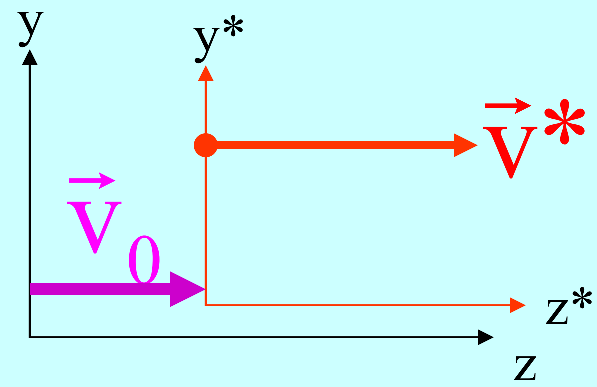
Linac-based Free-Electron Lasers

Jörg Rossbach
Hamburg University & DESY

Outline:

- Free Electron Laser by finger physics
- Free Electron Laser: Low Gain
- Free Electron Laser: High Gain, Start-up from noise (SASE)
- Experimental realization, technical challenges, future plans

Peak (instantaneous) power of accelerated charge q is Lorentz-invariant:



$$P_\gamma = \frac{q^2}{6\pi\varepsilon_0 c^3}\left(\dot{v}^*\right)^2 = P_\gamma^* \qquad (!)$$

* means: in co-moving instantaneous rest frame

i) remember: $\dot{v}^*$ depends strongly on γ if expressed in lab-frame quantities:

$$\dot{v}_z^* = \gamma^3 \dot{v}_z \,, \quad \dot{v}_y^* = \gamma^2 \dot{v}_y \quad \rightarrow \quad P_\gamma = \frac{q^2}{6\pi\varepsilon_0 c^3}\gamma^4 \dot{v}_y^2$$

ii) consider $\quad q = N \cdot e_0 \rightarrow P_\gamma = \frac{e_0^2}{6\pi\varepsilon_0 c^3} N^2 \gamma^4 \dot{v}_y^2$

$\rightarrow$ power per electron $\quad \frac{P_\gamma}{N} = \frac{e_0^2}{6\pi\varepsilon_0 c^3} N \cdot \gamma^4 \dot{v}_y^2 \quad$ "stimulated emission"

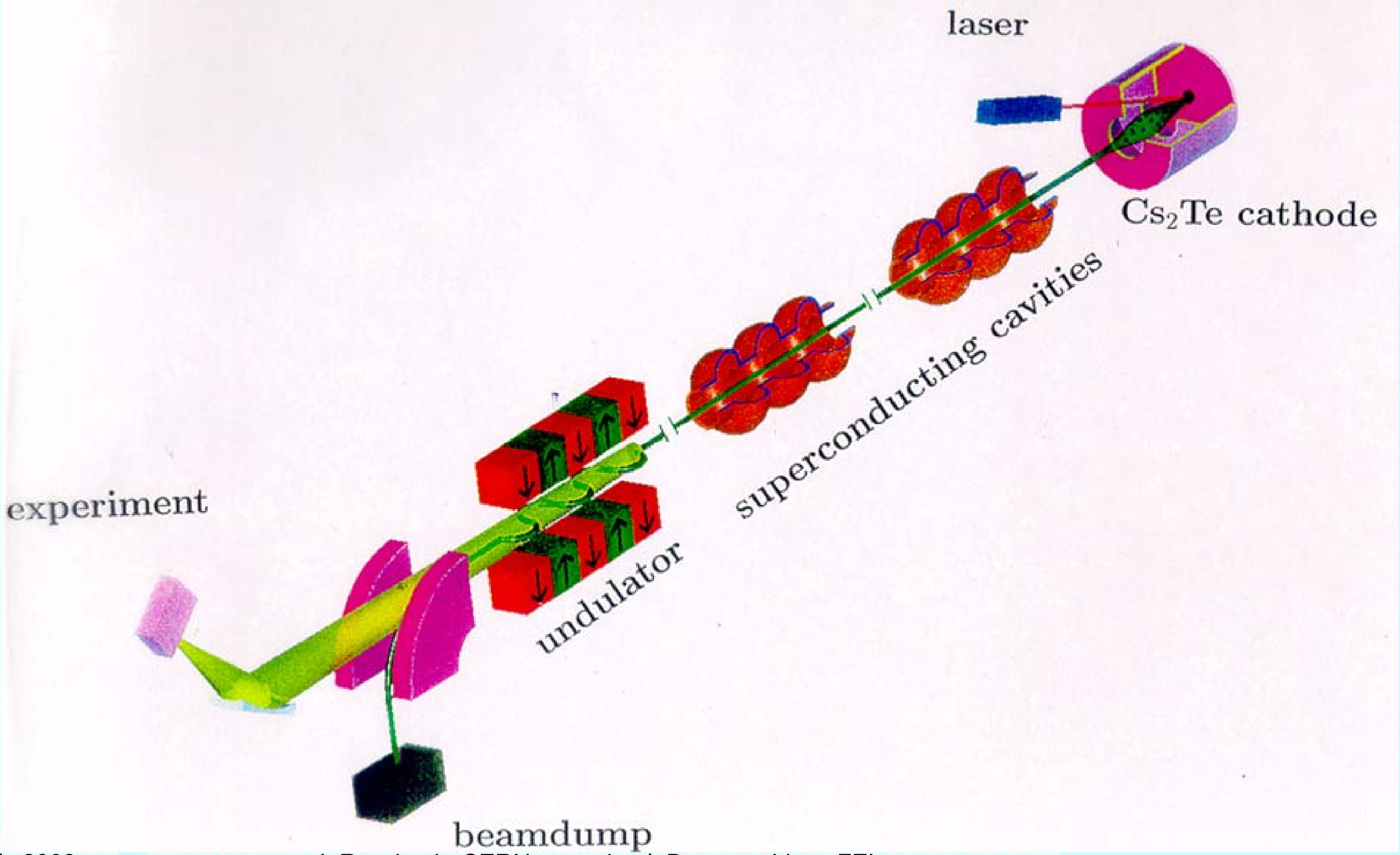
- assumes point-like charge q -

Free-Electron Laser:

Provides mechanism to concentrate electrons into bunches of length < l_{rad}

→ recovers factor N in power !

Schematic of a (single-pass) free electron laser (FEL)



Basic principle of a Free-Electron Laser (FEL)

A) Due to oscillation in undulator field, electron velocity receives (transverse) component parallel to electric field vector of e.m. wave

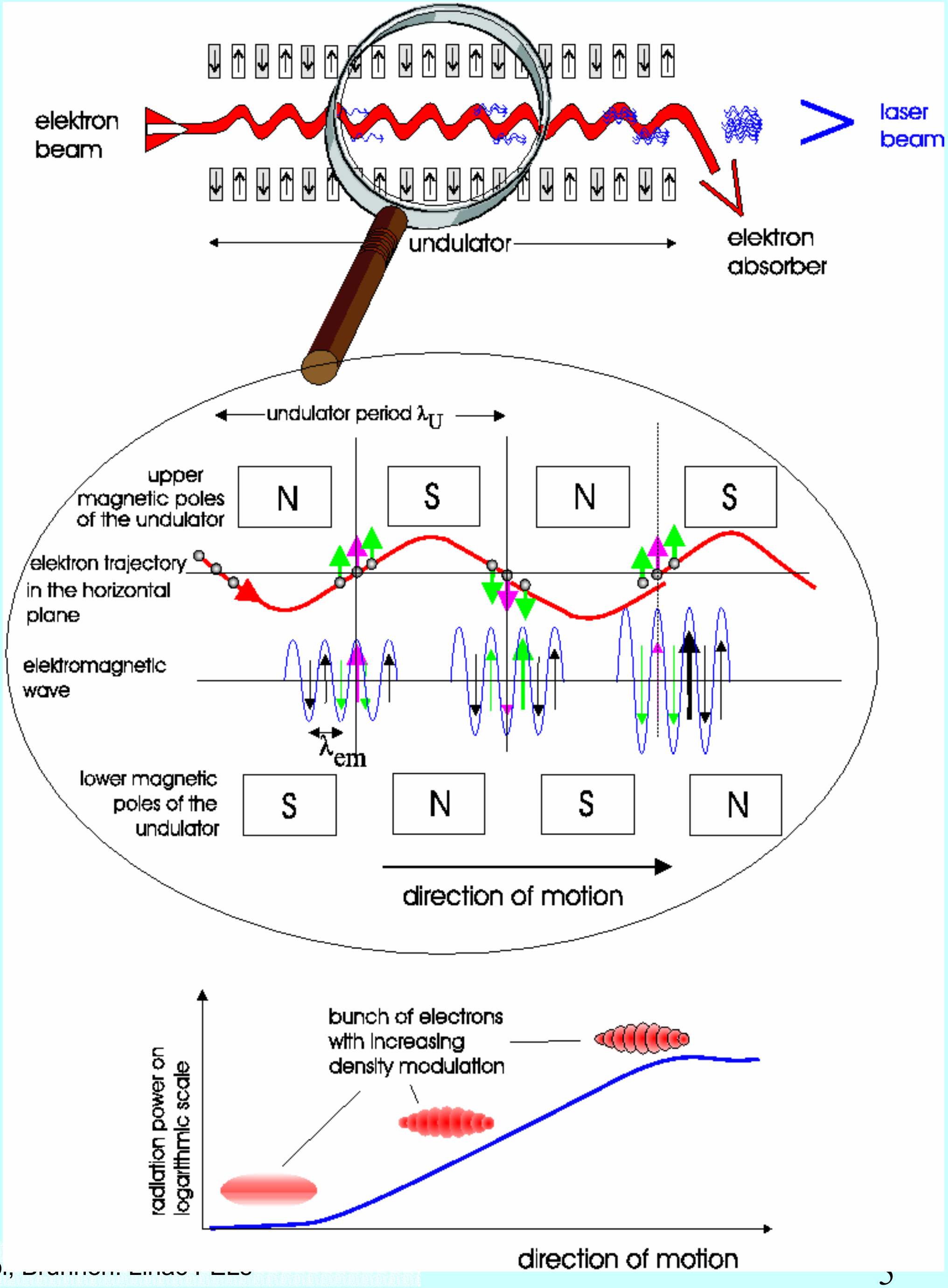
→ electrons may loose or gain energy, depending on relative phase between electron oscillation and e.m. wave.

→ For a certain combination of parameters, this effect is stationary within the electron bunch →

Resonance wavelength:

$$\lambda_{em} = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

B) Modulation of electron energy leads to longitudinal density modulation of electron bunch at the optical wavelength. Thus, radiation starts to scale ~ N^2 , eventually leading to exponential growth of rad. power.



Basic theory of free electron laser

1) Low gain approximation =

we assume an initial, external e.m. field that changes only slightly (few % in power) during FEL process

Step 1: electron motion in undulator

field of helical undulator with period λ_u: $\vec{B} = B\begin{pmatrix} -\sin(k_u z) \\ \cos(k_u z) \\ 0 \end{pmatrix} + O(r^2) \quad \left(\text{using } k_u = \frac{2\pi}{\lambda_u}\right)$

electron motion: $m\gamma\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = q\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \times \vec{B} = qB\begin{pmatrix} -\dot{z}\cdot\cos(k_u z) \\ -\dot{z}\cdot\sin(k_u z) \\ \dot{x}\cdot\cos(k_u z) + \dot{y}\cdot\sin(k_u z) \end{pmatrix}$

$\boxed{\exp(i\cdot x) = \cos x + i\cdot\sin x}$

One solution (prove it!) is a periodic, helical motion:

longitudinal motion: v_z=const., $z = v_z t = \beta_z ct$

(1)

transverse motion on a circle: $\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = c\frac{K}{\gamma}\begin{pmatrix} -\sin(k_u z) \\ \cos(k_u z) \end{pmatrix}$, or, using $w = x + iy$, $\dot{w} = ic\frac{K}{\gamma}\exp(ik_u z) \quad \rightarrow w = \frac{cK}{\gamma k_u v_z}\exp(ik_u z)$

$K = \frac{e\lambda_u B}{2\pi m_0 c}$ is called undulator parameter. It is typically $K \approx 1$ $\rightarrow$ opening angle of helical motion $\frac{v_\perp}{c} = \frac{K}{\gamma} \approx \frac{1}{\gamma}$

We can now determine $\beta_z = \frac{v_z}{c}$: $\quad \beta_z = \frac{1}{c}\sqrt{v^2 - \dot{x}^2 - \dot{y}^2} = \sqrt{\beta^2 - \left(\frac{K}{\gamma}\right)^2} = \sqrt{1 - \frac{1}{\gamma^2} - \left(\frac{K}{\gamma}\right)^2} \approx 1 - \frac{1}{2\gamma^2}\left(1 + K^2\right)$

External electromagnetic wave moving parallel to electron beam (i.e. in z-direction):

$$\vec{\mathbf{E}}_L = \mathbf{E}_0 \begin{pmatrix} \cos(\omega_L t - k_L z - \varphi_0) \\ \sin(\omega_L t - k_L z - \varphi_0) \\ 0 \end{pmatrix} \quad ; \quad \vec{B}_L = \frac{1}{c\omega_L}\dot{\vec{\mathbf{E}}}_L \quad ;$$

again: complex notation: $\mathbf{E}_L = \mathbf{E}_{L,x} + i\,\mathbf{E}_{L,y} \quad \rightarrow \quad \mathbf{E}_L = \mathbf{E}_0 \exp i(\omega_L t - k_L z - \varphi_0)$

Change of electron energy in presence of undulator and wave:

$$\frac{dE}{dt} = mc^2\frac{d\gamma}{dt} = \vec{v}\cdot\vec{F} = q\cdot\vec{v}\cdot\vec{\mathbf{E}}_L = q\left(\dot{x}\,\mathbf{E}_{L,x} + \dot{y}\,\mathbf{E}_{L,y}\right) = q\Re\left(\dot{w}\,\mathbf{E}_L^*\right) = -qc\frac{K\mathbf{E}_0}{\gamma}\sin\{(k_u + k_L)z - \omega_L t + \varphi_0\}$$

$$= -qc\frac{K\mathbf{E}_0}{\gamma}\sin\Psi \quad \text{with} \quad \boxed{\Psi = (k_u + k_L)z - \omega_L t + \varphi_0} = (k_u + k_L)z - \frac{\omega_L z}{\beta_z c} + \varphi_0 \quad (\text{using } z = v_z t = \beta_z ct)$$

$$\boxed{\frac{dE}{dz} = -\frac{q\mathbf{E}_0 K}{\gamma\beta_z}\sin\Psi} \qquad (2)$$

The energy dE is taken from or transferred to the radiation field. For most frequencies, dE/dt oscillates very rapidly. A significant energy transfer will only be accumulated if the phase difference Ψ between particle motion and e.m. wave stays constant with time.

$$\Psi = \text{const.} \rightarrow \frac{d\Psi}{dz} = (k_u + k_L) - \frac{\omega_L}{\beta_z c} = 0. \text{ Using } \omega_L = ck_L \text{ yields } k_u + k_L - \frac{k_L}{\beta_z} = 0$$

→ Resonance condition: $\lambda_L = \lambda_u \dfrac{1-\beta_z}{\beta_z} \approx \lambda_u(1-\beta_z) \approx \dfrac{\lambda_u}{2\gamma^2}(1+K^2)$

The same equation as for undulator radiation!

We have seen what happens <u>on</u> resonance.

For particles slightly <u>off</u> resonance energy, the phase Ψ will slip. By how much?

