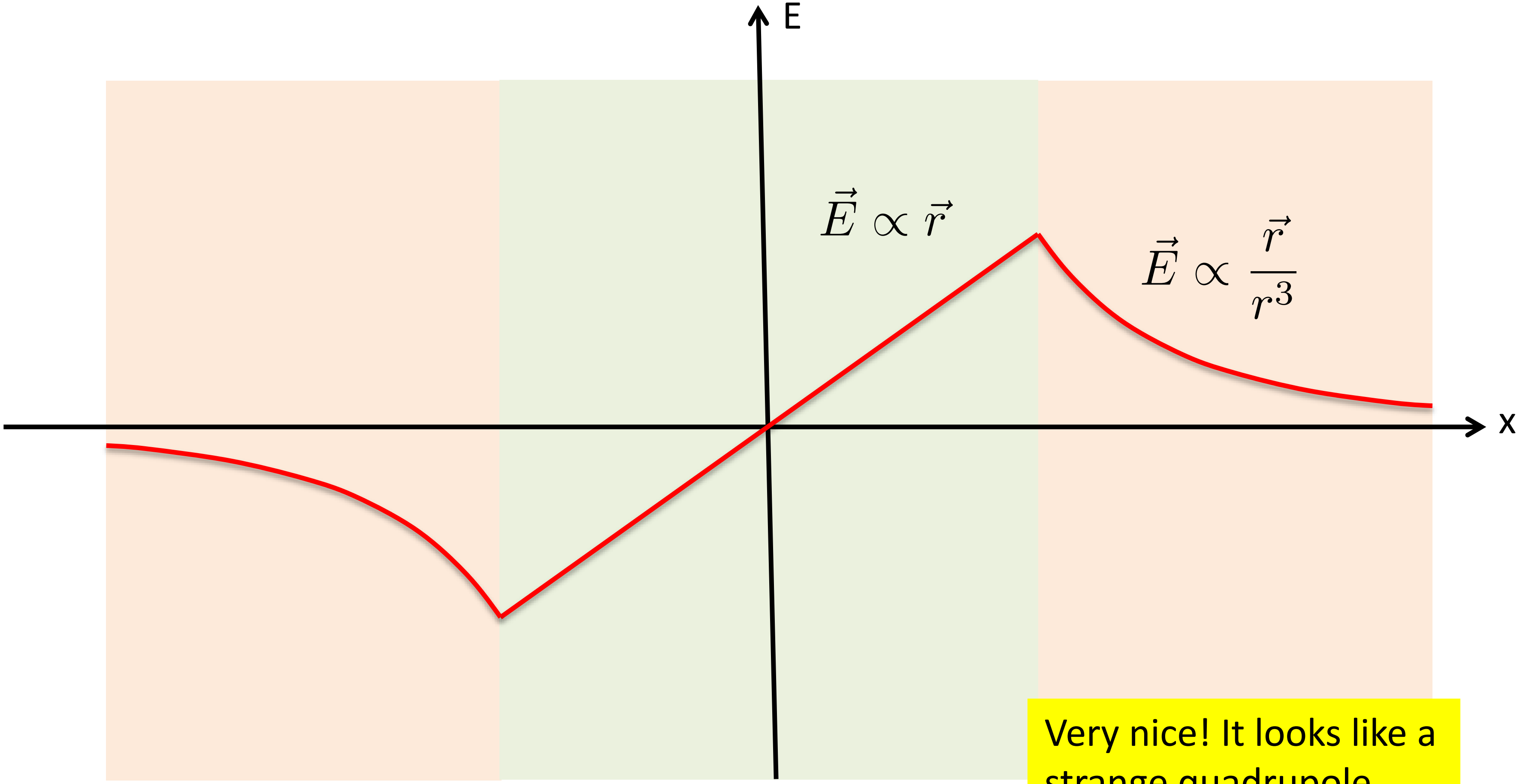
$$E = \frac{\rho}{3\epsilon_0} r$$

ρ = charge density

Outside the sphere

$$E = \frac{R^3}{3\epsilon_0} \frac{\rho}{r^2}$$

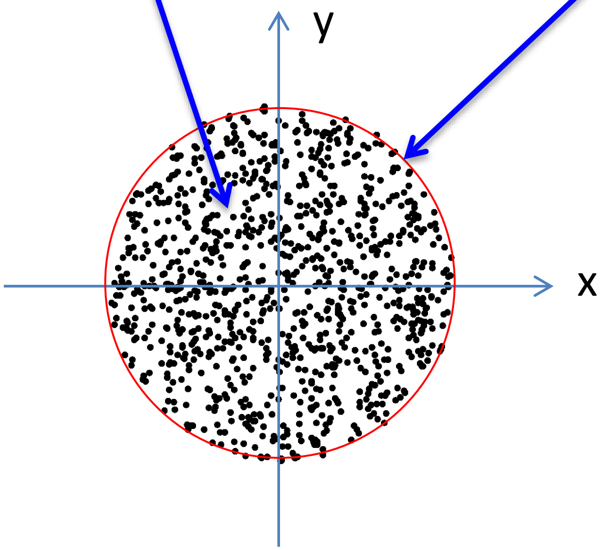
Radial Electric field (along x)



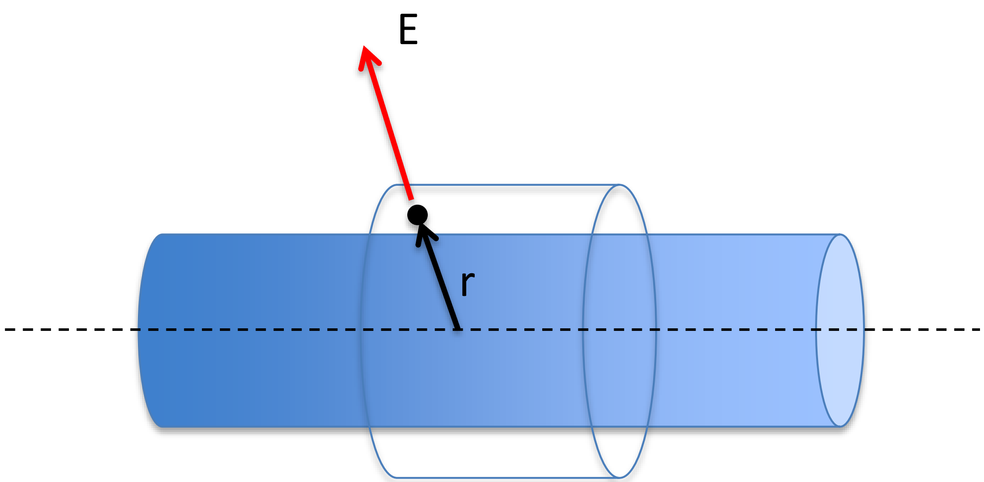
Very nice! It looks like a strange quadrupole

Beam distribution ansatz

We assume in this first discussion that the beam distribution in (x,y) is always uniform and the beam is round



Infinitely long uniform axi-symmetric cylinder



Longitudinal electric field is zero

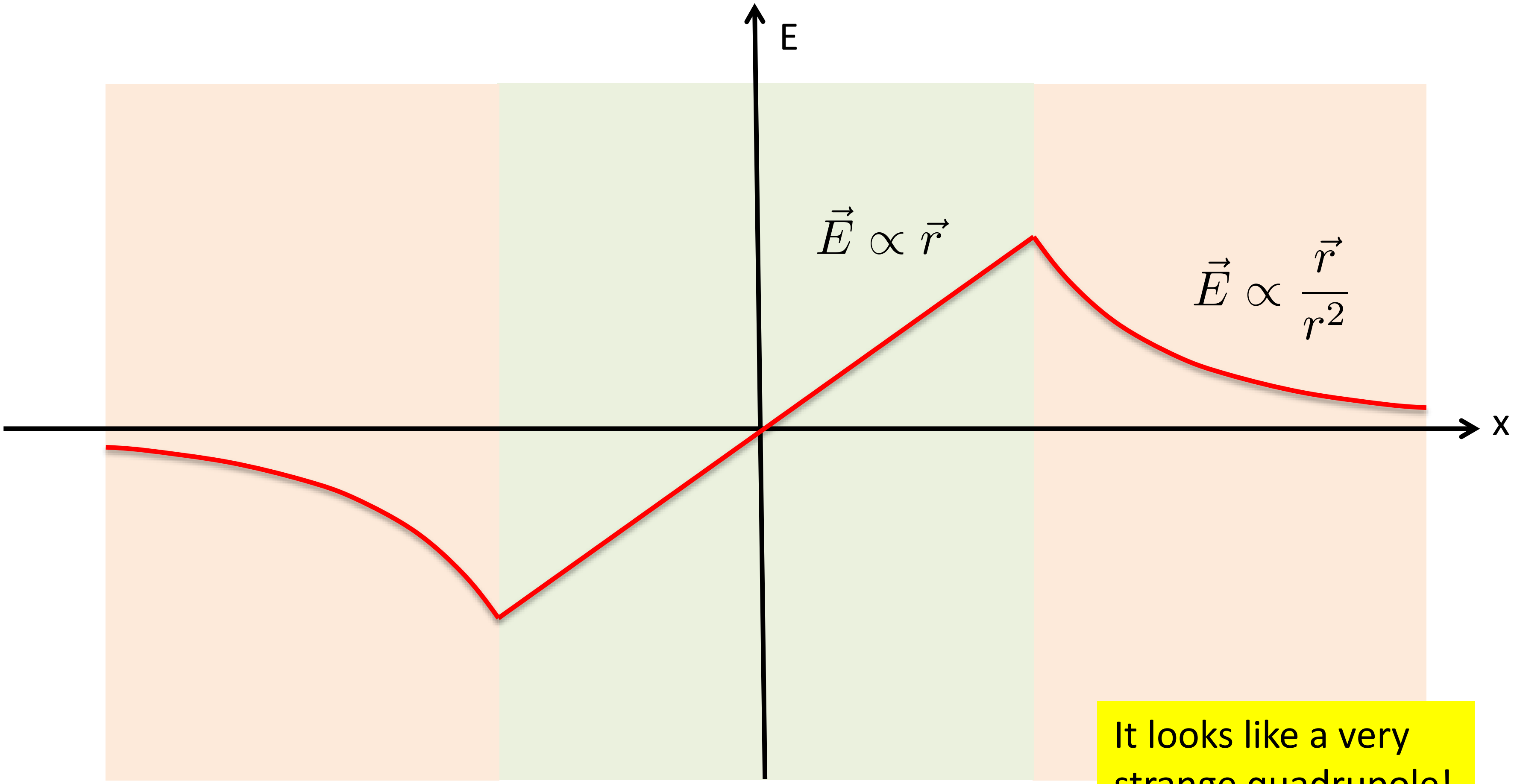
From Gauss law inside

$$E = \frac{\rho}{2\epsilon_0} r$$

Outside the cylinder

$$E = \frac{\rho R^2}{2\epsilon_0} \frac{1}{r}$$

Transverse Electric field



It looks like a very strange quadrupole!

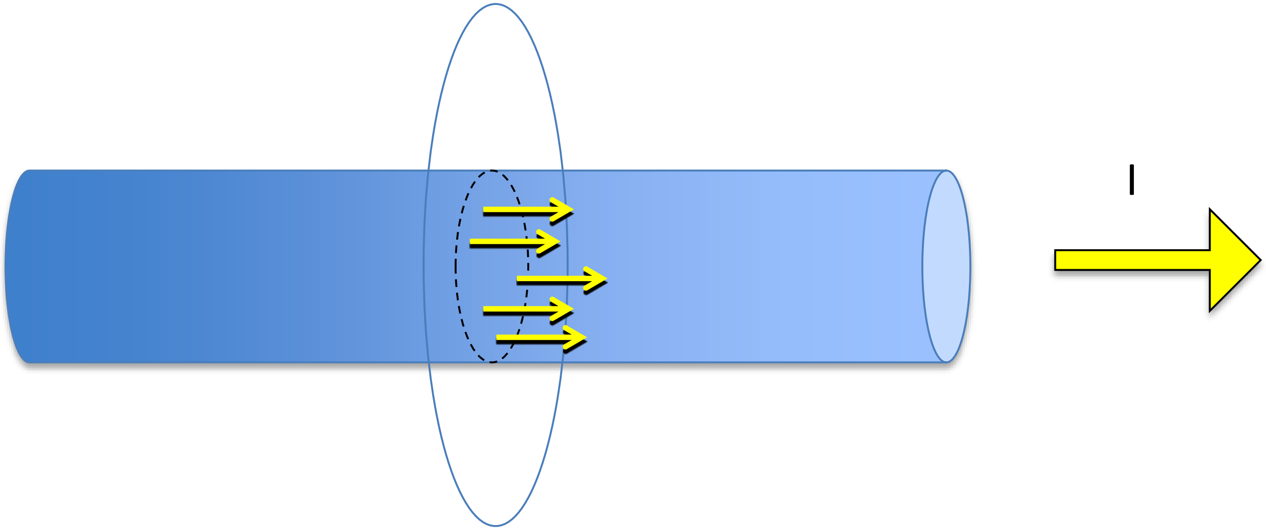
This is an approximation ... real beam infinitely long does not exists

Such a beam would require infinite energy…
in fact the energy a particle gain is infinite

$$\int_R^\infty E(r)dr = \int_R^\infty \frac{\rho R^2}{2\epsilon_0}\frac{1}{r}dr = \frac{\rho R^2}{2\epsilon_0}[\log(\infty) - \log(R)] \to \infty$$

Also $$\int_R^\infty E_r^2(r)r^2 dr \to \infty$$ the energy of the beam is infinite !

