# Electromagnetic Theory II

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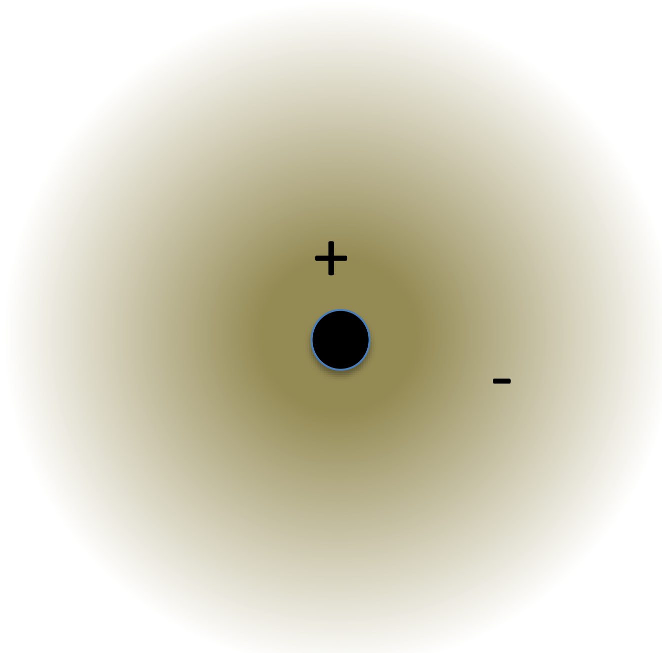
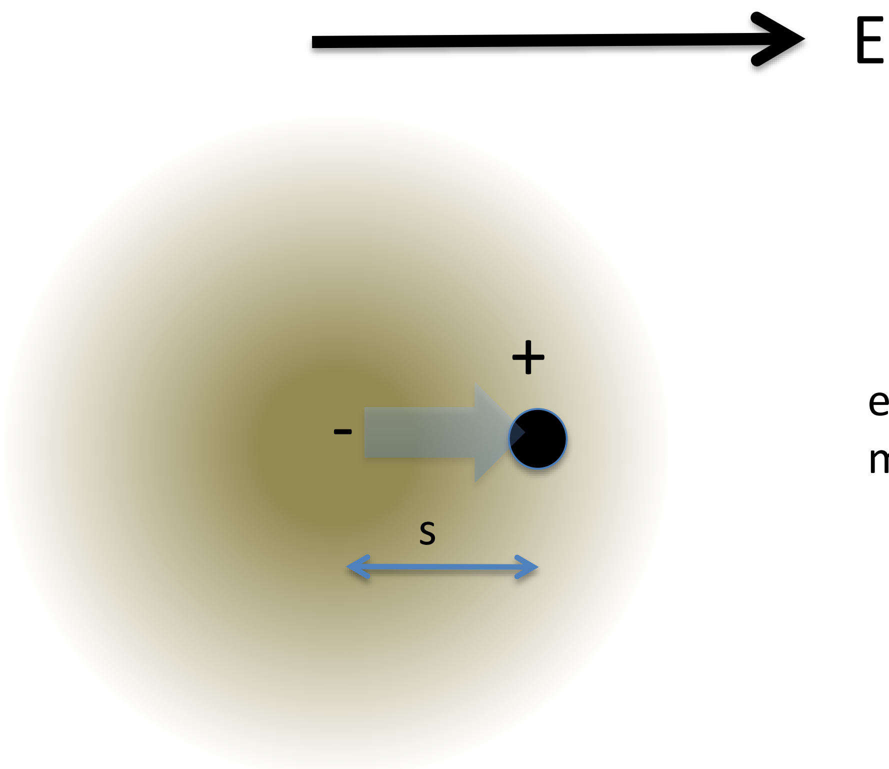
CERN Accelerator – School

2-14 / 10 / 2016

# Dielectrics

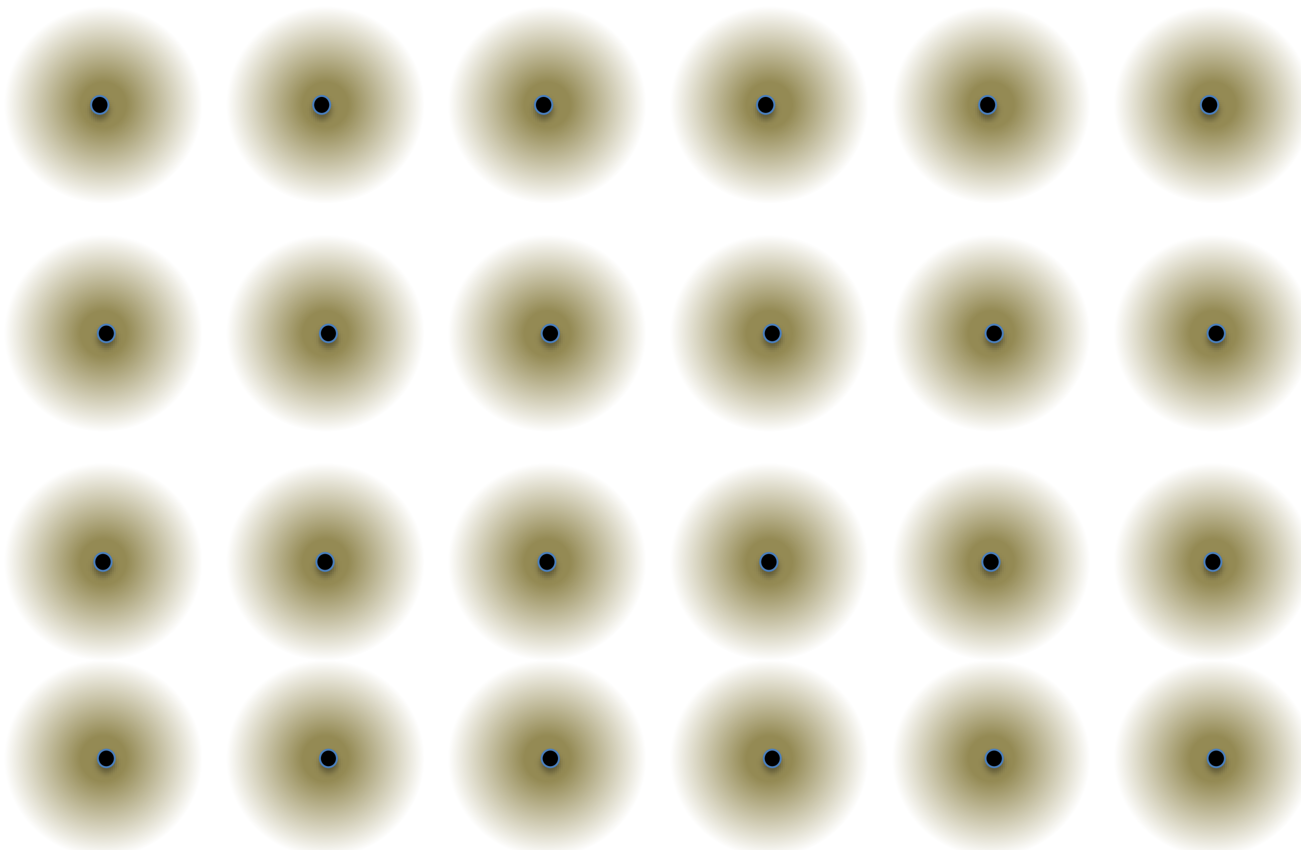


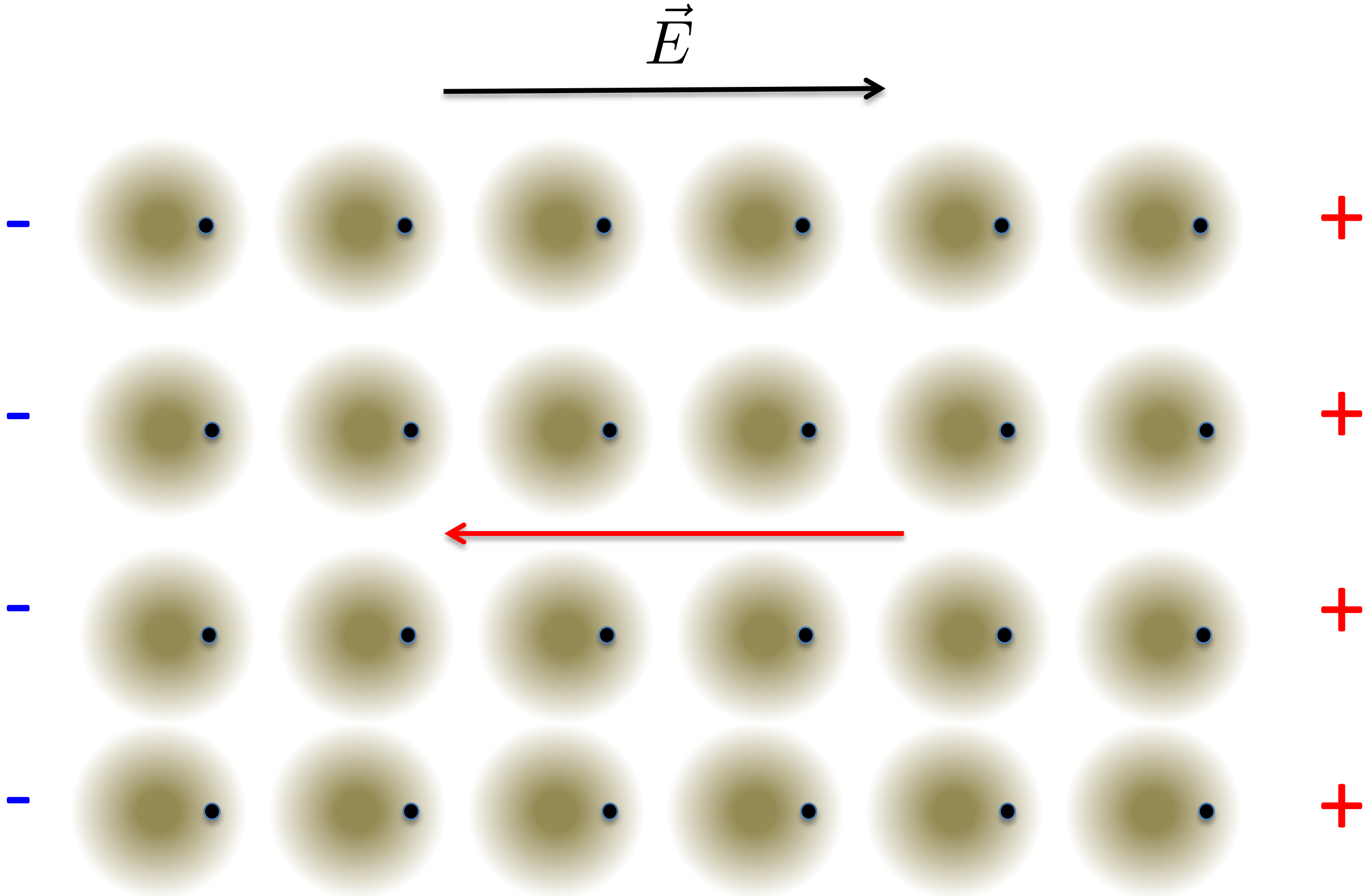
# Dielectrics and electric field




electric dipole moment

$$\vec{p} = q\vec{s}$$




$\vec{E}$
-
+
-
+
-
+
-
+

# Polarization

For homogeneous isotropic dielectrics

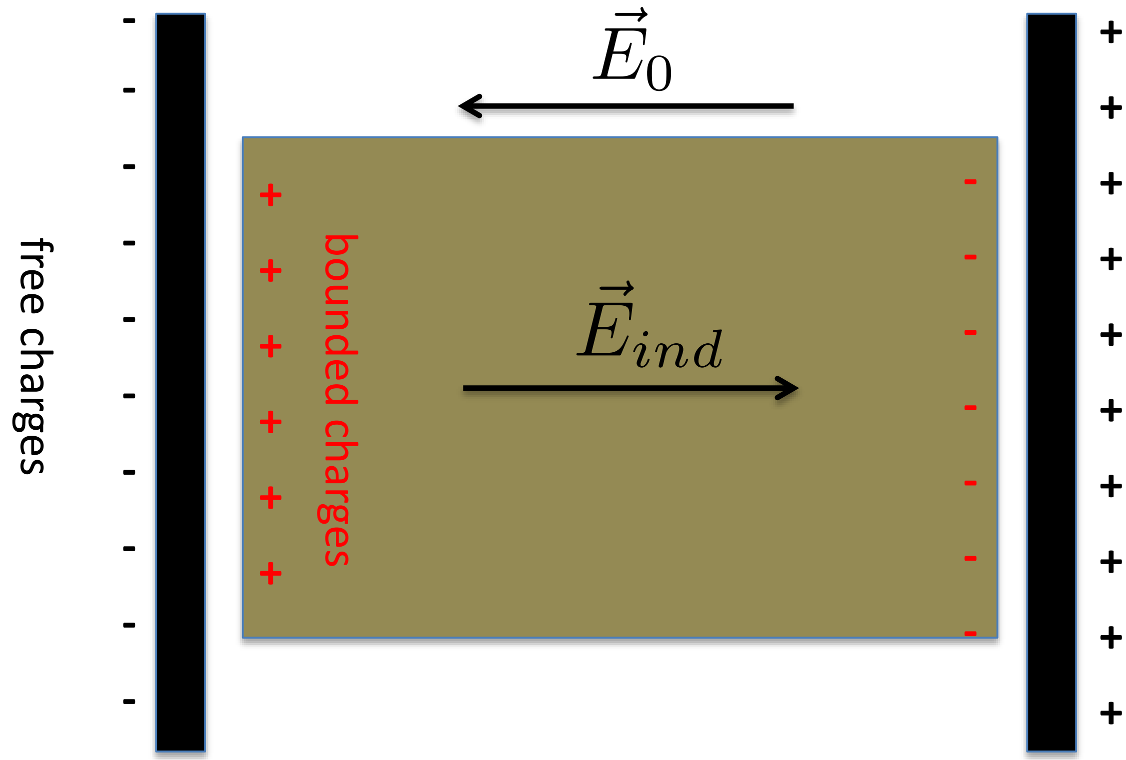
$$\vec{P} = \chi \epsilon_0 \vec{E}$$

$\vec{P}$ polarization → number of electric dipole moment per volume

$\chi$ dielectric susceptibility

$\vec{E}$ electric field “in” the dielectric

# Example




Total Electric field $\vec{E} = \vec{E}_0 + \vec{E}_{ind}$

$$\vec{E}_{ind} = -\frac{1}{\epsilon_0}\vec{P}$$

# Field internal to the capacitor

$$\vec{E} = \vec{E}_0 - \frac{1}{\epsilon_0}\chi\epsilon_0\vec{E} \quad \Rightarrow \quad \vec{E} = \frac{1}{\epsilon_r}\vec{E}_0$$

relative permettivity

$$\epsilon_r = 1 + \chi$$

| Material | $\varepsilon_r$ |
|---|---|
| Vacuum | 1 |
| Mica | 3-6 |
| Glass | 4.7 |
| water | 80 |
| Calcium copper titanate | 250000 |

