INSTABILITIES I
LONGITUDINAL INSTABILITIES

LONGITUDINAL DYNAMICS

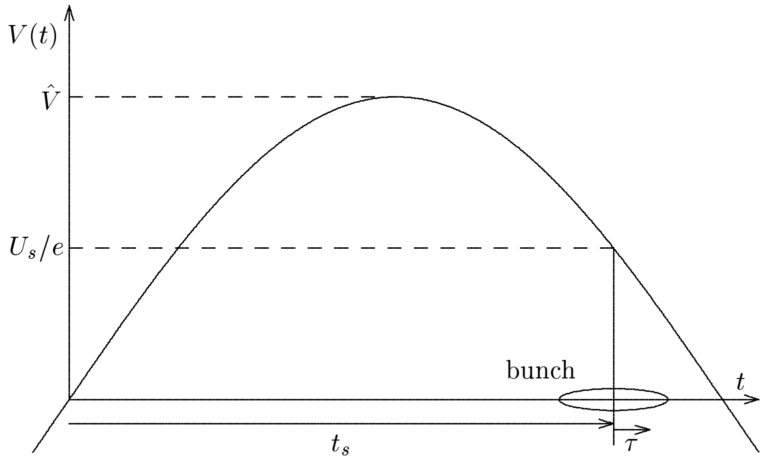
A particle with momentum deviation Δp has a different orbit length L, revolution time T_0 and revolution frequency ω_0

$$\frac{\Delta L}{L} = \alpha_c \frac{\Delta p}{p}\ , \ \ \frac{\Delta \omega_0}{\omega_0} = -\frac{\Delta T_0}{T_0} = -\left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p} = -\eta_c \frac{\Delta p}{p}$$

with momentum compaction α_c and $\eta_c = \alpha_c - 1/\gamma^2$. At the transition energy $E_T = m_0 c^2 \gamma_T$ with $\gamma_T = 1/\alpha_c^2$ the revolution frequency dependence on momentum (or energy) changes sign

$$E\ >\ E_T\ \rightarrow\ \frac{1}{\gamma^2} < \alpha_c\ \rightarrow\ \eta_c > 1\ \rightarrow\ \omega_0\,\mathrm{decreases\,with}\,\Delta E$$
$$E\ <\ E_T\ \rightarrow\ \frac{1}{\gamma^2} > \alpha_c\ \rightarrow\ \eta_c < 1\ \rightarrow\ \omega_0\,\mathrm{increases\,with}\,\Delta E\,.$$

For synchrotron radiation sources the electrons are ultra-relativistic and we approximate $\Delta p/p \approx \Delta E/E = \epsilon$, $\eta_c \approx \alpha_c$. For isochronous (low alpha) rings this has to be checked.



With an RF cavity of voltage $\hat{V}$ and frequency $\omega_{\mathsf{RF}} = h\omega_0$ and a SR energy loss per turn U the energy gain or loss δE is

$$\delta E = e\hat{V}\sin(h\omega_0(t_s+\tau)) - U$$

with t_s= synchronous arrival time at the cavity, τ= deviation from it and synchronous phase $\phi_s = h\omega_0 t_s$. For $h\omega_0\tau \ll 1$ we develop

$$\delta\epsilon = \delta\left(\frac{\Delta E}{E}\right) = \frac{e\hat{V}\sin(\phi_s)}{E} + \frac{h\omega_0 e\hat{V}\cos\phi_s}{E}\tau - \frac{U}{E}.$$

For $\delta\epsilon \ll 1$ we use a smooth approximation

$$\delta\epsilon = \dot{\epsilon}T_0 = \dot{\epsilon}\frac{2\pi}{\omega_0}\ ,\ \dot{\epsilon} = \frac{\omega_0 e\hat{V}\sin\phi_s}{2\pi E} + \frac{\omega_0^2 h e\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{U}{E}.$$

The energy loss U can depend on energy deviation ϵ

$$U(\epsilon,\tau) \approx U_0 + \frac{\partial U}{\partial E}\Delta E + \frac{\partial U}{\partial t}\tau.$$

giving for the derivative of the energy loss

$$\dot{\epsilon} = \frac{\omega_0 e\hat{V}\sin\phi_s}{2\pi E} + \frac{\omega_0^2 h e\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{U_0}{E} - \frac{\omega_0}{2\pi}\frac{dU}{dE}\epsilon - \frac{\omega_0}{2\pi}\frac{1}{E}\frac{dU}{dt}\tau.$$