Magnetic field generated by an infinitely long beam



Apply BIOT-SAVART law

Example for uniform, round beam

Outside the beam

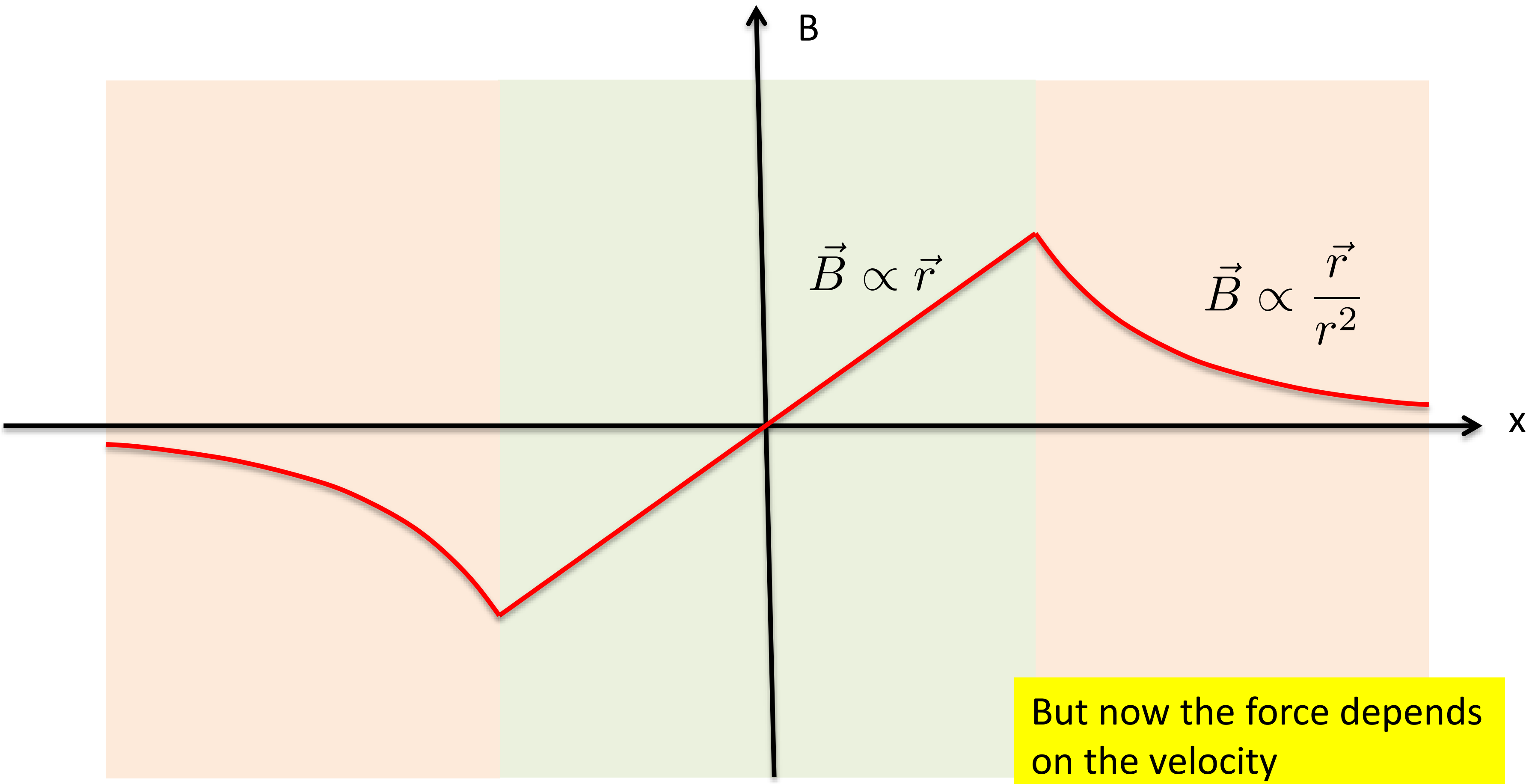
$$B = \frac{\mu_0}{2\pi}\frac{I}{r}$$

Inside the beam

$$B = \frac{\mu_0}{2\pi}\frac{I}{r}\frac{r^2}{R_b^2} = \frac{\mu_0}{2\pi}\frac{I}{R_b^2}r$$

Exactly the same dependence as for the electric field of a uniform coasting beam

Transverse Magnetic Field



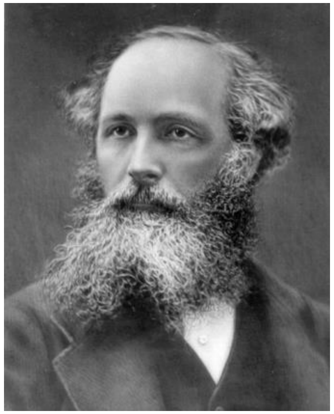
But now the force depends on the velocity

The electric + magnetic fields enter in the equation of motion as

$$\frac{d^2x}{ds^2} + k_x x = \frac{e}{m\gamma v_0^2} E_{b,x}(1 - v_z^2 \mu_0 \epsilon_0)$$

But the fundamental constants combines as follow

$$\boxed{\mu_0 \epsilon_0 = \frac{1}{c^2}}$$



therefore

$$\frac{d^2x}{ds^2} + k_x x = \frac{e}{m\gamma v_0^2} E_{b,x} \left(1 - \frac{v_z^2}{c^2}\right)$$

As $\quad |v_z| \simeq v_0 = |\vec{v}| \quad$ therefore we reach the result

$$\boxed{\frac{d^2x}{ds^2} + k_x x = \frac{e}{m\gamma^3 v_0^2} E_{b,x}}$$

Equation of motion for coasting beams axi-symmetric

$$\frac{d^2x}{ds^2} + k_x x = \frac{e}{m\gamma^3 v_0^2} E_{b,x}$$

$$\frac{d^2y}{ds^2} + k_y y = \frac{e}{m\gamma^3 v_0^2} E_{b,y}$$

result valid for any axi-symmetric distribution

Space charge is suppressed as $1/\gamma^2$

Uniform distribution

Suppose that the beam **"remains"** always uniform in x-y circle, then

$$I = v_z \pi R_b^2 \rho$$

only I is constant ! (not ρ, not R_b)

and the electric field becomes

$$E_x = \frac{\rho}{2\epsilon_0} x = \frac{1}{2\epsilon_0} \frac{I}{v_z \pi R_b^2} x$$

then ....

$$\frac{d^2x}{ds^2} + k_x x = \frac{e}{m\gamma^3 v_0^2} E_{b,x}$$

$$\frac{d^2x}{ds^2} + k_x x = \frac{eI}{2\pi\epsilon_0 m\gamma^3 v_0^2 v_z} \frac{x}{R_b^2}$$

but $eI/v_z > 0$ $\Rightarrow$ $eI/v_z \simeq |eI|/v_0$ (positive)

Perveance

It is convenient to define the quantity

$$K = \frac{eI}{2\pi\epsilon_0 m\gamma^3\beta^3c^3}$$

(positive)

General form of the transverse equation of motion for a uniform axi-symmetric coasting beam

$$\frac{d^2x}{ds^2} + k_x x = K\frac{x}{R_b^2}$$

Everything is linear !

$$\frac{d^2 x}{ds^2} + \left(k_x - \frac{K}{R_b^2} \right) x = 0$$

This is like a quadrupole with changed strength:
too beautiful to be true !!

Consequences for the motion of one particle

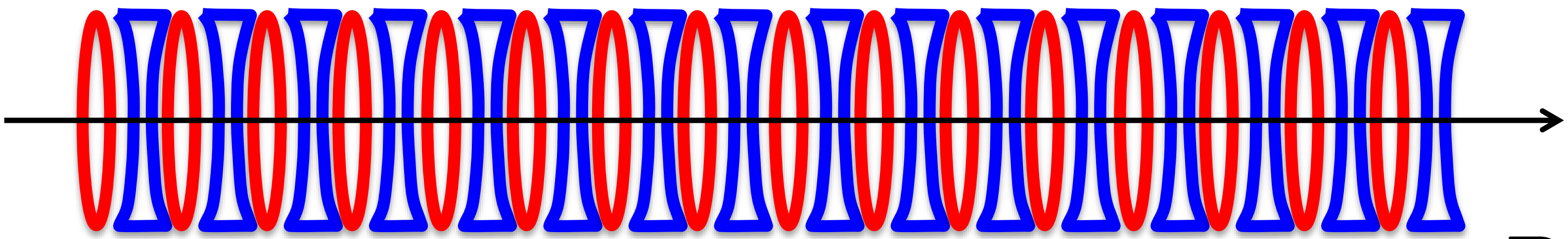
A particle experiences a modified optics

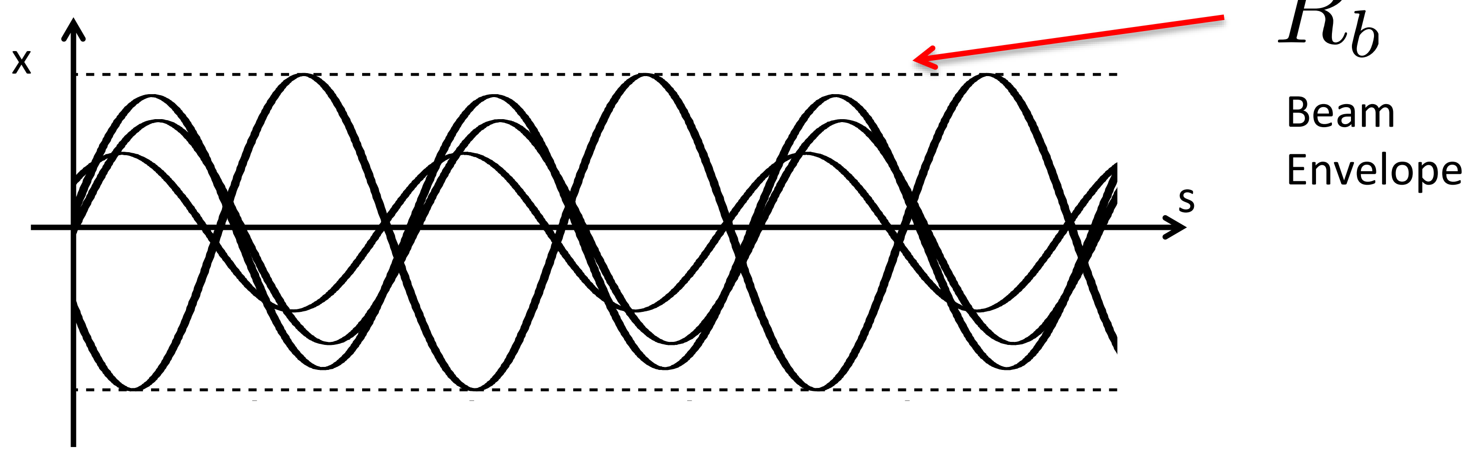
$$k_{x,eff}(s) = k_x(s) - \frac{K}{R_b^2}$$

$$k_{y,eff}(s) = k_y(s) - \frac{K}{R_b^2}$$

Is it R_b constant? Example with constant focusing lattice

We have to remember that the radius of the beam depends on the optics





But if there is a linear space charge we have a beta function that **depends also on the radius of the envelope**

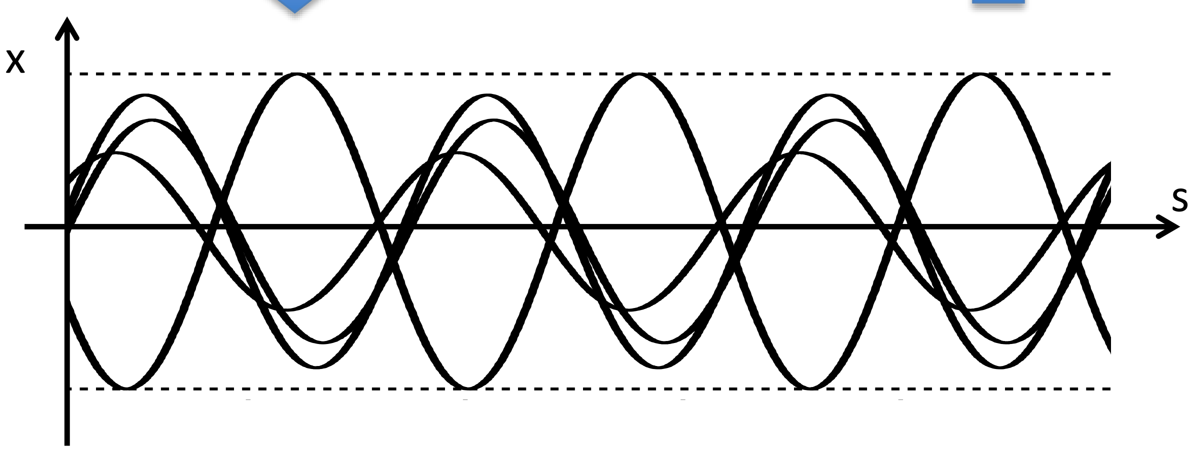
Strange situation

$$k_{x,eff}(s) = k_x(s) - \frac{K}{R_b^2}$$

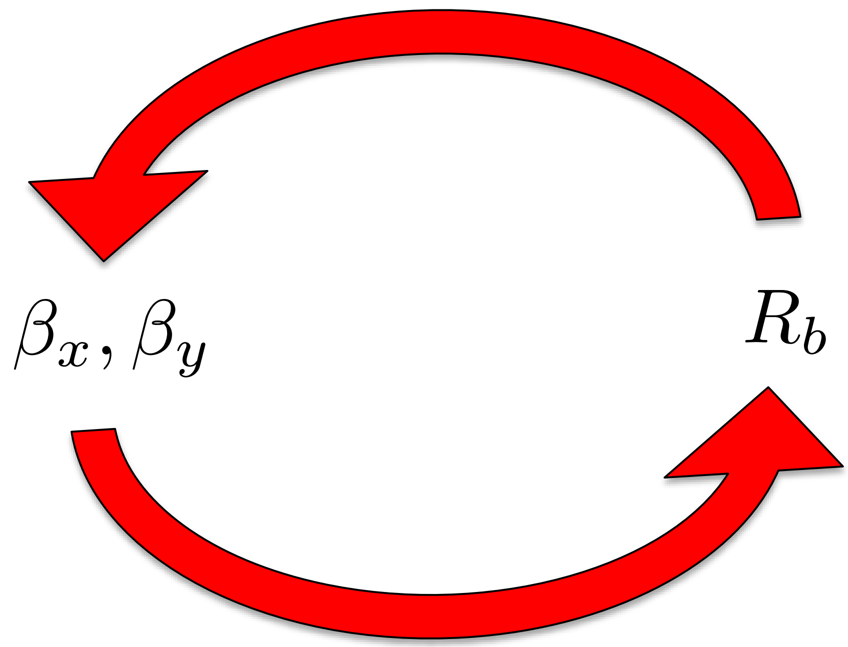
β_x, β_y

set optics:
this is taken
constant

R_b



Optics sets the beam → beam sets space charge → space charge sets the optics !



Is there a stationary solution ?

$$k_{x,eff}(s) = k_x(s) - \frac{K}{R_b^2}$$

For a constant focusing channel $$k_{x,eff} = \frac{1}{\beta_x^2}$$

and the beam radius is $$R_b^2 = \beta_x \epsilon_x$$ (ϵ_x = "beam emittance")

Therefore given k_x, K, ε_x $$\frac{1}{(\beta_x^*)^2} = k_x - \frac{K}{\beta_x^* \epsilon_x}$$

there is one β_x^* which creates a beam such that **space charge + linear optics** creates exactly β_x^*

What does it mean ?

