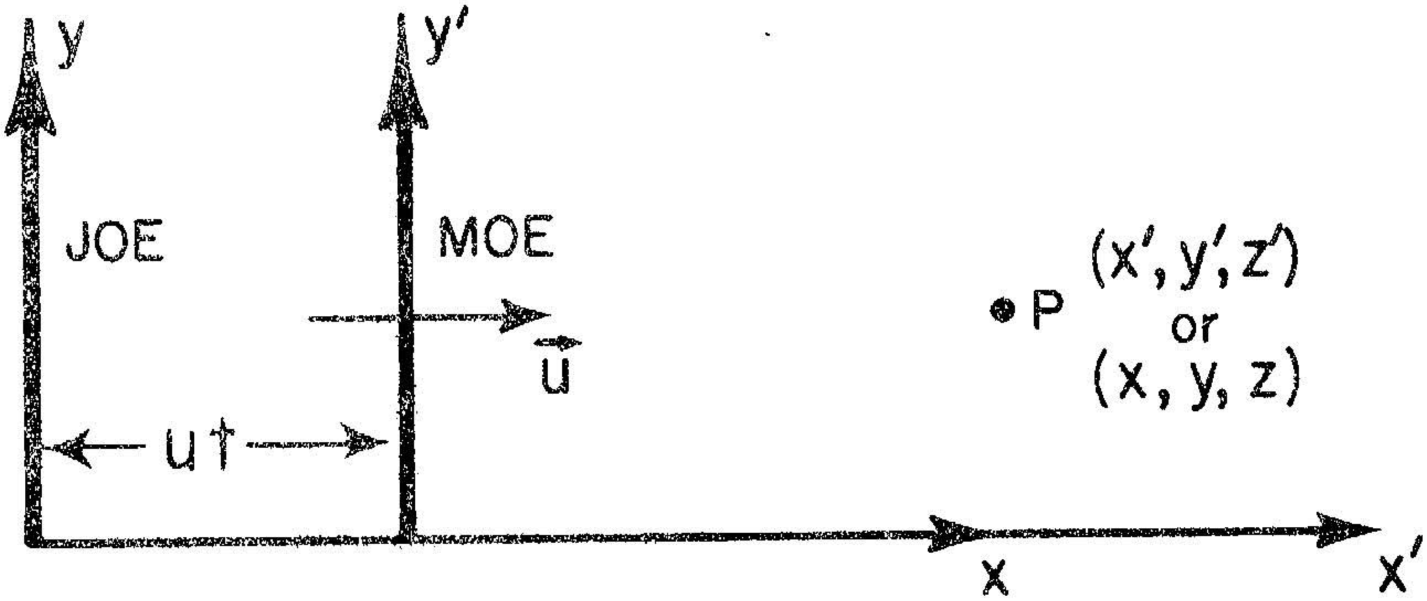
Special relativity – E. J. N. Wilson - CERN