# Electric Displacement

Field $E_0$ depends only on free charges

$$\epsilon_0 \vec{E}_0 = \epsilon_0 \vec{E} + \vec{P}$$

We give a special name: electric displacement

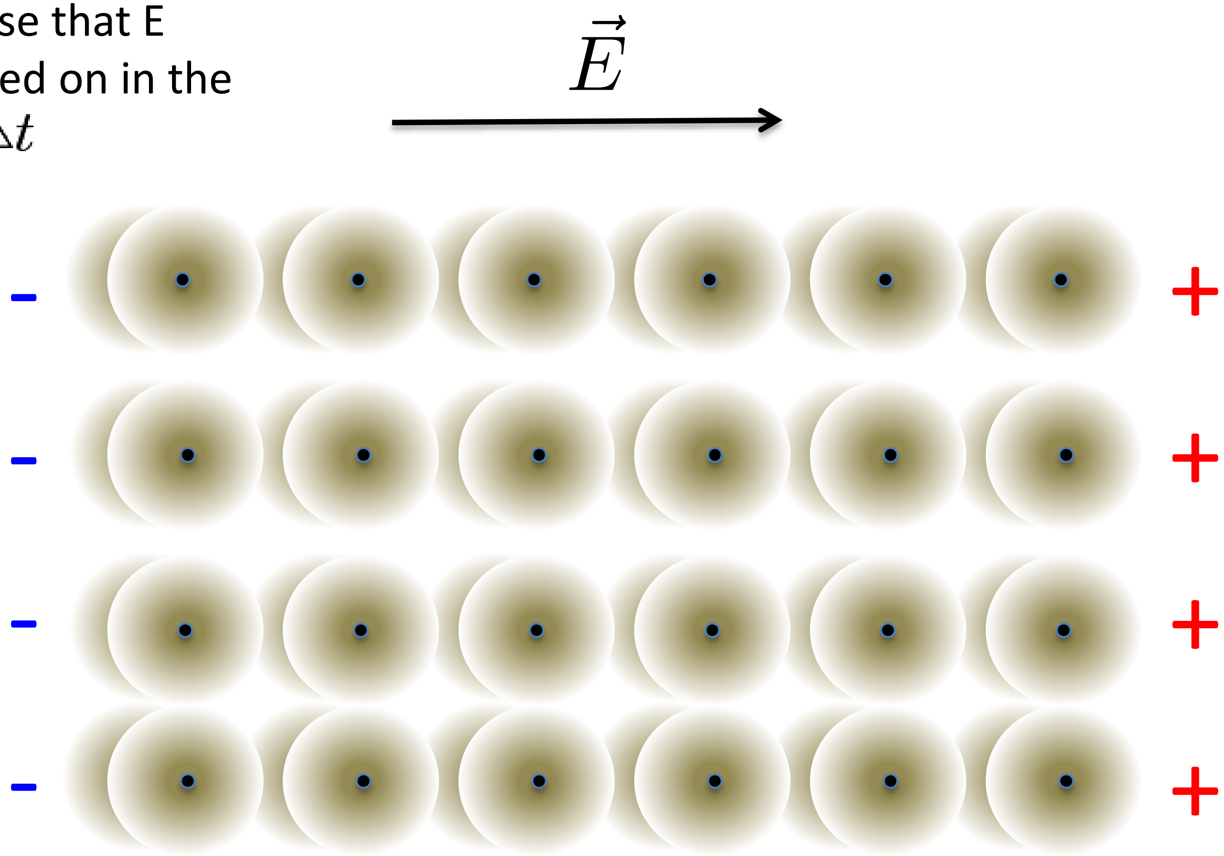
$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \quad \longrightarrow \quad \vec{D} = \epsilon \vec{E} = \epsilon_0 \epsilon_r \vec{E}$$

first Maxwell equation $\quad \vec{\nabla} \cdot \vec{D} = \rho_f$

Free charges

# Bounded current

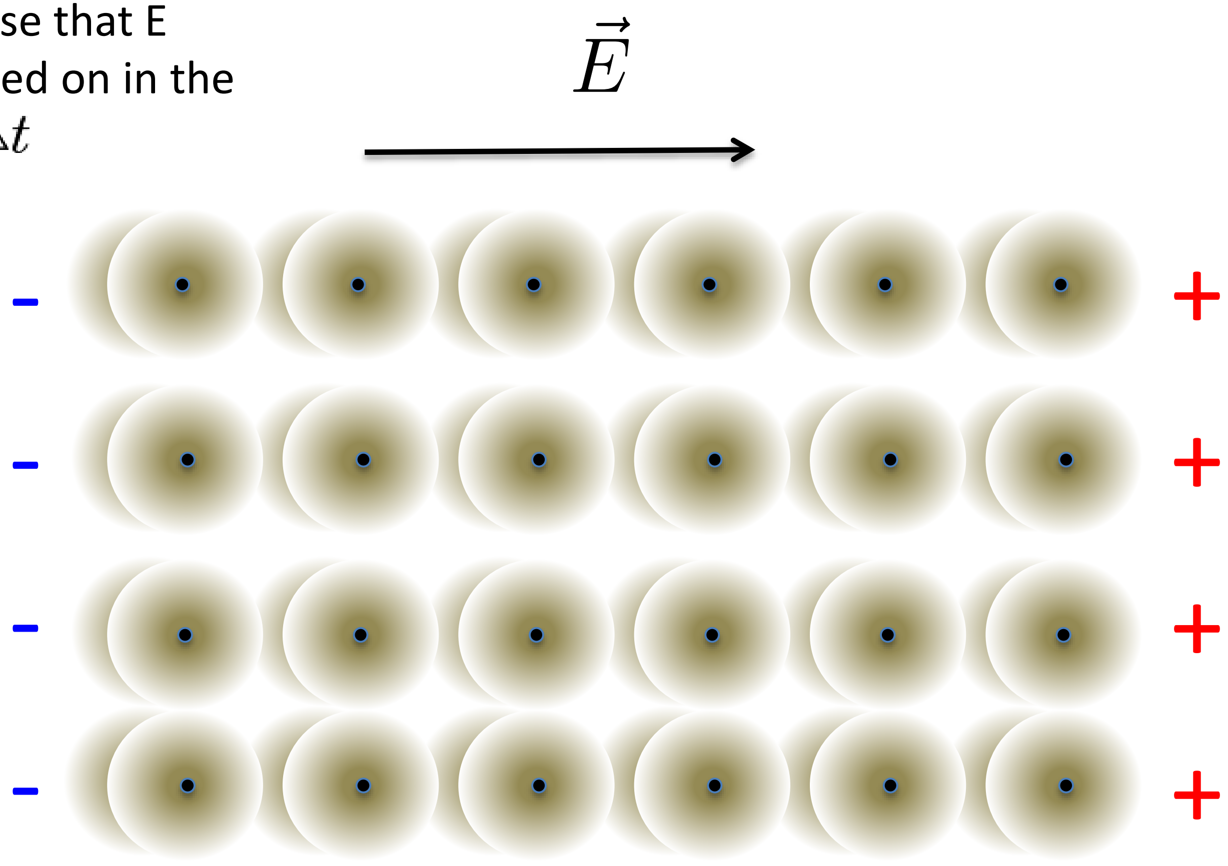
Suppose that E
is turned on in the
time $\Delta t$




The polarization **changes** with time

# Bounded current

Suppose that E
is turned on in the
time $\triangle t$




The polarization **changes** with time

Single electric dipole moment

$$\vec{p} = q\vec{s}$$

$$\frac{\partial}{\partial t}\vec{P} = Nq\frac{d\vec{s}}{dt} = \vec{J}_b$$

Density of current due to bounded charges

$N$ = number of dipole moments Per volume

It has to be included in the Maxwell equation

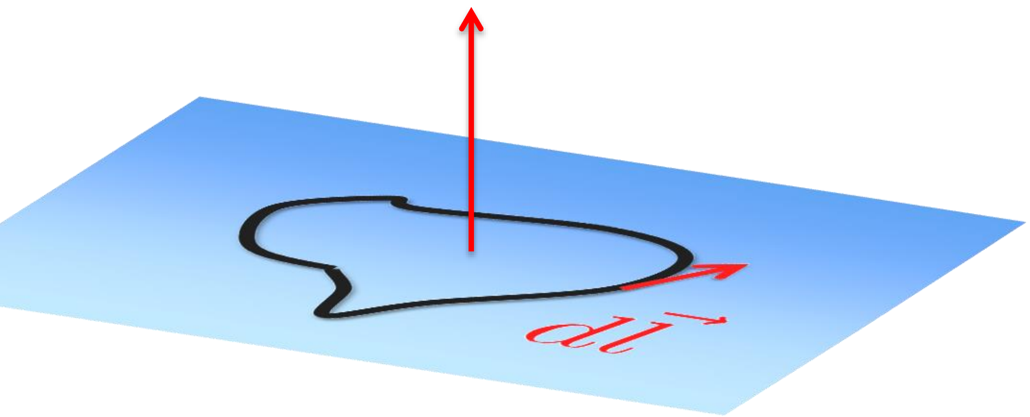
It is already in the definition of $\vec{D}$

# Magnetic field in matter

As $\vec{\nabla} \cdot \vec{B} = 0$ there are no magnetic charges

Magnetic phenomena are due to "currents"

# Magnetic moment




$$d\vec{F} = I d\vec{l} \times \vec{B}$$

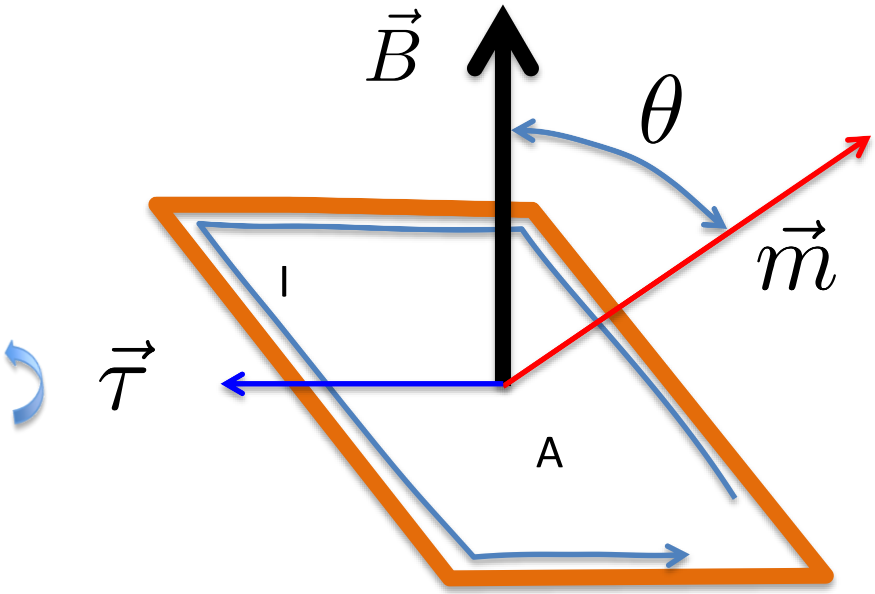
$$\oint_\Gamma d\vec{F} = I \left( \oint_\Gamma d\vec{l} \right) \times \vec{B} = 0$$

$$\oint_\Gamma \vec{r} \times d\vec{F} = I \oint \vec{r} \times \left( d\vec{l} \times \vec{B} \right) = \vec{m} \times \vec{B}$$