This means that we have to create a beam of radius

$$R_b^* = X^* = \sqrt{\beta_x^* \epsilon_x}$$

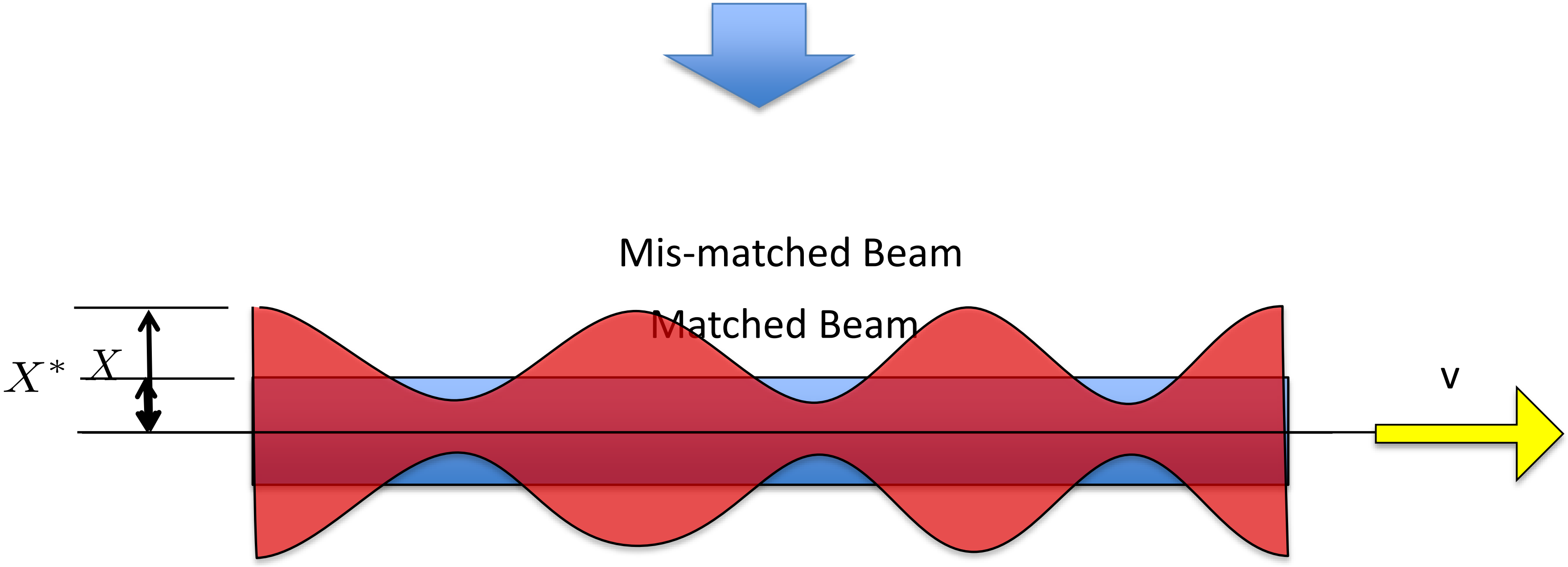
which is the only beam that, for an emittance of ϵ_x, lattice strength of k_x, perveance K, can create an effective optics with β_x^*

This beam is called **MATCHED** with the effective optics deriving from **linear optics + linear space charge forces**

When we inject a non matched beam

The optics created by the **lattice + space charge forces** makes the beam mismatched

Mismatch oscillations



Summary of finding for a uniform coasting beam

1) the lattice focusing strength is affected by space charge

2) there exists a beam that is matched

Important consequences of the modified optics (constant focusing)

	Equation of motion		tune
without space charge	$$\frac{d^2x}{ds^2} + k_x x = 0$$	$\Rightarrow$	$$Q_{x0} = \sqrt{k_x}\frac{L}{2\pi}$$
with space charge	$$\frac{d^2x}{ds^2} + \left(k_x - \frac{K}{R_b^2}\right)x = 0$$	$\Rightarrow$	$$Q_x = \sqrt{k_x - \frac{K}{R_b^2}\frac{L}{2\pi}}$$

Space charge tune-shift

$$\Delta Q_x = Q_x - Q_{x0}$$ is the space charge tune-shift

$$\Delta Q_x = \sqrt{k_x - \frac{K}{R_b^2}}\frac{L}{2\pi} - \sqrt{k_x}\frac{L}{2\pi}$$

for $K/(k_x R^2)$ small

$$\Delta Q_x = -Q_{x0}\frac{K}{2k_x R_b^2} = -Q_{x0}\frac{K}{2R_b^2}\frac{L^2}{4\pi^2 Q_{x0}^2}$$

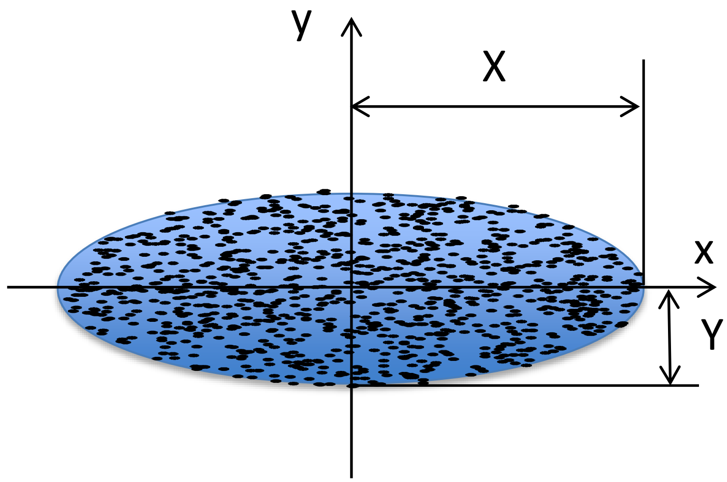
Detuning created by an axi-symmetric coasting beam, with weak intensity

$$\boxed{\Delta Q_x = -\frac{R_m^2}{2R_b^2}\frac{K}{Q_{x0}}}$$

R_m is the accelerator radius
R_b is the radius of the beam
Q_{x0} is the bare tune
K is the perveance

Non axi-symmetric uniform beams

For uniform beams the electric field becomes



Inside the beam

$$E_x = \frac{I}{\pi \epsilon_0 v} \frac{x}{X(X+Y)}$$

$$E_y = \frac{I}{\pi \epsilon_0 v} \frac{y}{Y(X+Y)}$$

Equation of motion

$$\frac{d^2x}{ds^2} + \left[k_x - \frac{2K}{X(X+Y)}\right]x = 0$$

$$\frac{d^2y}{ds^2} + \left[k_y - \frac{2K}{Y(X+Y)}\right]y = 0$$

Modified beta function

The lattice optics is modified in x, and y

$$k_{x,eff} = k_x - \frac{2K}{X(X+Y)} \Rightarrow \beta_x^*$$

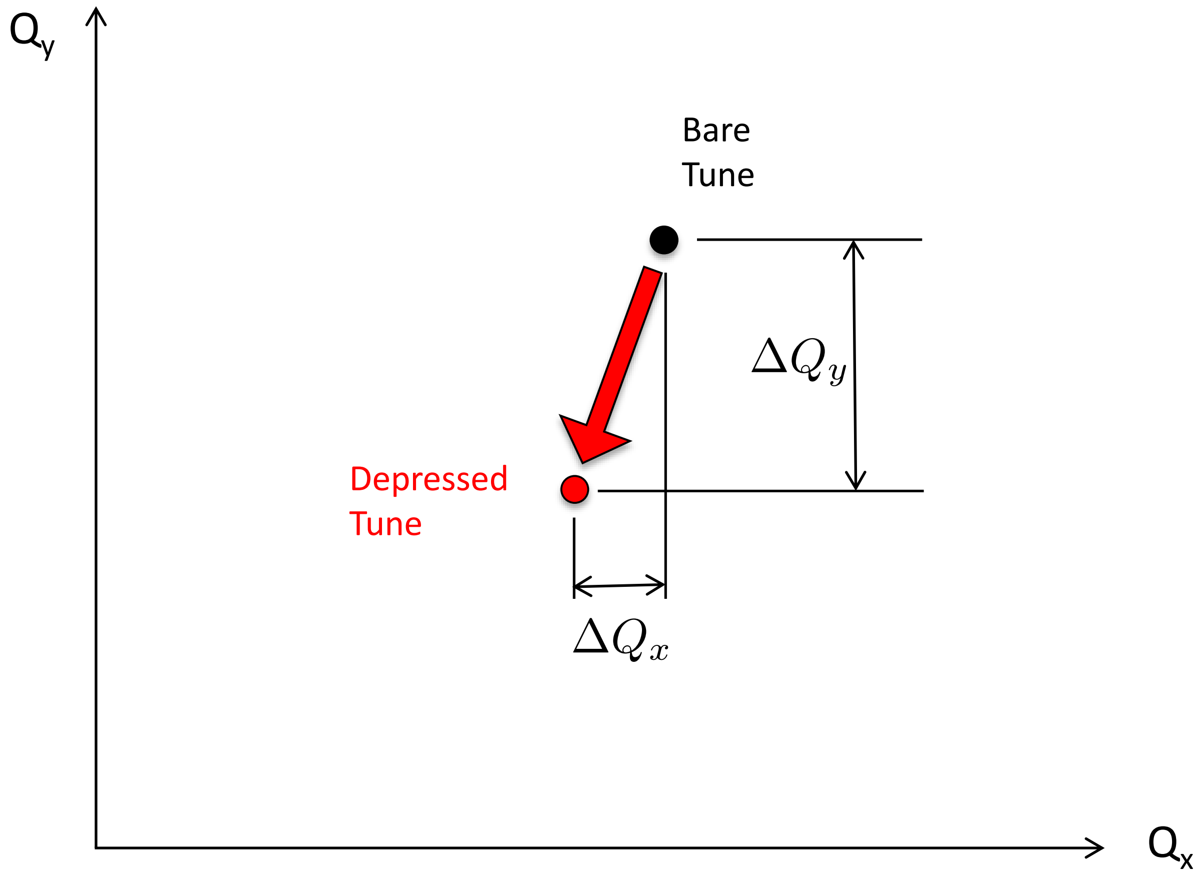
$$k_{y,eff} = k_y - \frac{2K}{Y(X+Y)} \Rightarrow \beta_y^*$$