For synchronous particle $\epsilon = 0\,,\ \tau = 0$ we have $U_0 = e\hat{V}\sin\phi_s$

$$\begin{aligned}\dot{\epsilon} &= \omega_0^2\frac{he\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{dU}{dE}\epsilon - \frac{1}{E}\frac{\omega_0}{2\pi}\frac{dU}{dt}\tau \\ \dot{\tau} &= \eta_c\epsilon\,.\end{aligned}$$

These two first order equations are combined into a second order one of a damped/growing oscillation

$$\ddot{\tau} + \frac{\omega_0}{2\pi}\frac{dU}{dE}\dot{\tau} - \frac{\omega_0^2 h\eta_c e\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\eta_c}{E}\frac{\omega_0}{2\pi}\frac{dU}{dt}\tau = 0.$$

Using

$$\omega_{s0}^2 = -\omega_0^2\frac{h\eta_c e\hat{V}\cos\phi_s}{2\pi E}\ ,\ \alpha_s = \frac{1}{2}\frac{\omega_0}{2\pi}\frac{dU}{dE}\,,$$

seeking a solution $\mathrm{e}^{j\omega t}$, and assuming $\alpha_s \ll \omega_{s0}$ we get

$$-\omega^2 + j\omega\alpha_s + (\omega_{s0}^2 + \frac{\omega_0}{2\pi}\frac{\eta_c}{E}\frac{dU}{dt}) = 0$$

$$\omega = j\alpha_s \pm \sqrt{(\omega_{s0}^2 + \frac{\omega_0}{2\pi}\frac{\eta_c}{E}\frac{dU}{dt}) - \alpha_s^2} \approx j\alpha_s \pm (\omega_{s0} + \frac{1}{2}\frac{\omega_0}{2\pi}\frac{\eta_c}{\omega_{s0}E}\frac{dU}{dt})\,.$$

Calling

$$\Delta\omega_i = \frac{1}{2}\frac{\omega_0}{2\pi}\frac{\eta_c}{\omega_{s0}E}\frac{dU}{dt}$$

gives

$$\epsilon = A\left(\mathrm{e}^{-\alpha_s + j(\omega_{s0}+\Delta\omega_i)t} + B\mathrm{e}^{-\alpha_s - j(\omega_{s0}+\Delta_i)t}\right).$$

For initial values $\epsilon(0) = \hat{\epsilon},\ \dot{\epsilon}(0) = -\alpha_s\hat{\epsilon}$ we get $A = B = \hat{\epsilon}/2$

$$\epsilon(t) = \hat{\epsilon}\ \mathrm{e}^{-\alpha_s t}\cos((\omega_{s0} + \Delta\omega_i)t).$$

With $\omega_s = \omega_{s0} = \omega_0\sqrt{h\eta_c e\hat{V}\cos\phi_s/2\pi E}$ for $U = 0$.
Stability if $\omega_{s0}^2 > 0$

$$E > E_T\ \eta_c < 0\ \rightarrow \cos\phi_s < 0\ \ ,\ \ E < E_T\ \eta_c > 0\ \rightarrow \cos\phi_s > 0.$$

For stability with energy loss U we also need

$$\alpha_s = \frac{1}{2}\frac{\omega_0}{2\pi}\frac{dU}{dE} > 0.$$

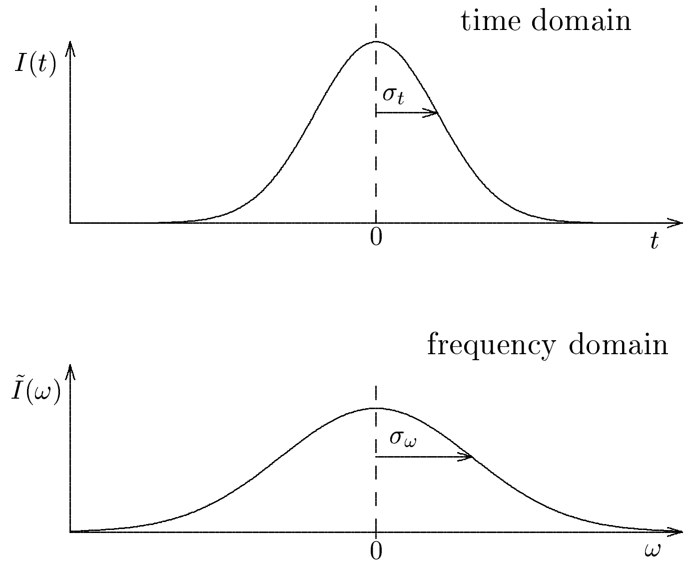
energy loss U has to increase for a positive energy deviation of the beam.