In $\Psi = (k_u + k_L) z - \frac{\omega_L z}{\beta_z c} + \varphi_0$ only $\beta_z \approx 1 - \frac{1}{2\gamma^2}(1+K^2)$ depends on energy. Writing $\gamma = \gamma_{res} + \Delta\gamma$ we get

$$\frac{d\Psi}{dz} = (k_u + k_L) - \frac{\omega_L}{c\left(1 - \frac{1+K^2}{2(\gamma_{res}+\Delta\gamma)^2}\right)} \approx k_u + k_L - \frac{\omega_L}{\beta_z(\gamma_{res})\cdot c} + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_{res}^3}\Delta\gamma = \frac{\omega_L}{c}\frac{1+K^2}{\gamma_{res}^3}\Delta\gamma = k_u \frac{2}{\gamma_{res}}\Delta\gamma \qquad (3)$$

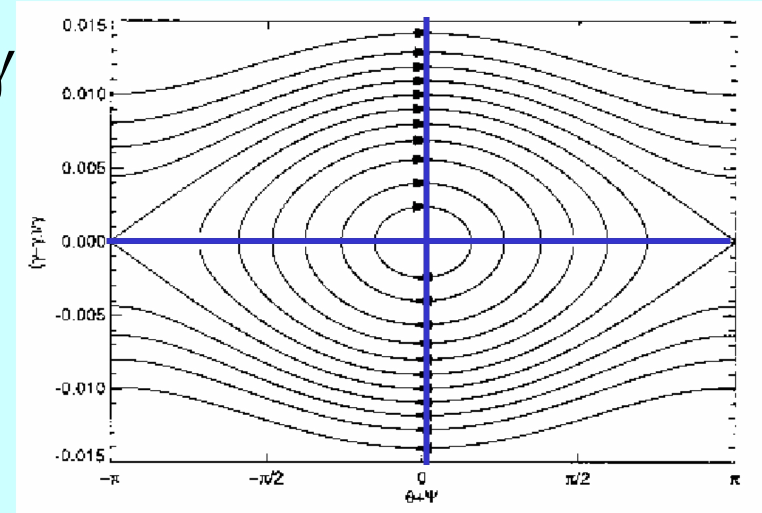
Deriving once more with respect to z yields: $\frac{d^2\Psi}{dz^2} = k_u \frac{2}{\gamma_{res}}\frac{d\gamma}{dz}$. Now using $\frac{d\gamma}{dz} = -\frac{q\mathbf{E}_0 K}{m_0 c^2 \gamma \beta_z}\sin\Psi$ (see Eq. 2)

we get

$$\boxed{\frac{d^2\Psi}{dz^2} = -\frac{2q}{m_0 c^2}\frac{\mathbf{E}_0 K k_u}{\gamma_{res}^2 \beta_z}\sin\Psi = -\Omega^2 \sin\Psi \quad \text{with} \quad \Omega^2 = \frac{2q}{m_0 c^2}\frac{\mathbf{E}_0 K k_u}{\gamma_{res}^2 \beta_z}}$$

This is a pendulum equation in the $\Delta\gamma$ - Ψ phase space:

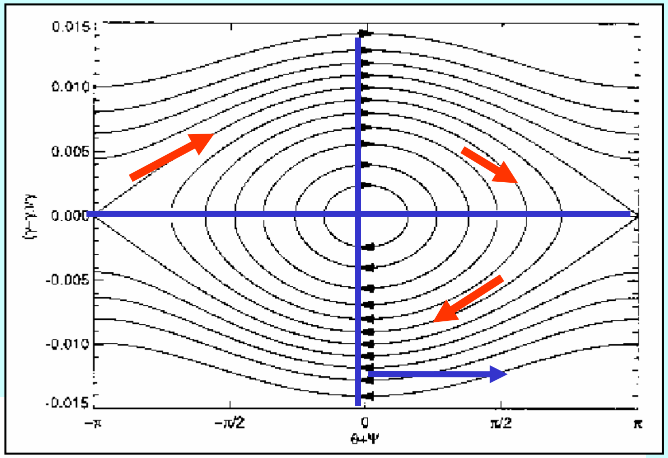
electrons with little deviation from synchronous phase or from resonance energy perform periodic oscillation.



Identical to synchrotron oscillation,
but „bucket“ length is now the optical wavength!
Particles within <u>separatrix</u> get bunched

Gain (or loss) in field energy per undulator passage,depending on where to start in phase space

$\Delta\gamma = \gamma - \gamma_{res}$



Ψ

Interpretation of separatrix:

Finding first integral:

Multiply $\Psi'' = -\Omega^2 \sin\Psi$ by $2\Psi'$ on both sides and use $2\Psi'\Psi'' = \frac{d}{dz}(\Psi')^2 \rightarrow$

$$\int 2\Psi'\Psi'' dz = (\Psi')^2 = 2\int -\Omega^2 \sin\Psi\,\Psi' dz = 2\int -\Omega^2 \sin\Psi\, d\Psi = 2\Omega^2 \cos\Psi + \text{const}$$

Eq.(3)

With $\Psi' = k_u \frac{2}{\gamma_{res}} \Delta\gamma$ this reads $\left(k_u \frac{2}{\gamma_{res}} \Delta\gamma\right)^2 = 2\Omega^2 \cos\Psi + \text{const}$, thus

$$\boxed{(\Delta\gamma)^2 - \frac{q\mathbf{E}_0 K}{m_0 c^2 k_u \beta_z} \cos\Psi = \text{const.}}$$ with const. determined by initial conditions.

Case 1: $\quad \text{const.} < \frac{q\mathbf{E}_0 K}{m_0 c^2 k_u \beta_z} \rightarrow \Delta\gamma = \sqrt{\text{const.} + \frac{q\mathbf{E}_0 K}{m_0 c^2 k_u \beta_z} \cos\Psi}$

has real solution only for limited range of phases $\rightarrow$ rotation within separatrix

Case 2: $\quad \text{const.} > \frac{q\mathbf{E}_0 K}{m_0 c^2 k_u \beta_z} \rightarrow \Delta\gamma = \sqrt{\text{const.} + \frac{q\mathbf{E}_0 K}{m_0 c^2 k_u \beta_z} \cos\Psi}$

all phases possible, but $\Delta\gamma = 0$ cannot be reached $\rightarrow$ libration outside separatrix

Separatrix: $\quad \text{const.} = \frac{q\mathbf{E}_0 K}{m_0 c^2 k_u \beta_z} \rightarrow (\Delta\gamma)^2 = \frac{q\mathbf{E}_0 K}{m_0 c^2 k_u \beta_z}(1 + \cos\Psi) \rightarrow$ height of separatrix is $\Delta\gamma_{max} - \Delta\gamma_{min} = \sqrt{\frac{2q\mathbf{E}_0 K}{m_0 c^2 k_u \beta_z}}$

Gain calculation:

$$G_i = \frac{\text{gain of field energy produced by electron i}}{\text{total field energy}} = \frac{-mc^2\left(\gamma_i(z=L_u)-\gamma_i(0)\right)}{\frac{\varepsilon_0}{2}\mathbf{E}_0^2\cdot V}$$

$$= \frac{-mc^2\gamma_{res}\left(\Psi_i'(z=L_u)-\Psi_i'(0)\right)}{\varepsilon_0\,\mathbf{E}_0^2\cdot Vk_u} \quad \text{(remember Eq. 3: } \Delta\gamma = \frac{\gamma_{res}}{2k_u}\Psi'\text{)}$$

$\rightarrow$ requires solution of pendulum equation for $\Psi(z)$.

$$\frac{d^2\Psi}{dz^2} = \Psi'' = -\Omega^2\,\mathbf{sin}\,\Psi$$

Solution by iteration: Ansatz: $\Psi(z)=\Psi_0+\Psi_0'\cdot z+\delta\Psi(z)$,where $\delta\Psi(z)$ is the higher order term.

Step 1: $\delta\Psi(z)=0$

$$\Psi'^{(1)} = \Psi_0' - \Omega^2\int_0^z \mathbf{sin}\left(\Psi_0+\Psi_0'\cdot\tilde z\right)d\tilde z = \Psi_0' - \Omega^2\frac{1}{\Psi_0'}\left[\mathbf{cos}\,\Psi_0 - \mathbf{cos}\left(\Psi_0+\Psi_0'\cdot z\right)\right]$$

Average gain of the total beam (N_p particles): $G=\sum_i G_i = \left\langle G_i\right\rangle_{\Psi_{i0}}\cdot N_p$

$$\left\langle \Psi'^{(1)}-\Psi_0'\right\rangle_{\Psi_0} = \left\langle -\Omega^2\frac{1}{\Psi_0'}\left[\mathbf{cos}\,\Psi_0-\mathbf{cos}\left(\Psi_0+\Psi_0'\cdot z\right)\right]\right\rangle_{\Psi_0} = 0$$ In first order, avg. gain is 0.

Step 2 (use result of Step 1): $$\delta\Psi(z) = -\Omega^2\frac{1}{\Psi_0'}\int_0^z\left[\mathbf{cos}\,\Psi_0-\mathbf{cos}\left(\Psi_0+\Psi_0'\cdot\tilde z\right)\right]d\tilde z = -\Omega^2\frac{1}{\Psi_0'}\left[z\cdot\mathbf{sin}\,\Psi_0-\frac{1}{\Psi_0'}\mathbf{sin}\left(\Psi_0+\Psi_0'\cdot z\right)+\frac{1}{\Psi_0'}\mathbf{sin}\,\Psi_0\right]$$

$$\Psi'^{(2)}-\Psi_0' = -\Omega^2\int_0^z\mathbf{sin}\left(\Psi_0+\Psi_0'\cdot\tilde z+\delta\Psi(\tilde z)\right)d\tilde z \approx -\Omega^2\int_0^z\left[\mathbf{sin}\left(\Psi_0+\Psi_0'\cdot\tilde z\right)+\mathbf{cos}\left(\Psi_0+\Psi_0'\cdot\tilde z\right)\delta\Psi(\tilde z)\right]d\tilde z \quad \text{(note: } \delta\Psi(\tilde z)\ll\pi\text{)}$$

Plugging in $\delta\Psi(\tilde z)$ and averaging yields:

$$\left\langle\Psi'^{(2)}-\Psi_0'\right\rangle_{\Psi_0} = \Omega^4\frac{1}{\Psi_0'^2}\left\langle\int_0^{Lu}\left[\tilde z\Psi_0'\,\mathbf{cos}^2\,\Psi_0\cdot\mathbf{cos}\left(\Psi_0'\cdot\tilde z\right)-\mathbf{sin}^2\,\Psi_0\cdot\mathbf{sin}\left(\Psi_0'\cdot\tilde z\right)\right]d\tilde z\right\rangle \quad \text{(having used } \left\langle\mathbf{cos}(\alpha+\beta)\,\mathbf{sin}(\alpha+\beta)\right\rangle_\alpha = 0\text{)}$$

$$\left\langle \Psi'^{(2)} - \Psi'_0 \right\rangle_{\Psi_0} = \frac{\Omega^4}{2\Psi_0'^2} \int_0^{Lu} \left[\tilde{z}\Psi'_0 \cos\left(\Psi'_0 \cdot \tilde{z}\right) - \sin\left(\Psi'_0 \cdot \tilde{z}\right) \right] d\tilde{z}$$

$$= \frac{\Omega^4}{2\Psi_0'^2} \left[L_u \sin\left(\Psi'_0 L_u\right) - 2\int_0^{Lu} \sin\left(\Psi'_0 \cdot \tilde{z}\right) d\tilde{z} \right] \quad \text{(by partial integration)}$$

$$= \frac{\Omega^4}{2\Psi_0'^2} \left[L_u \sin\left(\Psi'_0 L_u\right) - 2\int_0^{Lu} \sin\left(\Psi'_0 \cdot \tilde{z}\right) d\tilde{z} \right] = \frac{\Omega^4}{2\Psi_0'^2} \left[L_u \sin\left(\Psi'_0 L_u\right) + 2\frac{1}{\Psi'_0}\left(\cos\left(\Psi'_0 L_u\right) - 1\right) \right]$$

$$= \Omega^4 \left(\frac{1}{\Psi_0'^2} L_u \sin\frac{\Psi'_0 L_u}{2} \cos\frac{\Psi'_0 L_u}{2} - \frac{2}{\Psi_0'^3} \sin^2\frac{\Psi'_0 L_u}{2} \right) = \Omega^4 \frac{d}{d\Psi'_0} \left\{ \frac{\sin^2 \frac{\Psi'_0 L_u}{2}}{\Psi_0'^2} \right\} \quad \text{(prove it!)}$$

Remember: $\Psi'_0 = \dfrac{2k_u}{\gamma_{res}} \Delta\gamma$ is determined by the initial off-resonance energy

$$G = N_p \frac{-mc^2\gamma_{res}\left(\Psi'_i\left(z = L_u\right) - \Psi'_i(0)\right)}{\varepsilon_0 \mathbf{E}_0^2 \cdot V k_u} = -\frac{mc^2\gamma_{res} N_p}{\varepsilon_0 E_0^2 \cdot V k_u} \Omega^4 \frac{d}{d\Psi'_0} \left\{ \frac{\sin^2 \frac{\Psi'_0 \cdot L_u}{2}}{\Psi_0'^2} \right\}$$

Now use $\boxed{\xi = \frac{\Psi'_0 \cdot L_u}{2},\ n_p = \frac{N_p}{V},\ L_u = N_u \lambda_u}$

$$\rightarrow \boxed{G = -\frac{\pi q^2 N_u^3 \lambda_u^2 K^2 n_p}{\varepsilon_0 mc^2 \gamma^3} \frac{d}{d\xi} \left\{ \frac{\sin^2 \xi}{\xi} \right\}}$$

(assumptions: helical undulator, perfect overlap electron/radiation field)

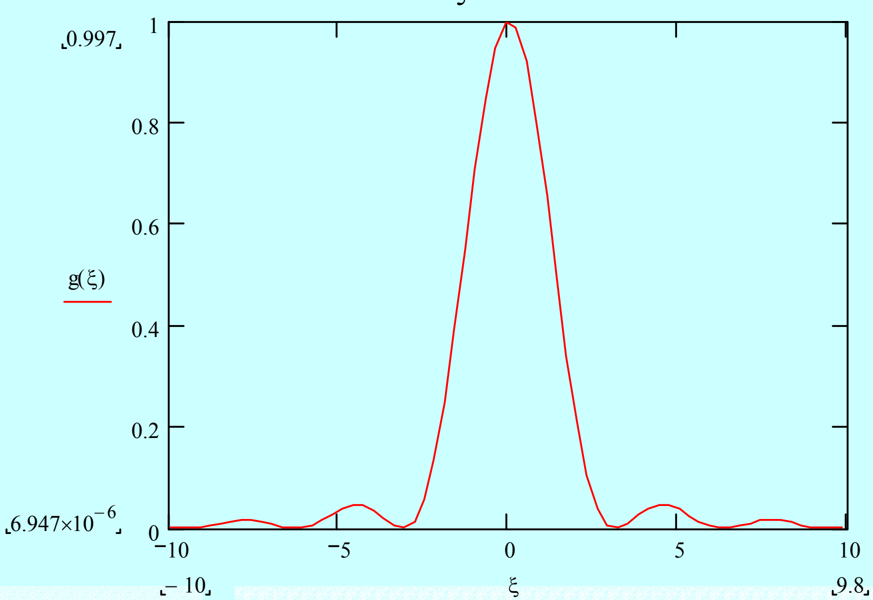
To first order in the iteration, there is no net gain (G=0), because motion in phase space is (almost) symmetric: As many particles move up as down.
In second order it is seen however that, for positive $\Delta\gamma$, the motion of particles with positive phase goes more rapidly downwards than the motion of the others goes upwards.