torque acting
on the coil

Magnetic moment

# Example




$$\vec{m} = IA\hat{v}$$

$$\vec{\tau} = \vec{m} \times \vec{B}$$

The effect of the magnetic field is to create a torque on the coil

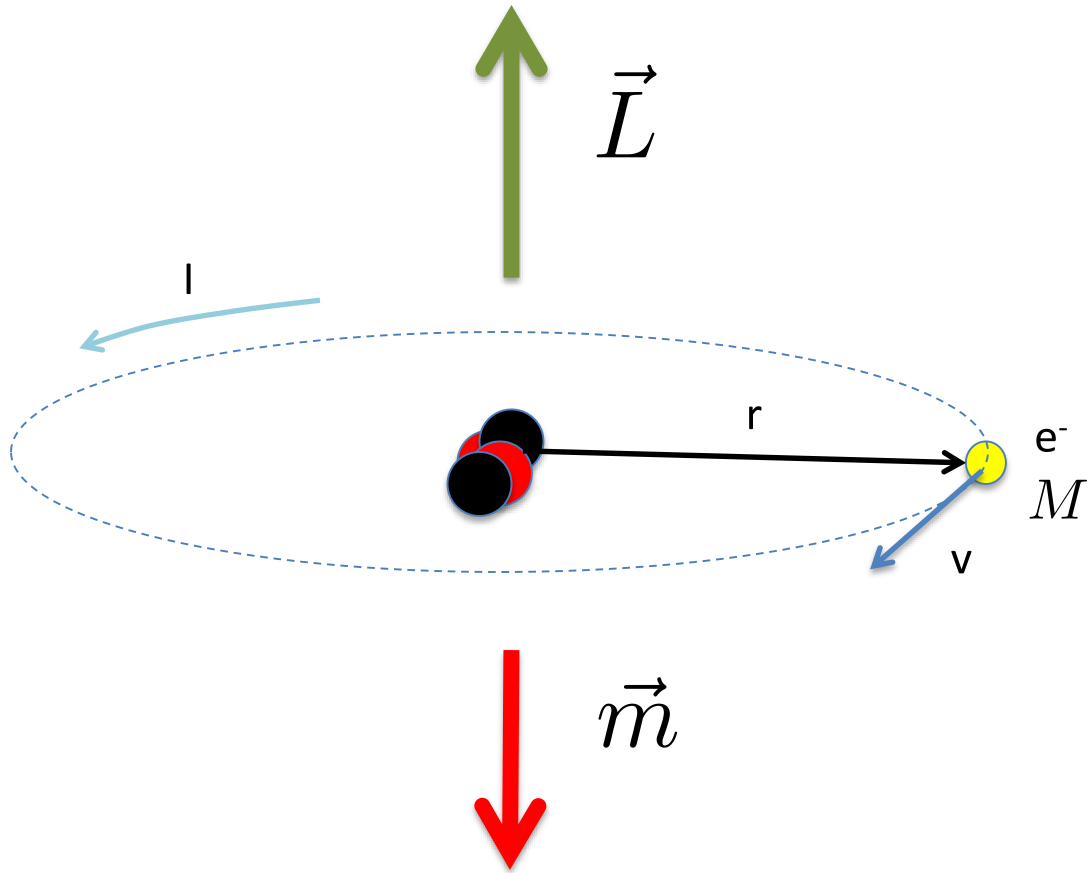
# Magnetic moments in matters

$$I = \frac{ev}{2\pi r}$$

$$m = \frac{rev}{2}$$

$$\vec{m} = -\frac{e}{2M}\vec{L}$$

Orbits of electrons



# Intrinsic magnetic moments: ferromagnetism

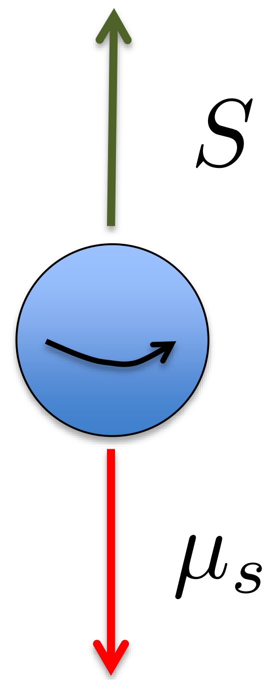
Spin of electrons

$$\vec{\mu} = -g_s \frac{e}{2m_e} \vec{L}$$

$$L = \frac{\hbar}{2}$$

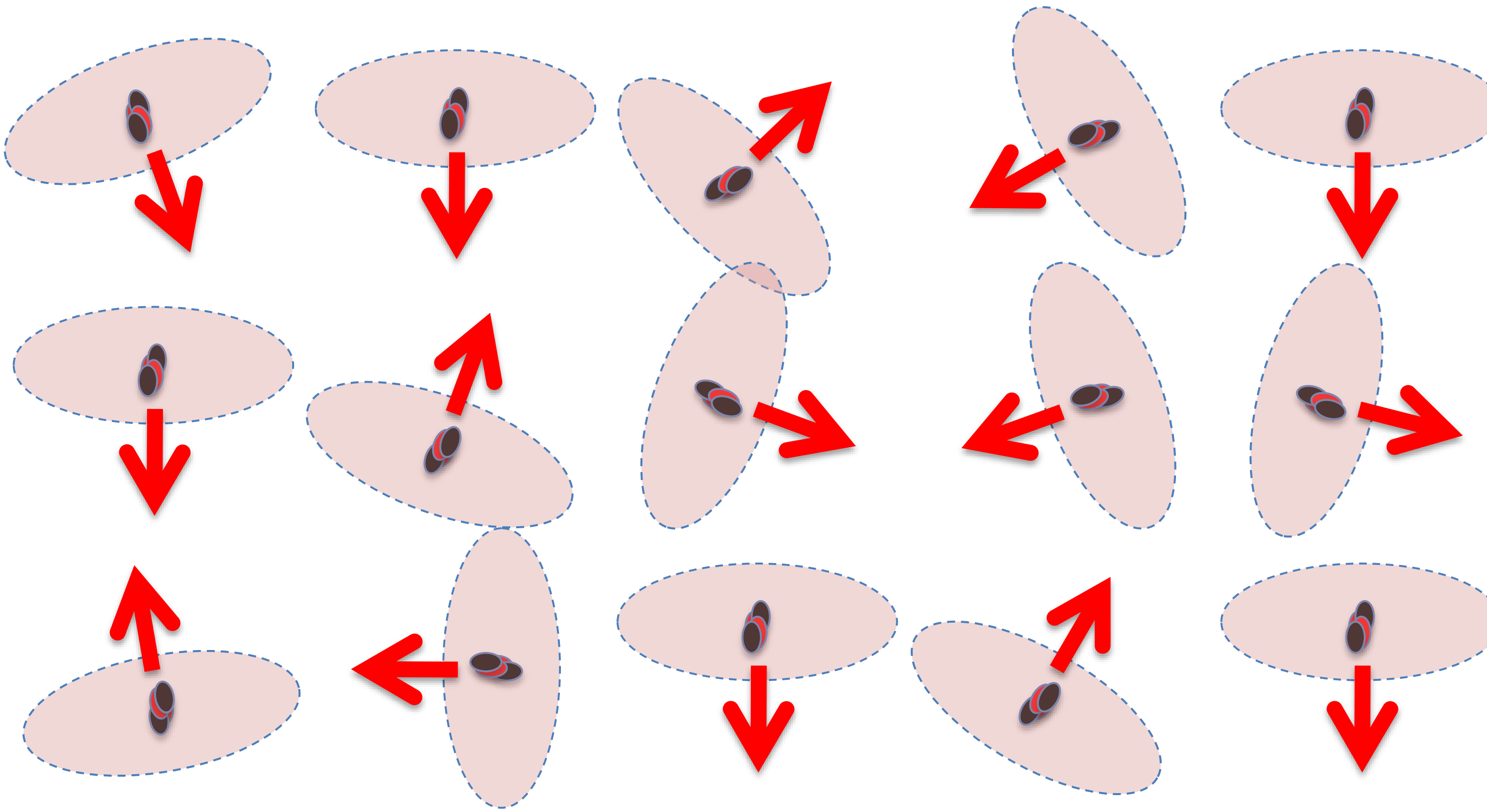
$$g_s \simeq 2$$

$$\mu_s = \mu_B = \frac{e\hbar}{2m_e}$$



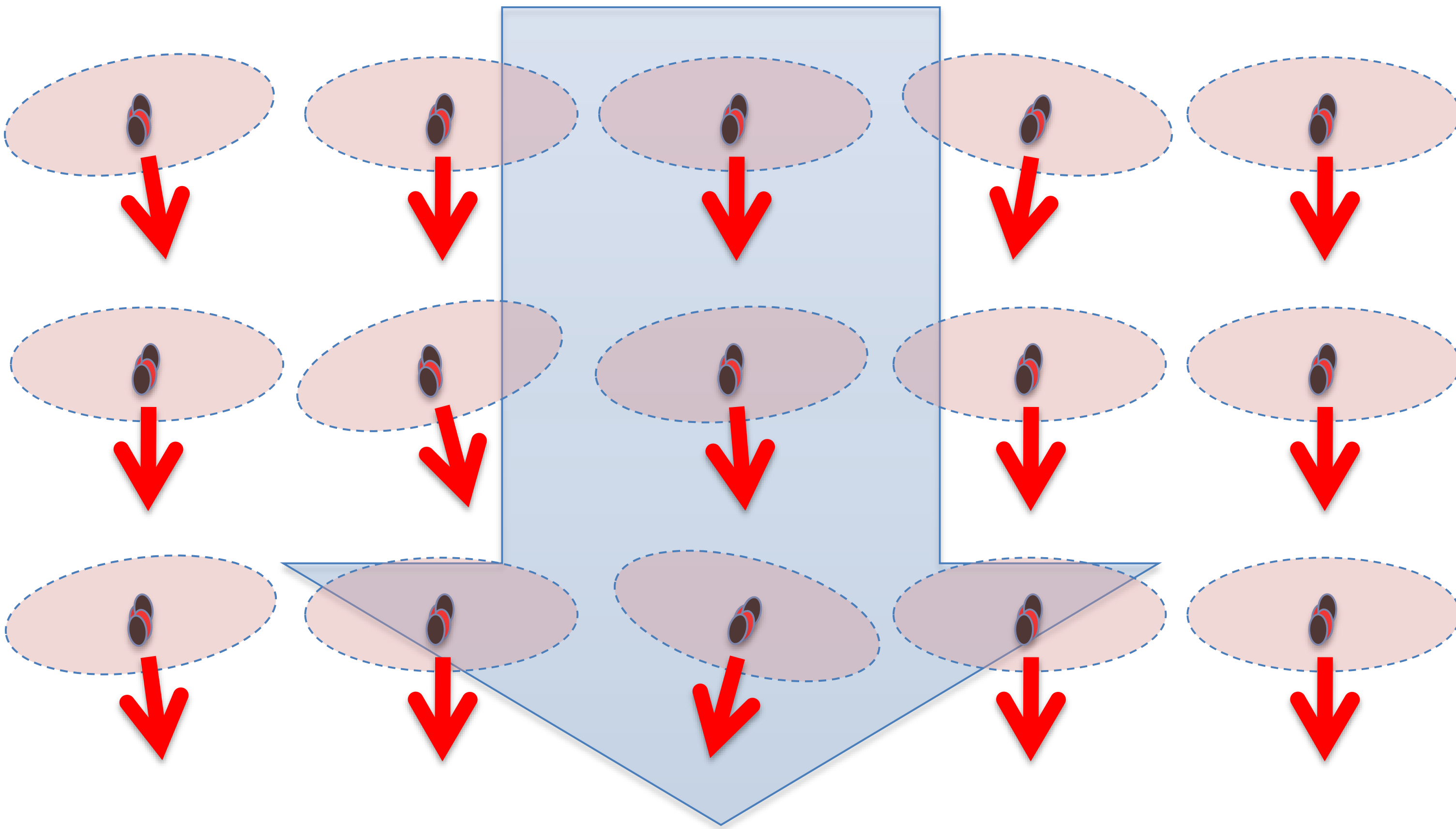
# Without external magnetic field

Random orientation (due to thermal motion)



# Without external magnetic field

dipoles moment of atoms orientates according to the external magnetic field

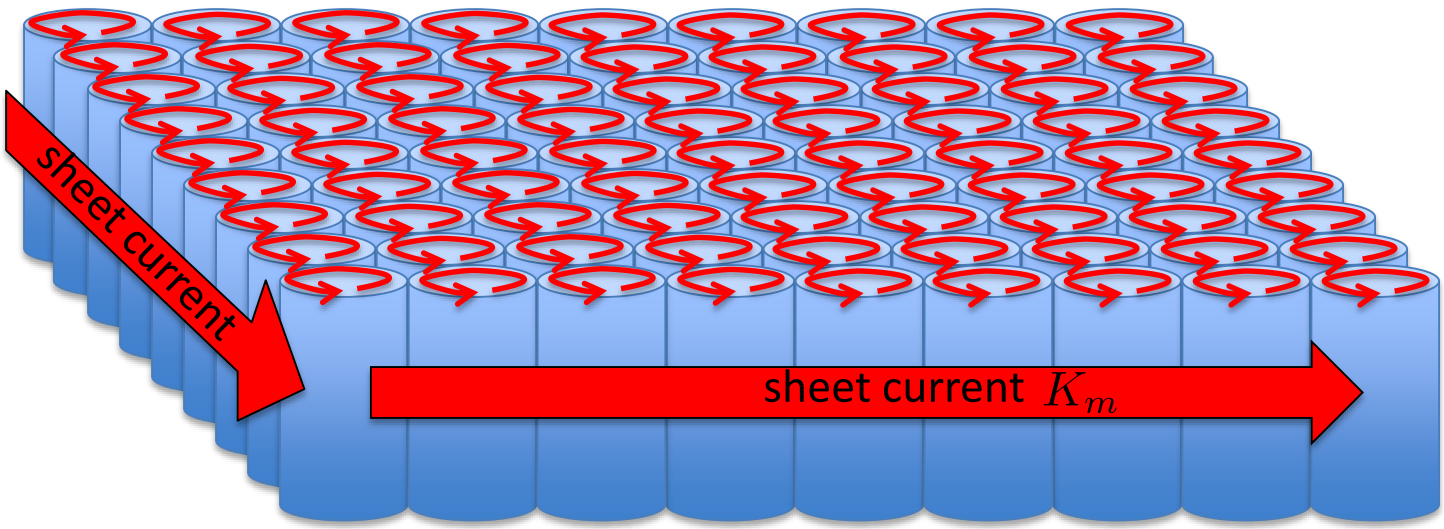


# Magnetization

$$\vec{M} = \overline{\chi}_m \vec{B}$$

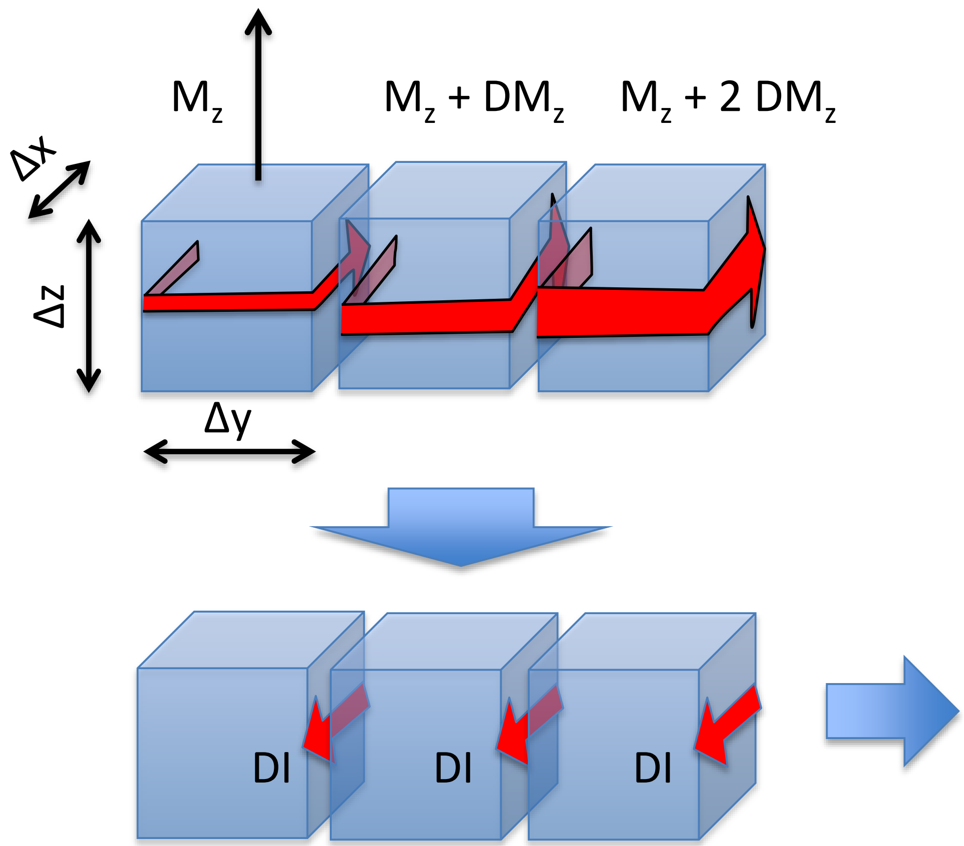
B is the macroscopic magnetic field in the matter

$\overline{\chi}_m$ = magnetic susceptibility



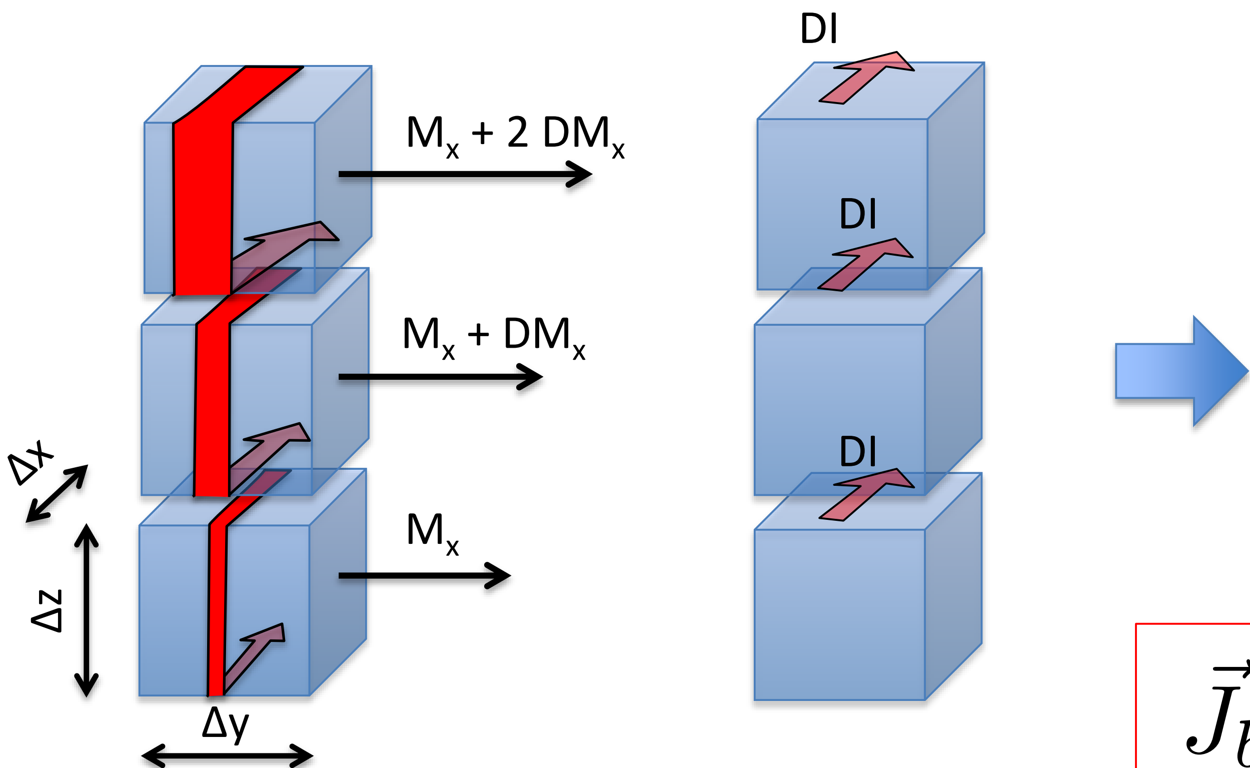

This surface current produces the magnetic field produced by magnetized matter $\vec{K}_m = \hat{n} \times \vec{M}$

# Non uniform magnetization




$$J_x = \partial_x M_z$$

# Non uniform magnetization




$$J_x = -\partial_z M_x$$

$$\boxed{\vec{J}_b = \vec{\nabla} \times \vec{M}}$$

# Free currents and bounded currents

The bounded currents are given by $$\vec{J}_b = \vec{\nabla} \times \vec{M}$$

This current should be included in Ampere's Law $$\vec{\nabla} \times \vec{B} = \mu_0(\vec{J}_f + \vec{J}_b)$$

$$\vec{\nabla} \times (\vec{B} - \mu_0 \vec{M}) = \mu_0 \vec{J}_f$$

Define $$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$ $$\Rightarrow \quad \boxed{\vec{\nabla} \times \vec{H} = \vec{J}_f}$$

# Magnetic susceptibility

$$\vec{M} = \overline{\chi}_m \vec{B}$$

this field depends
on all free and bounded
currents: NOT PRACTICAL

$$\vec{M} = \chi_m \vec{H}$$

this field depends
only on the current
that I create

$$\vec{B} = \mu_0(1 + \chi_m)\vec{H}$$

$$\vec{B} = \mu_0 \mu_r \vec{H} = \mu \vec{H}$$

$$\mu_r = 1 + \chi_m$$

relative permeability

| Material | $\chi_m$ | $\mu_r$ | $\mu$ |
|---|---|---|---|
| Vacuum | 0 | 1 | $4\pi \times 10^{-7}$ |
| water | $-8.0\times10^{-6}$ | 0.999992 | $1.2566\times10^{-6}$ |
| Iron (pure) | | 5000 | $6.3\times10^{-3}$ |
| Superconductors | -1 | 0 | 0 |

# Maxwell equation in matter

$$\vec{\nabla} \cdot \vec{D} = \rho_f$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{j}_f + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{D} = \epsilon \vec{E}$$

$$\epsilon = \epsilon_0 \epsilon_r$$

$$\vec{H} = \frac{1}{\mu} \vec{B}$$

$$\mu = \mu_0 \mu_r$$

$$\vec{E} = \sigma \vec{j}$$

# Summary of quantities

$$\vec{E} = \text{electric field}$$

$$\vec{D} = \text{electric displacement}$$

$$\vec{H} = \text{magnetic field}$$

$$\vec{B} = \text{magnetic flux density}$$

$$\vec{\rho} = \text{electric charge density}$$

$$\vec{j} = \text{current density}$$

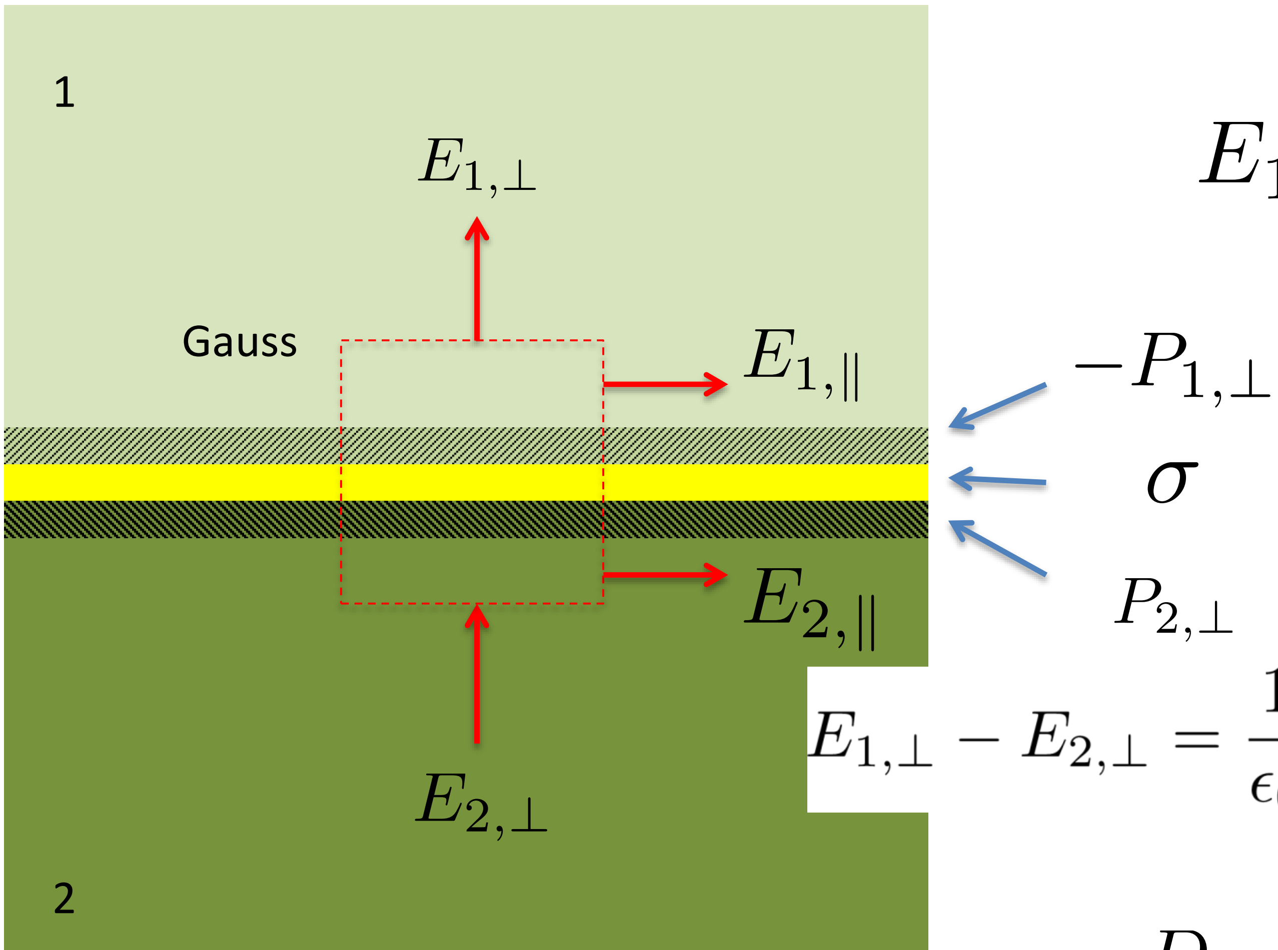
$$\vec{E} = \text{electric displacement}$$