STATIONARY BUNCH WITH IMPEDANCE

Spectrum of a stationary bunch

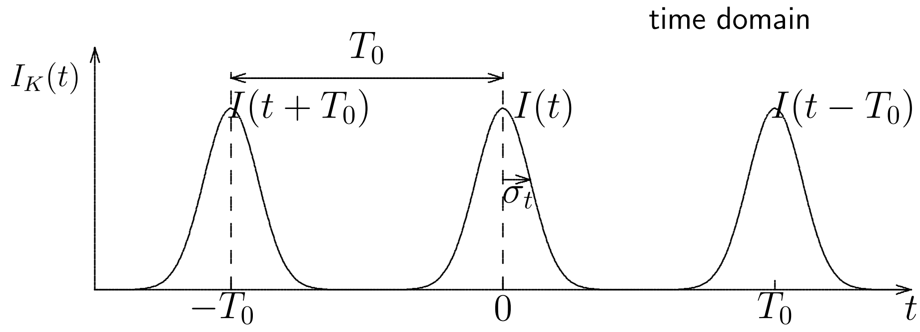
We consider a single traversal of a bunch of charge q and current $I(t)$ which is (for convenience) symmetric in t, $I(t) = I(-t)$. Its spectrum is given by the **Fourier transform**

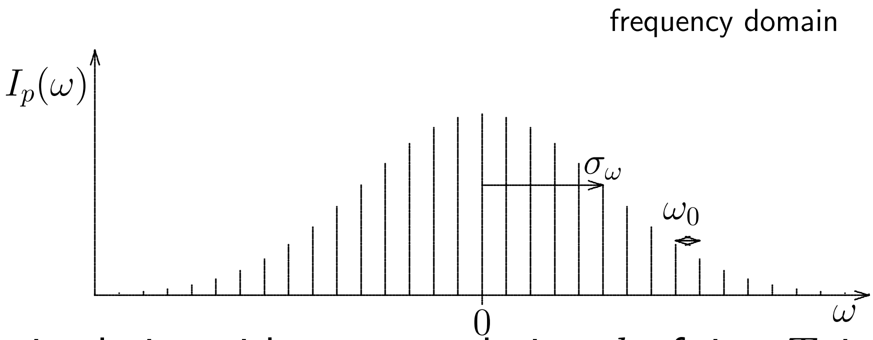
$$I(t)\ ,\ \tilde{I}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} I(t) \mathrm{e}^{-j\omega t} dt.$$



Gaussian bunch

$$I(t) = \frac{q}{\sqrt{2\pi}\sigma_t} \mathrm{e}^{-\frac{t^2}{2\sigma_t^2}}\ ,\ \tilde{I}(\omega) = \frac{q}{\sqrt{2\pi}} \mathrm{e}^{-\frac{\omega^2}{2\sigma_\omega^2}}\ ,\ \sigma_\omega = \frac{1}{\sigma_t}$$





A bunch circulating with many revolutions k of time T_0 in a ring represents a periodic current can be express by a Fourier series representing a line spectrum

$$I_K(t) = \sum_{k=-\infty}^{k=\infty} I(t-kT_0) = \sum_{p=-\infty}^{\infty} I_p \mathrm{e}^{jp\omega_0 t} = I_0 + 2\sum_{p=1}^{\infty} I_p \cos(p\omega_0 t)$$

with

$$I_p = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} I(t)\mathrm{e}^{jp\omega_0 t} = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} I(t)\cos(p\omega_0 t) = \frac{\omega_0}{\sqrt{2\pi}}\tilde{I}(p\omega_0).$$

Since we assumed a symmetric bunch $I(t) = I(-t)$ we have I_p real and only cosine terms. With this definition we have $I_p \approx I_0$ at low frequencies.