Using $\frac{\Delta\omega}{2\omega_{res}} = \frac{\Delta\gamma}{\gamma_{res}}$, we can write: $\quad \text{Gain} \propto -\frac{d}{d\gamma} \frac{\sin^2\left(2\pi N_u \frac{\Delta\gamma}{\gamma_{res}}\right)}{(\Delta\gamma)^2} \propto -\frac{d}{d\omega} \frac{\sin^2\left(\pi N_u \frac{\Delta\omega}{\omega_{res}}\right)}{(\Delta\omega)^2}$

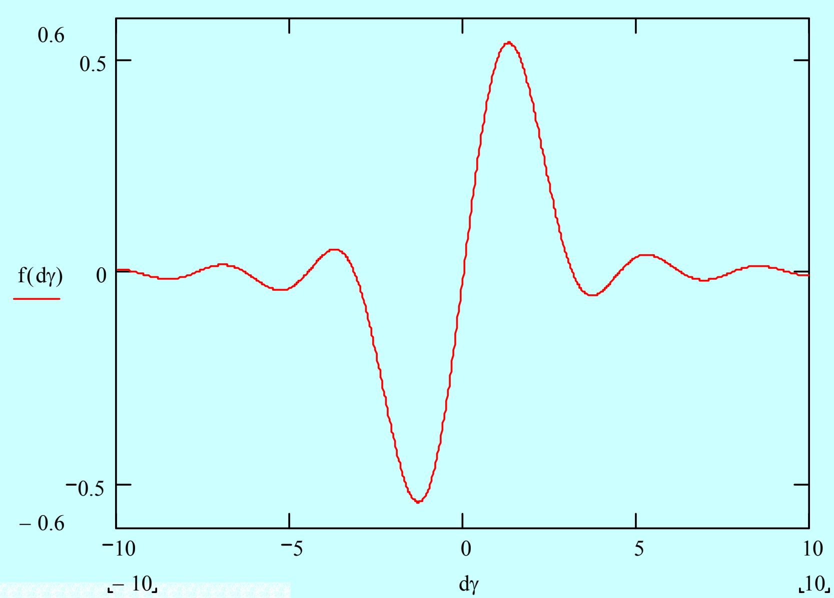
The line shape function of low gain FEL emission is the derivative of the line shape of spontaneous undulator radiation

Madey-Theorem

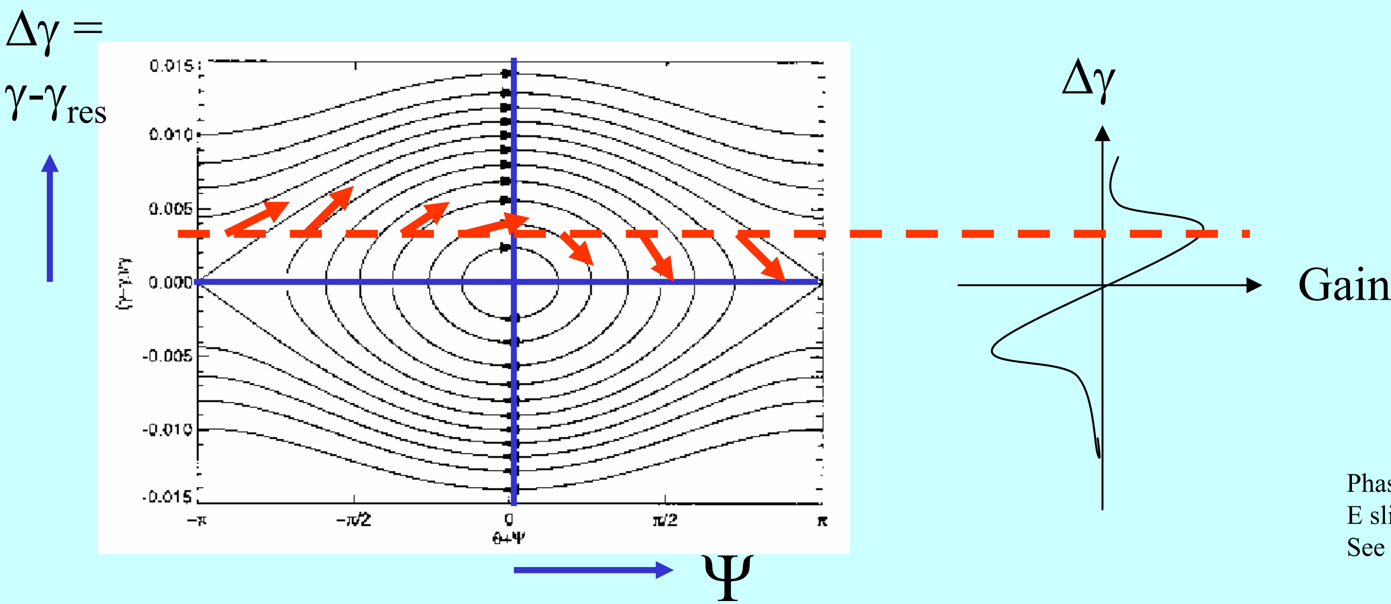
$$\frac{\sin^2\xi}{\xi^2}$$



$$-\frac{d}{d\xi}\frac{\sin^2\xi}{\xi^2} = \frac{1}{\xi^3}\left(1-\cos(2\xi)-\xi\sin(2\xi)\right)$$



e.m. field is amplified if electron energy is slightly above resonance



Phase space simulation of low gain FEL
E slightly above resonance →
See electron bunch losing energy in averave

Radiation energy produced per undulator passage is $\Delta E = G \cdot E_i$ (field energy before passage of undulator).

Note that $G \propto N_p$, like power of spontaneous radiation!

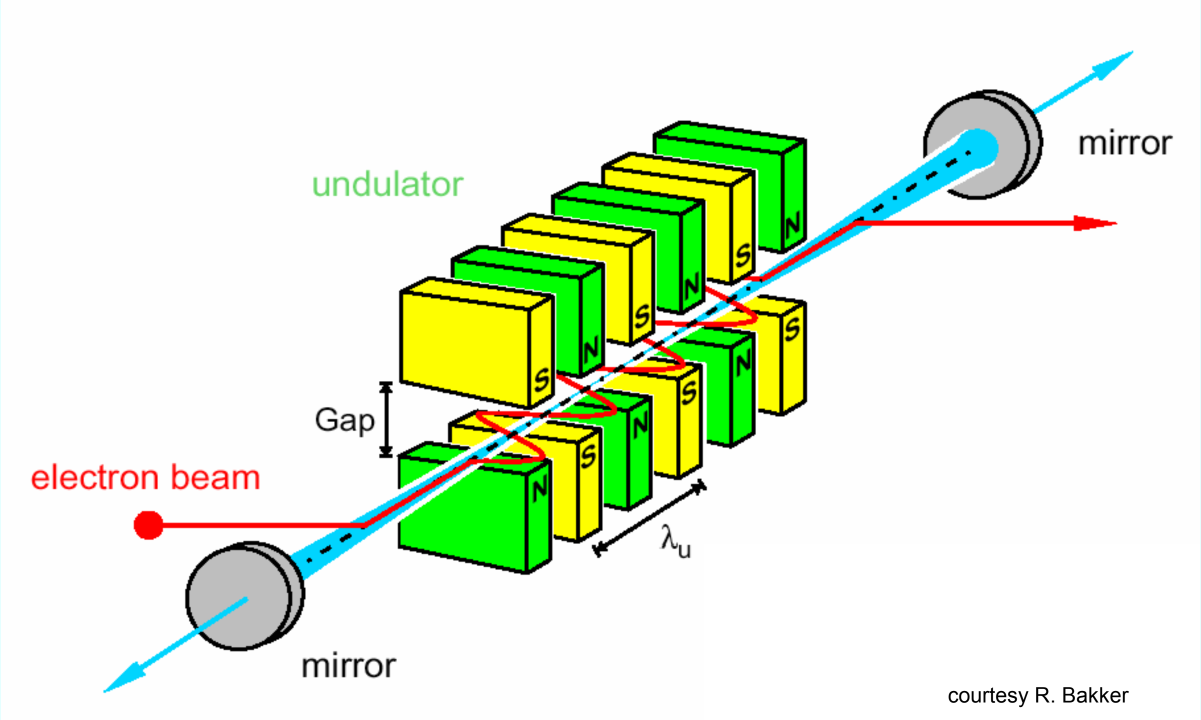
BUT: 1. ΔE adds to spontaneous radiation

2. $\Delta E \propto E_i$ i.e. electrons are stimulated to emit due to presence of E_i

3. ΔE may become arbitrarily large if only E_i is large enough

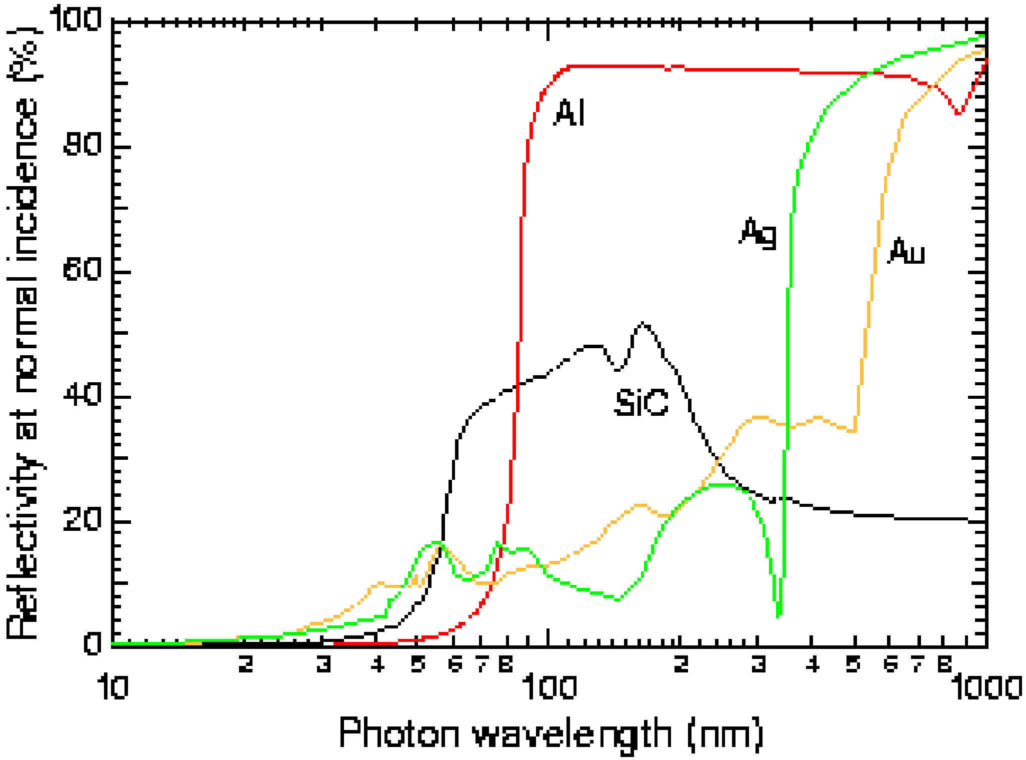
For applications, a few % power gain (i.e. a low gain FEL) don't seem to be of interest. However, with a pair of mirrors, one can multiply the gain, if on each round trip of radiation there is a fresh electron bunch available.

After N round trips, $G_{total} = G^N$, and the e.m. field is so strong that microbunching is almost perfect. → **saturation**



Only few % of radiation intensity is extracted per electron passage (mirror reflectivity) to keep stored field high

Very nice scheme.
But what if we want wavelength < approx. 200nm where no good mirrors exist?



Reflectivity of most surfaces at normal incidence drops drastically at wavelengths below 100 – 200 nm.

2) High gain FEL =

we take into account that the initial, external e.m. field changes during FEL process

Closely following Ref. 3 Saldin et al.

Maxwell equations

$$\left(\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right)\vec{A} = -\mu_0 \vec{I}$$

$$\left(\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right)\phi = -\frac{\rho}{\varepsilon_0}$$

$$\rightarrow \vec{\mathbf{E}}_L = -\mathrm{grad}\phi - \frac{\partial \vec{A}}{\partial t}$$

e.m. fields are generated by charges and currents.

For a purely transverse e.m. field we get:

$$\boxed{\frac{\partial^2 \mathbf{E}_\perp}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}_\perp}{\partial t^2} = \mu_0 \frac{\partial j_\perp}{\partial t} + \frac{1}{\varepsilon_0}\nabla_\perp \rho} \qquad (4)$$

The term $4\pi\nabla_\perp\rho$ can be neglected, because its contribution to radiation generation is small in all practical cases see Ref. 5, Chapter 4.1.

A) What can we say about the e.m. field $\mathbf{E}_\perp$?

1. Again: Only transverse components → complex notation:

$$\vec{\mathbf{E}}_\perp = \mathbf{E}_0 \begin{pmatrix} \cos(\omega_L t - k_L z - \varphi_0) \\ \sin(\omega_L t - k_L z - \varphi_0) \\ 0 \end{pmatrix} \;;\; \vec{B}_\perp = \frac{1}{c\omega_L}\dot{\vec{\mathbf{E}}}_\perp \;;\; \mathbf{E} = \mathbf{E}_{\perp,x} + i\mathbf{E}_{\perp,y} \quad \rightarrow \quad \mathbf{E} = \mathbf{E}_0 \exp i(\omega_L t - k_L z - \varphi_0)$$

2. But now: Amplitude $\mathbf{E}_0$ and phase φ_0 (which we call ψ_E now) may vary with z (though slowly compared to ω_L) →
$\mathbf{E} = \mathbf{E}_0(z)\exp i(\omega_L t - k_L z - \psi_E) = \mathbf{E} = \tilde{\mathbf{E}}_0^*(z)\exp i(\omega_L t - k_L z)$ with the slow part $\tilde{\mathbf{E}}_0(z) = \mathbf{E}_0(z)\exp i\psi_E$ and $\tilde{\mathbf{E}}_0^*(z)$ its c.c.

Eq. 4 re-written for $\mathbf{E}_{\perp,x} + i\mathbf{E}_{\perp,y} = \tilde{\mathbf{E}}_0^*(z)\exp i(\omega_L t - k_L z)$ reads:

$$\frac{\partial^2(\mathbf{E}_{\perp,x}+i\mathbf{E}_{\perp,y})}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2(\mathbf{E}_{\perp,x}+i\mathbf{E}_{\perp,y})}{\partial t^2}=\frac{\partial^2}{\partial z^2}\tilde{\mathbf{E}}_0^*(z)\,\mathbf{exp}\,i(\omega_L t-k_L z)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\tilde{\mathbf{E}}_0^*(z)\,\mathbf{exp}\,i(\omega_L t-k_L z)=$$

$$\frac{\partial}{\partial z}\left[\tilde{\mathbf{E}}_0^*(z)\frac{\partial}{\partial z}\mathbf{exp}\,i(\omega_L t-k_L z)+\mathbf{exp}\,i(\omega_L t-k_L z)\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z)\right]-\frac{1}{c^2}\frac{\partial}{\partial t}\left[\tilde{\mathbf{E}}_0^*(z)\frac{\partial}{\partial t}\mathbf{exp}\,i(\omega_L t-k_L z)+\mathbf{exp}\,i(\omega_L t-k_L z)\cancel{\frac{\partial}{\partial t}\tilde{\mathbf{E}}_0^*(z)}\right]=$$

$$\frac{\partial}{\partial z}(\tilde{\mathbf{E}}_0^*(z))\cdot(-ik_L)\,\mathbf{exp}\,i(\omega_L t-k_L z)+\tilde{\mathbf{E}}_0^*(z)(-k_L^2)\,\mathbf{exp}\,i(\omega_L t-k_L z)+(-ik_L)\,\mathbf{exp}\,i(\omega_L t-k_L z)\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z)+\mathbf{exp}\,i(\omega_L t-k_L z)\frac{\partial^2}{\partial z^2}\tilde{\mathbf{E}}_0^*(z)$$

$$\cancel{-\frac{1}{c^2}\frac{\partial}{\partial t}(\tilde{\mathbf{E}}_0^*(z))\cdot i\omega_L\,\mathbf{exp}\,i(\omega_L t-k_L z)}-\frac{1}{c^2}\tilde{\mathbf{E}}_0^*(z)(-\omega_L^2)\,\mathbf{exp}\,i(\omega_L t-k_L z)+0=$$ (neglecting second derivative of slowly varying $\tilde{\mathbf{E}}_0^*(z)$)

$$-\left[2ik_L\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z)\right]\mathbf{exp}\,i(\omega_L t-k_L z)-k_L^2\,\tilde{\mathbf{E}}_0^*(z)\,\mathbf{exp}\,i(\omega_L t-k_L z)-\frac{(-\omega_L^2)}{c^2}\tilde{\mathbf{E}}_0^*(z)\,\mathbf{exp}\,i(\omega_L t-k_L z)=$$ (using $\frac{\omega_L}{c}=k_L$) =

$$=\boxed{-\left[2ik_L\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z)\right]\mathbf{exp}\,i(\omega_L t-k_L z)=\mu_0\frac{\partial(j_{\perp,x}+i\cdot j_{\perp,y})}{\partial t}=\ i\mu_0\frac{K}{\gamma}\mathbf{exp}(ik_u z)\ \frac{\partial j_z}{\partial t}}\qquad(5)$$

Note: Because of $\vec{j}=\rho\vec{v}$, we can write $(j_x+i\cdot j_y)=(v_x+i\cdot v_y)\frac{j_z}{v_z}=$ (see Eq.1) $=i\frac{c}{v_z}\frac{K}{\gamma}\mathbf{exp}(ik_u z)j_z\approx\ i\frac{K}{\gamma}\mathbf{exp}(ik_u z)\ j_z$

Collecting the rapidly oscillation term rewrites eq. 5, using $\Psi=(k_u+k_L)z-\omega_L t$:

$$\boxed{-\left[2k_L\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z)\right]=\ \mu_0\frac{K}{\gamma}\frac{\partial j_z}{\partial t}\mathbf{exp}\,i(k_u z+k_L z-\omega_L t)=\mu_0\frac{K}{\gamma}\frac{\partial j_z}{\partial t}\mathbf{exp}\,i\Psi}\qquad(6)$$

Electromagnetic field amplitude is generated by time dependent current

Major steps to derive the 3rd order Diff. Eq. for High Gain FEL

Maxwell Eq.

purely transv.

slowly varying ampl. & phase (6)

Kinematic Eqs. in γ/Ψ

Vlasov ⇔ Liouville

Space charge

Evolution of current density (8)

Maxwell Eq.:
Space charge $E_z \rightarrow j_1 \rightarrow E_{trans}$

Eq. (10) ff:
self-consistent integro-differential Eq.
For arbitrary inital energy distribution

Cold beam

3rd order differential Eq.