Space charge tune-shift

$$\Delta Q_x = -\frac{K}{X(X+Y)}\frac{R_m^2}{Q_{x0}}$$

$$\Delta Q_y = -\frac{K}{Y(X+Y)}\frac{R_m^2}{Q_{y0}}$$

Situation in a tune diagram

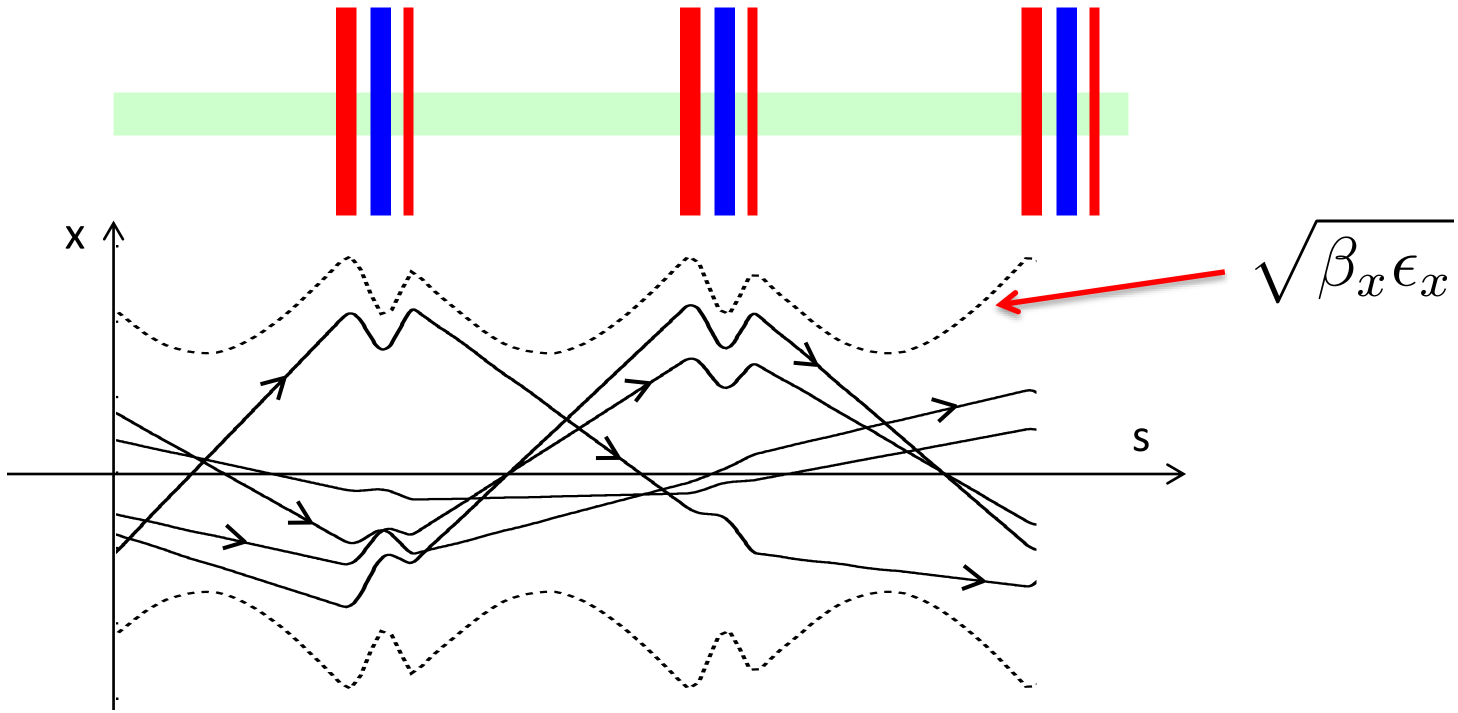


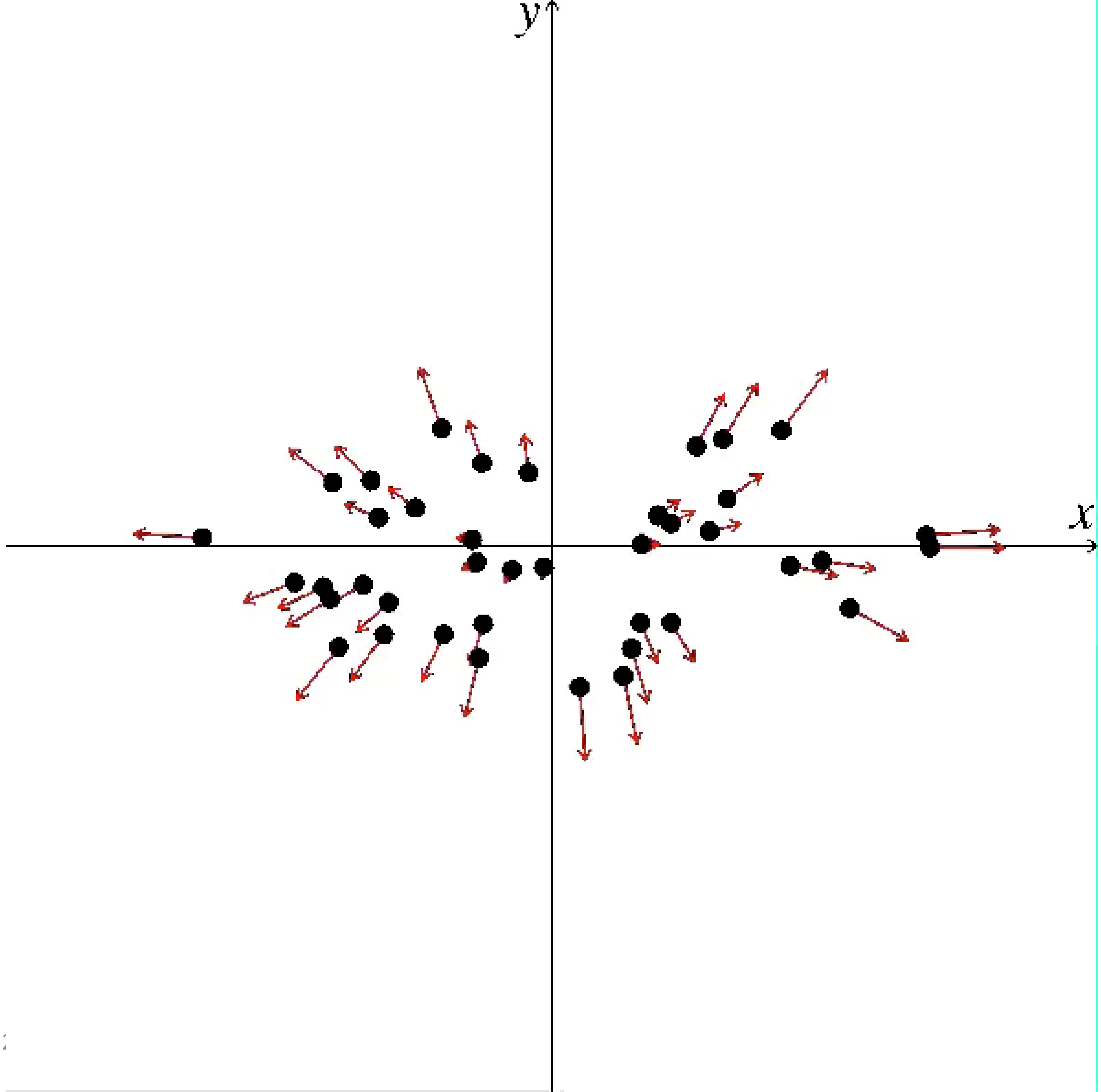
Conclusion for the constant focusing

Space charge changes the particle tune, in both planes
according to the beam sizes, and the optics:
we find formulas that predicts incoherent space charge
tune-shifts for a “matched” beam

For varying focusing

All formulation remains the same, but the difference is in what happens to the beta functions and the detuning




y
x

New optics

We continue to keep the ansatz that the beam remains uniform, and with the same transverse emittances

$$\beta_{x,0}(s), \beta_{y,0}(s)$$

$$\frac{d^2x}{ds^2} + \left[k_x - \frac{2K}{X(X+Y)}\right]x = 0$$

$$\frac{d^2y}{ds^2} + \left[k_y - \frac{2K}{Y(X+Y)}\right]y = 0$$

$$\beta_{x,1}(s), \beta_{y,1}(s)$$

Go on until $\beta_{x,n}(s), \beta_{y,n}(s)$ converges

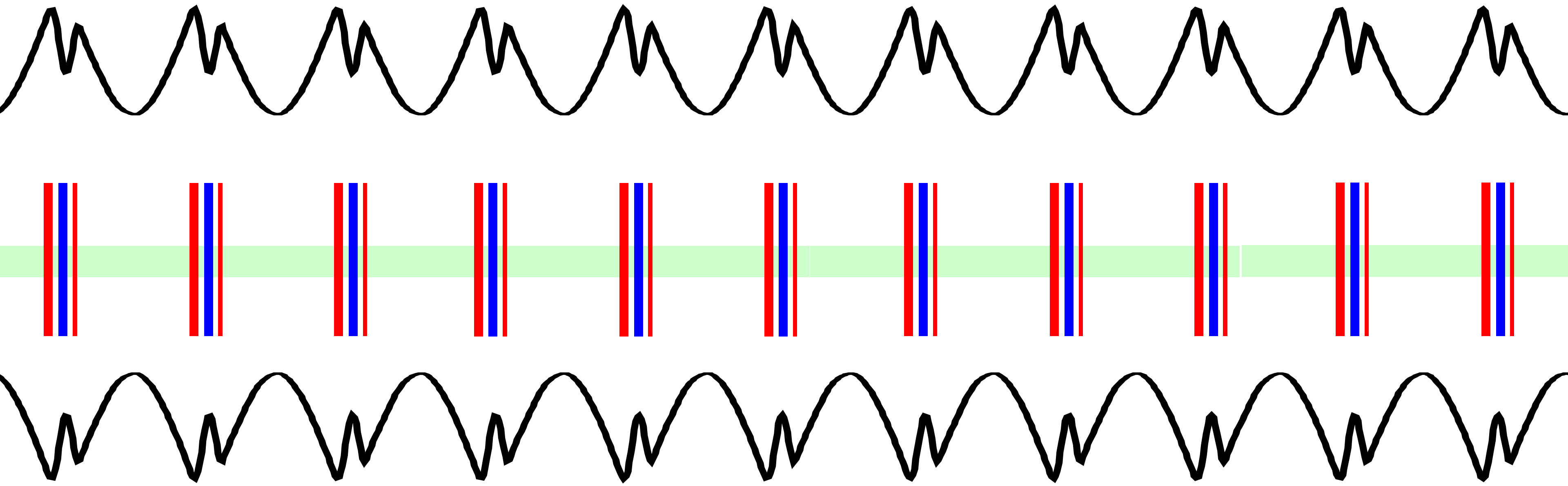
Space charge tune-shift

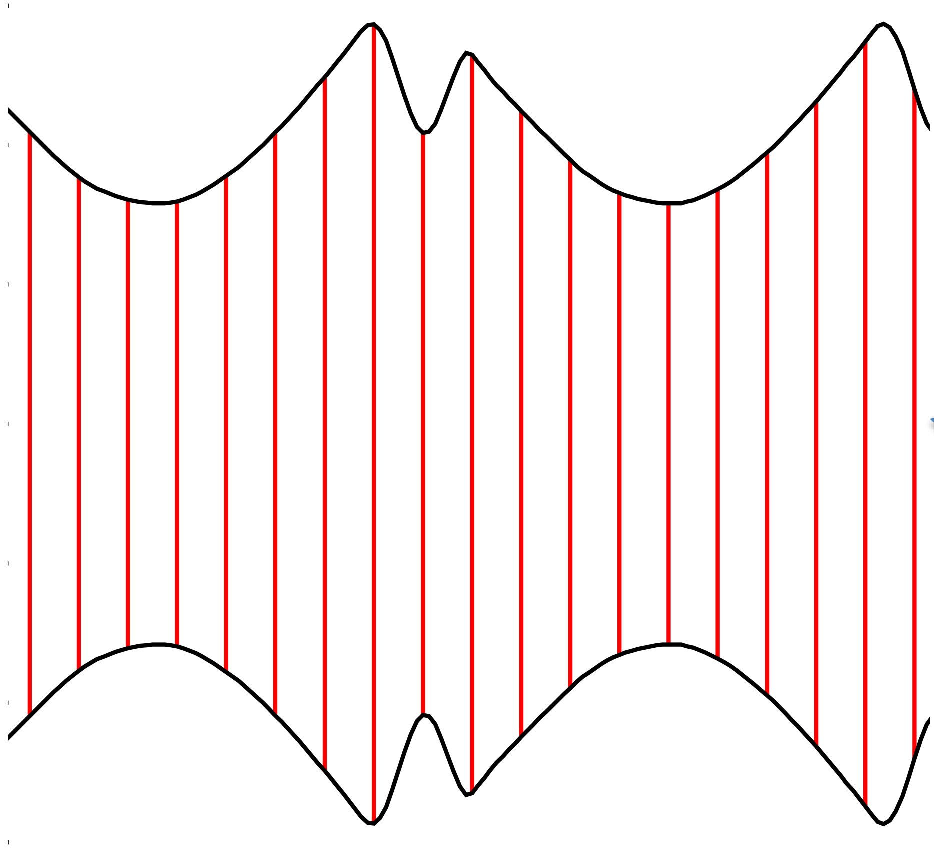
Now we have a matched optics for a beam with perveance K, and transverse emittances $\varepsilon_x, \varepsilon_y$. Therefore injecting a beam matched with

$$\beta_x^*(s), \alpha_x^*(s), \beta_y^*(s), \alpha_y^*(s)$$

will create a matched optical function.

Now you can look at the space charge as a distribution of many space charge "kicks"



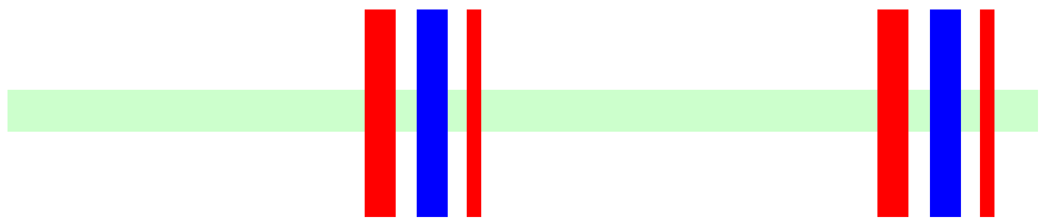


$$\frac{d^2x}{ds^2} + \left[k_x - \frac{2K}{X(X+Y)}\right] x = 0$$

Space charge kick

$$ds\ \mathcal{E}_x = \frac{2K}{X(X+Y)} x\ ds$$

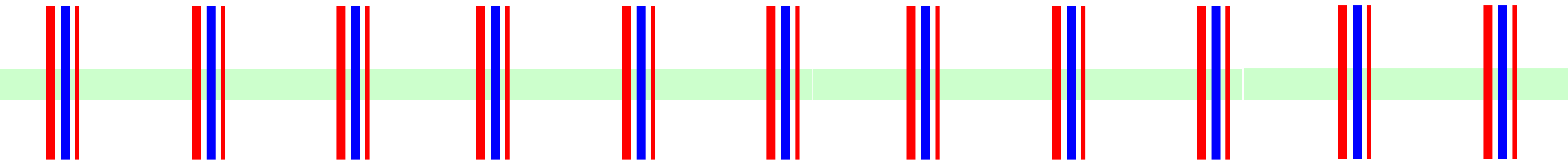
$$ds\ \mathcal{E}_y = \frac{2K}{Y(X+Y)} y\ ds$$



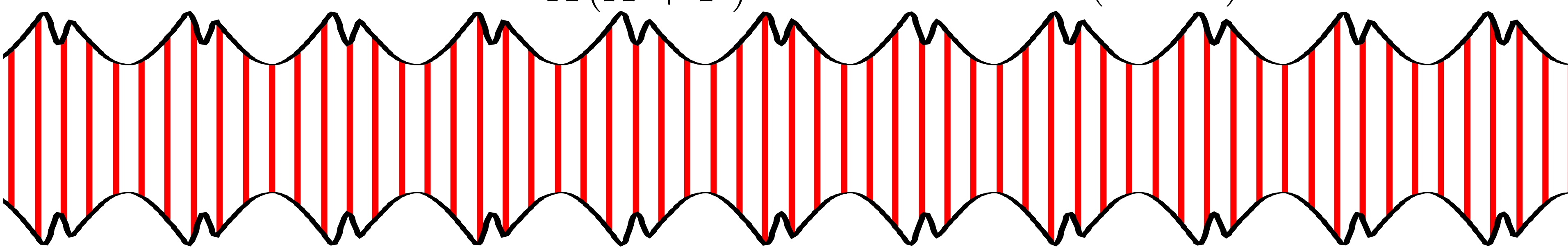
in units of the equation of motion

Situation

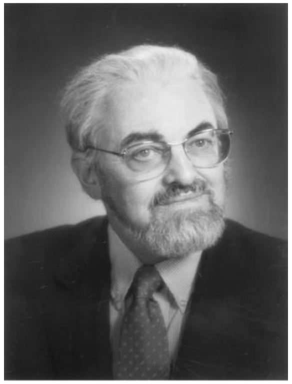
Linear optics $$\frac{d^2x}{ds^2} + k_x x = 0 \quad \frac{d^2y}{ds^2} + k_y y = 0$$



Space charge kicks $$ds\,\mathcal{E}_x = \frac{2K}{X(X+Y)}x\;ds \qquad ds\,\mathcal{E}_y = \frac{2K}{Y(X+Y)}y\,ds$$



E. Courant



$$\Delta\nu = \frac{\Delta\mu}{2\pi} = -\frac{\Delta(\cos\mu)}{2\pi\sin\mu_0} = \frac{1}{4\pi}\int_0^C \beta(s)k(s)\,ds.$$

$$\Delta Q_x = \frac{1}{4\pi}\int_0^C \beta_x(s)\,\mathcal{E}_x(s)ds = -\frac{1}{4\pi}\int_0^C \beta_x(s)\frac{2K}{X(s)(X(s)+Y(s))}ds$$

$$\Delta Q_y = \frac{1}{4\pi}\int_0^C \beta_y(s)\,\mathcal{E}_y(s)ds = -\frac{1}{4\pi}\int_0^C \beta_y(s)\frac{2K}{Y(s)(X(s)+Y(s))}ds$$

$$\Delta Q_x = -\frac{KR_m}{\epsilon_x}\left\langle \frac{1}{1+\sqrt{\frac{\epsilon_y \beta_y(s)}{\epsilon_x \beta_x(s)}}} \right\rangle$$

It is a usual approximation that

$$\left\langle \frac{1}{1+\sqrt{\frac{\epsilon_y \beta_y(s)}{\epsilon_x \beta_x(s)}}} \right\rangle \simeq \frac{1}{1+\sqrt{\frac{\epsilon_y \langle \beta_y \rangle}{\epsilon_x \langle \beta_x \rangle}}}$$

(not really obvious…)

Therefore

$$\Delta Q_x \simeq -\frac{KR_m}{\epsilon_x}\frac{1}{1+\sqrt{\frac{\epsilon_y\langle\beta_y\rangle}{\epsilon_x\langle\beta_x\rangle}}} = -KR_m\frac{\langle\beta_x\rangle}{\sqrt{\epsilon_x\langle\beta_x\rangle}(\sqrt{\epsilon_x\langle\beta_x\rangle}+\sqrt{\epsilon_y\langle\beta_y\rangle})}$$

Taking $\langle\beta_x\rangle \simeq \dfrac{R_m}{Q_{x0}}$

$$\Delta Q_x \simeq -K\frac{R_m^2}{Q_{x0}}\frac{1}{\sqrt{\epsilon_x\langle\beta_x\rangle}(\sqrt{\epsilon_x\langle\beta_x\rangle}+\sqrt{\epsilon_y\langle\beta_y\rangle})}$$

Exactly the same formula of the constant focusing channel

Ring with constant focusing

$$\Delta Q_x = -\frac{K}{X(X+Y)}\frac{R_m^2}{Q_{x0}}$$

Ring with AG focusing

$$\Delta Q_x \simeq \frac{K}{\sqrt{\epsilon_x\langle\beta_x\rangle}(\sqrt{\epsilon_x\langle\beta_x\rangle}+\sqrt{\epsilon_y\langle\beta_y\rangle})}\frac{R_m^2}{Q_{x0}}$$

What is the meaning?