Gaussian bunch

$$I_p = \frac{q\omega_0}{2\pi}\,\mathrm{e}^{-\frac{p\omega_0}{2\sigma_\omega}} = \frac{q}{T_0}\,\mathrm{e}^{-\frac{p\omega_0}{2\sigma_\omega}}.$$

Voltage induced by the stationary bunch

In an impedance $Z(\omega) = Z_r(\omega) + jZ_i(\omega)$ the stationary bunch current induces a voltage

$$V_K(t) = \sum_{p=-\infty}^{\infty} Z(p\omega_0) I_p \mathrm{e}^{jp\omega_0 t}.$$

Combining positive and negative frequencies, using symmetry relations $Z_r(-\omega) = Z_r(\omega)$, $Z_i(-\omega) = -Z_i(\omega)$ and $Z(0) = 0$

$$V_K(t) = 2 \sum_{p=1}^{\infty} I_p \left[Z_r(p\omega_0) \cos(p\omega_0 t) - Z_i(p\omega_0) \sin(p\omega_0 t) \right].$$

Energy loss of a stationary circulating bunch

The energy lost of the whole stationary bunch in one turn in an impedance $Z(\omega)$ is

$$W_b = \int_{-T_0/2}^{T_0/2} I_K(t) V_K(t) dt.$$

The product in the above expression contains

$$I_K(t) = \sum_{p'=-\infty}^{\infty} I_{p'} \mathrm{e}^{jp'\omega_0 t} \ , \ V_K(t) = \sum_{p=-\infty}^{\infty} Z(p\omega_0) I_p \mathrm{e}^{jp\omega_0 t}.$$

contains integral $\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(p'\omega_0 t) \cos(p\omega_0 t) dt = \begin{matrix} \frac{T_0}{2} & \text{for } p' = p \\ 0 & \text{for } p' \neq p \end{matrix}$

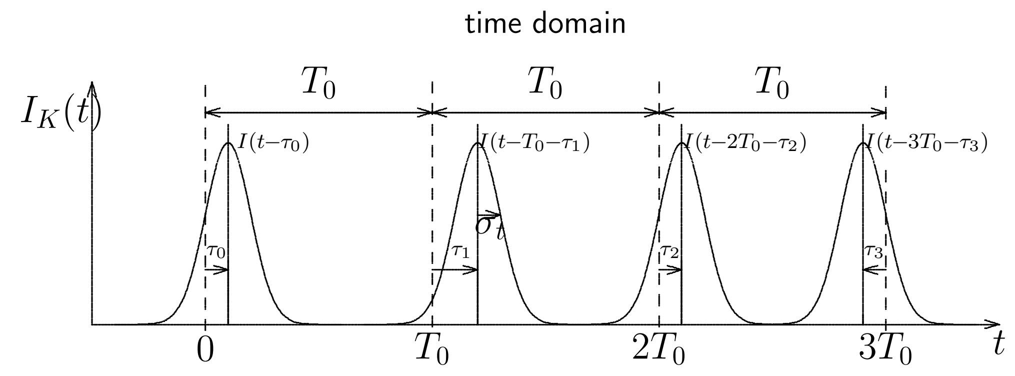
while the integral over $\cos(p'\omega_0 t) \sin(p\omega_0 t)$ always vanishes.

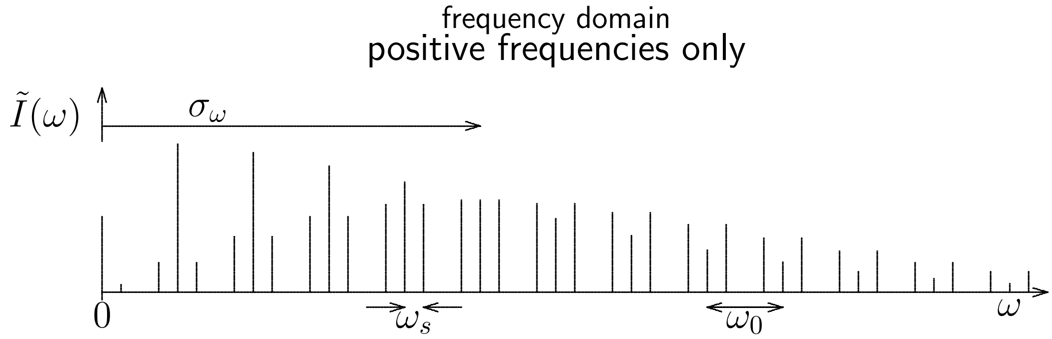
$$W_b = T_0 \sum_{p=-\infty}^{\infty} I_p^2 Z(p\omega_0) = 2T_0 \sum_1^{\infty} I_p^2 Z_r(p\omega_0).$$