▭	equation
◯	approximation step

B) What can we say about the current density $j_\perp$?

$j_\perp$ is determined by initial charge distribution and its evolution in presence of e.m. and undulator field. We know that electron dynamics is governed by Hamiltonian

$$H(p_z, z, t) = \left[(p_z c - qA_z)^2 + q^2 (A_\perp + A_u)^2 + m^2 c^4 \right]^{1/2} + q\phi$$

A_u describing the undulator field, and A_z, ϕ the space charge.

Applying a canonical transformation we can change from canonical pair of coordinates, z/p_z to Ψ / γ (actually $\frac{\omega_L}{m_0 c^2} \cdot \Psi / \gamma$, but $\frac{\omega_L}{m_0 c^2}$ is constant), $\Psi = (k_u + k_L) z - \omega_L t$, $\gamma m_0 c^2$ kinetic energy of electron.

A consequence of Hamiltonian mechanics is Liouville's Theorem, i.e. phase space density along the particle's motion is constant. In coordinates z, γ, Ψ this reads:

$$\boxed{\frac{df}{dz} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \Psi}\frac{\partial \Psi}{\partial z} + \frac{\partial f}{\partial \gamma}\frac{\partial \gamma}{\partial z} = 0 \quad \text{(also called Vlasov equation)}} \qquad (7)$$

We have seen before (Eq. 2) that $\boxed{\frac{d\gamma}{dz} = -\frac{q\mathbf{E}_0 K}{m_0 c^2 \gamma_0 \beta_z} \mathbf{sin}(\Psi + \psi_E) + \frac{q\mathbf{E}_z}{m_0 c^2}}$ - but now, the electric field strength $\mathbf{E}_0$ isn't constant any more, and we have included the energy gain due to space charge field $\mathbf{E}_z$.

Also (Eq.3) $\frac{d\Psi}{dz} = k_u + k_L - \frac{\omega_L}{\beta_z(\gamma) \cdot c} + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma$, but now we allow for a small deviation of γ from γ_{res} described by the detuning parameter $C(\gamma) = k_u + k_L - \frac{\omega_L}{\beta_z(\gamma) \cdot c}$, i.e. $C(\gamma_{res}) = 0$. $\rightarrow \boxed{\frac{d\Psi}{dz} = C + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma}$

$$\boxed{\text{Eq. (7) now reads (from now on } \beta_z \approx 1\text{):} \quad \frac{\partial f}{\partial z} + \left(C + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma \right)\frac{\partial f}{\partial \Psi} + \left(-\frac{q\mathbf{E}_0 K}{m_0 c^2 \gamma_0} \mathbf{sin}(\Psi + \psi_E) + \frac{q\mathbf{E}_z}{m_0 c^2} \right)\frac{\partial f}{\partial \gamma} = 0} \qquad (8)$$

Ansatz: $f(z,\gamma,\Psi) = f_0(\gamma) + f_1(z,\gamma)\cos(\Psi+\psi_0)$ i.e. we assume a density modulation at the optical wavelength, growing with z (in a way to be calculated).

$$f_1(z,\gamma)\cos(\Psi+\psi_0) = \tfrac{f_1}{2}e^{i(\Psi+\psi_0)} + \tfrac{f_1}{2}e^{-i(\Psi+\psi_0)} = \tfrac{f_1}{2}e^{i\psi_0}e^{i\Psi} + C.C. = \tilde{f}_1(z,\gamma)e^{i\Psi} + C.C.$$

Similar for space charge force: $\mathbf{E}_z = E_z(z)\cos(\Psi+\psi_s) = \tilde{\mathbf{E}}_z(z)e^{i\Psi} + C.C.$

From here on, we keep all expressions on space charge in blue color.

$$\frac{\partial f}{\partial z} + \left(C + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma\right)\frac{\partial f}{\partial \Psi} + \left(-\frac{q\mathbf{E}_0 K}{m_0c^2\gamma_0}\sin(\Psi+\psi_E) + \frac{q\mathbf{E}_z}{m_0c^2}\right)\frac{\partial f}{\partial\gamma} =$$

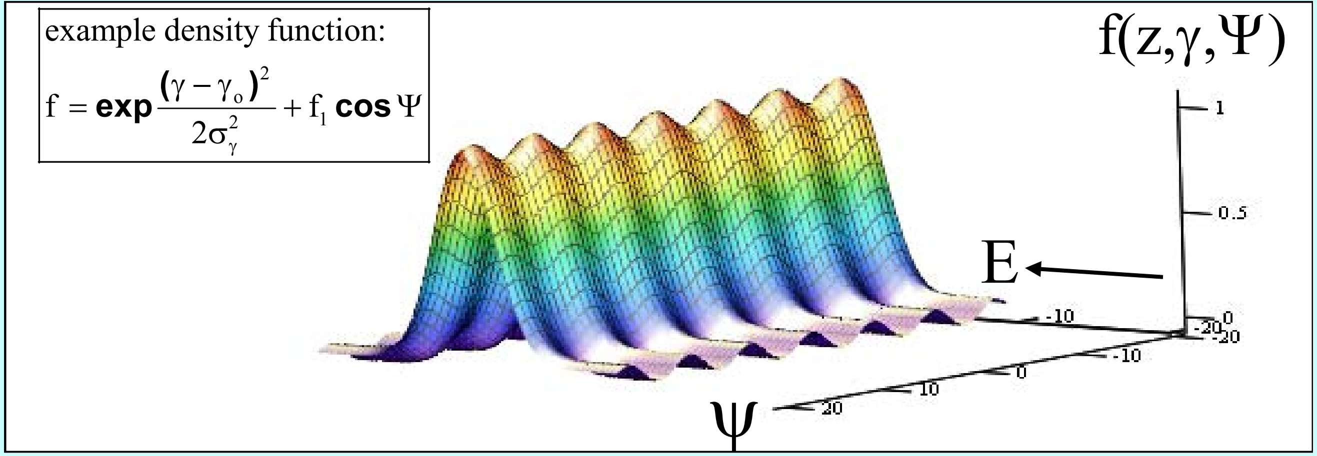
$$\frac{\partial\tilde{f}_1}{\partial z}e^{i\Psi} + \frac{\partial\tilde{f}_1^*}{\partial z}e^{-i\Psi} + \left(C + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma\right)\left(i\tilde{f}_1e^{i\Psi} - i\tilde{f}_1^*e^{-i\Psi}\right) +$$

$$+\left[\frac{q\mathbf{E}_0 K}{m_0c^2\gamma_0}\frac{i}{2}\left(e^{i(\Psi+\psi_E)} - e^{-i(\Psi+\psi_E)}\right) + \frac{q}{m_0c^2}\left(\tilde{\mathbf{E}}_z e^{i\Psi} + \tilde{\mathbf{E}}_z^* e^{-i\Psi}\right)\right]\left(\frac{\partial f_0}{\partial\gamma} + \frac{\partial\tilde{f}_1}{\partial\gamma}e^{i\Psi} + \frac{\partial\tilde{f}_1^*}{\partial\gamma}e^{-i\Psi}\right) = \qquad (\text{use } \tilde{\mathbf{E}}_0 = \mathbf{E}_0 e^{i\psi_E})$$

$$e^{i\Psi}\left\{\frac{\partial\tilde{f}_1}{\partial z} + i\left(C + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma\right)\tilde{f}_1 + \left(i\frac{q\tilde{\mathbf{E}}_0 K}{2m_0c^2\gamma_0} + \frac{q}{m_0c^2}\tilde{\mathbf{E}}_z\right)\frac{\partial f_0}{\partial\gamma} + \cancel{\left[i\frac{q\mathbf{E}_0 K}{2m_0c^2\gamma_0}\left(e^{i(\Psi+\psi_E)} - e^{-i(\Psi+\psi_E)}\right) + \frac{q}{m_0c^2}\left(\tilde{\mathbf{E}}_z e^{i\Psi} + \tilde{\mathbf{E}}_z^* e^{-i\Psi}\right)\right]\frac{\partial\tilde{f}_1}{\partial\gamma}}\right\} +$$

$$+c.c. = 0$$

Thus, $\{\ \} = 0 \approx \boxed{\frac{\partial\tilde{f}_1}{\partial z} + i\left(C + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma\right)\tilde{f}_1 + \left(i\frac{q\tilde{\mathbf{E}}_0 K}{2m_0c^2\gamma_0} + \frac{q}{m_0c^2}\tilde{\mathbf{E}}_z\right)\frac{\partial f_0}{\partial\gamma} = 0}$



$$\frac{\partial \tilde f_1(z,\gamma)}{\partial z} + i\left(C+\frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma\right)\tilde f_1(z,\gamma) + \left[i\frac{q\tilde{\mathbf{E}}_0(z)K}{2m_0c^2\gamma_0} + \frac{q}{m_0c^2}\tilde{\mathbf{E}}_z(z)\right]\frac{\partial f_0(\gamma)}{\partial\gamma} = 0$$ is a diff. eq. in z of the type $\frac{df(z)}{dz} + i\alpha f(z) = g(z)$

A solution is $f(z)=\int_0^z g(z')\,\mathbf{exp}\left[i\alpha(z'-z)\right]dz'$ (prove it!)

Thus: $$\tilde f_1(z,\gamma) = \int_0^z dz'\left[i\frac{q\tilde{\mathbf{E}}_0(z')K}{2m_0c^2\gamma_0} + \frac{q}{m_0c^2}\tilde{\mathbf{E}}_z(z')\right]\frac{\partial f_0(\gamma)}{\partial\gamma}\,\mathbf{exp}\left[i\left(C+\frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma\right)(z'-z)\right]\;;\quad \tilde f_1^*(z,\gamma) = \text{c.c.}$$ (9)

We can now calculate the current density

$$j_z = \rho v_z \approx \rho c = qc\int f(z,\gamma,\Psi)d\gamma = qc\int f_0(\gamma)d\gamma + e^{i\Psi}qc\int \tilde f_1(z,\gamma)d\gamma + e^{-i\Psi}qc\int \tilde f_1^*(z,\gamma)d\gamma = j_0 + \tilde j_1 e^{i\Psi} + \tilde j_1^* e^{-i\Psi}$$

With this definition, Eq.(6) reads $-\left[2k_L\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z)\right] = \mu_0\frac{K}{\gamma_0}\frac{\partial j_z}{\partial t}e^{i\Psi} = \mu_0\frac{K}{\gamma_0}\frac{\partial\left(j_0+\tilde j_1 e^{i\Psi}+\tilde j_1^* e^{-i\Psi}\right)}{\partial t}e^{i\Psi} =$

(use $\Psi = (k_u + k_L)z - \omega_L t$ and note that $\tilde j_1$ is "almost" independent of time)

$$\approx \mu_0\frac{K}{\gamma_0}\left[(-i\omega_L)\tilde j_1 e^{i\Psi} + (i\omega_L)\tilde j_1^* e^{-i\Psi}\right]e^{i\Psi} = i\mu_0\frac{\omega_L K}{\gamma_0}\left(-\tilde j_1 e^{2i\Psi} + \tilde j_1^*\right) \approx i\mu_0\frac{\omega_L K}{\gamma_0}\tilde j_1^*\;,$$

and equally: $$\boxed{i\mu_0\frac{\omega_L K}{\gamma_0}\tilde j_1 = 2k_L\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0(z)}$$ (10)

$$\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0(z) = i\frac{\mu_0 cK}{2\gamma_0}\tilde{j}_1 = i\frac{\mu_0 Kqc^2}{2\gamma_0}\int_1^\infty \tilde{f}_1(z,\gamma)d\gamma =$$

$$i\frac{\mu_0 Kqc^2}{2\gamma_0}\int_1^\infty d\gamma\int_0^z dz'\left[i\frac{q\tilde{\mathbf{E}}_0(z')K}{2m_0c^2\gamma_0} + \frac{q}{m_0c^2}\tilde{\mathbf{E}}_z(z')\right]\frac{\partial f_0(\gamma)}{\partial\gamma}\,\mathbf{exp}\left[i(C + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma)(z'-z)\right]$$

For our assumption on the space charge field: $\mathbf{E}_z = \tilde{\mathbf{E}}_z(z)e^{i\Psi} + C.C.$, long. component of 1st Maxwell Eq.reads

(note $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$) $\quad (\nabla\times H)_z = 0 = j + \varepsilon_0\frac{\partial}{\partial t}\mathbf{E}_z$ or $\frac{\partial}{\partial t}\mathbf{E}_z(z) = -\mu_0 c^2 j_z$ thus: $\tilde{\mathbf{E}}_z(z) \approx -\frac{i\mu_0 c^2}{\omega_L}\tilde{j}_1.$

With Eq. (10), this is related to the transverse e.m. field: $\tilde{j}_1 = -i\frac{2\gamma_0}{\mu_0 cK}\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0(z)$, thus $\tilde{\mathbf{E}}_z(z) \approx -\frac{2\gamma_0 c}{\omega_L K}\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0(z)$

$$\boxed{\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0(z) = i\frac{\mu_0 q^2K^2c^2}{4\gamma_0^2 m_0 c^2}\int_1^\infty d\gamma\int_0^z dz'\left[i\tilde{\mathbf{E}}_0(z') - \frac{4\gamma_0^2 c}{\omega_L K^2}\frac{\partial}{\partial z'}\tilde{\mathbf{E}}_0(z')\right]\frac{\partial f_0(\gamma)}{\partial\gamma}\,\mathbf{exp}\left[i(C + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma)(z'-z)\right]}$$

This is an integro-differential equation for the complex amplitude of the e.m. field.
Only for few non-trivial model functions of the initial energy distribution f_0,
the solution can be found analytically, using Laplace-transform technique.