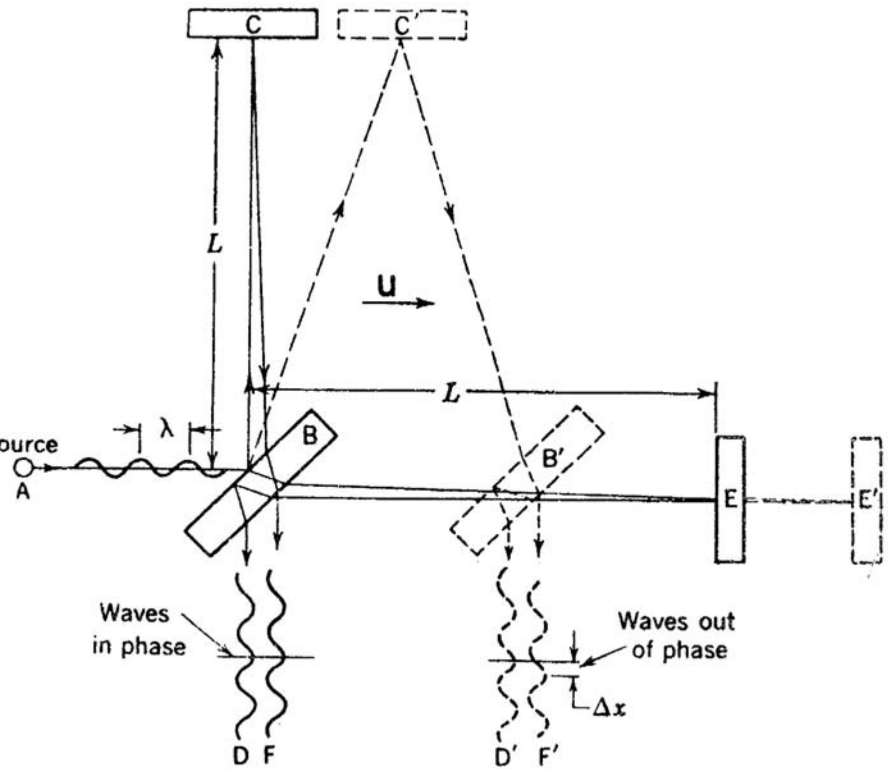
- Moving and rest frames
- Michelson-Morley
- Clocks
- Lorentz transformation
- Time dilation
- Space-time four vector
- Transforming velocity
- Momentum – energy four vector
- Transforming acceleration
- Transforming force
- Synchrotron radiation
- Electromagnetic field transformation

Observers in Laboratory and Moving Frames

- JOE is an observer in the "laboratory" frame and uses unprimed coordinates to describe P
- MOE is an observer in the "moving frame" and uses primed coordinates to describe P
- The relative velocity MOE-JOE is the vector u



Michelson Morley Experiment (1887) points to space contraction



- Suppose device is moving with velocity, u, relative to the "ether" while light velocity in ether is constant, c
- Mirror E moves a distance ut in time t for light to pass from B to E
- t =L/(c-u) and L/(c+u) on return
- total time is

$$2Lc/(c^2-u^2)$$

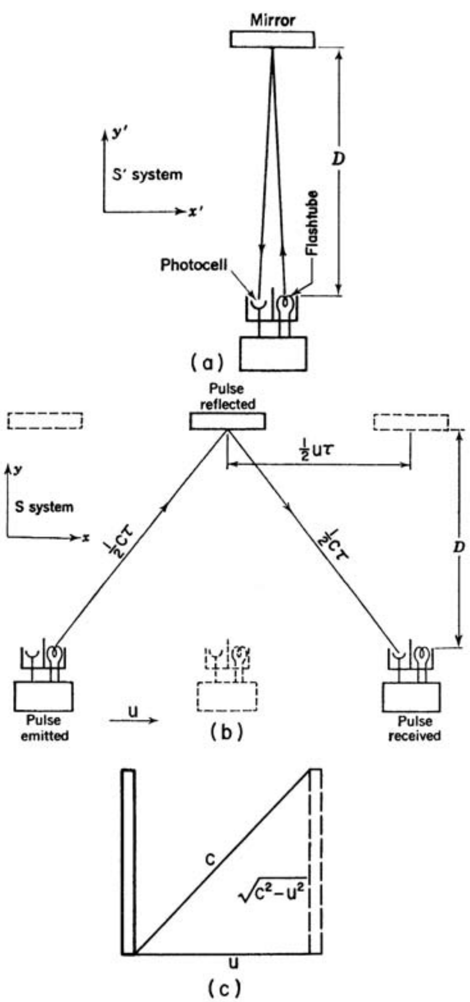
- Forth and back to C covers a longer distance (hypotenuse of a triangle) and total time is:

$$2Lc/\sqrt{c^2-u^2}$$

- They changed u and found no interference?
- But suppose BE shrinks as

$$L_{BE}=L_{BC}/\sqrt{c^2-u^2}$$

The light clock explains time dilation



- This clock ticks every time a photon travels back and forth falling on a photocell which sends another photon off. The interval is 2L/c