$$\mu_0 = \text{permeability of free space}, 4\pi \times 10^{-7}$$

$$\epsilon_0 = \text{permittivity of free space}, 8.854 \times 10^{-12}$$

$$c = \text{speed of light}, 2.99792458 \times 10^{8}$$

# Boundary conditions

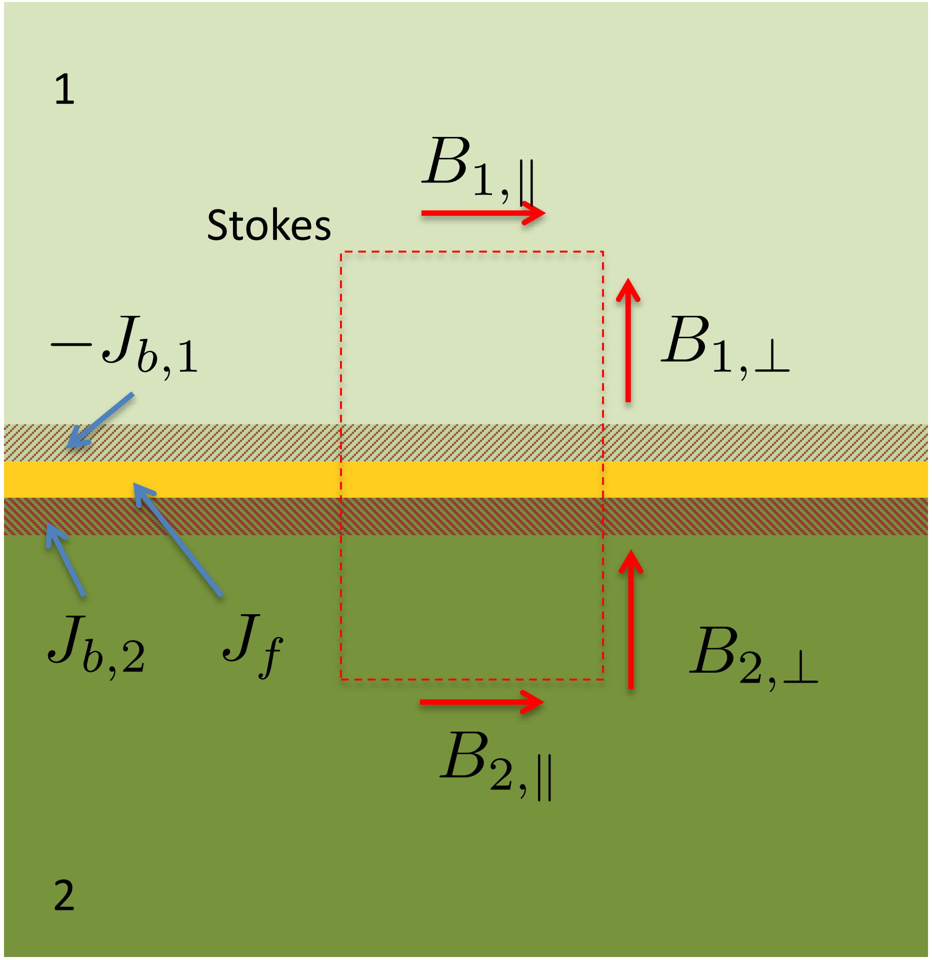



$$E_{1,\parallel} = E_{2,\parallel}$$

$$E_{1,\perp} - E_{2,\perp} = \frac{1}{\epsilon_0}(\sigma - P_{1,\perp} + P_{2,\perp})$$

$$D_{1,\perp} - D_{2,\perp} = \sigma$$

# Boundary conditions



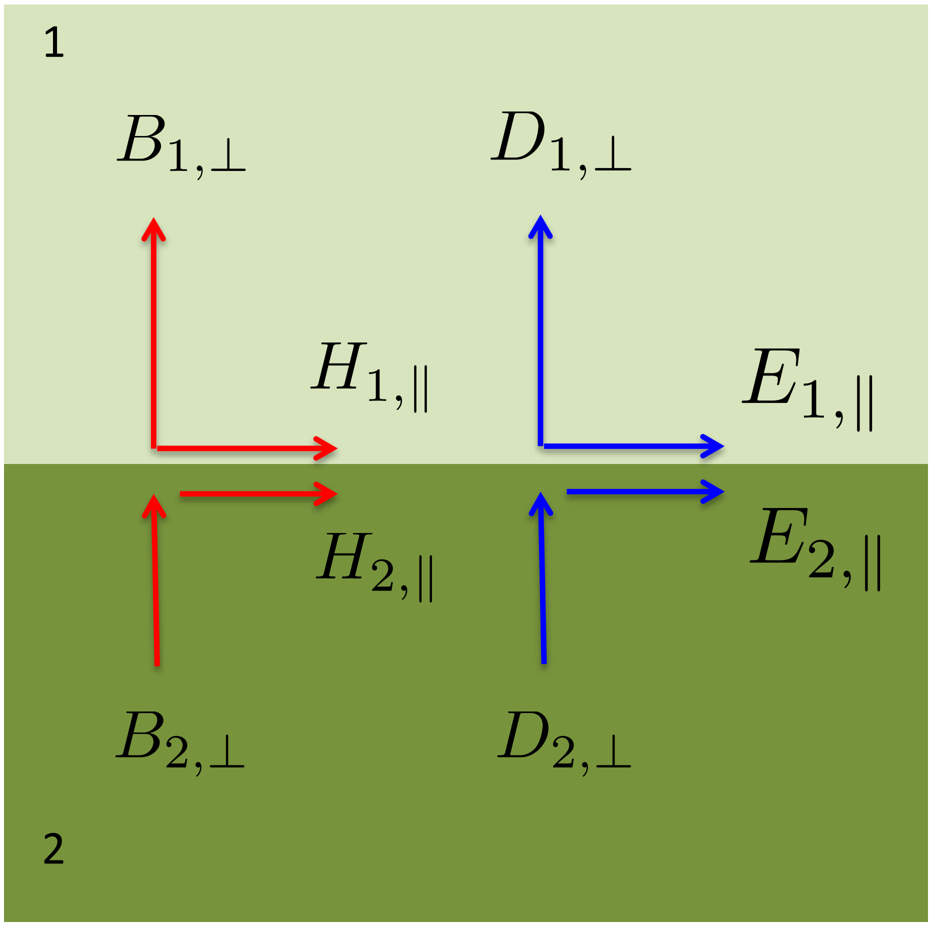

$$B_{1,\perp} = B_{2,\perp}$$

$$B_{1,\parallel} - B_{2,\parallel} = \mu_0(-J_{b,1} + J_f + J_{b,2})$$

$$B_{1,\parallel} - B_{2,\parallel} = \mu_0(-M_{1,\parallel} + j_f + M_{2,\parallel})$$

$$H_{1,\parallel} - H_{2,\parallel} = j_f$$

# Summary boundary conditions



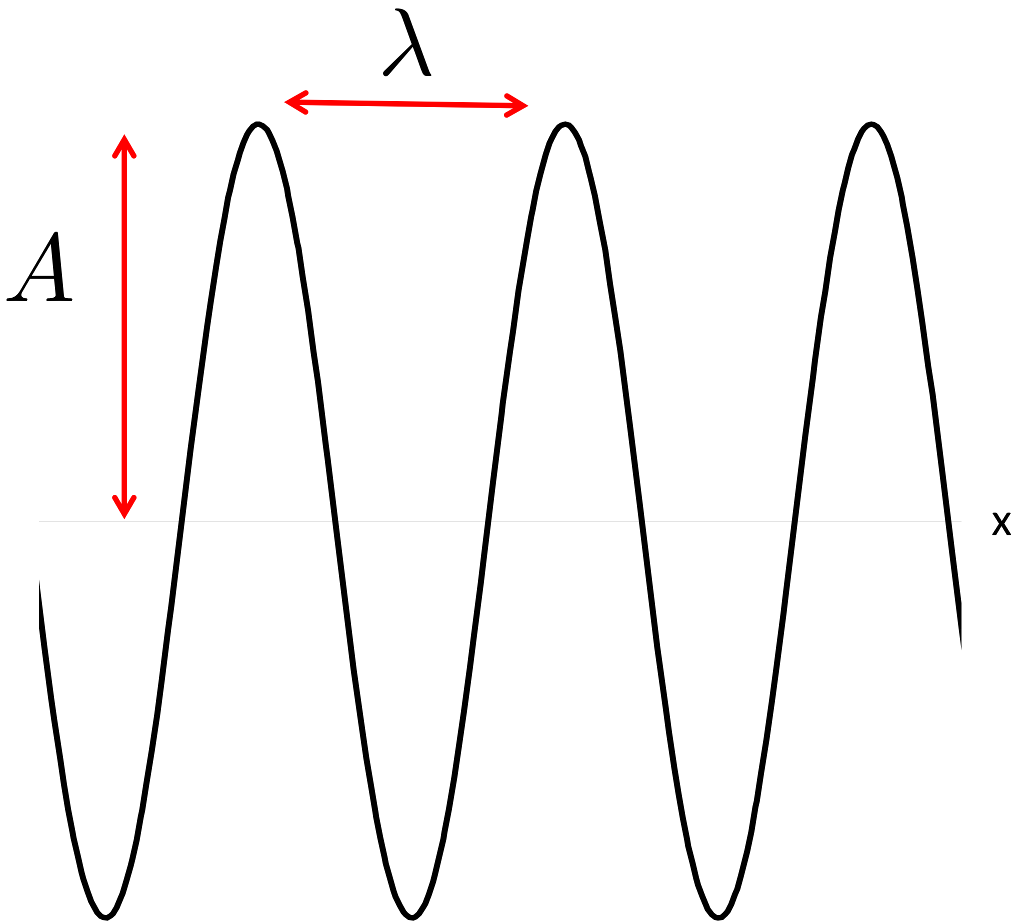

$$E_{1,\parallel} = E_{2,\parallel}$$

$$D_{1,\perp} - D_{2,\perp} = \sigma$$

$$H_{1,\parallel} - H_{2,\parallel} = j_f$$

$$B_{1,\perp} = B_{2,\perp}$$

# Waves




At t=0

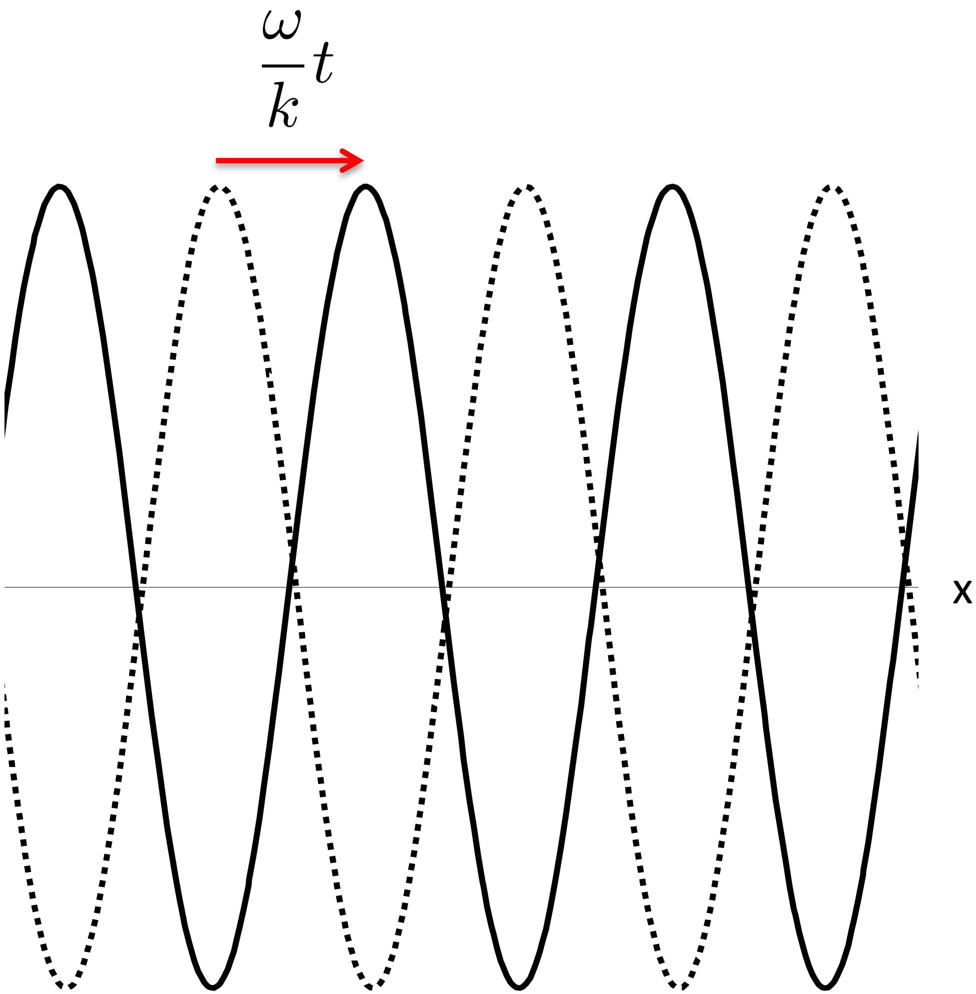
$$y = A\sin\left(\frac{2\pi}{\lambda}x\right)$$

wave number

$$k = \frac{2\pi}{\lambda}$$

$$y = A\sin(kx)$$

# Waves




At time t

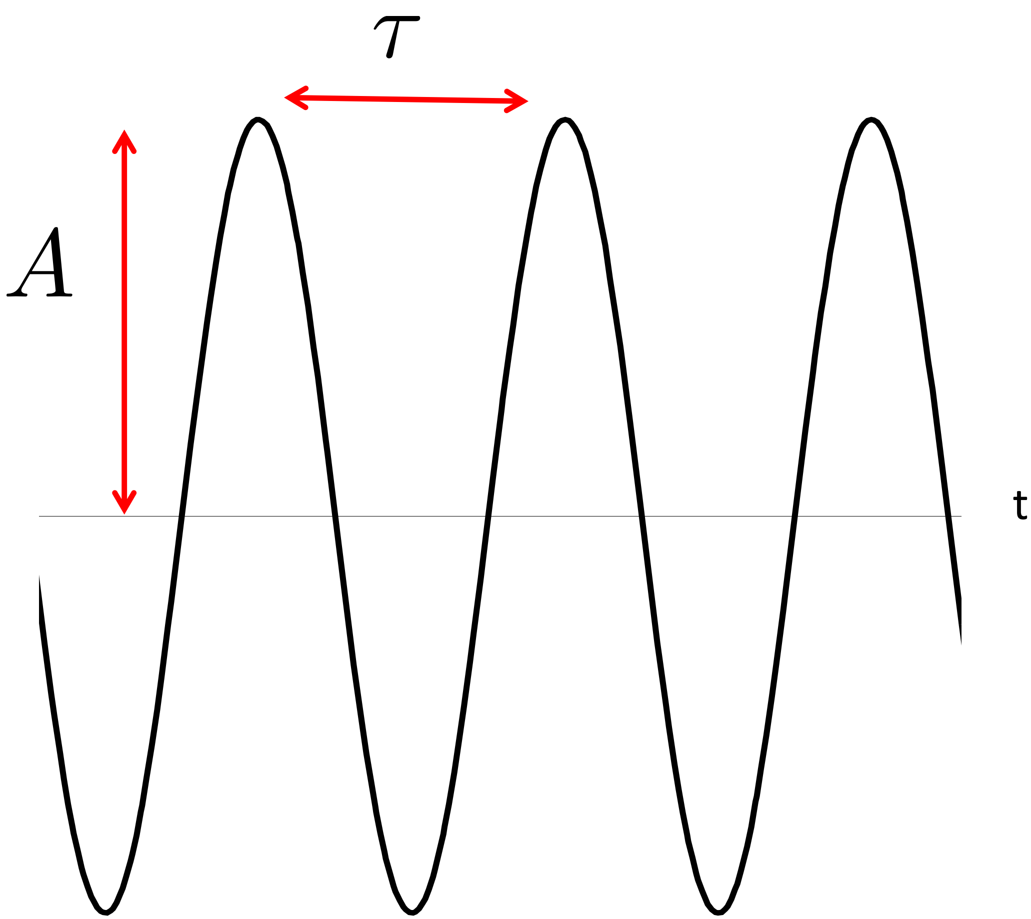
$$y = A\sin(kx - \omega t)$$

the wave travels of

$$\frac{\omega}{k}t$$

wave velocity $v = \dfrac{\omega}{k}$

# Waves




At fixed x

$$y = A \sin(kx - \omega t)$$

y oscillates with period $\tau$

$$\omega = \frac{2\pi}{\tau}$$

# Wave equation

$$\frac{\partial^2}{\partial t^2} f = v^2 \nabla^2 f$$

$$f = A \sin(\vec{k} \cdot \vec{x} - \omega t)$$

$$v^2(k_x^2 + k_y^2 + k_z^2) = \omega^2$$

The vector $\frac{\vec{k}}{|\vec{k}|}$ gives the direction of propagation of the wave

the velocity of propagation is $v = \frac{\omega}{|\vec{k}|}$

# Electromagnetic waves

Maxwell equations in vacuum

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \vec{E}$$

$$\frac{\partial^2}{\partial t^2} \vec{E} = \frac{1}{\epsilon_0 \mu_0} \nabla^2 \vec{E}$$

$$\frac{\partial^2}{\partial t^2} \vec{B} = \frac{1}{\epsilon_0 \mu_0} \nabla^2 \vec{B}$$

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = c$$

speed of light !!