Most simple case: monoenergetic („cold“) beam:

$$\boxed{f_0(\gamma) = n_0\delta(\gamma-\gamma_0),\ \text{i.e. } \Delta\gamma=0,\quad \text{with charge density } n_0,\ \text{i.e. } j_0 = qc\int_{-\infty}^{\infty} n_0\delta(\gamma-\gamma_0)d\gamma = qcn_0}$$

Integration over energy can then be executed, using partial integration:

$$\int_1^\infty \frac{d\delta(\gamma-\gamma_0)}{d\gamma}F(\gamma)d\gamma = \left[\delta(\gamma)F(\gamma)\right]_1^\infty - \int_1^\infty \delta(\gamma-\gamma_0)\frac{dF(\gamma)}{d\gamma}d\gamma, \quad \text{thus:}$$

$$\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0(z) = i\frac{\mu_0 n_0 q^2 K^2}{4\gamma_0^2 m_0}\int_0^z dz'\int_1^\infty d\gamma\,\delta(\gamma-\gamma_0)\left[i\tilde{\mathbf{E}}_0(z') - \frac{4\gamma^2 c}{\omega_L K^2}\frac{\partial}{\partial z'}\tilde{\mathbf{E}}_0(z')\right]\left(i\frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}(z'-z)\right)\mathbf{exp}\left[i(C+\frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma)(z'-z)\right] =$$

$$-\frac{\mu_0 n_0 q^2 K^2(1+K^2)\omega_L}{4\gamma_0^5 m_0 c}\int_0^z dz'\left[i\tilde{\mathbf{E}}_0(z') - \frac{4\gamma^2 c}{\omega_L K^2}\frac{\partial}{\partial z'}\tilde{\mathbf{E}}_0(z')\right](z'-z)\mathbf{exp}\left[iC(z'-z)\right] =$$

$$-\Gamma^3\int_0^z dz'\left[i\tilde{\mathbf{E}}_0(z') - \frac{k_p^2}{\Gamma^3}\frac{\partial}{\partial z'}\tilde{\mathbf{E}}_0(z')\right](z'-z)\mathbf{exp}\left[iC(z'-z)\right]$$

with abbreviations:

$$\Gamma^3 = \frac{\mu_0 n_0 q^2 K^2(1+K^2)\omega_L}{4\gamma_0^5 m_0 c} = \frac{\pi j_0 K^2(1+K^2)\omega_L}{I_A c\gamma_0^5} \quad \left(I_A = \frac{4\pi m_0 c}{\mu_0 q} = 17\text{kA "Alven current"}\right)$$ Γ is called gain parameter.

$$k_p^2 = \frac{4\pi j_0(1+K^2)}{I_A\gamma_0^3} = \Gamma^3\frac{4\gamma^2 c}{\omega_L K^2}$$ k_p is wave number of longitudinal plasma oscillation.

We have ended with an ordinary integro-differential equation:

$$\frac{d}{dz}\tilde{\mathbf{E}}_0(z) = -\Gamma^3\int_0^z dz'\left[i\tilde{\mathbf{E}}_0(z') - \frac{k_p^2}{\Gamma^3}\frac{d}{dz'}\tilde{\mathbf{E}}_0(z')\right](z'-z)\,\mathbf{exp}\left[iC(z'-z)\right]$$

Deriving one more time: $\left(\text{Prove it, using } \frac{d}{dz}\int_0^z dz' g(z)h(z') = \frac{d}{dz}\left[g(z)\int_0^z dz' h(z')\right] = \frac{d}{dz}g(z)\int_0^z dz' h(z') + g(z)h(z)\right)$

$$\frac{d^2}{dz^2}\tilde{\mathbf{E}}_0 = -iC\frac{d}{dz}\tilde{\mathbf{E}}_0 + \Gamma^3\int_0^z dz'\left[i\tilde{\mathbf{E}}_0(z') - \frac{k_p^2}{\Gamma^3}\frac{d}{dz'}\tilde{\mathbf{E}}_0(z')\right]\mathbf{exp}\left[iC(z'-z)\right]$$

and once more:

$$\frac{d^3}{dz^3}\tilde{\mathbf{E}}_0 = -iC\frac{d^2}{dz^2}\tilde{\mathbf{E}}_0 + \Gamma^3\left[i\tilde{\mathbf{E}}_0(z) - \frac{k_p^2}{\Gamma^3}\frac{d}{dz}\tilde{\mathbf{E}}_0(z)\right] - iC\Gamma^3\int_0^z dz'\left[i\tilde{\mathbf{E}}_0(z') - \frac{k_p^2}{\Gamma^3}\frac{d}{dz'}\tilde{\mathbf{E}}_0(z')\right]\mathbf{exp}\left[iC(z'-z)\right]$$

$$= -iC\frac{d^2}{dz^2}\tilde{\mathbf{E}}_0 + \Gamma^3\left[i\tilde{\mathbf{E}}_0(z) - \frac{k_p^2}{\Gamma^3}\frac{d}{dz}\tilde{\mathbf{E}}_0(z)\right] - iC\left(iC\frac{d}{dz}\tilde{\mathbf{E}}_0 + \frac{d^2}{dz^2}\tilde{\mathbf{E}}_0\right) = -2iC\frac{d^2}{dz^2}\tilde{\mathbf{E}}_0 + \Gamma^3\left[i\tilde{\mathbf{E}}_0(z) - \frac{k_p^2}{\Gamma^3}\frac{d}{dz}\tilde{\mathbf{E}}_0(z)\right] + C^2\frac{d}{dz}\tilde{\mathbf{E}}_0$$

or rearranging:

$$\frac{d^3}{dz^3}\tilde{\mathbf{E}}_0 + 2iC\frac{d^2}{dz^2}\tilde{\mathbf{E}}_0 + \left(k_p^2 - C^2\right)\frac{d}{dz}\tilde{\mathbf{E}}_0 = i\Gamma^3\,\tilde{\mathbf{E}}_0(z), \qquad (11)$$

A linear third-order differential equation for the complex field amplitude.

$$\frac{d^3}{dz^3}\tilde{\mathbf{E}}_0 + 2iC\frac{d^2}{dz^2}\tilde{\mathbf{E}}_0 + \left(k_p^2 - C^2\right)\frac{d}{dz}\tilde{\mathbf{E}}_0 = i\Gamma^3\tilde{\mathbf{E}}_0(z)$$

Ansatz: $\mathbf{E} = A\,\mathbf{exp}(\Lambda z)$ $\rightarrow$ characteristic equation: $\Lambda^3 + 2iC\Lambda^2 + \left(k_p^2 - C^2\right)\Lambda = i\Gamma^3$

$$\Lambda\left(\Lambda^2 + 2iC\Lambda - C^2 + k_p^2\right) = \Lambda\left[\left(\Lambda + iC\right)^2 + k_p^2\right] = i\Gamma^3 \qquad (12)$$

This equation has 3 roots, and the general solution of Eq.(11) is constructed from three independent partial solutions:

$$\tilde{\mathbf{E}}(z) = A_1\,\mathbf{exp}(\Lambda_1 z) + A_2\,\mathbf{exp}(\Lambda_2 z) + A_3\,\mathbf{exp}(\Lambda_3 z)$$

The amplitudes $A_{1,2,3}$ are determined by the initial conditions.

most simple case: No detuning $C = 0$, and negligible space charge $k_p^2 = \dfrac{4\pi j_0 (1 + K^2)}{I_A \gamma_0^3} \rightarrow 0$ (i.e. very high beam energy γ_0):

$$\Lambda^3 = i\Gamma^3 \Rightarrow \Lambda_1 = -i\Gamma;\ \Lambda_2 = \frac{i + \sqrt{3}}{2}\Gamma;\ \Lambda_3 = \frac{i - \sqrt{3}}{2}\Gamma$$

The general solution is thus: $\mathbf{E}(z) = A_1\,\mathbf{exp}(-i\Gamma z) + A_2\,\mathbf{exp}\left(\dfrac{i + \sqrt{3}}{2}\Gamma z\right) + A_3\,\mathbf{exp}\left(\dfrac{i - \sqrt{3}}{2}\Gamma z\right)$

All contributions to solution oscillate or vanish, except for:

For an undulator much longer than $1/\Gamma$, this part of solution dominates.

The most practical ways to specify the intial conditions (in fact we have to determine 3 independent conditions!)

is to specify $\tilde{\mathbf{E}}_0(z=0), \dfrac{d}{dz}\tilde{\mathbf{E}}_0(z=0), \dfrac{d^2}{dz^2}\tilde{\mathbf{E}}_0(z=0)$, or $\left(\text{using Eq. 11: } \dfrac{d}{dz}\tilde{\mathbf{E}}_0 \propto \tilde{j}_1\right)$: $\tilde{\mathbf{E}}_0(z=0), \tilde{j}_1(z=0), \dfrac{d}{dz}\tilde{j}_1(z=0)$

With these initial condition, the amplitudes $A_{1,2,3,}$ can be calculated as follows:

We write $\tilde{\mathbf{E}}(z) = A_1 \,\mathbf{exp}(\Lambda_1 z) + A_2 \,\mathbf{exp}(\Lambda_2 z) + A_3 \,\mathbf{exp}(\Lambda_3 z)$ in the form

$\tilde{\mathbf{E}}(z) = A_1 \tilde{\mathbf{E}}_1(z) + A_2 \tilde{\mathbf{E}}_2(z) + A_3 \tilde{\mathbf{E}}_3(z)$ with $\tilde{\mathbf{E}}_1(z) = \mathbf{exp}(\Lambda_1 z)$, etc. and write $\frac{d}{dz}\tilde{\mathbf{E}} = \tilde{\mathbf{E}}'$, etc.

The general solution is written in the form

$$\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_z = \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3'' \end{pmatrix}_z \cdot \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

Since $\Lambda_{1,2,3}$ are known from characteristic Eq., all matrix elements are known..

Using initial conditions $\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_{z=0}$, we can determine $\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3'' \end{pmatrix}_{z=0}^{-1} \cdot \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_{z=0}$. Thus

$$\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_z = \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3'' \end{pmatrix}_z \cdot \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3'' \end{pmatrix}_{z=0}^{-1} \cdot \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_{z=0}$$

, or using $\tilde{\mathbf{E}}_1(z) = \mathbf{exp}(\Lambda_1 z)$, $\tilde{\mathbf{E}}_1'(z) = \Lambda_1 \,\mathbf{exp}(\Lambda_1 z)$, etc.

$$\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_z = \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3'' \end{pmatrix}_z \cdot \begin{pmatrix} 1 & 1 & 1 \\ \Lambda_1 & \Lambda_2 & \Lambda_3 \\ \Lambda_1^2 & \Lambda_2^2 & \Lambda_3^2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_{z=0}$$

Thus the General Solution is

$$\begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{E}}'\\ \tilde{\mathbf{E}}''\end{pmatrix}_z = \begin{pmatrix}\tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3\\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3'\\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3''\end{pmatrix}_z \cdot \overbrace{\begin{pmatrix}\frac{\Lambda_2\Lambda_3}{(\Lambda_1-\Lambda_2)(\Lambda_1-\Lambda_3)} & -\frac{\Lambda_2+\Lambda_3}{(\Lambda_1-\Lambda_2)(\Lambda_1-\Lambda_3)} & \frac{1}{(\Lambda_1-\Lambda_2)(\Lambda_1-\Lambda_3)}\\ \frac{\Lambda_1\Lambda_3}{(\Lambda_2-\Lambda_1)(\Lambda_2-\Lambda_3)} & -\frac{\Lambda_1+\Lambda_3}{(\Lambda_2-\Lambda_1)(\Lambda_2-\Lambda_3)} & \frac{1}{(\Lambda_2-\Lambda_1)(\Lambda_2-\Lambda_3)}\\ \frac{\Lambda_2\Lambda_1}{(\Lambda_3-\Lambda_2)(\Lambda_3-\Lambda_1)} & -\frac{\Lambda_2+\Lambda_1}{(\Lambda_3-\Lambda_2)(\Lambda_3-\Lambda_1)} & \frac{1}{(\Lambda_3-\Lambda_2)(\Lambda_3-\Lambda_1)}\end{pmatrix}}^{\begin{pmatrix}1 & 1 & 1\\ \Lambda_1 & \Lambda_2 & \Lambda_3\\ \Lambda_1^2 & \Lambda_2^2 & \Lambda_3^2\end{pmatrix}^{-1}} \cdot \begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{E}}'\\ \tilde{\mathbf{E}}''\end{pmatrix}_{z=0}$$

For the simple case C=0, $k_p = 0$ we got $\Lambda_1 = -i\Gamma; \Lambda_2 = \frac{i+\sqrt{3}}{2}\Gamma; \Lambda_3 = \frac{i-\sqrt{3}}{2}\Gamma$, thus

$$\begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{E}}'\\ \tilde{\mathbf{E}}''\end{pmatrix}_z = \begin{pmatrix}\tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3\\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3'\\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3''\end{pmatrix}_z \cdot \begin{pmatrix}\frac{1}{3} & \frac{i}{3\Gamma} & -\frac{1}{3\Gamma^2}\\ \frac{1}{3} & \frac{1}{3\Gamma}\mathbf{exp}\left(-i\frac{\pi}{6}\right) & \frac{1}{3\Gamma^2}\mathbf{exp}\left(-i\frac{\pi}{3}\right)\\ \frac{1}{3} & \frac{-1}{3\Gamma}\mathbf{exp}\left(i\frac{\pi}{6}\right) & \frac{1}{3\Gamma^2}\mathbf{exp}\left(i\frac{\pi}{3}\right)\end{pmatrix} \cdot \begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{E}}'\\ \tilde{\mathbf{E}}''\end{pmatrix}_{z=0} \quad (13)$$

First example: Seeding by input e.m. wave

In this case: $\tilde{\mathbf{E}}(z=0)=\mathbf{E}_{ext}$, $\tilde{j}_1(z=0)=0$, $\frac{d}{dz}\tilde{j}_1(z=0)=0$ (i.e. no current modulation at the beginning) $\rightarrow \begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{E}}'\\ \tilde{\mathbf{E}}''\end{pmatrix}_{z=0}=\begin{pmatrix}\mathbf{E}_{ext}\\ 0\\ 0\end{pmatrix}$

Thus

$$\begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{E}}'\\ \tilde{\mathbf{E}}''\end{pmatrix}_z=\begin{pmatrix}\tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3\\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3'\\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3''\end{pmatrix}_z\cdot\begin{pmatrix}\frac{1}{3} & \frac{i}{3\Gamma} & -\frac{1}{3\Gamma^2}\\ \frac{1}{3} & \frac{1}{3\Gamma}\mathbf{exp}\left(-i\frac{\pi}{6}\right) & \frac{1}{3\Gamma^2}\mathbf{exp}\left(-i\frac{\pi}{3}\right)\\ \frac{1}{3} & \frac{-1}{3\Gamma}\mathbf{exp}\left(i\frac{\pi}{6}\right) & \frac{1}{3\Gamma^2}\mathbf{exp}\left(i\frac{\pi}{3}\right)\end{pmatrix}\cdot\begin{pmatrix}\mathbf{E}_{ext}\\ 0\\ 0\end{pmatrix}=\begin{pmatrix}\tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3\\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3'\\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3''\end{pmatrix}_z\cdot\begin{pmatrix}\frac{1}{3}\mathbf{E}_{ext}\\ \frac{1}{3}\mathbf{E}_{ext}\\ \frac{1}{3}\mathbf{E}_{ext}\end{pmatrix}=\frac{1}{3}\mathbf{E}_{ext}\begin{pmatrix}\tilde{\mathbf{E}}_1+\tilde{\mathbf{E}}_2+\tilde{\mathbf{E}}_3\\ \tilde{\mathbf{E}}_1'+\tilde{\mathbf{E}}_2'+\tilde{\mathbf{E}}_3'\\ \tilde{\mathbf{E}}_1''+\tilde{\mathbf{E}}_2''+\tilde{\mathbf{E}}_3''\end{pmatrix}=$$