- When the clock moves in a spaceship it ticks at the same rate to MOE but in the laboratory the light must clearly travel a longer distance and the interval between ticks will be :

$$2Lc/\sqrt{c^2-u^2}=2(L/c)/\sqrt{1-(u/c)^2}$$

- Clearly this is a slower tick rate by the factor.

$$1/\sqrt{1-(u/c)^2}=1/\sqrt{1-\beta^2}=\gamma$$

Transformations
(between observers with relative velocity u)

Galileo (1630)

(Newton is unchanged but Maxwell equations change)

$$x' = x - ut$$

$$y' = y,$$

$$z' = z,$$

$$t' = t$$

Maxwell (1880)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho,$$

$$\nabla \cdot \mathbf{B} = 0$$

Lorentz (1900)

(Maxwell equations unchanged)

$$x' = \frac{x - ut}{\sqrt{1 - u^2/v^2}},$$

$$y' = y,$$

$$z' = z,$$

$$t' = \frac{t - ux/c^2}{\sqrt{1 - u^2/v^2}}$$

Foretaste

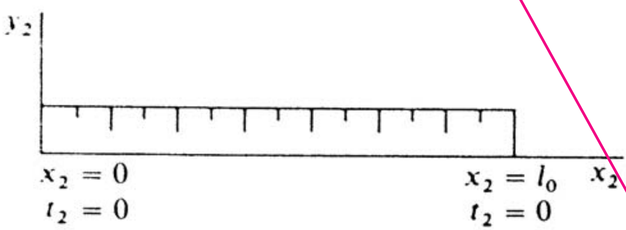
$$\frac{1}{\sqrt{1-\beta^2}} = \gamma$$

Lorentz found this, feeling in the dark for a transformation which did not spoil Maxw

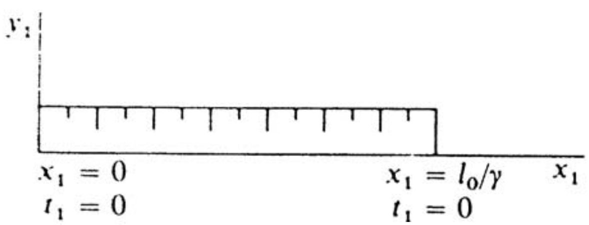
Lorentz transformation (slightly different notation)

$$x_2 = \frac{x_1 - vt_1}{\sqrt{1 - v^2/c^2}}, \qquad y_2 = y_1, \qquad z_2 = z_1, \qquad t_2 = \frac{t_1 - vx_1/c^2}{\sqrt{1 - v^2/c^2}}$$

- **Here Joe measures suffix one in the lab and Moe suffix 2 in frame moving with velocity v with respect to Joe.**
- **Lorentz did not know where to put the velocity of light so we did this for him**



MOE's view



JOE's view

MOE lays down a ruler length: $\boxed{l_2}$

JOE in the lab, compares the position of the ends at the same time (t=0 in both systems) with marks on his bench (perhaps by a photo) and concludes MOE's ruler is shorter:

$$l_1 = l_2\sqrt{1 - v^2/c^2} = l_2/\gamma$$

Called Lorentz contraction

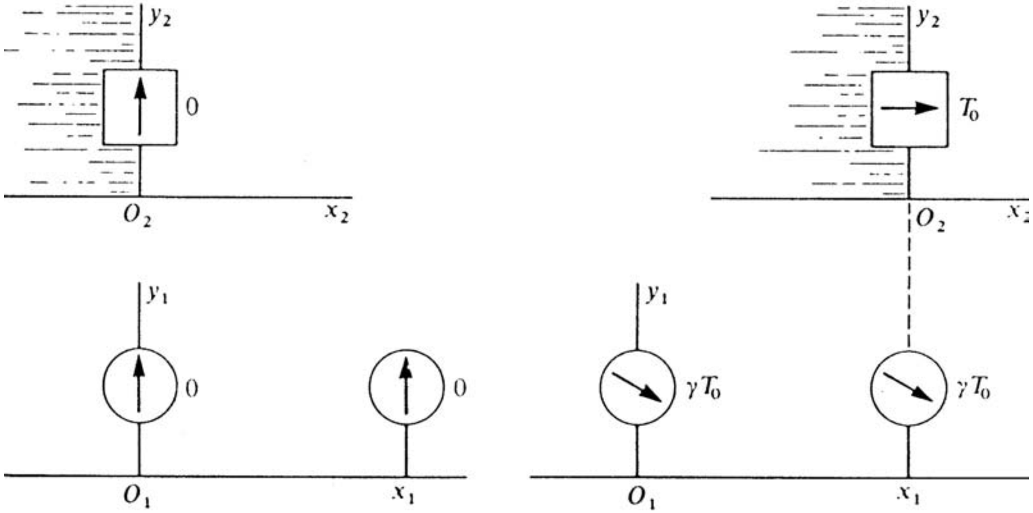
Time dilation

$$\boxed{x_2 = \frac{x_1 - vt_1}{\sqrt{1-v^2/c^2}}, \qquad y_2 = y_1, \quad z_2 = z_1, \quad t_2 = \frac{t_1 - vx_1/c^2}{\sqrt{1-v^2/c^2}}}$$

- **The three clocks are identical and start at time zero.**
- **When MOE's reaches JOE's second clock MOE's has not advanced as much as JOE's**
- **If we arbitrarily choose** $\boxed{x_1 = \frac{vt_2}{\sqrt{1-v^2/c^2}}}$ **then** $\boxed{t_2 = \frac{t_1 - vx_1/c^2}{\sqrt{1-v^2/c^2}}}$

MOE

JOE



Gives:

$$\boxed{t_1 = \frac{t_2}{\sqrt{1-v^2/c^2}} = \gamma t_2}$$

Explains long life of cosmic muons

Four vector of space-time

$$x_2 = x_1 \cos\theta + y_1 \sin\theta,$$
$$y_2 = -x_1 \sin\theta + y_1 \cos\theta$$

- **This transformation is a rotation of a vector of constant length:** $\sqrt{x^2 + y^2}$

- **The Lorentz transformation:**

$$\boxed{x_2 = \frac{x_1 - vt_1}{\sqrt{1 - v^2/c^2}}, \qquad y_2 = y_1, \quad z_2 = z_1, \quad t_2 = \frac{t_1 - vx_1/c^2}{\sqrt{1 - v^2/c^2}}}$$

Rotates the 4-vector: $(x, y, z, -ct)$

So that its "length" is an invariant $\sqrt{x^2 + y^2 + z^2 - c^2t^2}$

Quantities that are invariant under Lorentz transformation are at the heart of physics

Lorentz matrix

$$\begin{pmatrix} x \\ y \\ z \\ -ct \end{pmatrix}_2 = \gamma \begin{pmatrix} 1 & 0 & 0 & \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ -ct \end{pmatrix}_1$$

$$\beta = v / c$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\begin{pmatrix} x \\ y \\ z \\ -ct \end{pmatrix}_1 = \gamma \begin{pmatrix} 1 & 0 & 0 & -\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ -ct \end{pmatrix}_2$$

System 2 moves in the x direction with velocity *v* with respect to the stationary system 1

Transforming a velocity

The relative velocity is now written υ

$$v_{x1} = \frac{dx_1}{dt_1} = \frac{dx_1}{dt_2}\frac{dt_2}{dt_1}$$

$$\frac{dx_1}{dt_2} = \frac{1}{\sqrt{1 - v^2/c^2}}\frac{d}{dt_2}\left(x_2 - \upsilon t_2\right) = \gamma\frac{d}{dt_2}\left(x_2 + \upsilon t_2\right) = \gamma\left(v_{2x} + \upsilon\right)$$