The loss is only due to Z_r. This is the energy loss of the whole bunch containing N_b particle. More relevant is the one per electron U

$$U = \frac{W_b}{N_b} = \frac{2T_0}{N_b} \sum_1^{\infty} I_p^2 Z_r(p\omega_0) = \frac{2e}{I_0} \sum_1^{\infty} I_p^2 Z_r(p\omega_0).$$

OSCILLATING BUNCH WITH A CAVITY
Spectrum of an oscillating bunch





A bunch executes a synchrotron oscillation with frequency $\omega_s = \omega_0 Q_s$ and amplitude $\hat{\tau}$. It gives a modulation of its passage time t_k at a cavity in successive turn k

$$I_K(t) = \sum_{k=-\infty}^{\infty} I(t-kT_0-\tau_k)\ , \quad \text{with } \tau_k = \hat{\tau}\cos(2\pi Q_s k) \approx \hat{\tau}\cos(\omega_s t)$$

leading to a phase modulation of the current

$$I_K(t) = \sum_{p=-\infty}^{\infty} I_p\, \mathrm{e}^{jp\omega_0(t-\hat{\tau}\cos(\omega_s t))} = I_0 + 2\sum_{\omega>0} I_p \cos(p\omega_0(t-\hat{\tau}\cos(\omega_s t))).$$

We omit the dc-part I_0 call $\omega_s = Q_s\omega_0$ and develop for $p\omega_0\hat{\tau} \ll 1$

$$I_K(t) \;=\; 2\sum_{\omega>0} I_p \left[\cos(p\omega_0 t) + p\omega_0\hat{\tau}\sin(p\omega_0 t)\cos(\omega_s t)\right]$$

$$\approx 2\sum_{\omega>0} I_p \left[\cos(p\omega_0 t) + \frac{p\omega_0\hat{\tau}}{2}(\sin((p+Q_s)\omega_0 t) + \sin((p-Q_s)\omega_0 t))\right].$$

The modulation by the synchrotron oscillation results in sidebands in the spectrum. The are out of phase and increase first with frequency.

Voltage induced by an oscillating bunch

The voltage induced in an impedance $Z(\omega) = Z_r(\omega) + jZ_i(\omega)$ by an oscillating current is

$$\begin{aligned} I(t) &= \hat{I}\cos(\omega t) \;\rightarrow\; V(t) = \hat{I}\left(Z_r(\omega)\cos(\omega t) - Z_i(\omega)\sin(\omega t)\right) \\ I(t) &= \hat{I}\sin(\omega t) \;\rightarrow\; V(t) = \hat{I}\left(Z_r(\omega)\sin(\omega t) + Z_i(\omega)\cos(\omega t)\right). \end{aligned}$$

The current of the oscillating bunch is

$$I_K(t) = 2\sum_{\omega>0}^{\infty} I_p \left[\cos(p\omega_0 t) + \frac{p\omega_0\hat{\tau}}{2}\left(\sin((p+Q_s)\omega_0 t) + \sin((p-Q_s)\omega_0 t)\right)\right].$$

We start with a **resistive** impedance Z_r and get the voltage

$$\begin{aligned} V_{Kr}(t) \;=\; & 2\sum_{\omega>0}^{\infty} I_p \left[Z_r(p\omega_0)\cos(p\omega_0 t)\right. \\ & \left. + \frac{p\omega_0\hat{\tau}}{2}\left(Z_r((p+Q_s)\omega_0)\sin((p+Q_s)\omega_0) + Z_r((p-Q_s)\omega_0)\sin((p-Q_s)\omega_0 t)\right)\right]. \end{aligned}$$

We split the trigonometric functions

$$\begin{aligned} V_{Kr}(t) = & 2\sum_{\omega>0}^{\infty} I_p \left[Z_r(p\omega_0)\cos(p\omega_0 t)\right. \\ & + \frac{p\omega_0\hat{\tau}}{2}\left[Z_r((p+Q_s)\omega_0)\left(\sin(p\omega_0 t)\cos(\omega_s t) + \cos(p\omega_0 t)\sin(\omega_s t)\right)\right. \\ & \left.\left. Z_r((p-Q_s)\omega_0)\left(\sin(p\omega_0 t)\cos(\omega_s t) - \cos(p\omega_0 t)\sin(\omega_s t)\right)\right]\right] \end{aligned}$$