$\boxed{\text{(use } 1+i\sqrt{3}=2\,\mathbf{exp}\,i\frac{\pi}{3}\text{)}}$

$$\frac{1}{3}\mathbf{E}_{ext}\begin{pmatrix}\mathbf{exp}(\Lambda_1 z)+\mathbf{exp}(\Lambda_2 z)+\mathbf{exp}(\Lambda_3 z)\\ \Lambda_1\,\mathbf{exp}(\Lambda_1 z)+\Lambda_2\,\mathbf{exp}(\Lambda_2 z)+\Lambda_3\,\mathbf{exp}(\Lambda_3 z)\\ \Lambda_1^2\,\mathbf{exp}(\Lambda_1 z)+\Lambda_2^2\,\mathbf{exp}(\Lambda_2 z)+\Lambda_3^2\,\mathbf{exp}(\Lambda_3 z)\end{pmatrix};\ \text{explicitly:}\ \boxed{\tilde{\mathbf{E}}(z)=\frac{1}{3}\mathbf{E}_{ext}\left[\mathbf{exp}(-i\Gamma z)+\mathbf{exp}\left(\frac{i+\sqrt{3}}{2}\Gamma z\right)+\mathbf{exp}\left(\frac{i-\sqrt{3}}{2}\Gamma z\right)\right]}$$

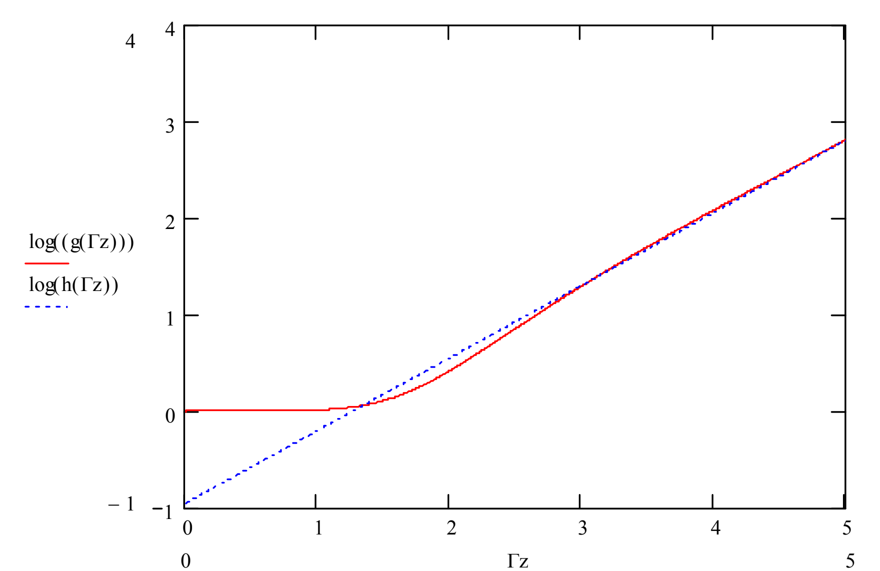
for $z \gg 1/\Gamma$: $\boxed{\tilde{\mathbf{E}}(z)=\frac{1}{3}\mathbf{E}_{ext}\,\mathbf{exp}\left(\frac{i+\sqrt{3}}{2}\Gamma z\right)}$ (14)

The power gain is given by (prove it!)

$$G=\frac{\left|\tilde{\mathbf{E}}\right|^2}{\mathbf{E}_{ext}^2}=\frac{1}{9}\left[1+4\,\mathbf{cosh}\frac{\sqrt{3}}{2}\Gamma z\left(\mathbf{cosh}\frac{\sqrt{3}}{2}\Gamma z+\mathbf{cos}\frac{3}{2}\Gamma z\right)\right]$$

$\rightarrow$ (for $z \gg 1/\Gamma$): $\boxed{G=\frac{1}{9}\mathbf{exp}\sqrt{3}\Gamma z}$

The factor 1/9 describes the coupling of the incoming e.m. field to FEL gain process

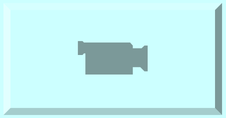


$$P_{rad} = \frac{1}{9} P_{in} \, \mathbf{exp}\left(\sqrt{3}\Gamma z\right) .$$

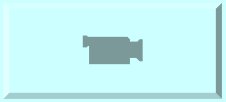
$$L_G = \frac{1}{\sqrt{3}\Gamma} = \frac{1}{\sqrt{3}} \left(\frac{I_A c \gamma^5}{\pi j_0 K^2 (1+K^2) \omega_L} \right)^{1/3} \quad \text{or, using} \quad \omega_L = \frac{4\pi c \gamma^2}{\lambda_u (1+K^2)} \quad \text{and } j_0 \approx \frac{\hat{I}}{\pi \sigma_r^2} ,$$

$$L_G = \frac{1}{\sqrt{3}} \left(\frac{I_A \gamma^3 \sigma_r^2 \lambda_u}{4\pi \hat{I} K^2} \right)^{1/3} \quad \text{is called power gain length.}$$

Also widely used : $\rho = \frac{\lambda_u \Gamma}{4\pi}$ "FEL - parameter" $\rho = \frac{1}{4\pi\sqrt{3}} \frac{\lambda_u}{L_G} = \frac{1}{4\pi\sqrt{3}} \frac{1}{N_{Gain}}$ (15)



Perseo/L. Gianessi SASE phase space



Bunching at SASE FEL seen in y/z coordinates

Saturation

Our treatment was based on the assumption $\frac{\left|\tilde{j}_1\right|}{j_0} \ll 1$.

When $\left|\tilde{j}_1\right|$ approaches j_0 (full modulation), e.m. field cannot further grow and our linear approximation breaks down.

→ needs numerical simulation

Electrons lose quickly so much energy that, due to particle motion in phase space, e.m. field may even pump some energy back to electron beam.

Also, the energy spread of the electron beam increases (thus the frequence spread).

Potential cure: undulator tapering

Detailed numerical analysis needed.

Phase space simulation of SASE FEL

Very rough estimate of saturation power:

We assume $\left|\tilde{j}_1\right| = j_0$ (full modulation). Using Eq. (10) $\frac{d}{dz}\tilde{\mathbf{E}}_0(z) = i\mu_0 \frac{cK}{2\gamma_0}\tilde{j}_1$, we estimate

$\left|\tilde{\mathbf{E}}_0(z = L_g)\right| \approx \frac{d}{dz}\tilde{\mathbf{E}}_0(L_g) \times L_g \approx \mu_0 \frac{cK}{2\gamma_0} j_0 L_g$ i.e. major part of radiation is generated in last gain length.

Plug in L_g and get

$$\boxed{P_{sat} = \frac{\varepsilon_0 c}{2}|E|^2 \times \text{Area} \approx \frac{\varepsilon_0 c \sigma_r^2}{2}|E_{sat}|^2 \approx \frac{\mu_0 c}{120}\left(\frac{\hat{I}^2 I_A K \lambda_u}{\sigma_r}\right)^{2/3}} \qquad (16)$$

Note: P_{sat} does not depend on beam energy!

typical numbers:

$\hat{I} = 1000$ A, K=1, $\lambda_u = 0.03$ m, $\sigma_r = 0.1$ mm $\rightarrow P_{sat} \approx 2$ GW

What is the power efficiency? Use Eqs. 15 and 16 to see:

$$\boxed{\frac{P_{sat}}{P_{beam}} = \frac{P_{sat}}{\gamma_o m_0 c^2 \hat{I} / q} \approx \rho}$$

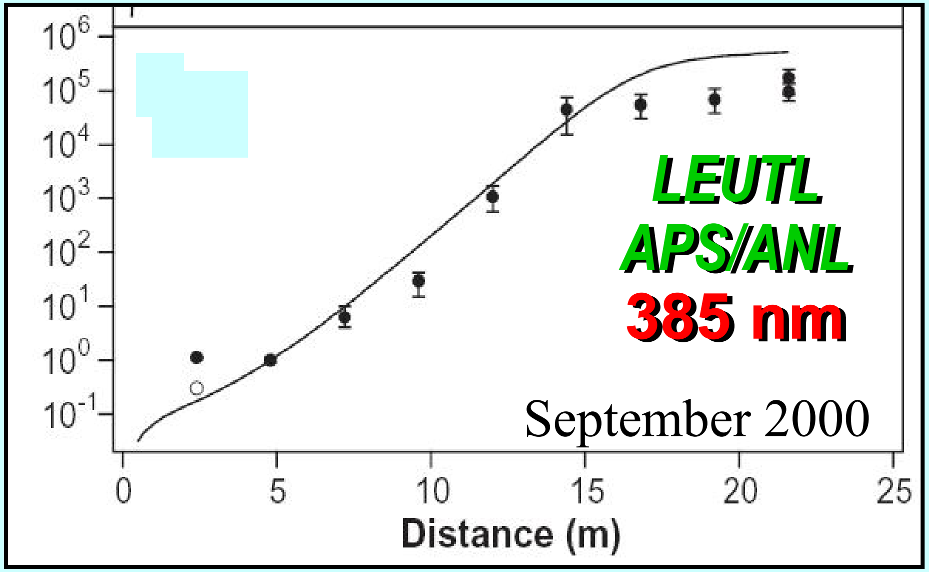
Note: This is the gain of the amplifier, it is NOT the gain compared to the spontaneous emission power. The total power of spontaneous radiation may be almost as big as P_{sat} **!!**

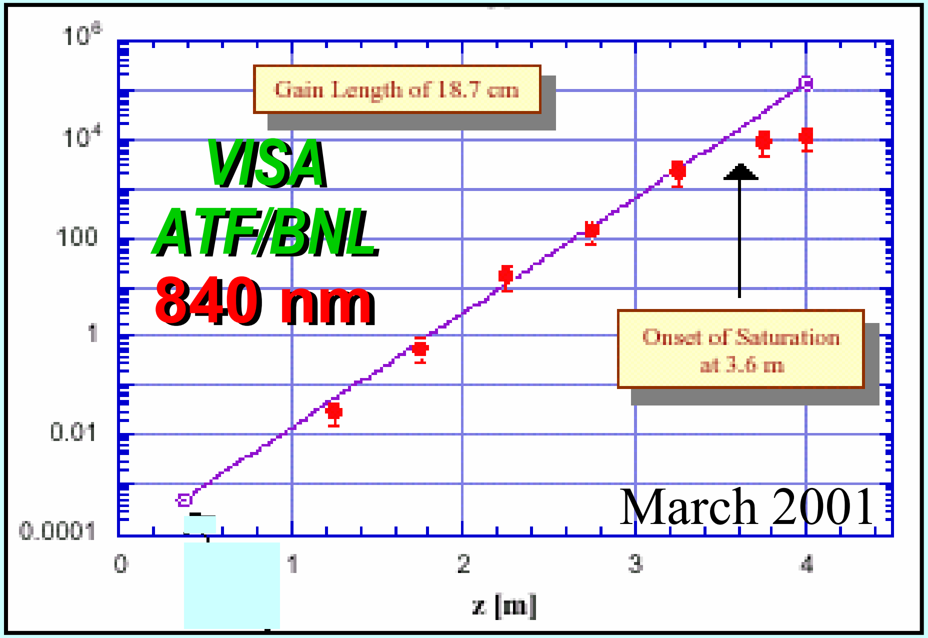
The typical saturation length is approx. 22 * (power gain length).
For $\lambda_{ph} = 0.1$ nm, L_{sat} may be as long as 200 m.

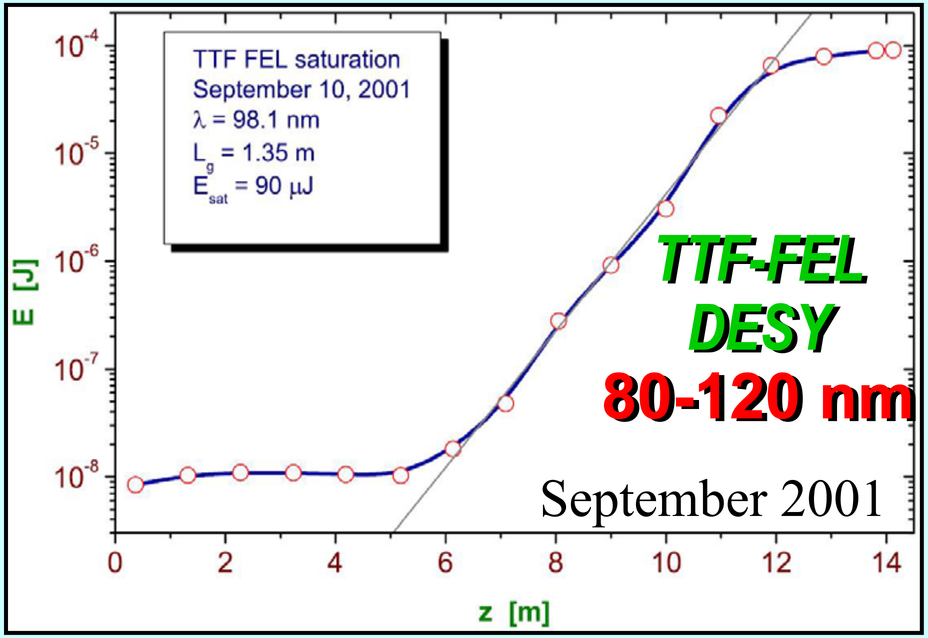
Peak power 1 GW

SASE FELs: State of the art

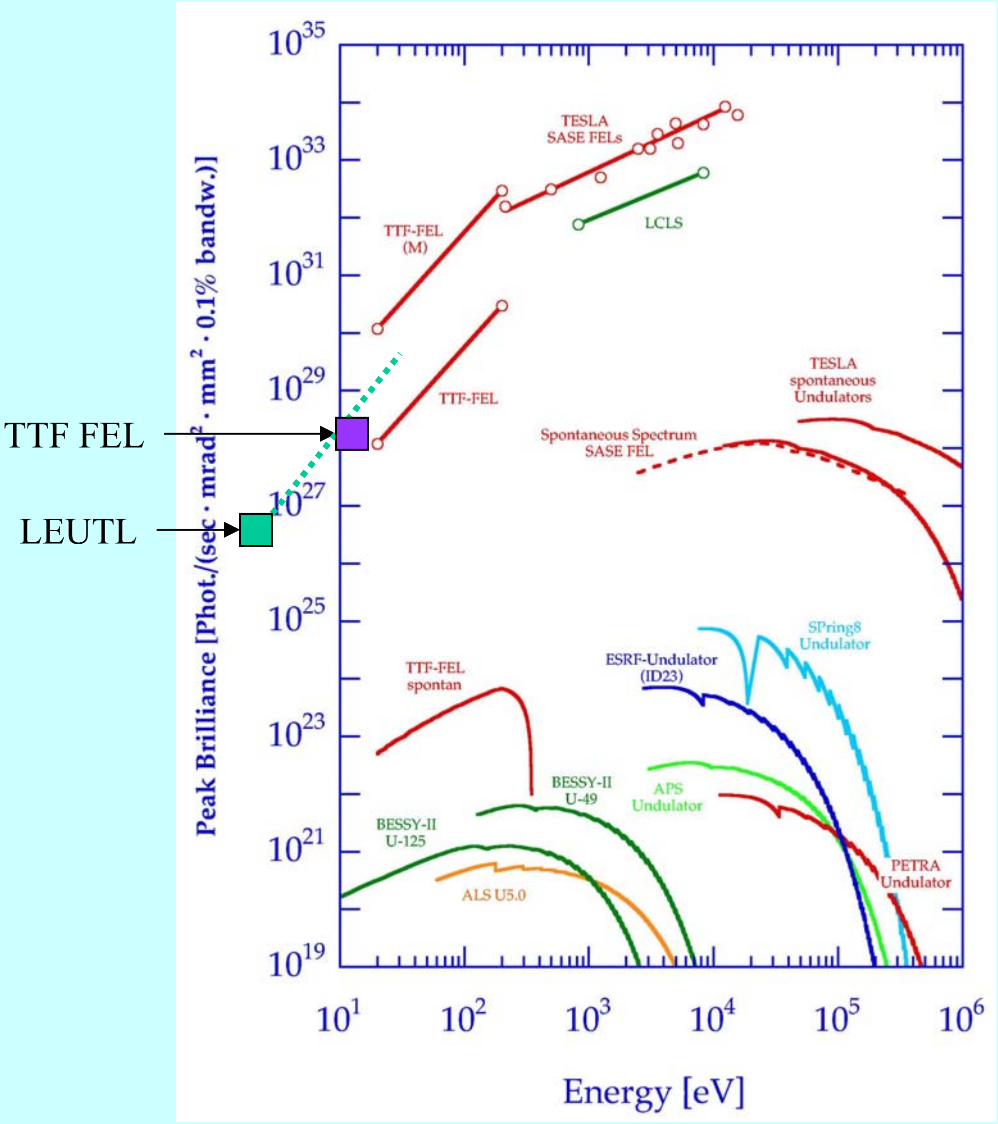
All observations agree with theor. expectations/ computer models