# Planar waves

Starting ansatz $$\vec{E} = \vec{E}_0 f(\vec{k} \cdot \vec{x} - \omega t)$$

From 1st equation

$$\vec{\nabla} \cdot \vec{E} = \vec{k} \cdot \vec{E}_0 f'(\vec{k} \cdot \vec{x} - \omega t) = 0$$

$$\Downarrow$$

$$\vec{k} \cdot \vec{E}_0 = 0$$

The electric field is orthogonal to the direction of wave propagation

From 3rd equation

$$\vec{\nabla} \times \vec{E} = \vec{k} \times \vec{E}_0 f'(\vec{k}\cdot\vec{x} - \omega t) = -\frac{\partial}{\partial t}\vec{B}$$

Integrating over time

$$\vec{B} = \frac{\vec{k}}{\omega} \times \vec{E}_0 f(\vec{k}\cdot\vec{x} - \omega t)$$

This satisfy the 2nd equation, in fact

$$\vec{\nabla} \cdot \vec{B} = \vec{k} \cdot \frac{\vec{k}}{\omega} \times \vec{E}_0 f'(\vec{k}\cdot\vec{x} - \omega t) = 0$$

The 4th equation is satisfied too

$$\vec{\nabla} \times \vec{B} = \vec{k} \times \left[ \frac{\vec{k}}{\omega} \times \vec{E}_0 f'(\vec{k} \cdot \vec{x} - \omega t) \right]$$

$$\vec{\nabla} \times \vec{B} = \left( \vec{k} \cdot \vec{E}_0 \vec{k} - k^2 \vec{E}_0 \right) \left[ \frac{1}{\omega} f'(\vec{k} \cdot \vec{x} - \omega t) \right]$$

$$\vec{\nabla} \times \vec{B} = -k^2 \vec{E}_0 \left[ \frac{1}{\omega} f'(\vec{k} \cdot \vec{x} - \omega t) \right]$$

$$\vec{\nabla} \times \vec{B} = -\frac{k^2}{\omega^2} \vec{E}_0 \omega f'(\vec{k} \cdot \vec{x} - \omega t)$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \left[ -\omega \vec{E}_0 f'(\vec{k} \cdot \vec{x} - \omega t) \right] = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \vec{E}$$

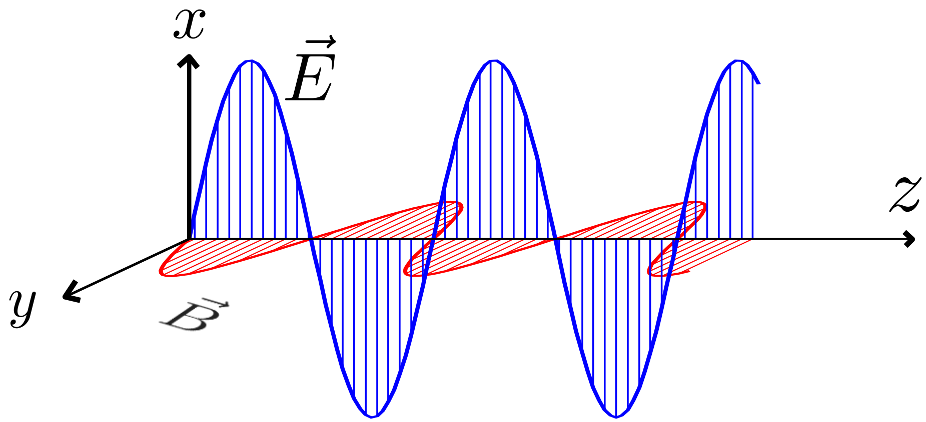
# Planar wave solution

$$\vec{E} = \vec{E}_0 f(\vec{k} \cdot \vec{x} - \omega t)$$

$$\vec{B} = \vec{B}_0 f(\vec{k} \cdot \vec{x} - \omega t)$$

$$\vec{B}_0 = \frac{1}{c} \hat{k} \times \vec{E}_0 \qquad \omega / k = c$$

# Sinusoidal example

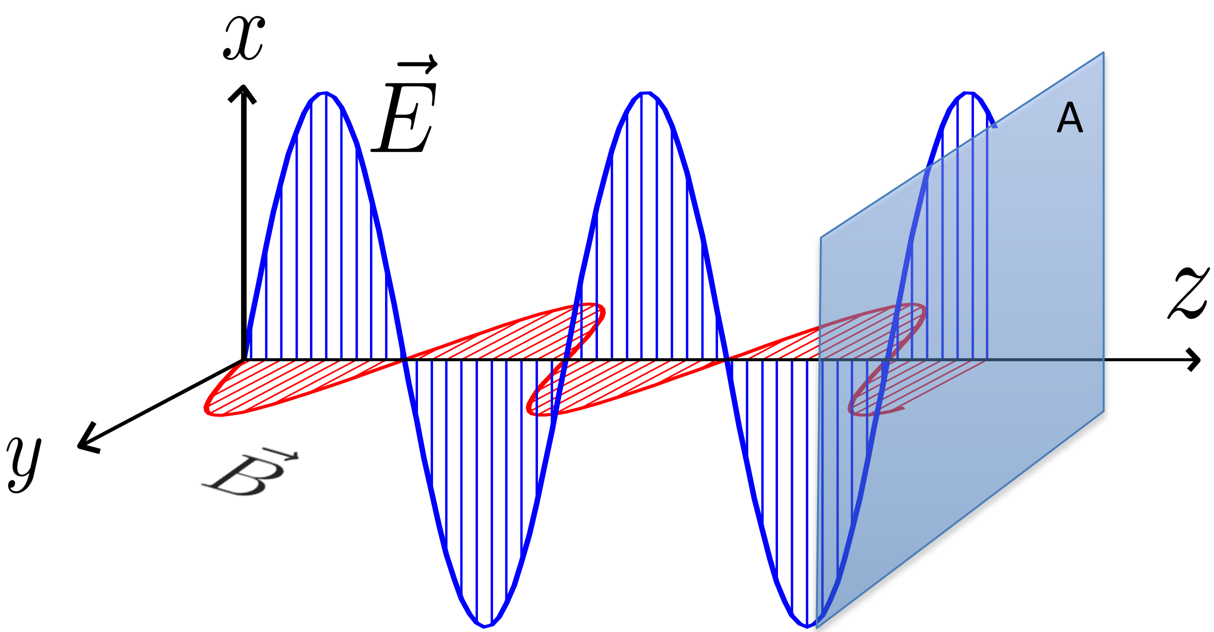



$$\vec{E} = \hat{x} E_0 \sin(\vec{k} \cdot \vec{x} - \omega t)$$

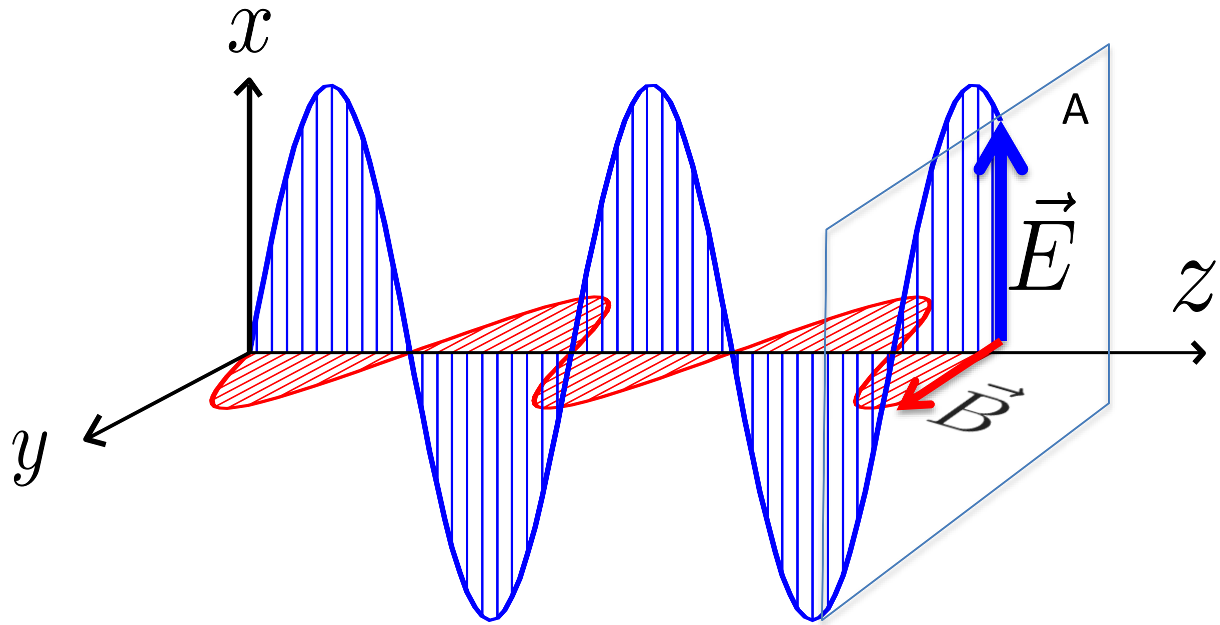
$$\vec{B} = \hat{y} B_0 \sin(\vec{k} \cdot \vec{x} - \omega t)$$

with $\vec{k} = \hat{z}\dfrac{\omega}{c}$ $\qquad B_0 = \dfrac{E_0}{c}$

# Poynting vector




What is the flux of energy going through the surface A ?




Electric field density energy $\epsilon_0 \dfrac{E^2}{2}$

Magnetic field density energy $\dfrac{B^2}{2\mu_0}$

Energy through A in time Dt

$$\Delta E = Ac\Delta t \left( \epsilon_0 \frac{E^2}{2} + \frac{B^2}{2\mu_0} \right)$$

# Energy flux: Poynting vector

Energy flux $$S = \frac{\Delta E}{A \Delta t} = c\left(\epsilon_0 \frac{E^2}{2} + \frac{B^2}{2\mu_0}\right)$$

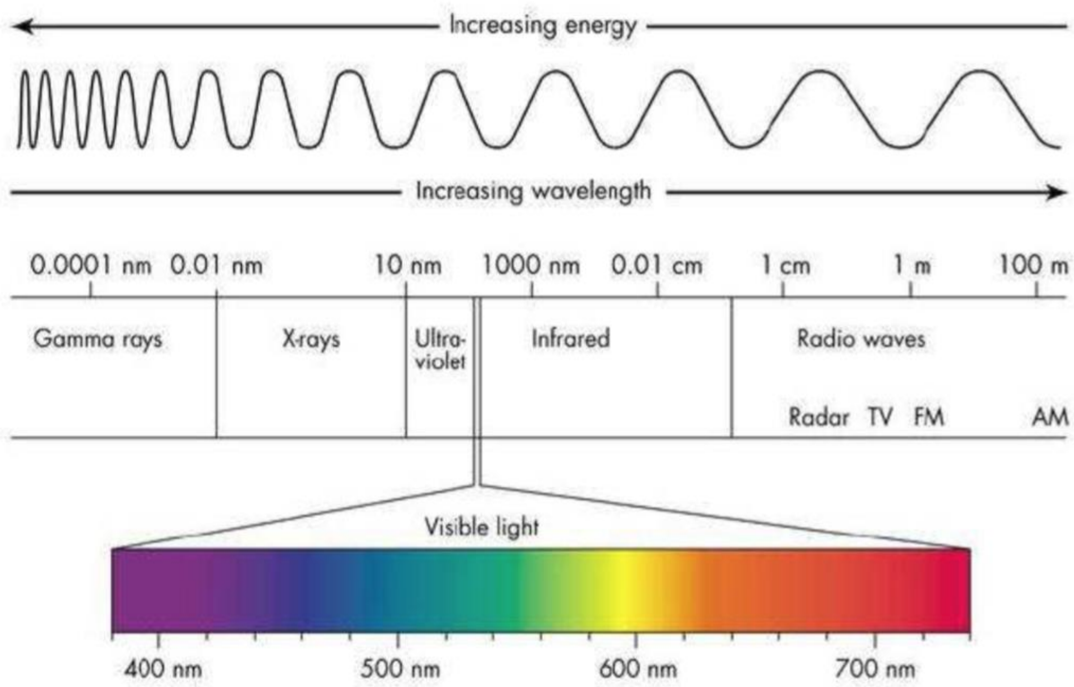
But for EM wave → $B = E/c$ $$\frac{B^2}{2\mu_0} = \frac{E^2}{2\mu_0 c^2} = \epsilon_0 \frac{E^2}{2}$$

Poynting vector

$$S = \epsilon_0 E^2 c = \epsilon_0 E B c^2$$

$$\boxed{\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}}$$

# Spectrum of Electromagnetic waves

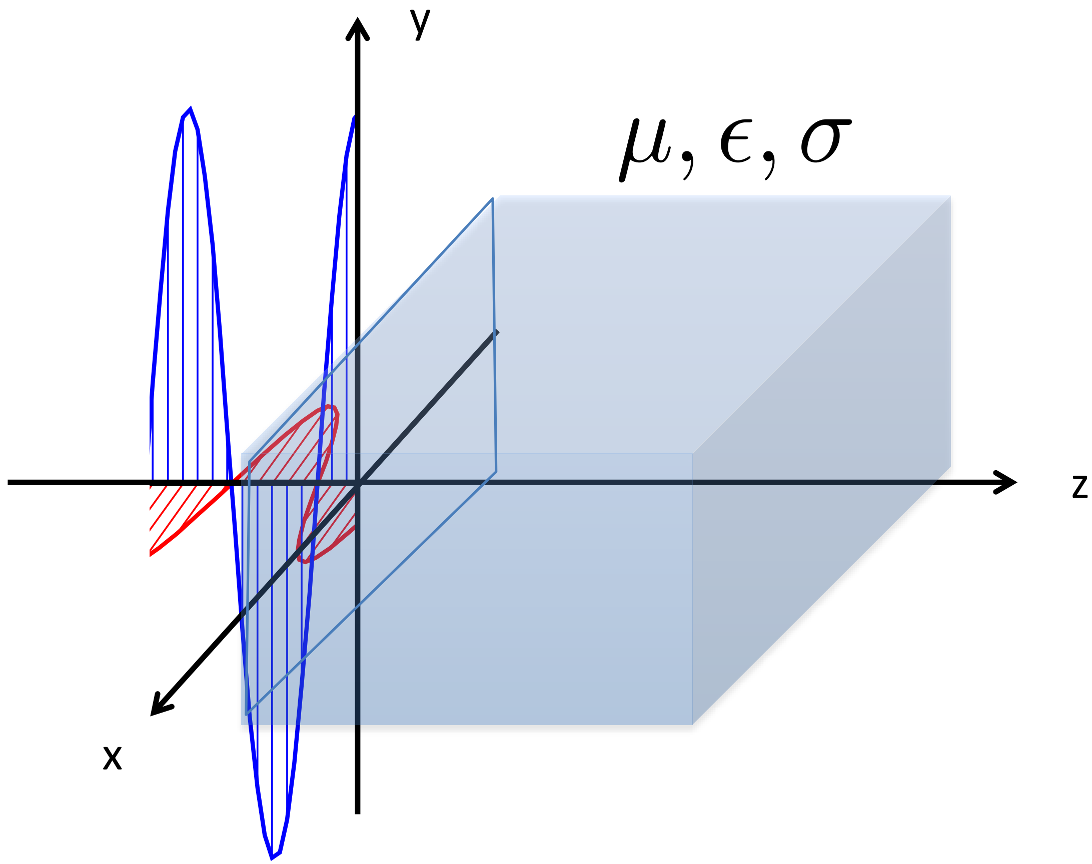



**Example: yellow light** → $\approx 5 \cdot 10^{14}$ **Hz (i.e.** $\approx$ **2 eV !)**

**gamma rays** → $\leq 3 \cdot 10^{21}$ **Hz (i.e.** $\leq$ **12 MeV !)**

**LEP (SR)** → $\leq 2 \cdot 10^{20}$ **Hz (i.e.** $\approx$ **0.8 MeV !)**

# Interaction with conductors



# EM wave in a conducting media

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{H} = 0$$

$$\nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \vec{H}$$

$$\nabla \times \vec{H} = \sigma \vec{E} + \epsilon \frac{\partial}{\partial t} \vec{E}$$

Ohm's Law

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial}{\partial t} \vec{E} + \mu\epsilon \frac{\partial^2}{\partial t^2} \vec{E}$$

Similar relation is found for H

Starting ansatz $\vec{E} = \hat{x} E_0 f(z) \sin(kz - \omega t)$

$$\vec{E} = \hat{x} E_0 e^{-\alpha z} \sin(kz - \omega t)$$

$$\alpha = \omega \sqrt{\frac{\mu\epsilon}{2}} \sqrt{-1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}}$$

$$k = \omega \sqrt{\frac{\mu\epsilon}{2}} \sqrt{1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}}$$