Using two partial differentials we obtain for :

$$\frac{dt_2}{dt_1} = \gamma\frac{d}{dt_1}\left[t_1 - \left(\upsilon/c^2\right)x_1\right] = \gamma\left[1 - \left(\upsilon/c^2\right)v_{1x}\right]$$

Finally :

$$v_{1x} = \frac{v_{2x} + \upsilon}{1 + \left(v_{1x}\,\upsilon/c^2\right)} \qquad v_{2x} = \frac{c\left(v_{1x} - c\beta\right)}{c - v_{1x}\beta}$$

moving frame's veloci

If $v_{1x} = c\beta$ then $v_{2x} = 0$

If $u = c$ then $v_{2x} = c$

for all v's

A small step to redefine momentum and energy

$$p = mv = \frac{m_0 v}{\sqrt{1-(v/c)^2}} = \frac{m_0 v}{\sqrt{1-\beta^2}}$$

where m_0 is the mass at rest

and $\beta = v/c$

Now the moving frame is a particle whose velocity is and whose rest energy is

$$E = \gamma E_0 = m_0 c^2 \gamma = T + E_0$$

where the rest energy is $E_0 = m_0 c^2$

T is the kinetic energy and $\gamma = E/E_0$

$$E^2 - (pc)^2 = \left(m_0 c^2\right)^2 \quad \text{which is invariant}$$

$$\gamma = E/E_0, \quad \beta = pc/E, \quad \beta\gamma = pc/E_0$$

Transformation of a momentum

$$\begin{pmatrix} E \\ -p_x c \\ -p_y c \\ -p_z c \end{pmatrix}_1 = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E \\ -p_x c \\ -p_y c \\ -p_z c \end{pmatrix}_2$$

- **With the invariant rest energy**

$$E^2 - (pc)^2 = \left(m_0 c^2\right)^2$$

$$\beta = v / c$$

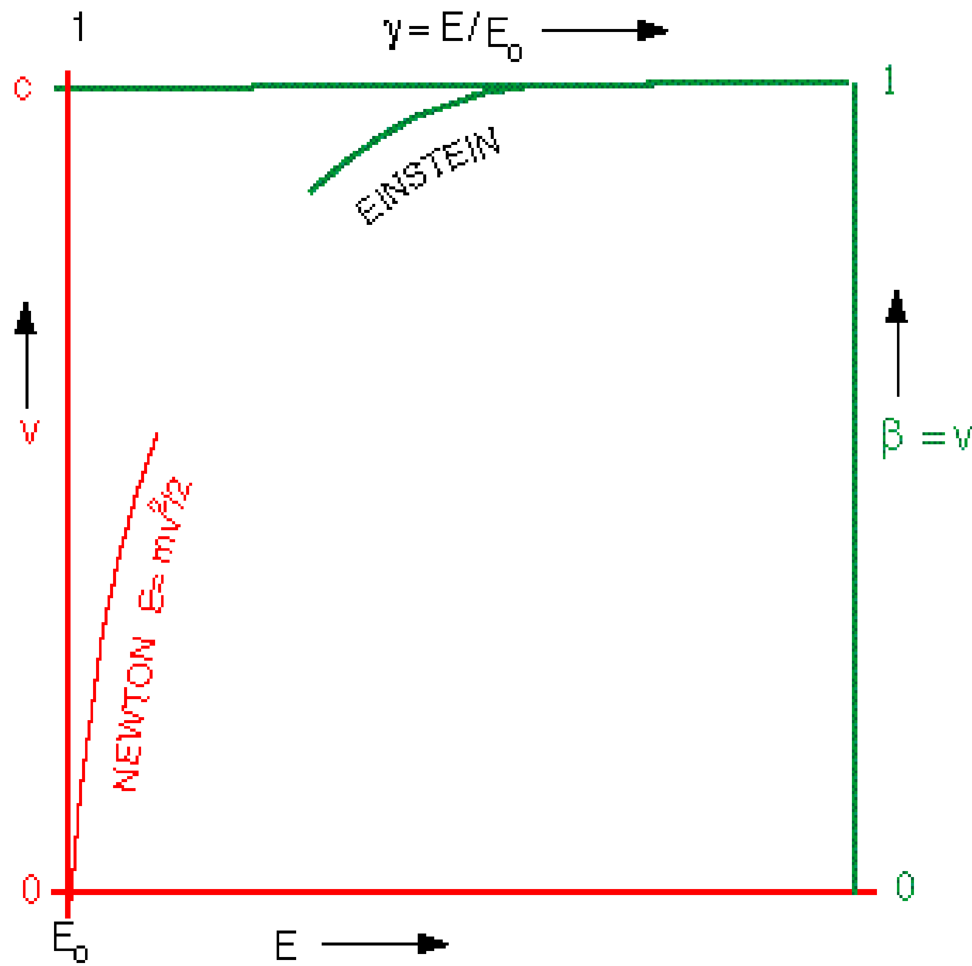
$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