Peak brilliance



Second example: NO seeding, but density modulation at undulator entrance

$\tilde{\mathbf{E}}(z=0)=0,\ \tilde{j}_1(z=0)\neq 0$ (current modulation at the beginning), $\frac{d}{dz}\tilde{j}_1(z=0)=0$ (modulation is stationary at beginning)

$$\rightarrow\ \tilde{\mathbf{E}}'(z=0)=\tilde{\mathbf{E}}'_0=i\mu_0\frac{cK}{2\gamma_0}\tilde{j}_1(z=0),\quad \tilde{\mathbf{E}}''(z=0)=0 \qquad \begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{E}}'\\ \tilde{\mathbf{E}}''\end{pmatrix}_{z=0}=\begin{pmatrix}0\\ i\mu_0\frac{cK}{2\gamma_0}\tilde{j}_1\\ 0\end{pmatrix}_{z=0}=\begin{pmatrix}0\\ \tilde{\mathbf{E}}'_0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{E}}'\\ \tilde{\mathbf{E}}''\end{pmatrix}_z=\begin{pmatrix}\tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3\\ \tilde{\mathbf{E}}'_1 & \tilde{\mathbf{E}}'_2 & \tilde{\mathbf{E}}'_3\\ \tilde{\mathbf{E}}''_1 & \tilde{\mathbf{E}}''_2 & \tilde{\mathbf{E}}''_3\end{pmatrix}_z\cdot\begin{pmatrix}\frac{1}{3} & \frac{i}{3\Gamma} & -\frac{1}{3\Gamma^2}\\ \frac{1}{3} & \frac{1}{3\Gamma}\mathbf{exp}\left(-i\frac{\pi}{6}\right) & \frac{1}{3\Gamma^2}\mathbf{exp}\left(-i\frac{\pi}{3}\right)\\ \frac{1}{3} & \frac{-1}{3\Gamma}\mathbf{exp}\left(i\frac{\pi}{6}\right) & \frac{1}{3\Gamma^2}\mathbf{exp}\left(i\frac{\pi}{3}\right)\end{pmatrix}\cdot\begin{pmatrix}0\\ \tilde{\mathbf{E}}'_0\\ 0\end{pmatrix}_{z=0}=\begin{pmatrix}\tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3\\ \tilde{\mathbf{E}}'_1 & \tilde{\mathbf{E}}'_2 & \tilde{\mathbf{E}}'_3\\ \tilde{\mathbf{E}}''_1 & \tilde{\mathbf{E}}''_2 & \tilde{\mathbf{E}}''_3\end{pmatrix}_z = \begin{pmatrix}\frac{i}{3\Gamma}\tilde{\mathbf{E}}'_0\\ \frac{1}{3\Gamma}\mathbf{exp}\left(-i\frac{\pi}{6}\right)\tilde{\mathbf{E}}'_0\\ -\frac{1}{3\Gamma}\mathbf{exp}\left(i\frac{\pi}{6}\right)\tilde{\mathbf{E}}'_0\end{pmatrix}$$

$$\rightarrow \tilde{\mathbf{E}}(z)=\frac{1}{3\Gamma}\tilde{\mathbf{E}}'_0\left[i\,\mathbf{exp}(\Lambda_1 z)+\mathbf{exp}\left(-i\frac{\pi}{6}\right)\mathbf{exp}(\Lambda_2 z)-\mathbf{exp}\left(i\frac{\pi}{6}\right)\mathbf{exp}(\Lambda_2 z)\right]$$

again: For $z \gg \frac{1}{\Gamma}$, Λ_2 – solution prevails: $\tilde{\mathbf{E}}'\left(z \gg \frac{1}{\Gamma}\right)\propto \mathbf{exp}(\Lambda_2 z)$.

i.e. We don't need any input ("seeding") e.m. wave, current modulation at optical, resonant wavelength is as good!

Resonance width

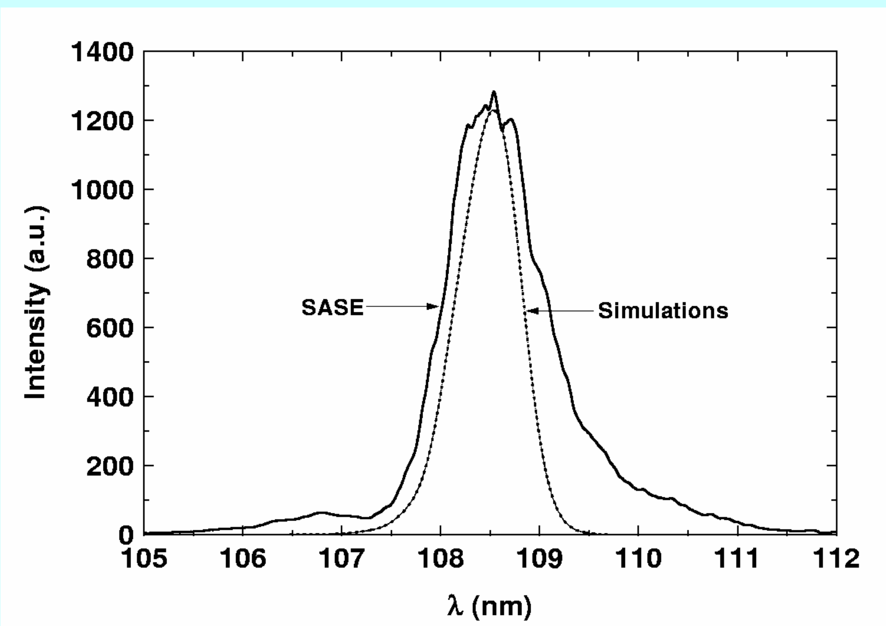
Analysis of Eq.(12) with $k_p = 0$, but $C \neq 0$ is straight-forward algebra. It is seen that

1. Maximum gain occurs for ON-resonance operation (i.e. for C=0) - in contrast to low gain!

2. Gain drops significantly when $|C|$ is increased to values corresponding to $\frac{\Delta\gamma}{\gamma} = \rho$

Because of $\lambda_{ph} \propto \frac{1}{\gamma^2}$, this means the bandwidth of a High-Gain FEL is $\boxed{\frac{\Delta\lambda_{ph}}{\lambda_{ph}} = 2\frac{\Delta\gamma}{\gamma} = 2\rho}$

TESLA FEL



All particles outside this energy window don't contribute to the gain process constructively

→ (uncorrelated, slice) energy spread should be $\frac{\Delta\gamma}{\gamma} \leq \rho$

→ Serious challenge for short λ_{ph}, low ρ

→ High-Gain FEL acts as a narrow-band amplified with bandwidth $\frac{\Delta\omega}{\omega} \leq 2\rho$

Start-up from noise = SASE (self-amplified spontaneous emission

Assume a perfectly smooth initial beam current, i.e.

- no initial current modulation
- Bunch longer than slippage of radiation w.r.t. electrons per gain length, i.e. $\sigma_z >> \frac{\lambda_{ph}}{\rho}$

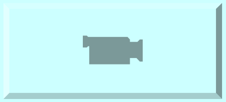
If, in addition, there is no initial e.m. field, Eq. (13) tells us that E = 0 forever, because there is nothing to be amplified.

BUT:

There is still current modulation in the electron beam, because is consists of many individual electrons randomly distributed in space and time.

PROOF: Intensity of spontaneous synchrotron radiation.

Equivalent current modulation at undulator entrance is given by that part of current noise spectrum that falls into the FELamplifies bandwidth.



Exponential growth of power at SASE FEL

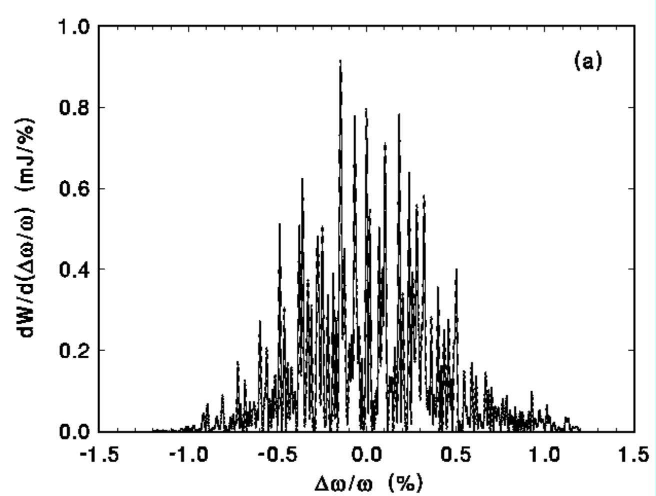
electron beam current in time domain: $I(t) = q_0 \sum_{k=1}^{N} \delta(t - t_k)$ (N = number of electrons)

In frequency domain: $I(\omega) = q_0 \sum_{k=1}^{N} \mathbf{exp}(\omega t_k)$, i.w. white noise.

Thus, within the FEL bandwidth, there will also be a white noise of current modulation frequencies. FEL amplifier is linear

→ all frequency components within bandwidth will be amplified in parallel (linear superposition).

→ We expect noisy spectrum at output.



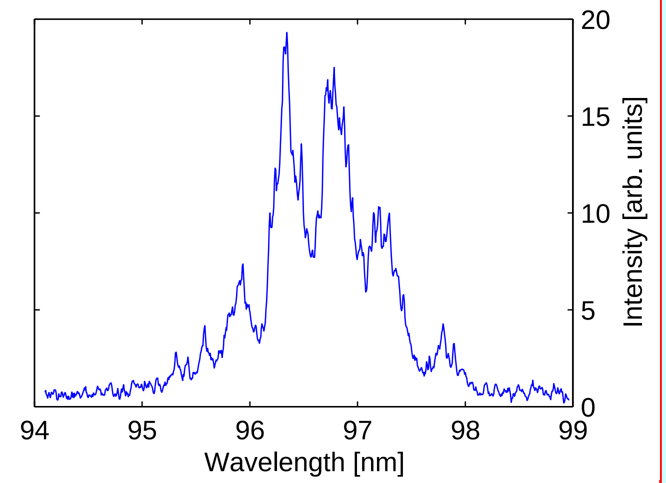
What is the frequency width of spikes?

While the radiation is amplified, wave packages develop of length $l_{coh} \approx \dfrac{\lambda_{ph}}{\pi\rho}$ (coherence length).

They add up with arbitrary phases, but the correlation length remains l_{coh}.

→ In time domain, we expect M wavepackages, with $\boxed{M = \dfrac{l_{bunch}\,(\text{FWHM bunch length})}{l_{coh}} = \text{number of long. modes}}$

→ We expect M spectral spikes of width $\Delta\omega \approx \dfrac{2\pi c}{l_{bunch}}$.



TTF FEL with
M=6 modes $\tau_{rad} \approx 100$ fs

The effective input power of shot noise can be estimated

at (Ref. 5, Eq. (6.95)): $P_{sh} \approx \dfrac{3}{N_c\sqrt{\pi \ln N_c}}\rho P_{beam}$.

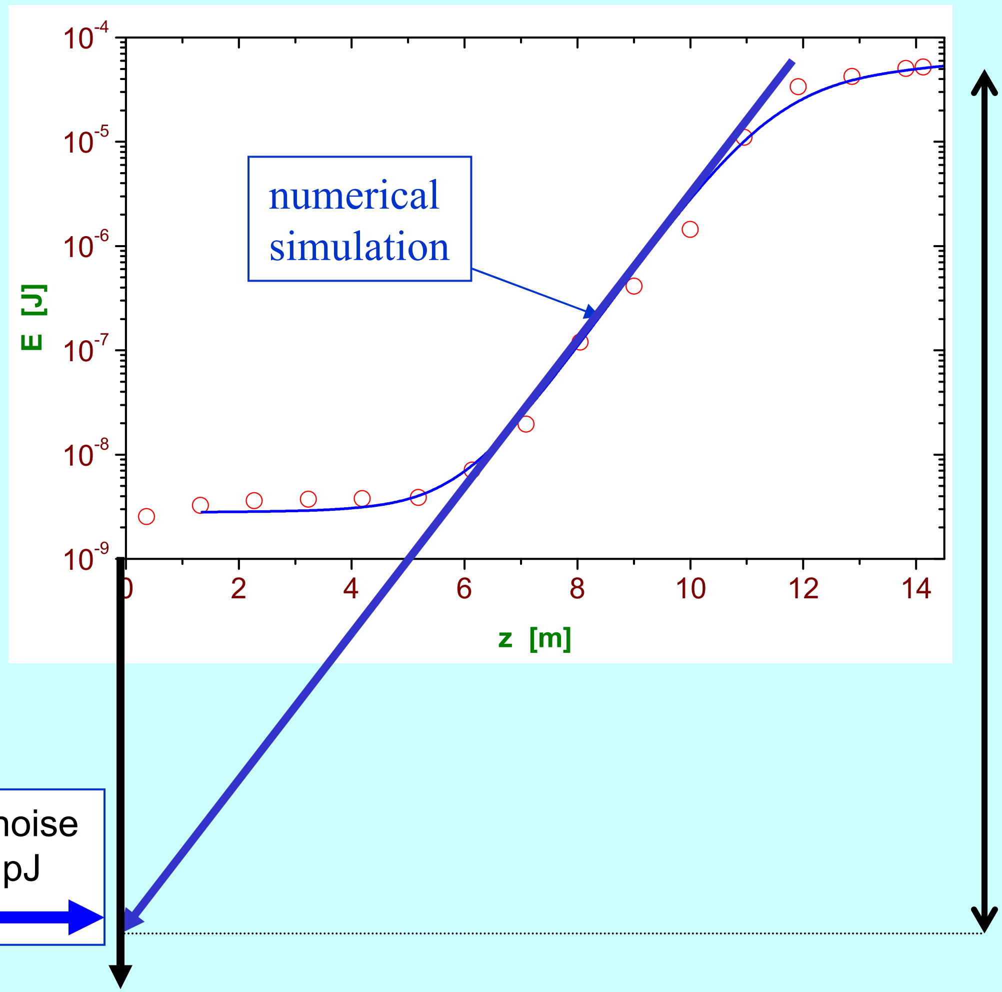
with N_c $0.5 \times$ number of electrons within coherence length.

Thus, the SASE power gain is

$$G = \frac{P_{sat}}{P_{sh}} = \frac{\rho P_{beam}}{P_{sh}} \approx \frac{1}{3} N_c \sqrt{\pi \ln N_c},$$

it is roughly given by the number of electrons in the cooperation length.

Note: This is the gain of the amplifier, it is NOT the gain compared to the spontaneous emission power.
The total power of spontaneous radiation may be almost as big as P_{sat} **!!**



Equivalent shot noise input energy 0.3 pJ

SASE power gain ~10^8

We can equivalently say, (a coherent part of) the spontaneous radiation from the first part of the undulator serves as input signal.

→Self-Amplified Spontaneous Emission (SASE) mode of operation

→ Most attractive for (short) wavelengths where no mirrors and no good (= powerful and tunable) input laser are available.

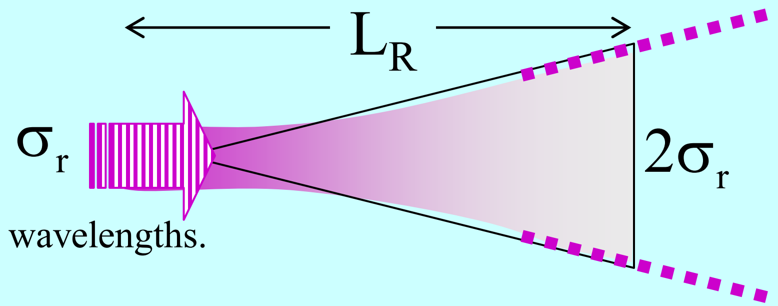
If you want to seed by external source (e.g. in order to improve long. Coherence, i.e. to avoid spikes), make sure your input power exceeds the equivalent input power of shot noise!