# Wave propagation

It depends on $\mu, \epsilon, \sigma$

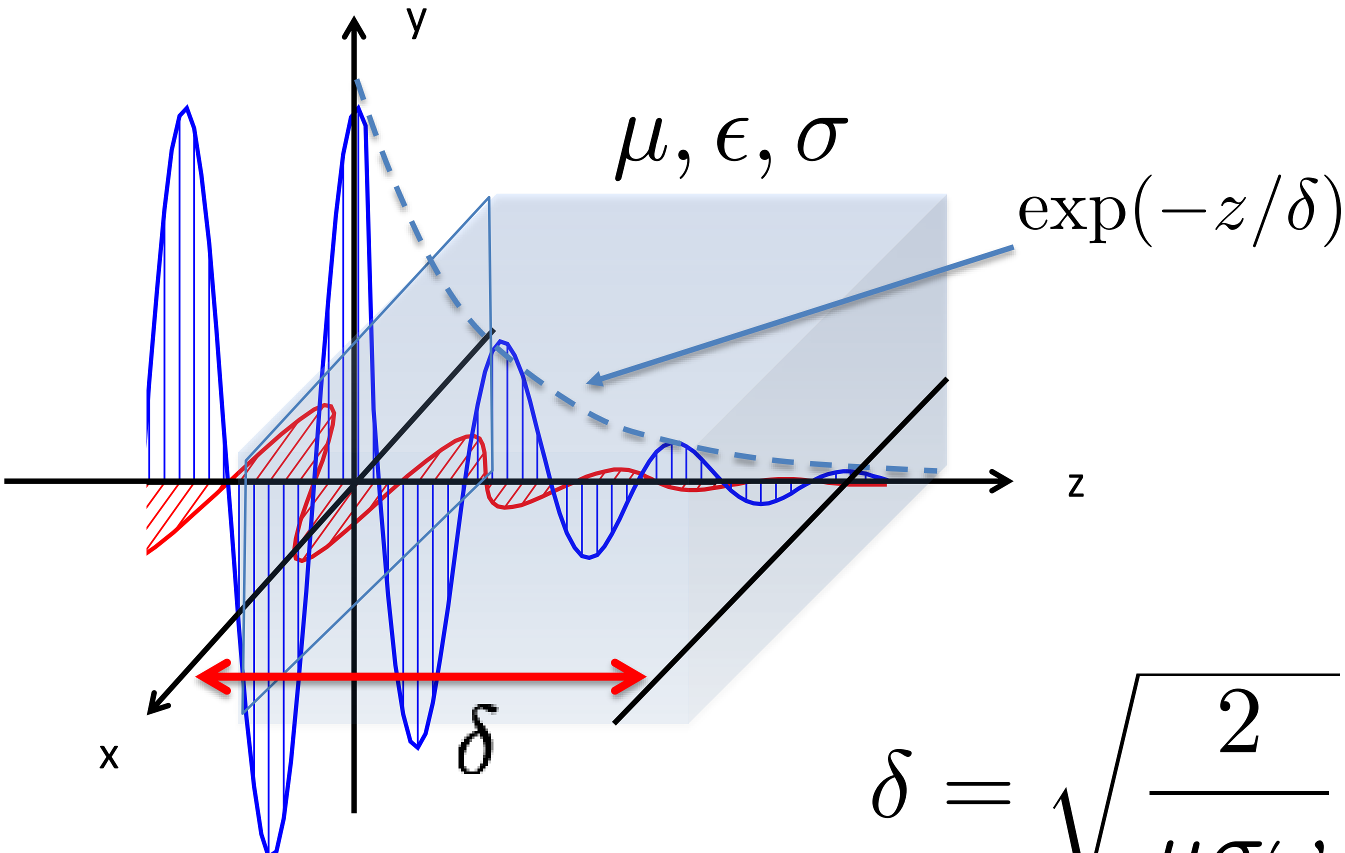
**Bad conductor**

If $\sigma \to 0$ then $\begin{cases} \alpha \to 0 & \text{wave is un-damped} \\ k \to \omega\sqrt{\mu\epsilon} \end{cases}$

**Good conductor**

if $\dfrac{\sigma}{\epsilon\omega} >> 1$ then $\left\{ \alpha = k = \sqrt{\dfrac{\mu\sigma\omega}{2}} \right.$

# Skin depth



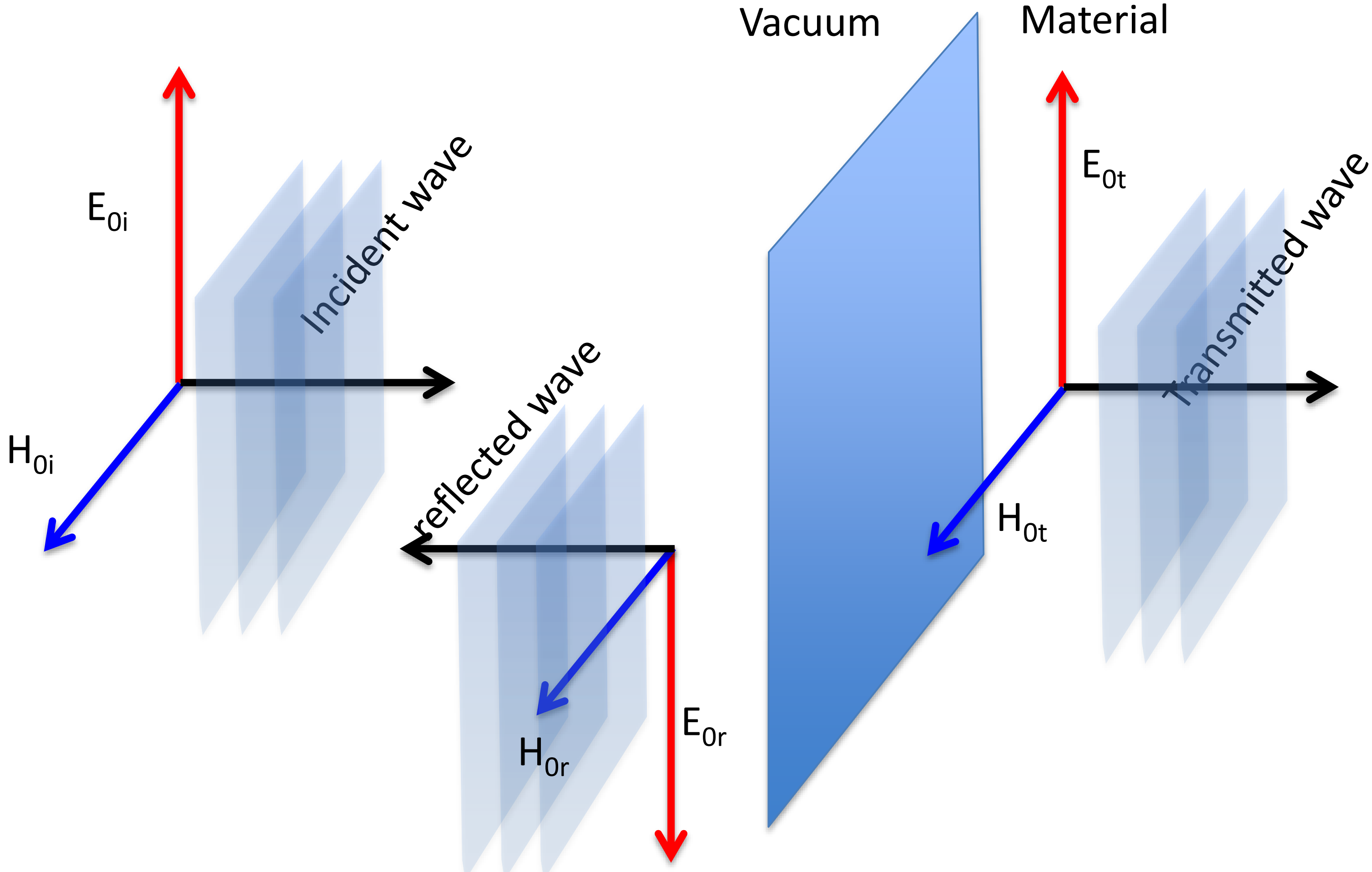

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$$

(drawing for not ideal conductor)

consider copper which has an electrical conductivity $\sigma = 5.8 \times 10^7 \, S/m \quad \mu \simeq \mu_0$ , and $\varepsilon \simeq \varepsilon_0$

| f | δ |
|---|---|
| 60 Hz | 8530 μm |
| 1 MHz | 66.1 μm |
| 10 MHz | 20.9 μm |
| 100 MHz | 6.6 μm |
| 1 GHz | 2.09 μm |

# Transmission, Reflection



At the interface between the two region the boundary condition are

$$E_{0i} + E_{0r} = E_{0t}$$
$$H_{0i} + H_{0r} = H_{0t}$$

$$\frac{E_{0i}}{H_{0i}} = \eta_1 \qquad \frac{E_{0r}}{H_{0r}} = \eta_1 \qquad \frac{E_{0t}}{H_{0t}} = \eta_2$$

$$\frac{E_{0r}}{E_{0i}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad \frac{E_{0t}}{E_{0i}} = \frac{2\eta_2}{\eta_2 + \eta_1}$$

$$\frac{H_{0r}}{H_{0i}} = \frac{\eta_1 - \eta_2}{\eta_2 + \eta_1} \qquad \frac{H_{0t}}{H_{0i}} = \frac{2\eta_2}{\eta_2 + \eta_1}$$

Perfect conductor $\sigma = \infty$ $\eta = \sqrt{\frac{\omega\mu}{\sigma}} \to 0$

Perfect dielectric $\sigma = 0$ $\eta = \sqrt{\frac{\mu}{\epsilon}}$

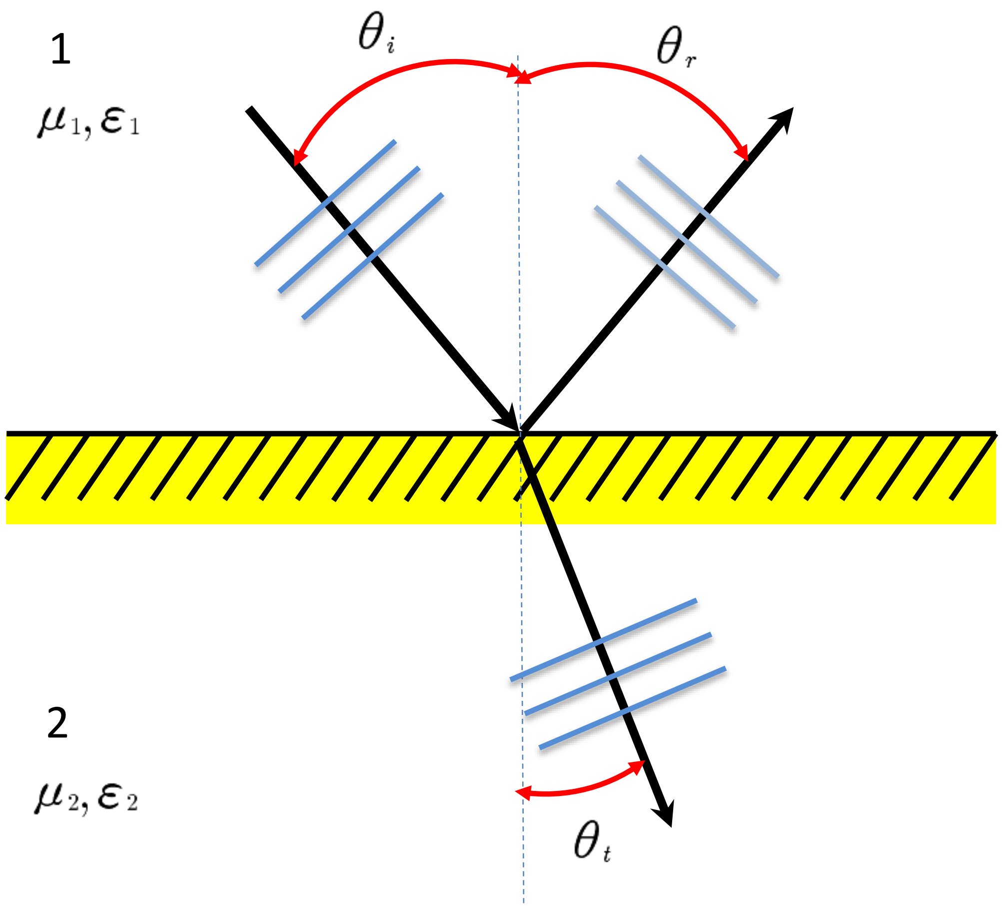
Perfect dielectric

$$\frac{E_{0r}}{E_{0i}} = -1$$

Perfect conductor

$$\frac{E_{0t}}{E_{0i}} = 0$$

# Snell's Law

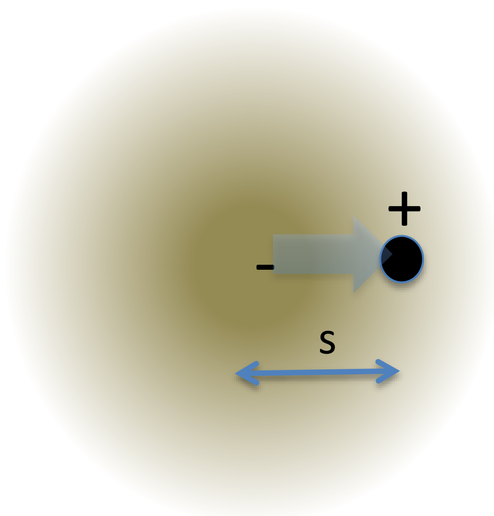



$$\frac{\sin\ \theta_i}{\sin\ \theta_t} = \sqrt{\frac{\mu_2\epsilon_2}{\mu_1\epsilon_1}}$$

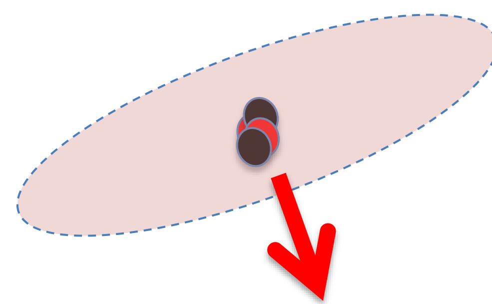
# EM in dispersive matter

Response to "external" electromagnetic field needs "time"

Electric field




Magnetic field



$$\epsilon = \epsilon(\omega) \qquad \mu = \mu(\omega)$$

Wave velocity depends on

$$v = \frac{1}{\sqrt{\epsilon(\omega)\mu(\omega)}}$$

The relation for k and $\omega$

$$v^2(k_x^2 + k_y^2 + k_z^2) = \omega^2$$

becomes more complicated because v depends on omega

Waves at different frequencies travels with different velocity → they "spread"

$$\omega = \omega(k)$$

Usually a pulse of electromagnetic wave is composed by several waves of different frequency -→

# Phase velocity and group velocity

A general wave can be decomposed in sum of harmonics

$$f(\vec{x},t) = \int A(\vec{k}) e^{i[\vec{k}\cdot\vec{x} - \omega(\vec{k})t]} dk^3$$

If $\omega$ is independent from $\vec{k}$ the wave does not get "dispersed"

If $A(k)$ is peaked around $k_0$ then $\omega$ can be expanded around $k_0$

$$\omega(k) = \omega_0 + \omega'(k - k_0)$$

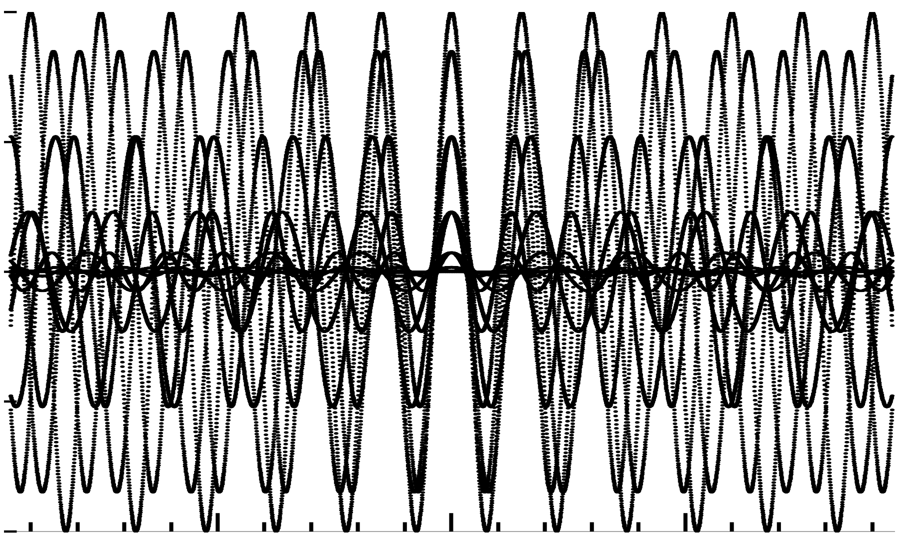
$$f(x,t) = \int A(k) e^{i[kx - \omega_0 t - \omega'(k-k_0)]} dk$$

$$f(x,t) = \underbrace{e^{i[k_0 x - \omega_0 t]}}_{\text{Fast wave}} \underbrace{\int A(k) e^{i(k-k_0)(x-\omega' t)} dk}_{\text{Slow wave modulating the fast wave}}$$