System 2 moves in the x direction with velocity *v* with respect to the stationary system 1

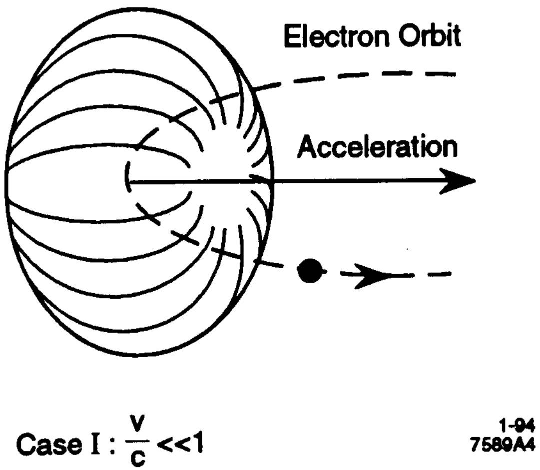
Newton & Einstein

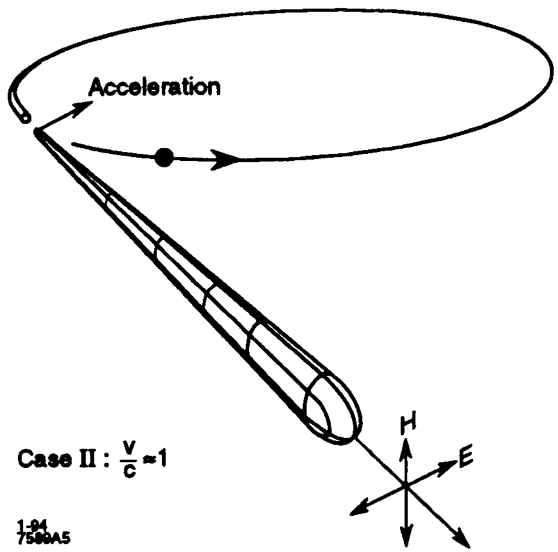
- **Almost all modern accelerators accelerate particles to speeds very close to that of light.**
- **In the classical Newton regime the velocity of the particle increases with the square root of the kinetic energy.**
- **As v approaches c it is as if the velocity of the particle "saturates"**
- **One can pour more and more energy into the particle, giving it a shorter De Broglie wavelength so that it probes deeper into the sub-atomic world**
- **Velocity increases very slowly and asymptotically to that of light**

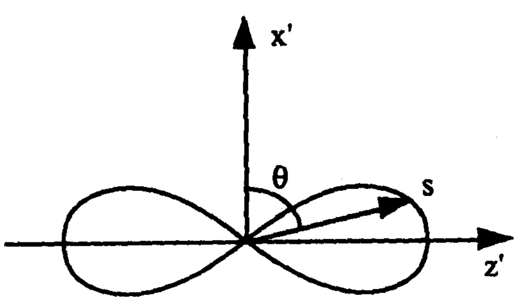
$$\frac{E}{m_0c^2} = \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \sqrt{1-\left(\frac{1}{\gamma}\right)^2} \Rightarrow 1$$