It seems that the space charge detuning is governed by the same type of law, provided we use some kind of "effective" beam size.

This **seems** to suggest that when two beams have the same "effective" size, and they are in a machine with the same radius, and the same tune, they have the same space charge detuning !!

(nice, but not obvious)

About the ansatz of the uniformity

Is it true that if we start with a beam distribution uniform, that is remains uniform ?

Beam distribution evolves according to the Vlasov equation

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) = 0$$

with $f(q, p, t) = \frac{\Delta N}{\Delta V}$ particle density in phase space

A very complex and difficult equation !!

Stationary distributions

Is there a distribution that does not change "functional shape" ?

That is, that it is not time dependent ?

Without space charge

for a linear uncoupled lattice → Answer: YES

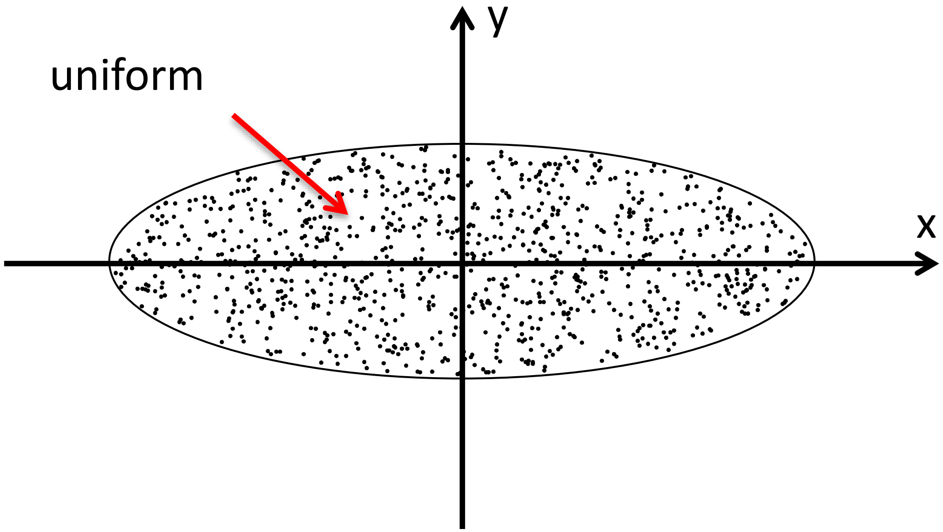
take $$f(x, x', y, y', t) = g(\epsilon_{0x}, \epsilon_{0y})$$

$$\epsilon_{0x} = \gamma_x x^2 + 2\alpha_x x x' + \beta_x^2 x'^2$$
$$\epsilon_{0y} = \gamma_y y^2 + 2\alpha_y y y' + \beta_y^2 y'^2$$

This type of distributions are all stationary → MATCHED with the lattice

Stationary distribution

If a distribution is x-y uniformly populated of particles



Forces (normalized) are linear

$$\mathcal{E}_x = \frac{2K}{X(X+Y)} x$$

$$\mathcal{E}_y = \frac{2K}{Y(X+Y)} y$$

But we are not sure that the x-y distribution remains uniform during beam propagation

KAPCHINSKY-VLADIMIRSKY (KV)

But any distribution $$f(x, x', y, y', t) = g(\epsilon_{0x}, \epsilon_{0y})$$

remains of the same type if forces are **linear**

But then, if we choose a distribution that creates linear space charge forces, then that distribution also will remain of the same type !

$$f = \delta\left(\frac{\epsilon_{0x}}{\mathcal{E}_x} + \frac{\epsilon_{0y}}{\mathcal{E}_y} - 1\right)$$

This distribution creates a uniform x-y distribution

it will remain of the same type !!

NON uniform distributions

Non-uniform beam distributions exhibits a more complex behaviour.

1) These distribution can be generated to be matched with a linear lattice without space charge

2) When the beam has space charge effects, these distributions are not stationary, hence they change with time, BUT for short time they keep their form.

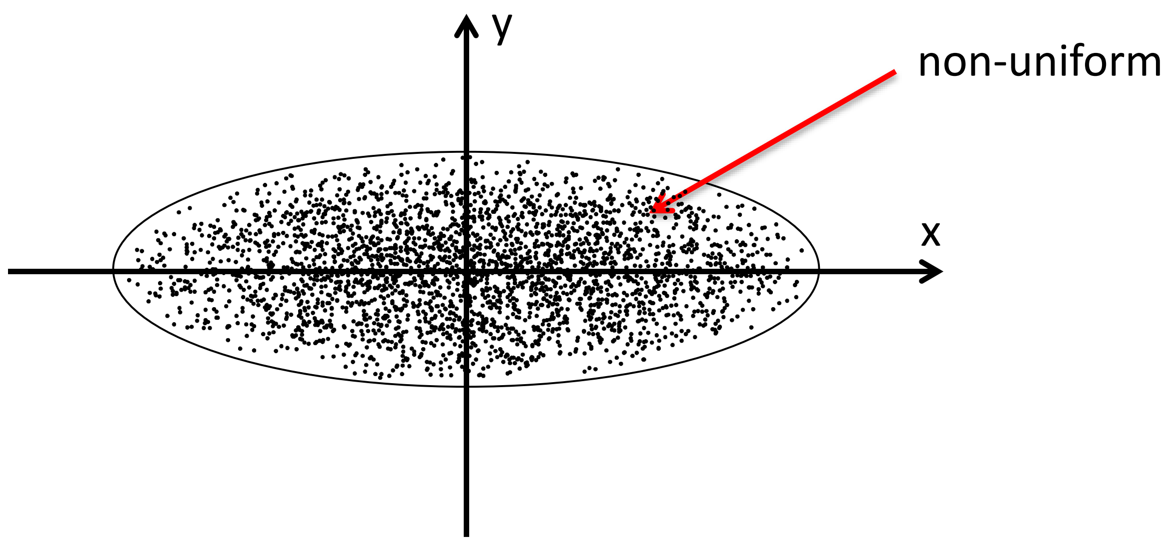
WATERBAG

$$f = \Theta\left(\frac{\epsilon_{0x}}{\mathcal{E}_x} + \frac{\epsilon_{0y}}{\mathcal{E}_y} - 1\right)$$

with $\Theta(x)$

the Heaviside function

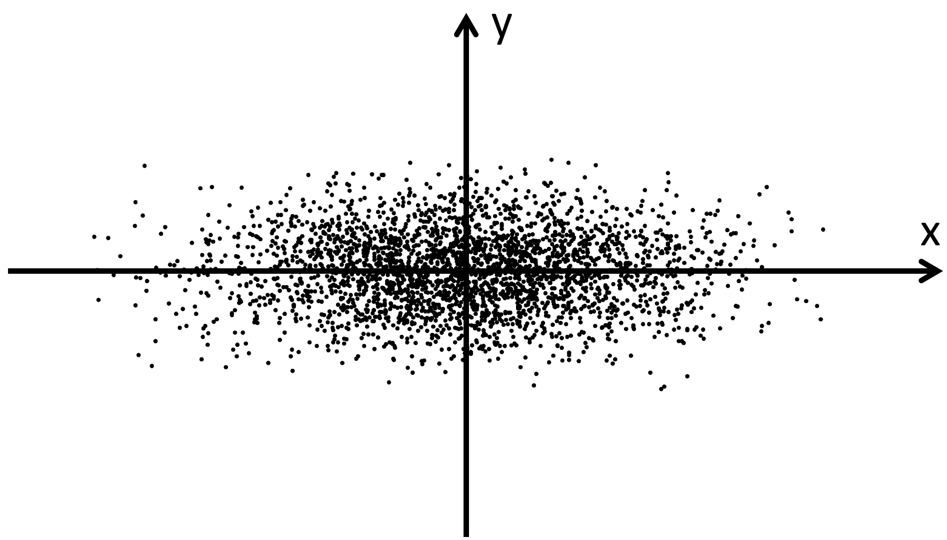
It is a 4D sphere completely filled



GAUSSIAN

$$f \propto e^{-\frac{1}{2}\left(\frac{\epsilon_{0x}}{\mathcal{E}_x}+\frac{\epsilon_{0y}}{\mathcal{E}_y}\right)}$$

The distribution is not bounded, and is the product of two 1D Gaussians



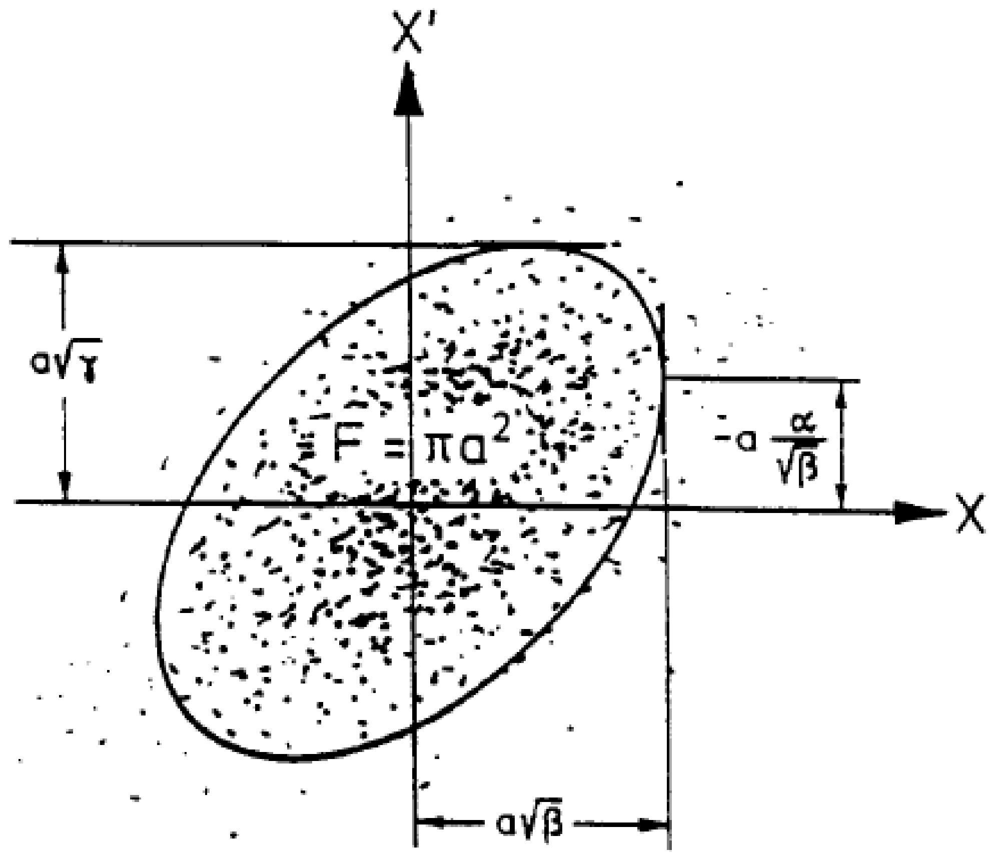
Moments

$$\langle x^2 \rangle = \frac{1}{N}\sum_{i=1}^{N} x_i^2$$

$$\langle p_x^2 \rangle = \frac{1}{N}\sum_{i=1}^{N} p_{x,i}^2$$

$$\langle x p_x \rangle = \frac{1}{N}\sum_{i=1}^{N} x_i\, p_{x,i}$$

$$\tilde{\epsilon}_x^2 = \langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2$$



RMS emittance depends
on the beam distribution

RMS envelope equation

Defining RMS envelope $\tilde{x} = \sqrt{\langle x^2 \rangle}$

$$\tilde{x}'' = \frac{\langle x x'' \rangle}{\tilde{x}} + \frac{\tilde{\epsilon}_x^2}{\tilde{x}^3}$$

Without space charge

$$x'' + k(s)x = 0 \quad \Rightarrow \quad \langle x x'' \rangle = -k(s)\langle x^2 \rangle$$

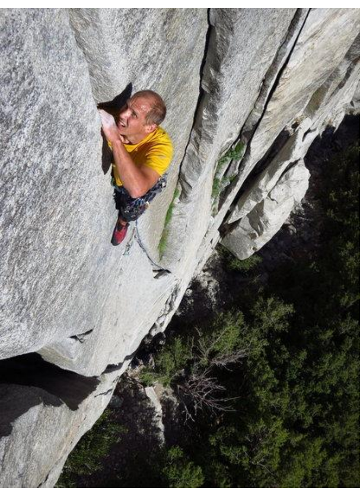
$$\tilde{x}'' = \frac{-k(s)\langle x^2 \rangle}{\tilde{x}} + \frac{\tilde{\epsilon}_x^2}{\tilde{x}^3}$$

RMS envelope equation without space charge (yields the equation of beta function)

$$\boxed{\tilde{x}'' + k(s)\tilde{x} - \frac{\tilde{\epsilon}_x^2}{\tilde{x}^3} = 0}$$

Including space charge

Frank Sacherer
1940 - 1978



Sacherer Cracker,
Yosemite (and 33 peaks climbed)

Equation of motion

Space charge force "scaled" in Equation of motion

$$x'' = -k(s)x + \mathcal{E}_x$$

Therefore

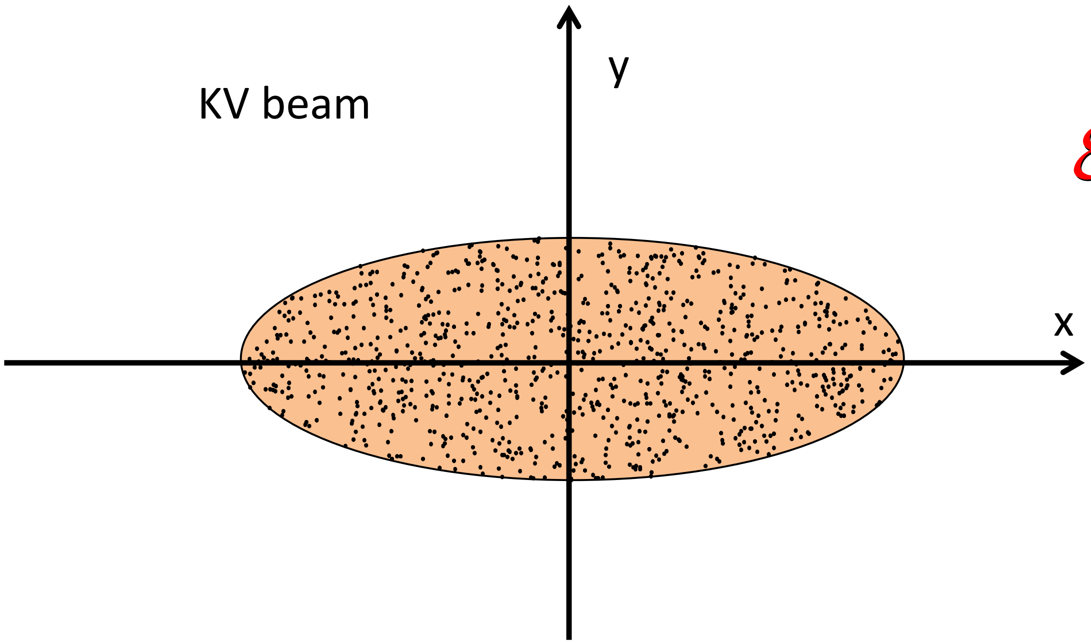
$$\langle xx'' \rangle = -k(s)\langle x^2 \rangle + \langle x\mathcal{E}_x \rangle$$

$$\tilde{x}'' + k(s)\tilde{x} - \frac{\langle x\mathcal{E}_x\rangle}{\tilde{x}} - \frac{\tilde{\epsilon}_x^2}{\tilde{x}^3} = 0$$

What is it $\langle x\mathcal{E}_x\rangle$?