Speed → $v_g = \frac{\partial \omega}{\partial k}|_{k_0}$

# Example

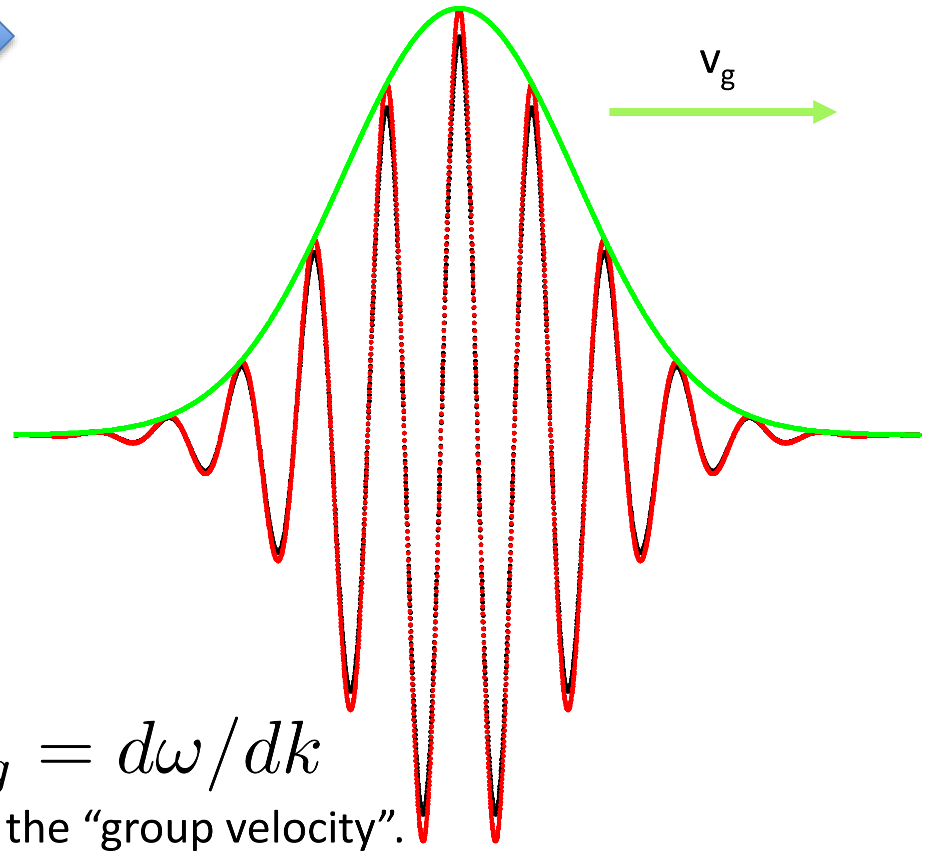
15 harmonics:
each with different
wave number, and
wavelength



Phase velocity

$$v_p = \omega(k)/k$$

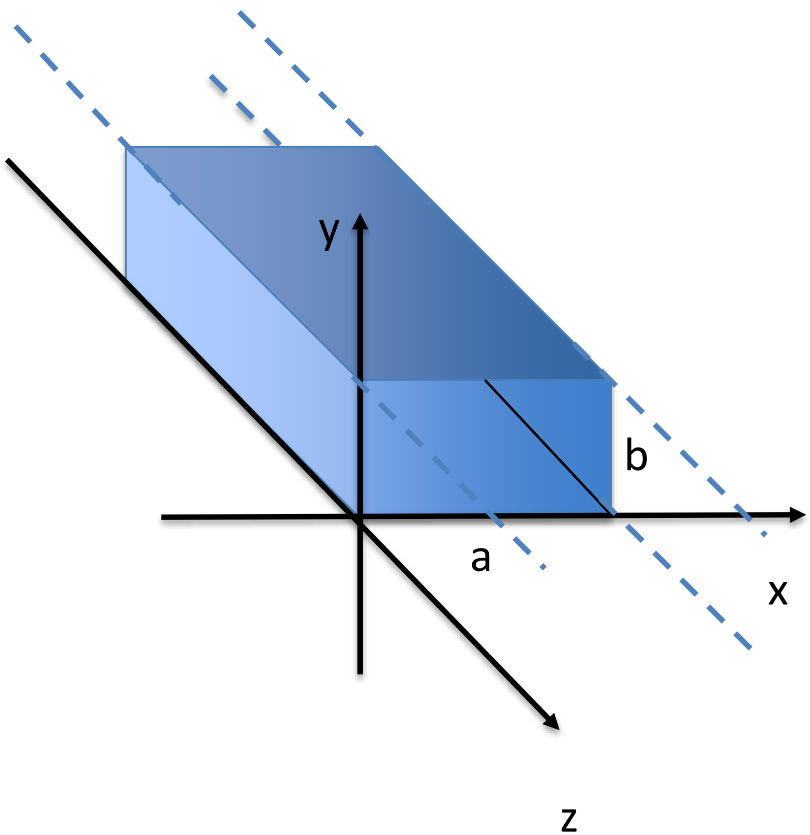
Wave packet




$$v_g = d\omega/dk$$

is the “group velocity”.
The speed of the wave packet

# Waveguides




Walls:
Perfect conductor $\sigma = \infty$

Inside the guide:
Perfect dielectric $\sigma = 0$

Boundary condition at the walls

$E_x(0,x,z) = 0$
$E_x(x,a,z) = 0$

$E_y(0,y,z)=0$
$E_y(b,y,z)=0$

# In the perfect dielectric

Maxwell equations

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{H} = 0$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t}$$

Working ansatz

$$\vec{E} = \vec{E}_0(x, y) e^{i(kz - \omega t)}$$

$$\vec{H} = \vec{H}_0(x, y) e^{i(kz - \omega t)}$$

$$E_{0y} = -i \frac{k}{k_c^2} \partial_y E_{0z} + i \frac{\omega\mu}{k_c^2} \partial_x H_{0z}$$

$$E_{0x} = -i \frac{k}{k_c^2} \partial_x E_{0z} - i \frac{\omega\mu}{k_c^2} \partial_y H_{0z}$$

$$H_{0y} = -i \frac{k}{k_c^2} \partial_y H_{0z} - i \frac{\omega\mu}{k_c^2} \partial_x E_{0z}$$

$$H_{0x} = -i \frac{k}{k_c^2} \partial_y H_{0z} + i \frac{\omega\mu}{k_c^2} \partial_y E_{0z}$$

Dispersion relation

$$k_c^2 = \omega^2 \epsilon\mu - k^2$$

# In the perfect dielectric

Only $E_{0z}$, and $H_{0z}$ are in the partial derivatives: → special solutions

Transverse electric wave TE ←→ $E_{0z}$ = 0

$$E_{0y} = \cancel{-i\frac{k}{k_c^2}\partial_y E_{0z}} + i\frac{\omega\mu}{k_c^2}\partial_x H_{0z}$$

$$E_{0x} = \cancel{-i\frac{k}{k_c^2}\partial_x E_{0z}} - i\frac{\omega\mu}{k_c^2}\partial_y H_{0z}$$

$$H_{0y} = -i\frac{k}{k_c^2}\partial_y H_{0z} - \cancel{i\frac{\omega\mu}{k_c^2}\partial_x E_{0z}}$$

$$H_{0x} = -i\frac{k}{k_c^2}\partial_y H_{0z} + \cancel{i\frac{\omega\mu}{k_c^2}\partial_y E_{0z}}$$

Transverse magnetic wave TM ←→ $H_{0z}$ = 0

$$E_{0y} = -i\frac{k}{k_c^2}\partial_y E_{0z} + \cancel{i\frac{\omega\mu}{k_c^2}\partial_x H_{0z}}$$

$$E_{0x} = -i\frac{k}{k_c^2}\partial_x E_{0z} - \cancel{i\frac{\omega\mu}{k_c^2}\partial_y H_{0z}}$$

$$H_{0y} = \cancel{-i\frac{k}{k_c^2}\partial_y H_{0z}} - i\frac{\omega\mu}{k_c^2}\partial_x E_{0z}$$

$$H_{0x} = \cancel{-i\frac{k}{k_c^2}\partial_y H_{0z}} + i\frac{\omega\mu}{k_c^2}\partial_y E_{0z}$$

# TE waves

Equations

$$E_{0y} = i\frac{\omega\mu}{k_c^2}\partial_x H_{0z}$$

$$E_{0x} = -i\frac{\omega\mu}{k_c^2}\partial_y H_{0z}$$

$$H_{0y} = -i\frac{k}{k_c^2}\partial_y H_{0z}$$

$$H_{0x} = -i\frac{k}{k_c^2}\partial_y H_{0z}$$

These eqs. + $\vec{\nabla} \cdot \vec{H} = 0$

$$\boxed{\partial_x^2 H_{0z} + \partial_y^2 H_{0z} = k_c^2 H_{0z}}$$

Automatically $\vec{\nabla} \cdot \vec{E} = 0$

Is satisfied

If you know $H_{0z}$, then you know everything

# Boundary conditions: modes

$$\partial_x^2 H_{0z} + \partial_y^2 H_{0z} = k_c^2 H_{0z}$$

Search for the solution

$$H_{0z} = X(x)Y(y)$$

$$\frac{X''}{X} = -k_x^2 \qquad \frac{Y''}{Y} = -k_y^2 \qquad \Rightarrow \qquad k_c^2 = k_x^2 + k_y^2$$

Boundary conditions

$E_x(0,x,z) = 0$
$E_x(x,a,z) = 0$

$E_y(0,y,z)=0$
$E_y(b,y,z)=0$

$$\Rightarrow \quad \boxed{H_{0z,nm} = H_{nm} \cos\left(\frac{\pi n}{a} x\right) \cos\left(\frac{\pi m}{b} y\right)}$$

# Cut-off frequency

Dispersion relation
$$\omega^2\epsilon\mu = k^2 + \left(\frac{\pi n_x}{a}\right)^2 + \left(\frac{\pi n_y}{b}\right)^2$$

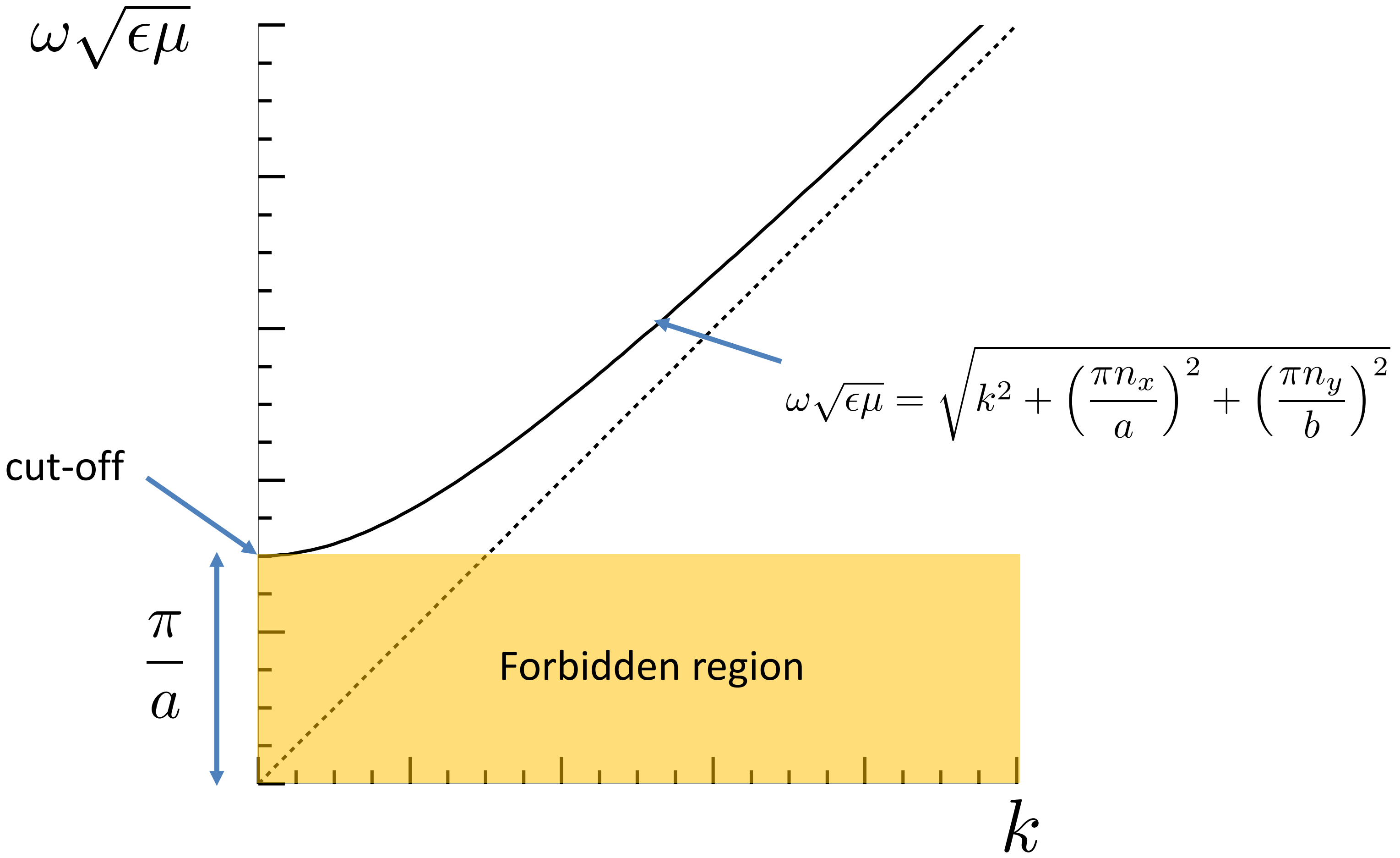
Only if k > 0 the wave can propagate without attenuation

Speed of wave
$$u = \frac{1}{\sqrt{\epsilon\mu}}$$

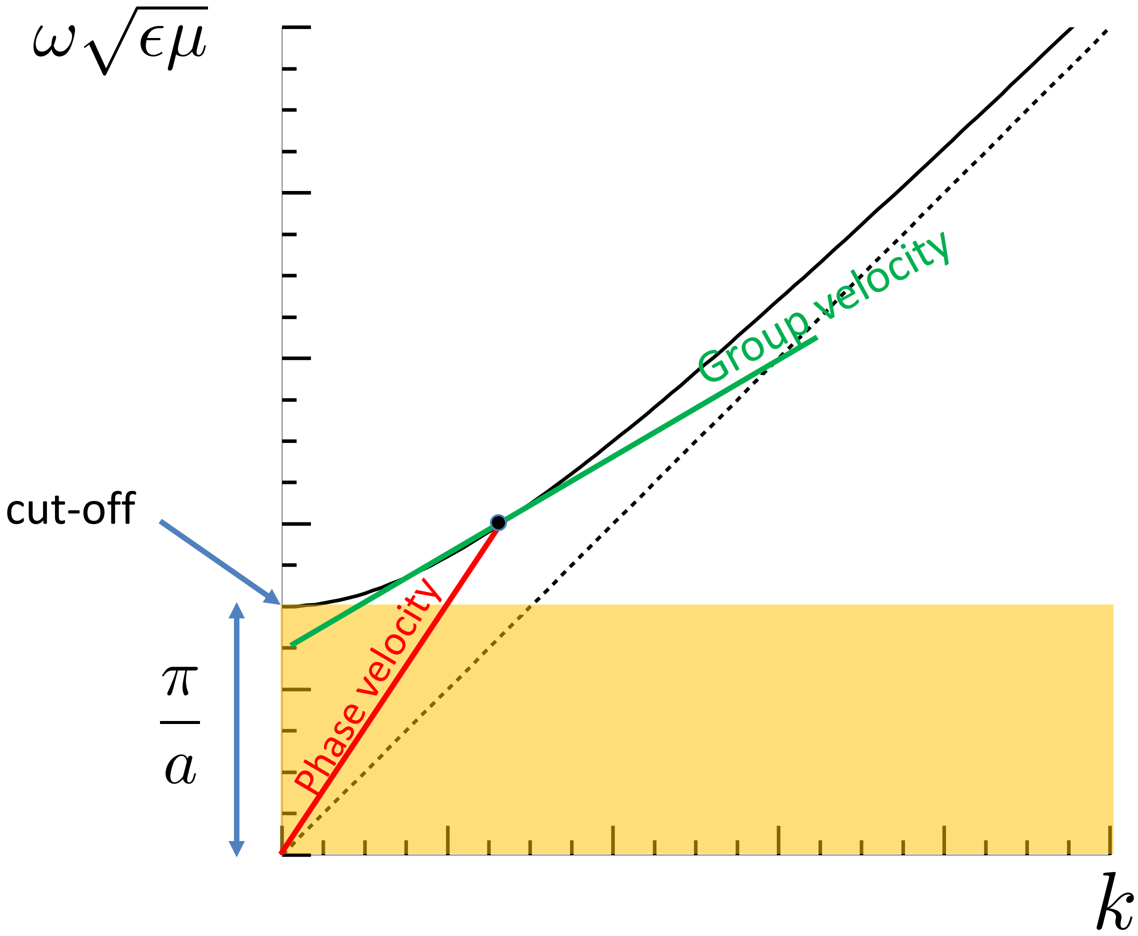
$$f > \underbrace{\frac{u}{2}\sqrt{\left(\frac{n_x}{a}\right)^2 + \left(\frac{n_y}{b}\right)^2}}_{\text{Cut-off frequency}}$$

Given the fix frequency of a wave, only a certain number of modes can exists in the waveguide