Present world record w.r.t. short wavelengths (100 nm):
Power gain $P_{rad} / P_{in} = 10^8$ demonstrated at DESY

3D-Effects

Key issue: Permanent, perfect overlap of e.m. wave and electron beam

A) Diffraction



Opening angle due to diffraction is described by Rayleigh-Length:

$$L_R = \frac{\pi\sigma_r^2}{\lambda_{ph}} \quad \rightarrow \quad \sigma_{r'} \approx \frac{2\sigma_r}{L_R} \approx \frac{\lambda}{2\sigma_r}$$ Diffraction less critical for short wavelengths.

Alternative approach: Transverse phase space volume of coh. source: $\sigma_r \cdot \sigma_{r'} = \frac{\lambda_{ph}}{2}$

$$\rightarrow \quad \sigma_{r'} = \frac{\lambda_{ph}}{2\sigma_r}$$

Example: LCLS: $\lambda_{ph} \approx 10^{-10}\,\mathrm{m},\ \sigma_r \approx 30\ \mu\mathrm{m},\ \rightarrow\ \sigma_{r'} \approx 2\ \mu\mathrm{rad}$

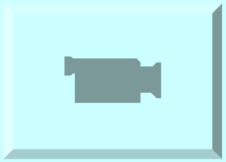
Note: This is much smaller than $\frac{1}{\gamma} \approx 30\ \mu\mathrm{rad}$!

Reason: FEL-radiation is no single-charge radiation but has directional characteristics like array of antennas.

For SASE, transv. coherence develops during gain process:

It starts with many transverse modes, but the axial one has highest gain and reaches saturation first.

At saturation, "normally" almost full transv. coherence.



Genesis simulation of 3D UCLA

B) 3D-properties of electron beam

i) Electron trajectory straightness must be perfect
to ensure permanent overlap with e.m. wave.
Very challenging due to small beam size. Tolerance for XFEL: few microns

ii) Emittance introduces longitudinal velocity spread, much like energy spread does. So in terms of FEL gain emittance is equivalent to additional energy spread:

$$\left.\frac{\Delta\gamma}{\gamma}\right|_{\text{èff}} \approx \frac{\gamma^2\varepsilon}{\beta\left(1+K^2\right)}$$ (β is the Twiss parameter of electron focusing)

We know the condition $\frac{\Delta\gamma}{\gamma} < \rho$ which keeps all particles inside amplifier bandwidth.

Thus: $\varepsilon \approx \frac{\beta\left(1+K^2\right)}{\gamma^2}\left.\frac{\Delta\gamma}{\gamma}\right|_{\text{èff}} < \frac{\beta\left(1+K^2\right)}{2\gamma^2}\rho$. (factor 2 makes sure the emittance contributes max. 50% of budget) (⊗)

From the diffraction effect there comes another condition:

Maintaining complete overlap AND maximum possible gain requires

$$L_R \approx L_g \;\rightarrow\; \frac{\pi\sigma_r^2}{\lambda_{ph}} = \frac{\pi\beta\varepsilon}{\lambda_{ph}} \approx \frac{1}{4\pi\sqrt{3}}\frac{\lambda_u}{\rho} \quad (\otimes\otimes)$$

Eliminating ρ from (⊗) and (⊗⊗) yields $\varepsilon < \frac{\lambda_{ph}}{\sqrt{4\sqrt{3}\pi}} \approx \frac{\lambda_{ph}}{4\pi}$

$\boxed{\varepsilon < \frac{\lambda_{ph}}{4\pi}}$ is a rather challenging condition for λ_{ph} in the nanometer range.

What helps (a little) is that $\varepsilon\gamma$ is conserved in linac acceleration, so $\varepsilon \propto \frac{1}{\gamma}$.

What are the challenges?Overview

Electron beam parameters needed for Self-Amplified-Spontaneous Emission (SASE)

Energy:

$$\lambda_{em} = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

für λ_{em}= 1 Å: E ≈ 20 GeV

Energy width:

Narrow resonance → $\sigma_E/E \leq \rho \sim 10^{-4}$

⇔ Small distortion by wakefields

⇒ super conducting linac ideal!

Straight trajectory in undulator:
ultimately < 10 µm over 100 m

Gain Length: $$L_g = \frac{1}{\sqrt{3}}\left[\frac{2mc}{\mu_0 e}\frac{\gamma^3\sigma_r^2\lambda_u}{K^2\hat{I}}\right]^{1/3}$$

Beam size:

$\sigma_r \approx 40$ µm ⇔ high electron desity for maximum interaction with radiation field

Emittance $\varepsilon \leq \lambda/4\pi$

need special electron source to accelerate the beam before it explodes due to Coulomb forces

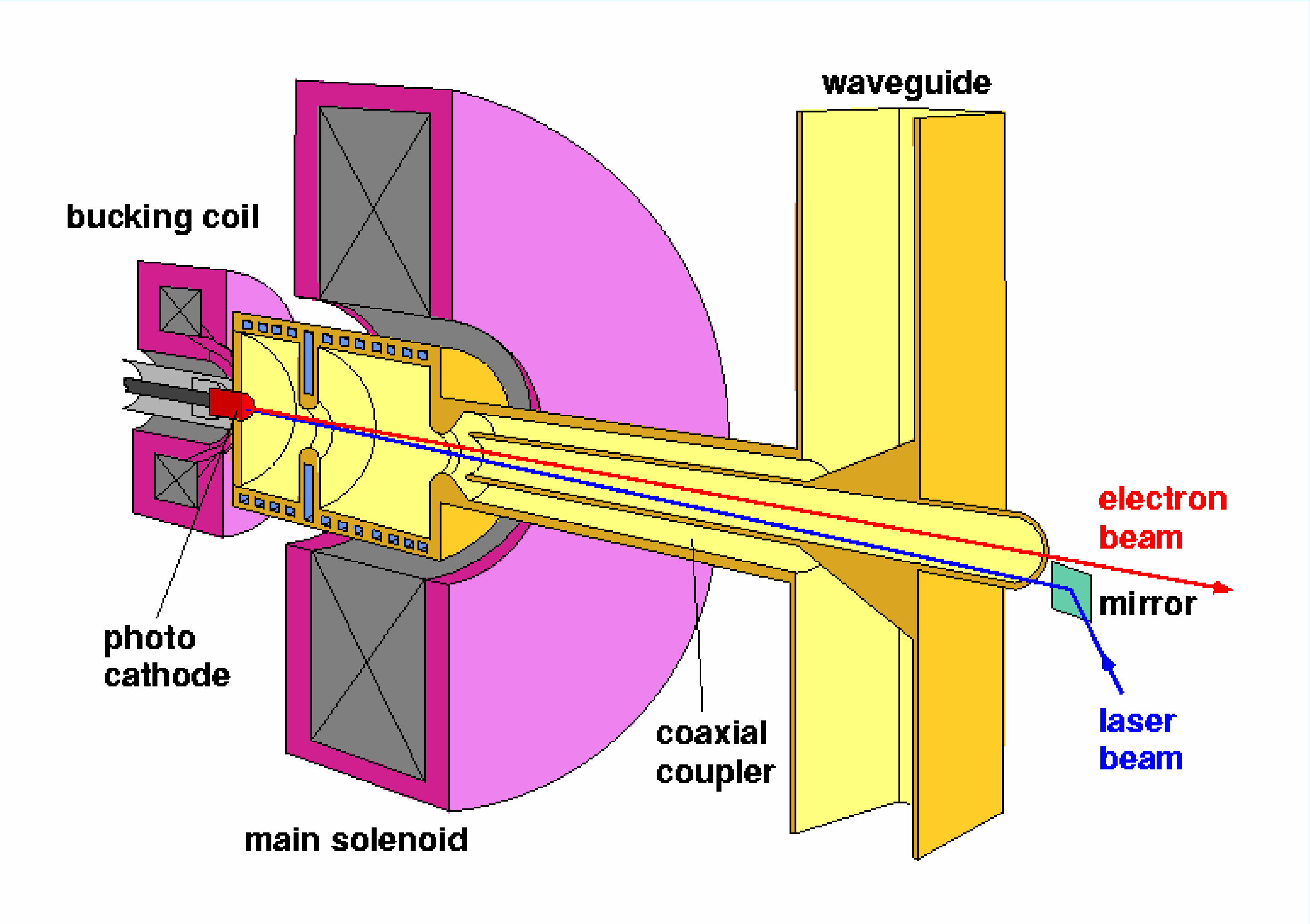
Peak current inside bunch:

$\hat{I} > 1$ kA

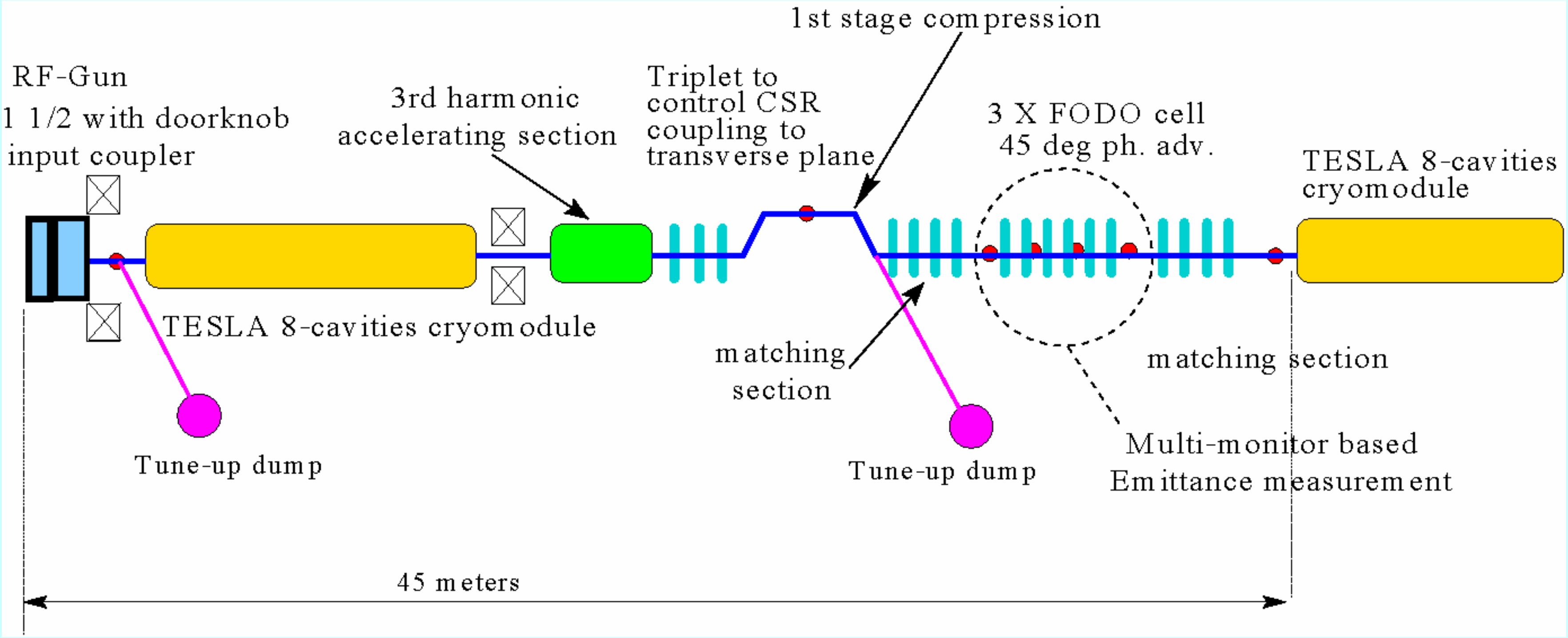
feasible only at ultrarelativistic energies, otherwise ruins emittance ⇒ bunch compressor

What are the challenges? RF gun

TESLA FEL photoinjector for small and short electron bunches

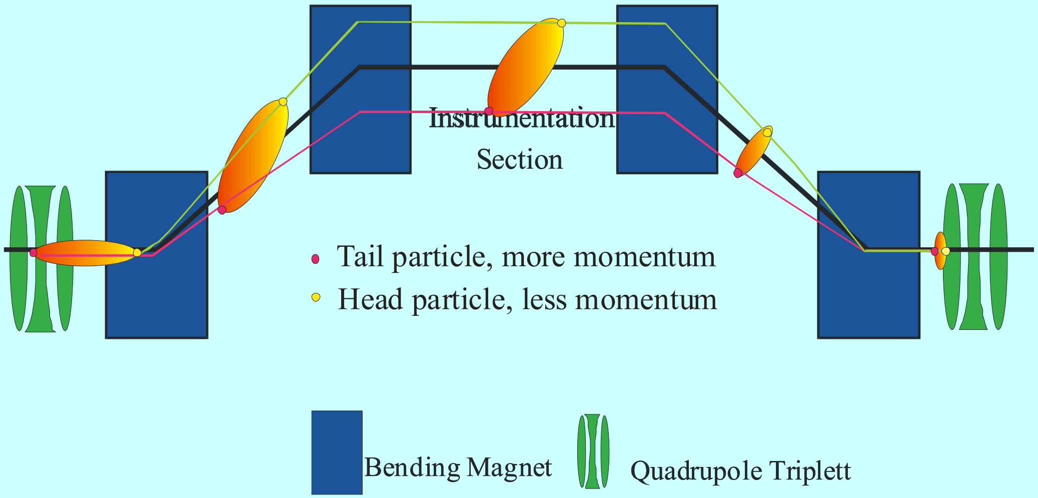


Layout of integrated injector/compressor for TTF2 and TESLA FEL



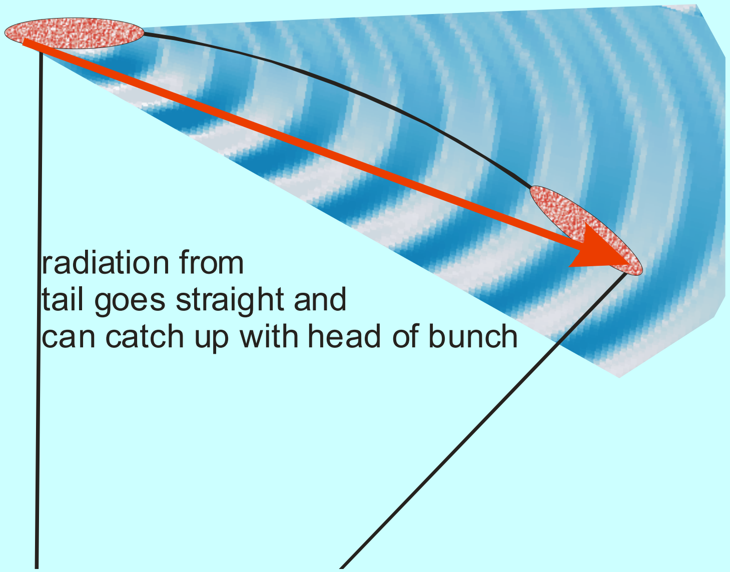
What are the challenges? Bunch compression

Magnetic bunch compression



Beware of
coherent synchrotron radiation (CSR)

very powerful microwave radiation
with λ >~ bunch length if
bunch length << size of vacuum chamber



Beam dynamics simulation must take into account combined
space charge and e.m. radiation in near-field. e.g.: TRAFIC4 by A. Kabel/SLAC

References:

Low gain FELs:

1. K. Wille: Introduction to accelerator physics

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   Introduction to FEL physics in: Laser Handbook Vol. 6 (North Holland)

High gain FELs:

3. E. Saldin, E. Schneidmiller, M. Yurkov: The Physics of FELs, Springer Verlag

4. J.B. Murphy, C. Pellegrini, in: Laser Handbook Vol. 6:
   Introduction to the physics of the FEL

5. S. Reiche/UCLA: GENESIS1.3, available via internet

6. L. Gianessi: PERSEO (MATHCAD package) c/o gianessi@frascati.enea.it