Well: If $\mathcal{E}_x = \lambda x \Rightarrow \langle x\mathcal{E}_x\rangle = \lambda\tilde{x}^2$

For a KV beam



$$\mathcal{E}_x = 2K\frac{x}{X(X+Y)}$$

$$\langle x\mathcal{E}_x\rangle = 2K\frac{\langle x^2\rangle}{X(X+Y)} = 2K\frac{X}{(X+Y)}$$

F. Sacherer: very surprising result

If the beam has transverse distribution

$$\rho \propto n\left(\frac{x^2}{X^2}+\frac{y^2}{Y^2}\right)$$

True for any distribution matched with the naked optics

$$\boxed{\langle x\mathcal{E}_x\rangle = 2K\frac{X}{(X+Y)}}$$

RMS envelope equation

Therefore the rms envelope follows the equation

$$\tilde{x}'' + k(s)\tilde{x} - \frac{K}{2(\tilde{x}+\tilde{y})} - \frac{\tilde{\epsilon}_x^2}{\tilde{x}^3} = 0$$

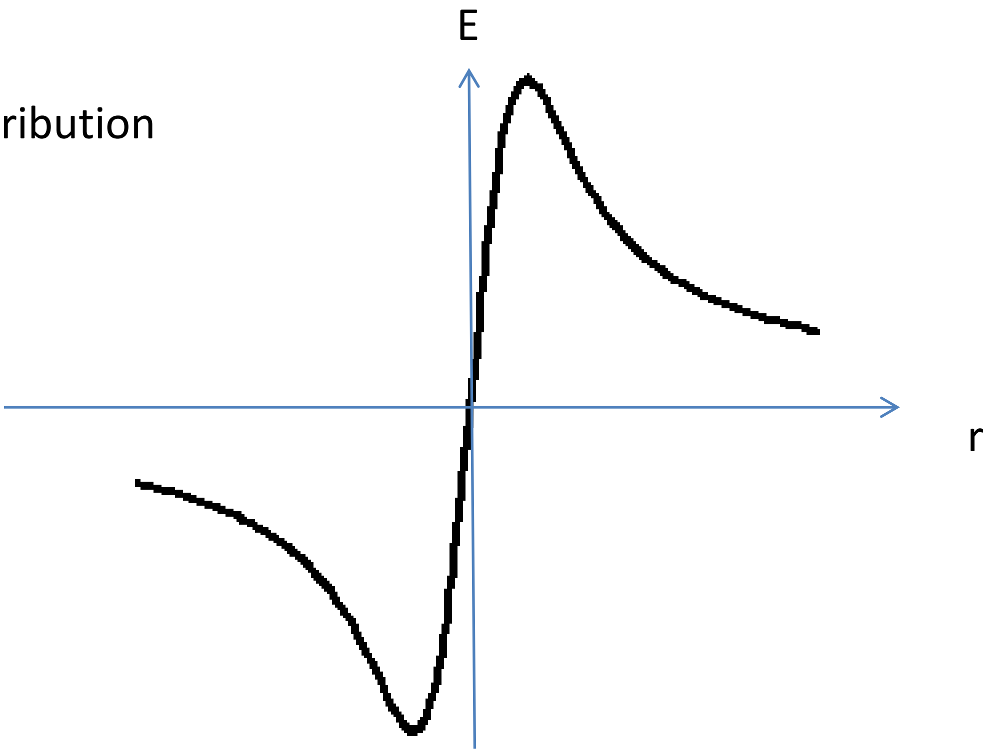
If different beams have the same rms sizes, the same rms emittance, the same perveance

All these beams have the same rms evolution

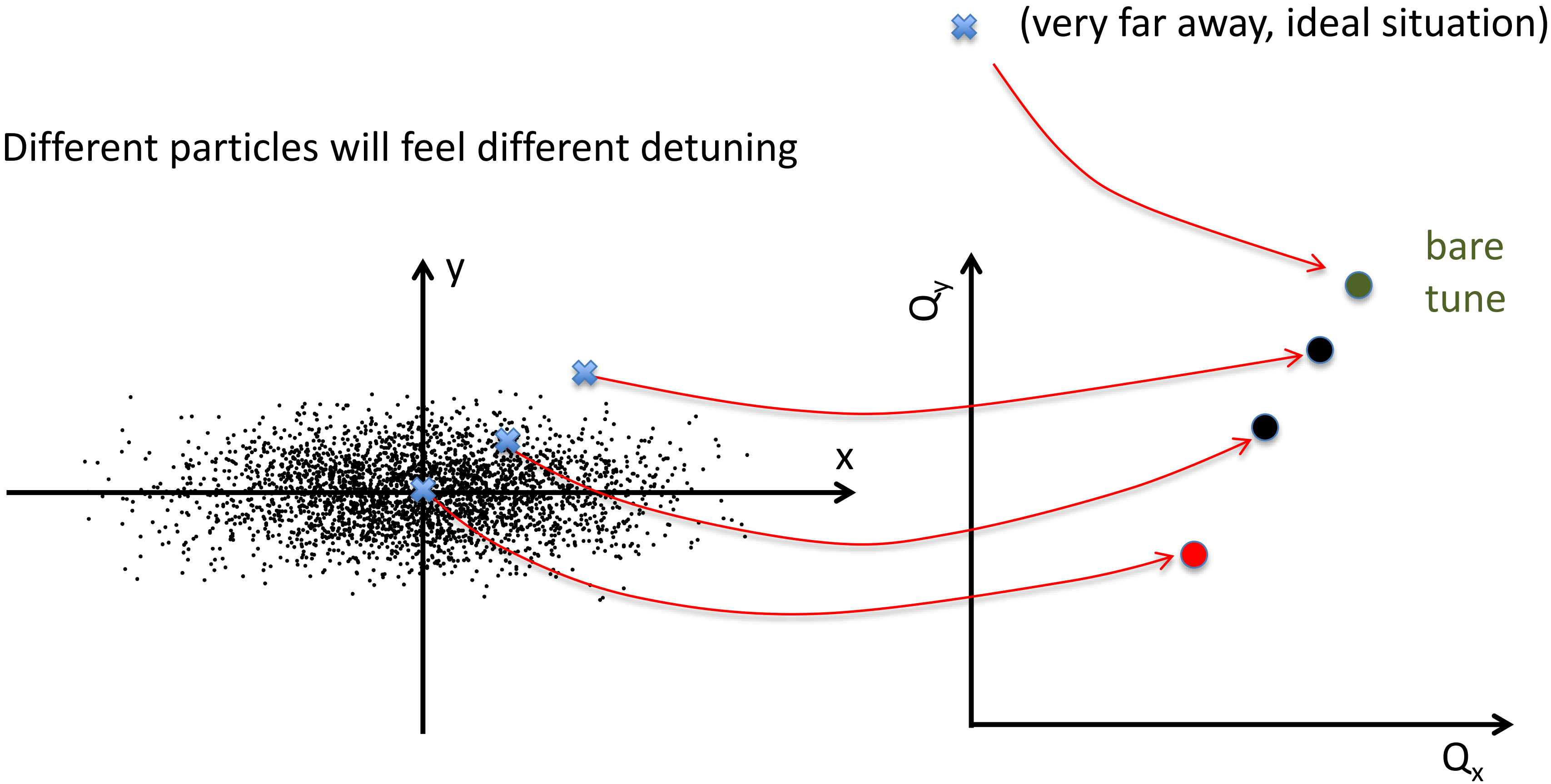
Space Charge Detuning of Non-uniform distribution

For WB, G distributions the expression of the space charge force is more complex.