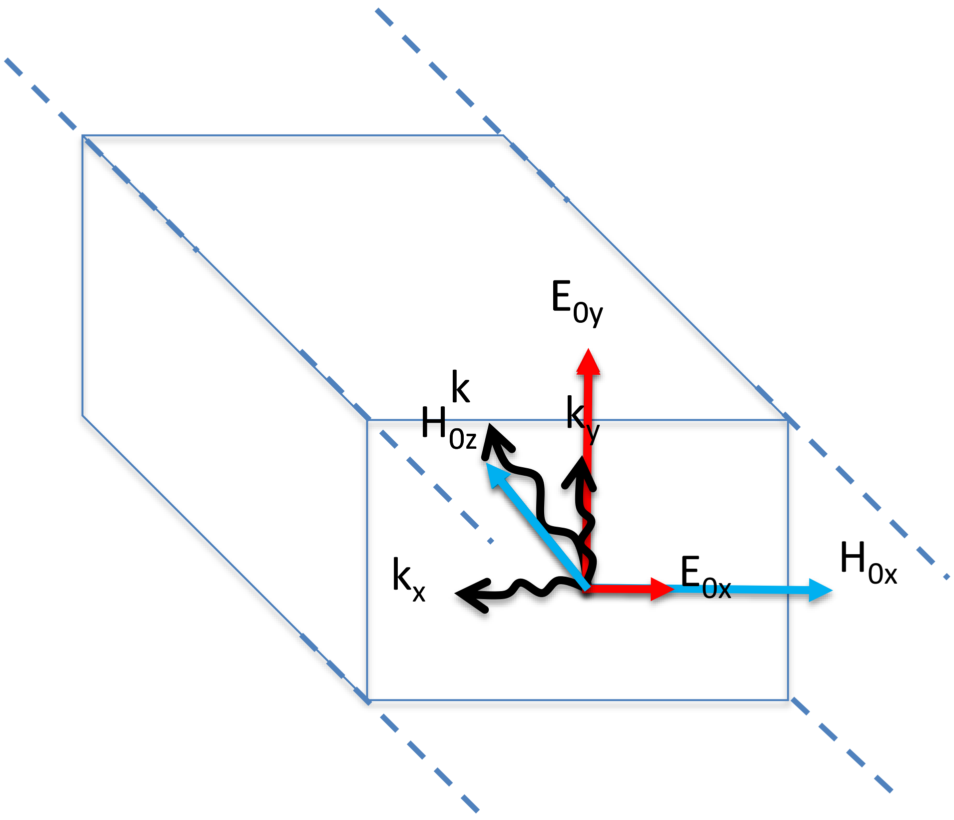
If a > b consider the TE10 mode



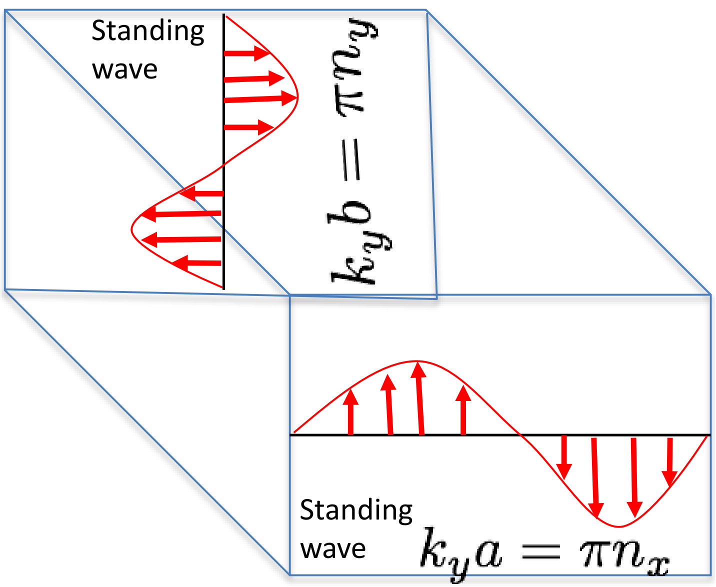
If a > b consider the TE10 mode



# visually TE

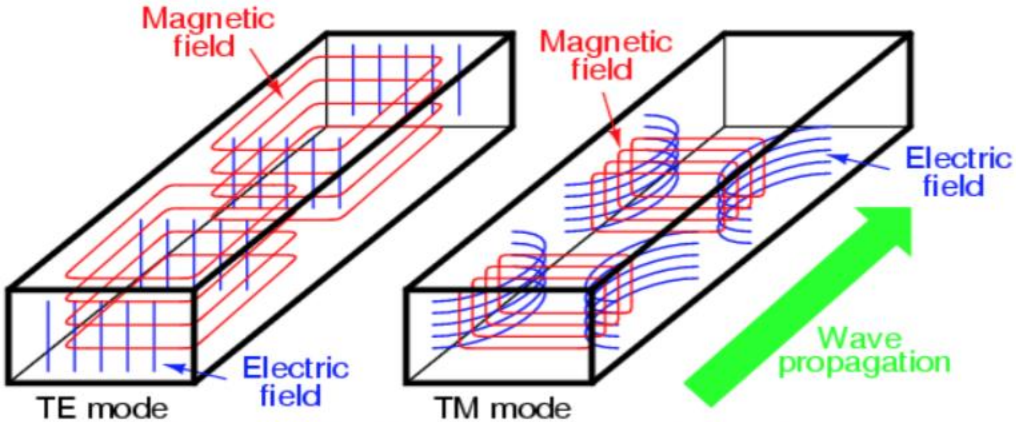


$E_{0y}$
k
$H_{0z}$
$k_y$
$k_x$
$E_{0x}$
$H_{0x}$

Standing wave
$k_y b = \pi n_y$
Standing wave
$k_y a = \pi n_x$

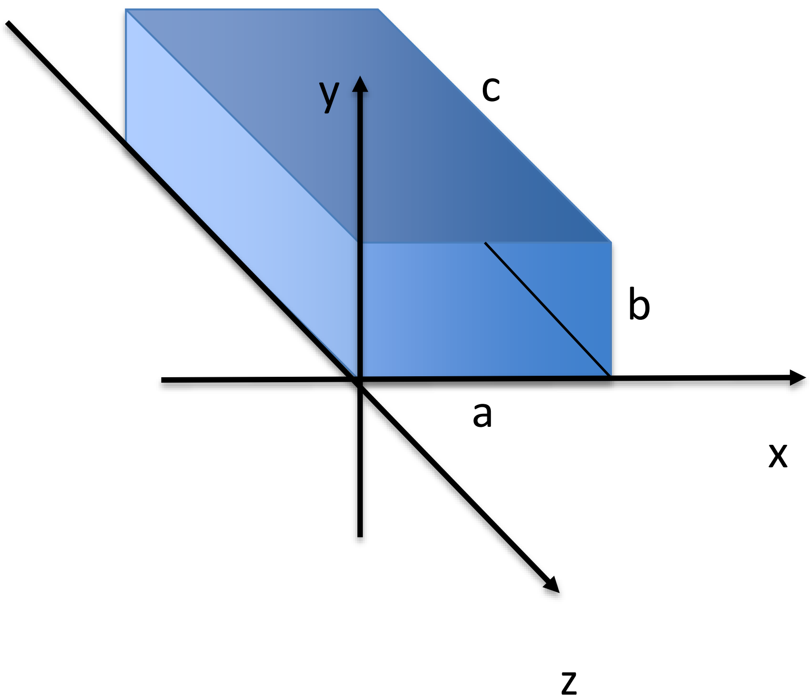
# The fields in wave guides




Magnetic flux lines appear as continuous loops
Electric flux lines appear with beginning and end points

- **Electric and magnetic fields through a wave guide**
- **Shapes are consequences of boundary conditions !**
- **Can be Transverse Electric (TE, no E-field in z-direction) or Transverse Magnetic (TM, no B-field in z-direction)**

# Cavity



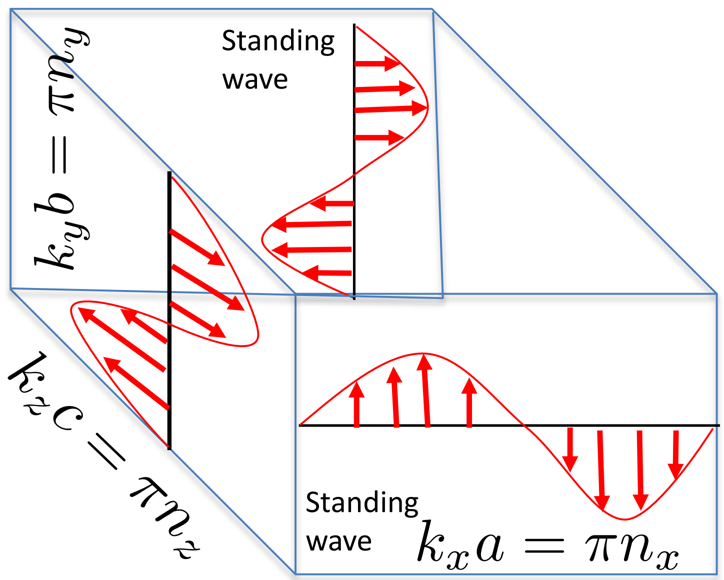

Walls:
Perfect conductor $\sigma = \infty$

Inside the guide:
Perfect dielectric $\sigma = 0$

Boundary condition at the walls

In every wall the
tangent electric field Is zero

# Cavity (rectangular)




**Boundary condition**

$$k_x a = \pi n_x$$

$$k_y b = \pi n_y$$

$$k_z c = \pi n_z$$

Normal modes are only standing waves

# Electromagnetic standing waves

$$E_x = E_{x0} \cdot cos(k_x x) \cdot sin(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$
$$E_y = E_{y0} \cdot sin(k_x x) \cdot cos(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$
$$E_z = E_{z0} \cdot sin(k_x x) \cdot sin(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$

$$B_x = \frac{i}{\omega}(E_{y0}k_z - E_{z0}k_y) \cdot sin(k_x x) \cdot cos(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$
$$B_y = \frac{i}{\omega}(E_{z0}k_x - E_{x0}k_z) \cdot cos(k_x x) \cdot sin(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$
$$B_z = \frac{i}{\omega}(E_{x0}k_y - E_{y0}k_x) \cdot cos(k_x x) \cdot cos(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$

Dispersion relation $\omega^2 = \frac{1}{\epsilon\mu}(k_x^2 + k_y^2 + k_z^2)$

# Final Observations

Potential vector was here presented for static field

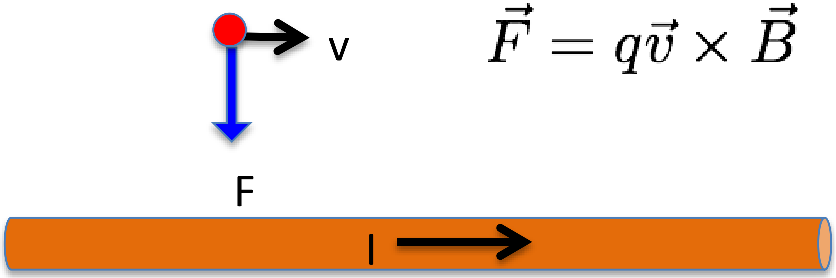
However one can also re-write the Maxwell equation in terms of the potential vector, and find electromagnetic wave of ”A”

Internal degree of freedom: Gauges

Potential vector $\vec{A} \to \vec{A} + \vec{\nabla} C$ ( A is defined not In unique way)

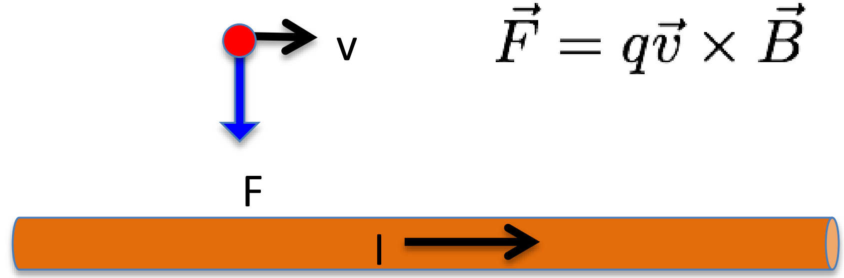
Electric potential $V \to V + c$ ( V is not unique)

# Reference frame ?




$$\vec{F} = q\vec{v} \times \vec{B}$$

$$\vec{B} = \frac{\mu_0}{2\pi}\frac{I}{r}\hat{v}$$

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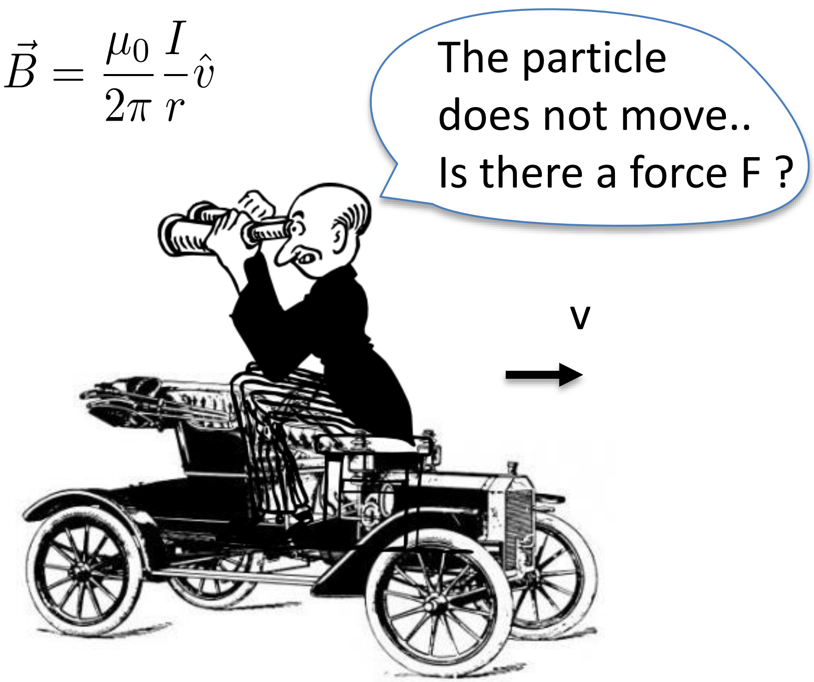

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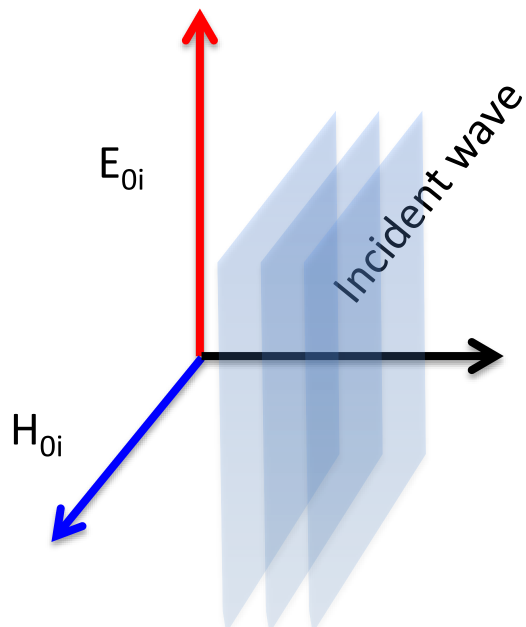
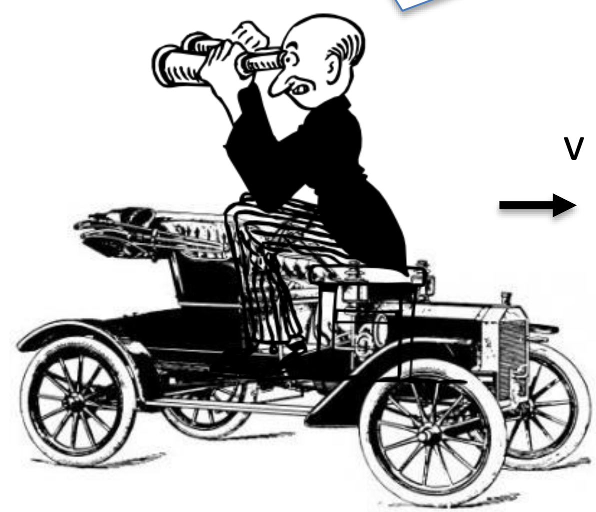
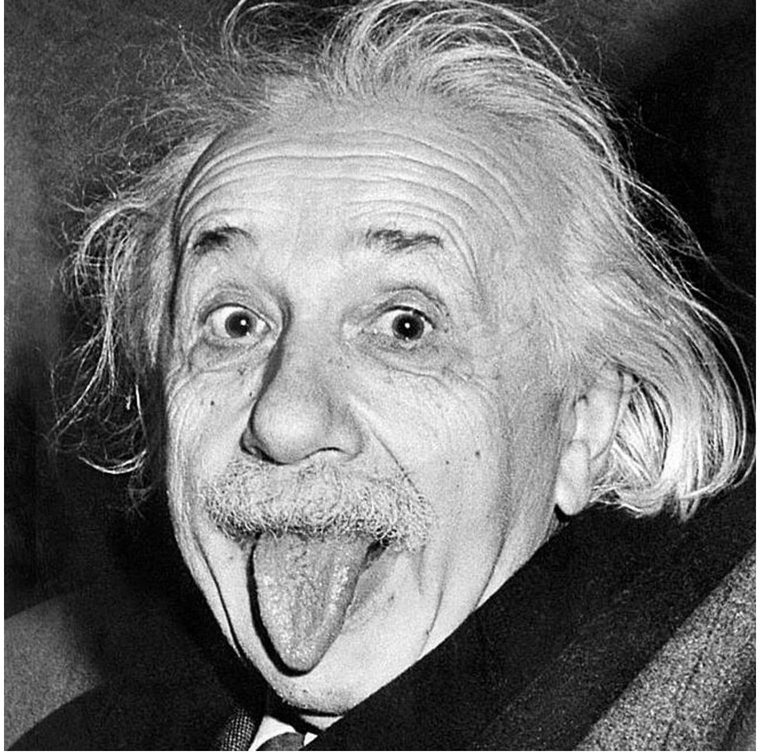



Maxwell equations tells me the speed is c

Maxwell equations tells me the speed is c ( but I move, mm)





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