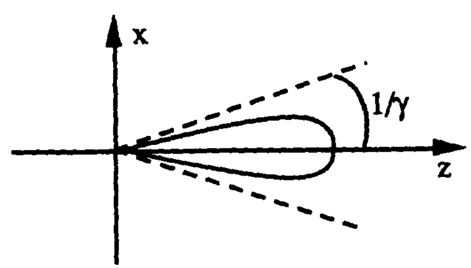


Synchrotron radiation









In moving frame :

$E = m_0 c^2 , p_x = 0$

Use matrix to transform to

lab frame :

$p_x = \beta \gamma m_o c^2 , p_y$ is unchang

Transforming acceleration

We can differentiate to find the acceleration

$$a_{1x} = \frac{dv_{1x}}{dt_1} = \frac{dv_{1x}}{dt_2}\frac{dt_2}{dt_1}$$

Again after using two partial differentials we obtain for :

$$a_{2x} = \frac{a_{1x}}{\gamma^3\left[1 - v_{1x}\upsilon/c^2\right]^3}$$

If only $a_{1z} \neq 0$ and relative velocity $\upsilon = v_{1x}$

$a_{2z} = \frac{1}{\gamma^2} a_{1z}$ hence rapid rise in synchrotron ligh

$$a_{2z} = \frac{1}{\gamma^2\left[1 - v_{1x}\upsilon/c^2\right]^2}\left\{a_{1z} + \frac{v_{1y}\upsilon}{c^2 - v_{1x}\upsilon}a_{1x}\right\}$$

$$a_{2y} = \frac{1}{\gamma^2\left[1 - v_{1x}\upsilon/c^2\right]^2}\left\{a_{1y} + \frac{v_{1y}\upsilon}{c^2 - v_{1x}\upsilon}a_{1x}\right\}$$

Transforming a force

We express the force as three components (*X*, *Y*, *Z*)

$$\boxed{X_2 = X_1 - \frac{\upsilon}{c^2 - v_{1x}\upsilon}\left(v_{1y}Y_1 + v_{1z}Z_1\right)}$$

$$\boxed{Y_2 = \frac{Y_1}{\gamma\left[1 - v_{1x}\upsilon/c^2\right]^2}}$$

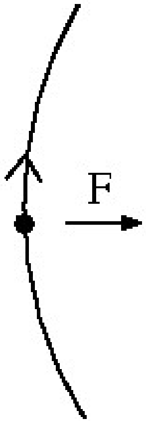
$$\boxed{Z_2 = \frac{Z_1}{\gamma\left[1 - v_{1x}\upsilon/c^2\right]^2}}$$

Why is synchrotron radiation so γ dependent?

- **Synchrotron radiation is simply dipole radiation from a moving charge like an electron circulating in a magnetic field. Larmor solved this problem and it is easy to calculate that the power radiated is :**

$$P = \frac{1}{6\pi\varepsilon_0}\frac{e^2}{c^3}\left(\ddot{z}\right)^2$$

Here we see the acceleration of the charge which is in the transverse direction



To be invariant this physical law must be modified

$$P = \frac{1}{6\pi\varepsilon_0}\frac{e^2}{c^3}\left(\ddot{z}\right)^2\gamma^4$$

This term is because the invariant transverse acceleration is

$$\ddot{z}\gamma^2$$

Transforming Electric and Magnetic Fields

$$\begin{pmatrix} E_{1x} \\ E_{1y} \\ E_{1z} \\ cB_{1x} \\ cB_{1y} \\ cB_{1z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 & 0 & \gamma\beta \\ 0 & 0 & \gamma & 0 & -\gamma\beta & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\gamma\beta & 0 & \gamma & 0 \\ 0 & \gamma\beta & 0 & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E_{2x} \\ E_{2y} \\ E_{2z} \\ cB_{2x} \\ cB_{2y} \\ cB_{2z} \end{pmatrix}$$

Special relativity Summary

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- **Michelson-Morley**
- **Clocks**
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