Example of a
Gaussian distribution

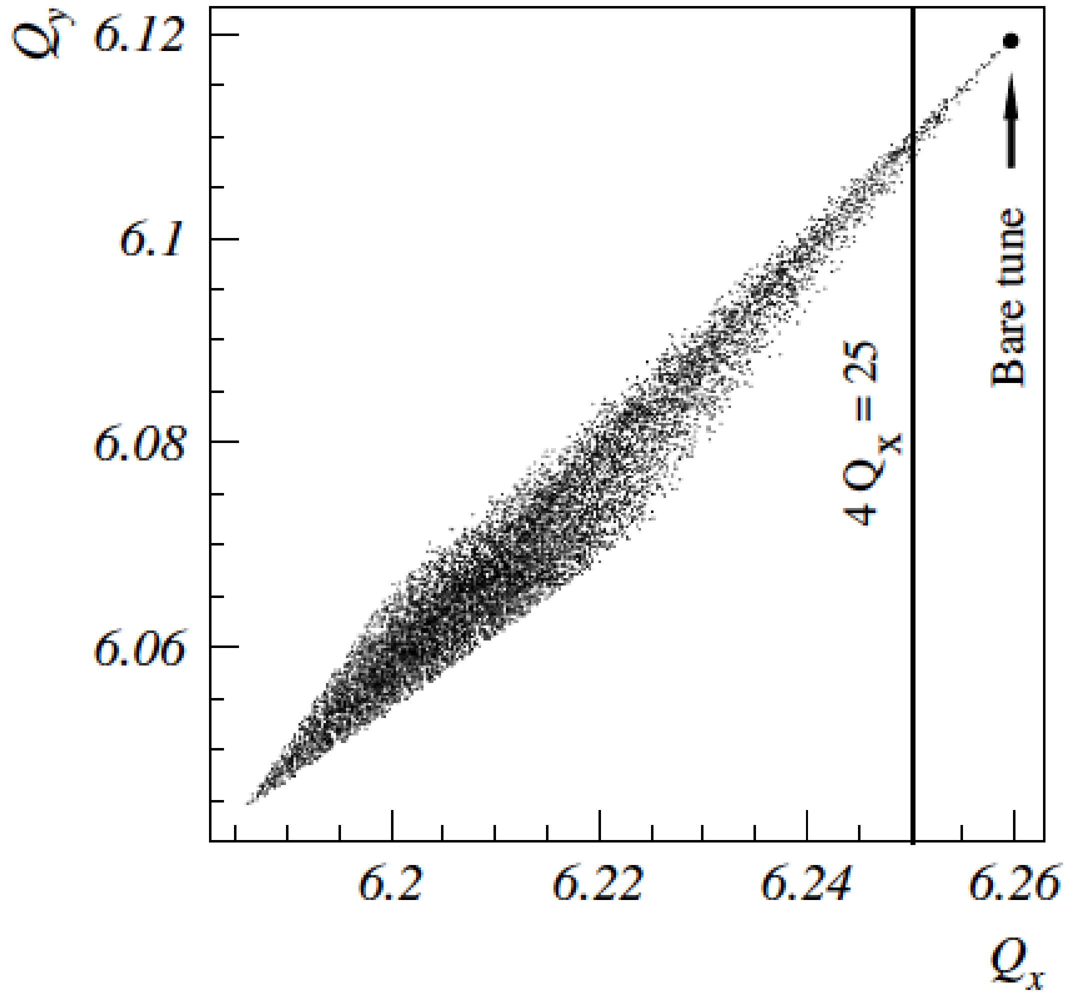


Different particles will feel different detuning



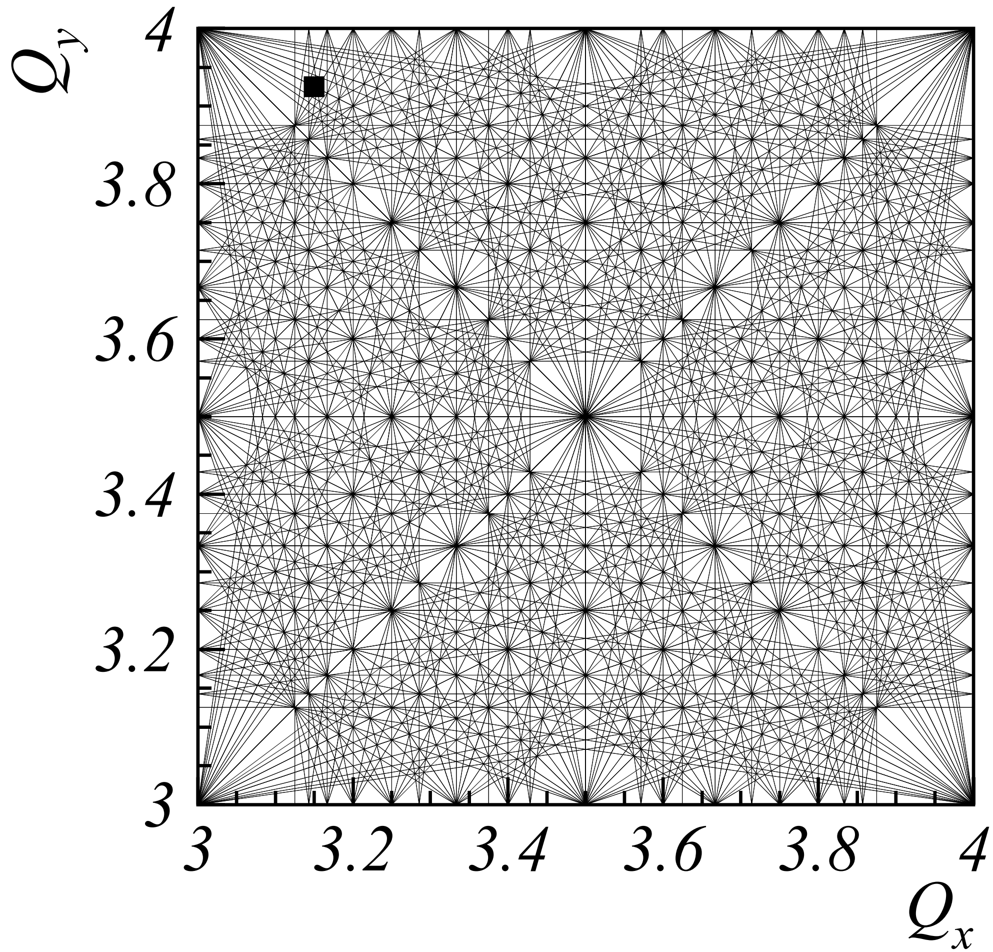
The space charge tune-spread

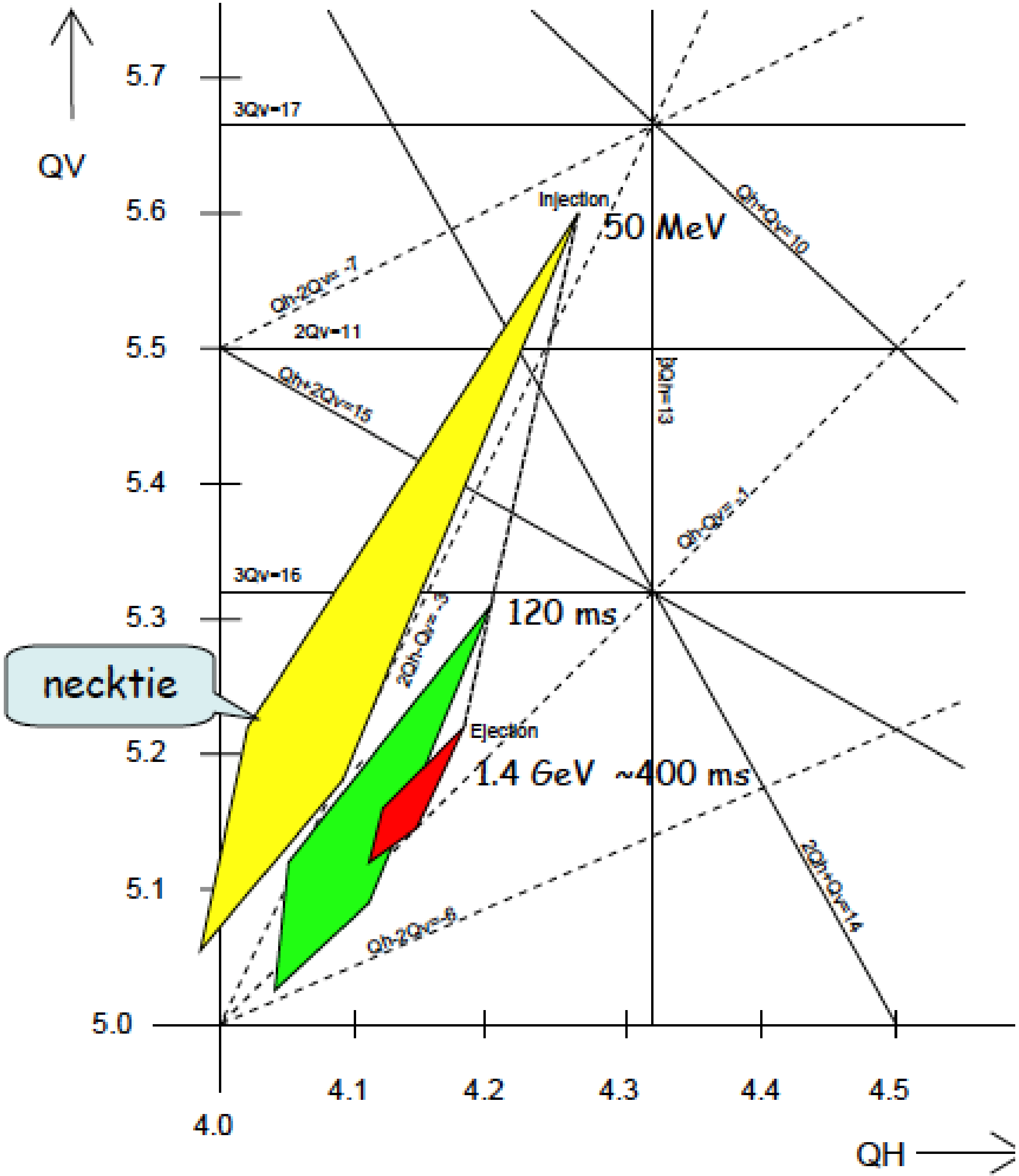
Example



Consequences

If the space charge induced tune-spread overlaps a machine resonance there is a problem





Issues

1) Space charge + resonances in coasting beams
2) Space charge + resonances in bunched beams
3) Collective beam response to direct space charge forces ?

Space charge in Linacs

Linac → low energy $\gamma \to 1$

Space charge forces are not damped by self magnetic field

↓

Much stronger effect on the dynamics

Collective modes excited by direct space charge are very important

Rings vs Linacs (example)

Usually beam intensity is limited to constrain the incoherent tuneshift $|\Delta Q_{x/y}| < 0.25$

Rings focusing strength typically provides large tunes $Q_{x/y} > 4$

Depressed tunes $Q_x/Q_{x0} > 0.95$

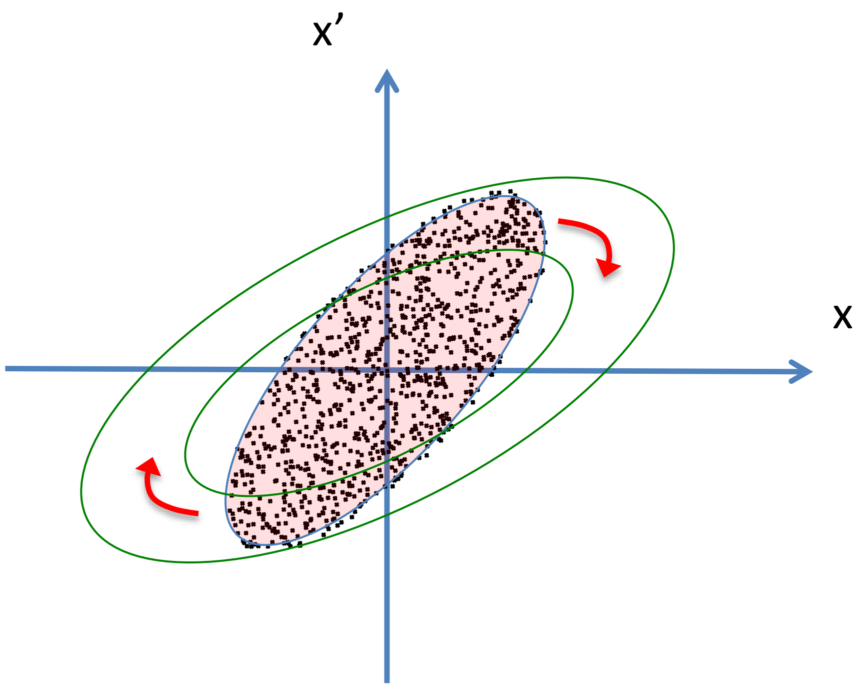
Depressed phase advance $\psi/\psi_0 \sim 0.5$

Direct space charge creates complex effects

Oscillation of mismatched beams

Without space charge

Small oscillation: a mismatched KV

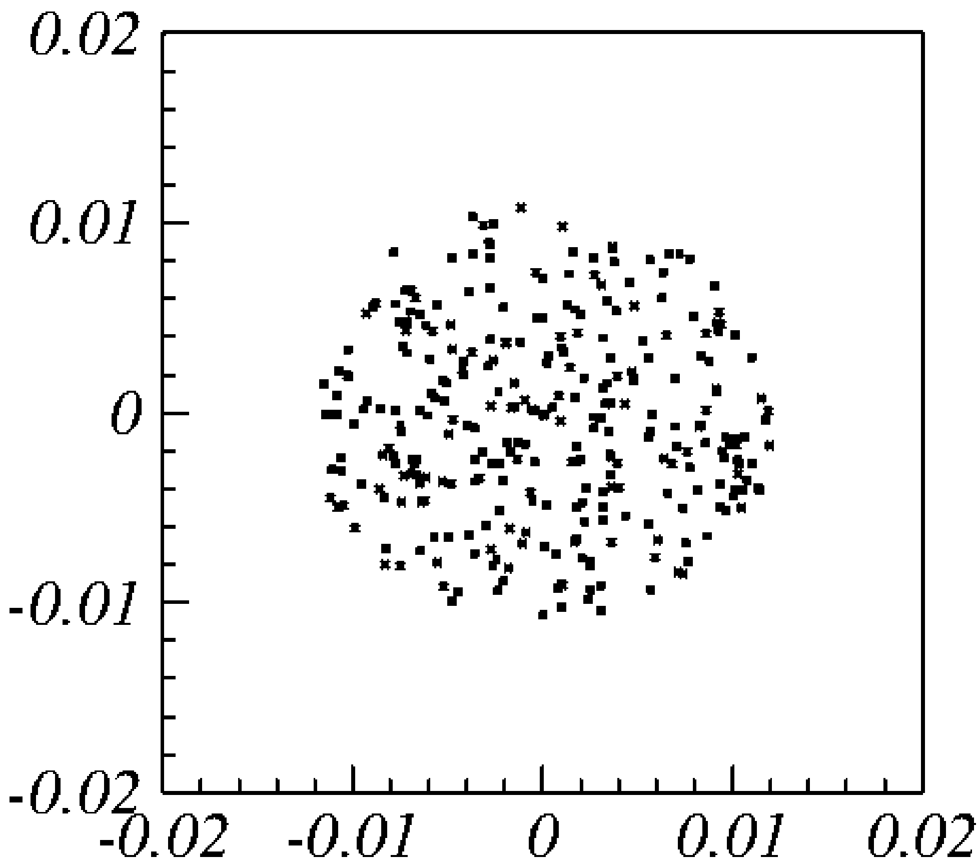


Number of oscillations per turn

$$2 \times Q_{0x}$$

Coherent frequencies

Example of coherent motion driven by an incoherent force (the lattice)
Matched beam kicked with a quadrupolar kick

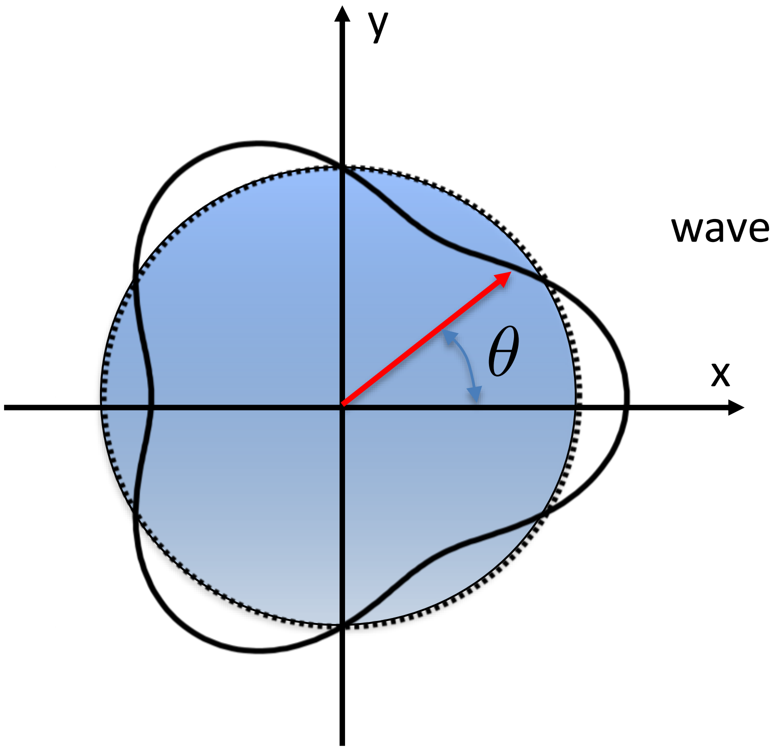


Coherent frequencies

without space charge
2 x Q_{0x}

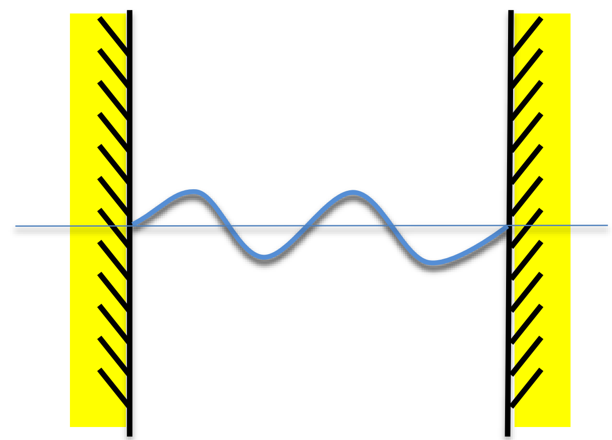
Coherent Modes

Transverse beam oscillations



$$f(k\theta - \omega s)$$

String between two walls

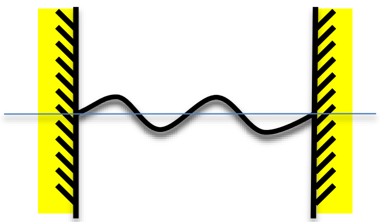


Any wave comes from a wave Equation → $f(kx - \omega t)$

Coherent Modes: stability/instability

String between two walls

$$f(kx - \omega t)$$



From wave equation

$$\frac{\partial^2}{\partial t^2} f = v^2 \nabla^2 f$$

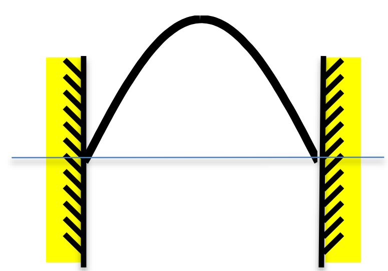
Dispersion relation

$$v^2(k_x^2 + k_y^2 + k_z^2) = \omega^2$$

Boundary condition →
Only special values of k are allowed

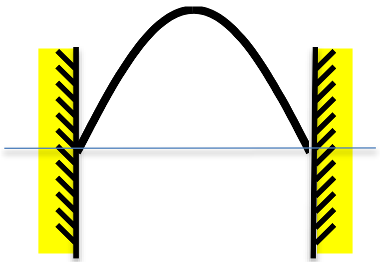
$$k_x L = n_x \pi$$

$$n_x = 0, 1, 2, 3, \ldots$$

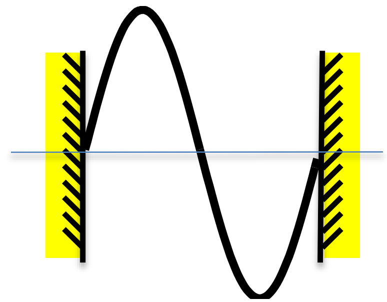


Mode n_x =1

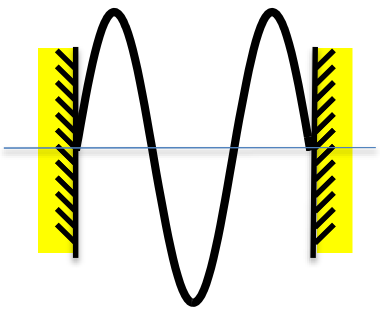
Coherent Modes: stability/instability



Mode $n_x = 1$



Mode $n_x = 2$



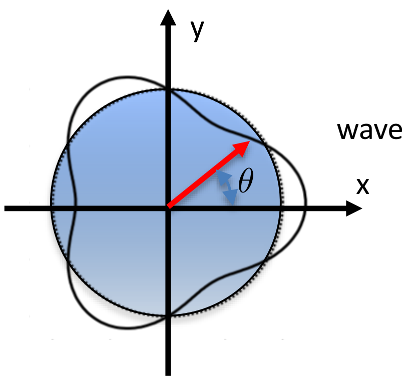
Mode $n_x = 3$

Modes are always stable

For the EM waveguide there is a cut-off frequency as a result of the boundary condition in x,y (if the wave propagates along z).

Coherent Modes: stability/instability

Transverse beam oscillations



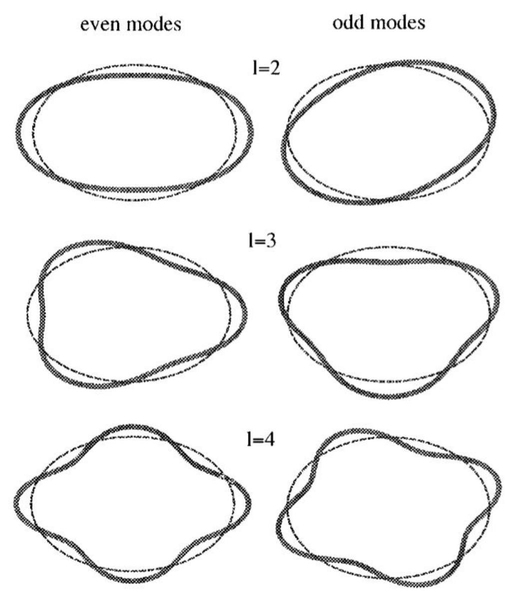
$$f(k\theta - \omega s)$$

Evolution of the "wave" is found from the Vlasov equation
→ Dispersion relation

Very complex

Normal modes $f = f(\theta)e^{-i\omega s}$

Frequency of the modes depends on the beam intensity (space charge tune-shift)



I.Hofmann, PRE.57, 4713

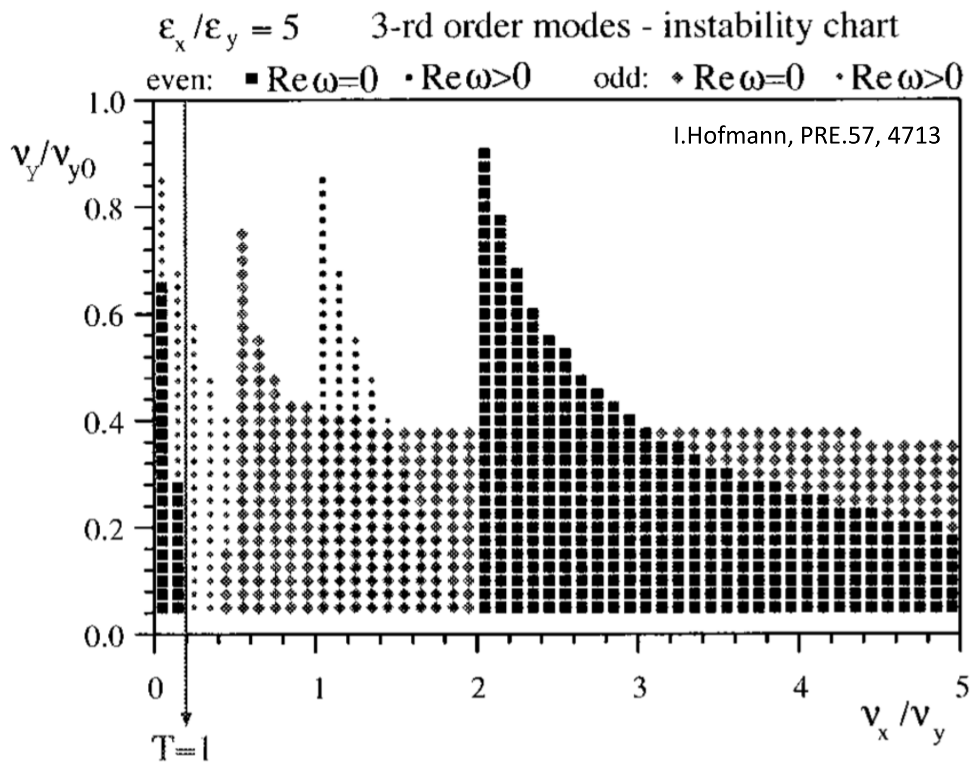
Dispersion relation

Modes can become unstable if ω is imaginary

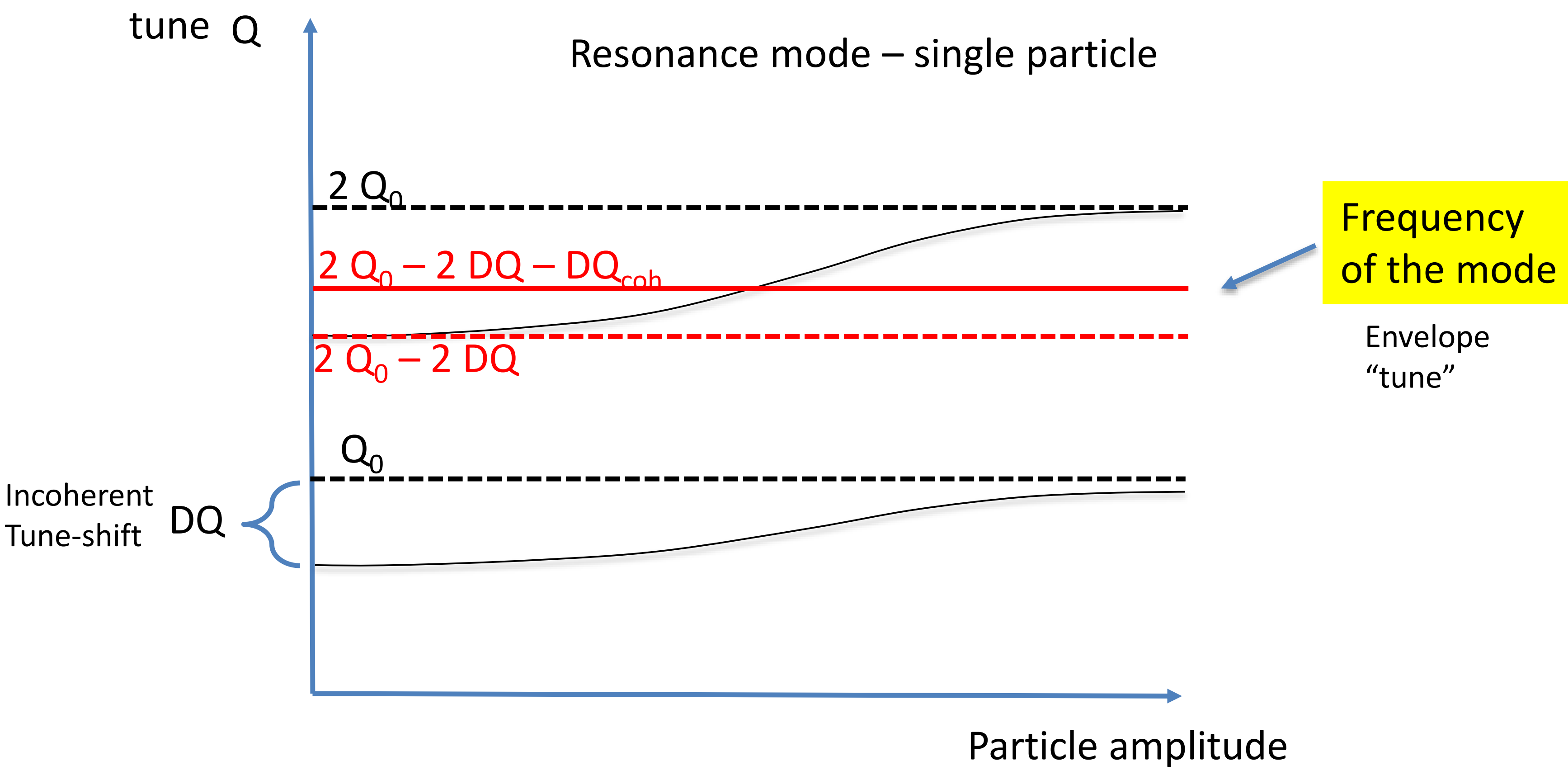
No damping, but Growth !

Instability charts

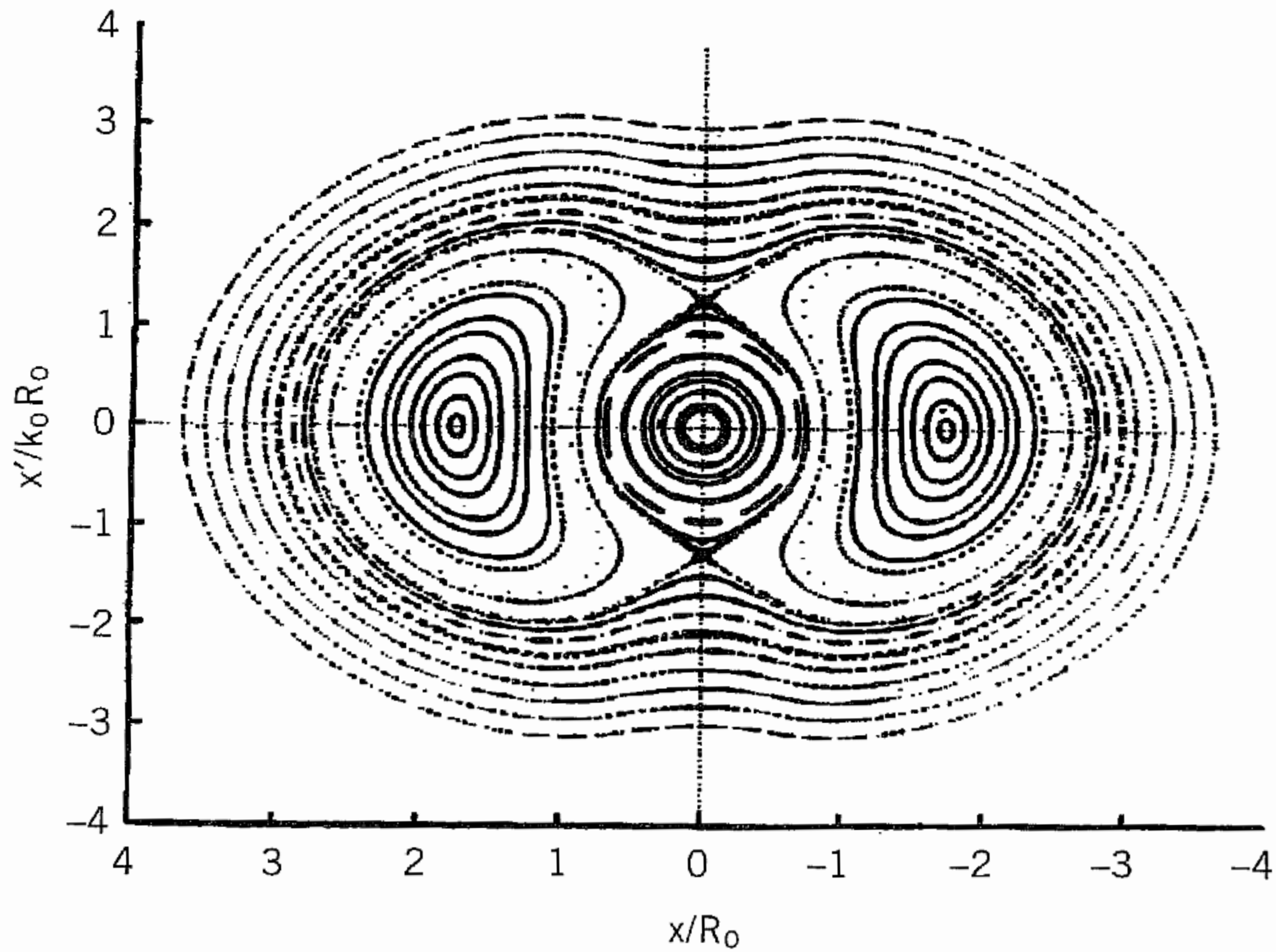
Example of instability charts



The 2:1 resonance



Halo formation



T.P. Wangler Principles of RF linear accelerators, Wiley 1998

Summary

1) Space charge is important at low energy
2) Space charge affect the optics
3) It requires a matched beam
4) It creates a tune-spread
5) Beams rms-equivalent behave similarly (in rms sense)
6) Mismatched beams oscillates (mismatch)
7) Self-consistency is important and desired
8) Space charge tune spread creates severe problem in case of resonance overlapping
10) The higher the space charge tune-spread the more difficult is to control the beam
11) Space charge in LINACS is much stronger
12) Space charge creates Halo
13) Collective space charge resonances shoud be avoided!

Next lecture → Image charge → Collective effects

Further readings

Theory and design of charged particle beams
Martin Reiser,   JOHN WILEY and Son, Inc., New York 1994

Principles of RF linear accelerators
T.P. Wangler, JOHN WILEY and Son, Inc., New York 1998

All previous CAS