The synchrotron motion, a modulation of the arrival time each revolution k, is approximated as a modulation in time

$$\tau_k = \hat{\tau}\cos(2\pi Q_s k) \quad\rightarrow\quad \tau = \hat{\tau}\cos(\omega_s t) \;\;,\;\; \dot{\tau} = -\omega_s\hat{\tau}\sin(\omega_s t).$$

and we get the voltage induced in the resistive impedance

$$\begin{aligned} V_{kr}(t) \;=\; & 2\sum_{\omega>0}^{\infty} I_p \left[Z_r(p\omega_0)\cos(p\omega_0 t)\right. \\ & + \frac{p\omega_0}{2}\left[Z_r((p+Q_s)\omega_0)\left(\sin(\omega_0 t)\tau - \cos(p\omega_0 t)\frac{\dot{\tau}}{\omega_s}\right)\right. \\ & \left.\left. + Z_r((p-Q_s)\omega_0)\left(\sin(p\omega_0 t)\tau + \cos(p\omega_0 t)\frac{\dot{\tau}}{\omega_s}\right)\right]\right] \end{aligned}$$

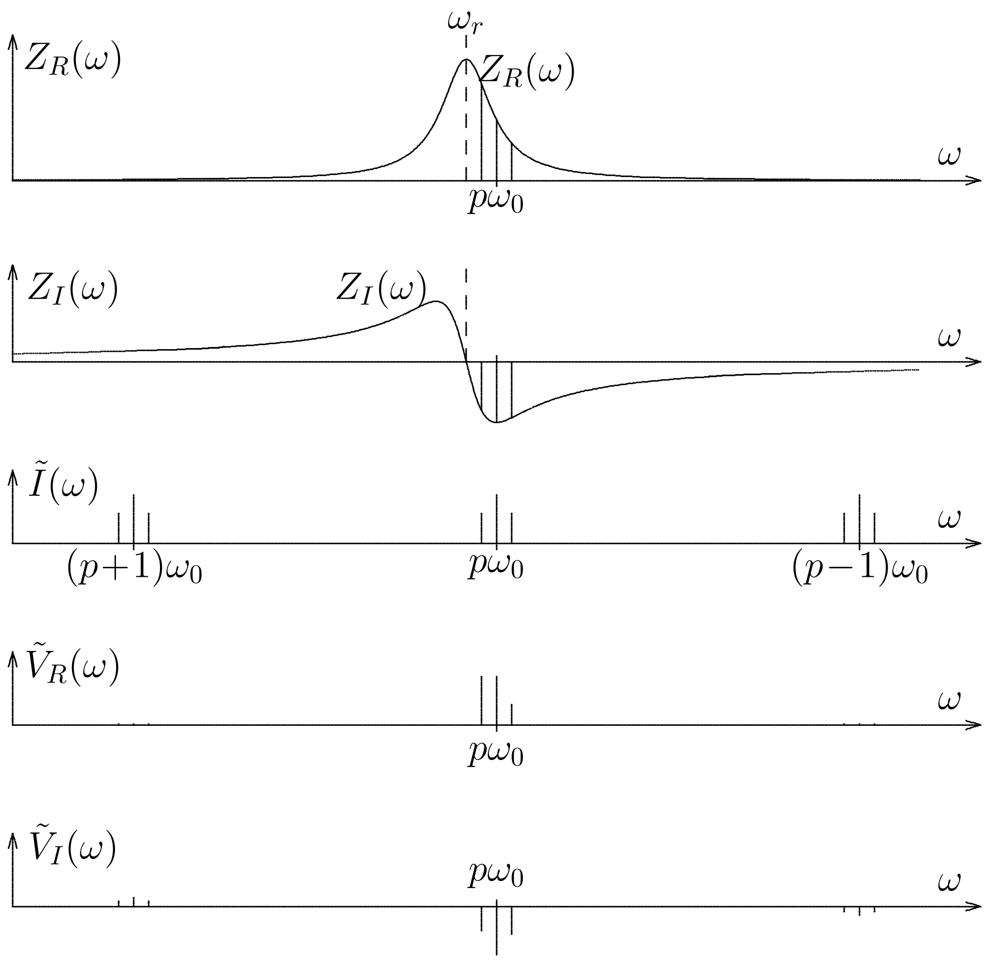
The voltage induced in the **reactive** (imaginary) part Z_i is

$$
\begin{aligned}
V_{Ki}(t) &= 2\sum_{\omega>0}^{\infty} I_p\left[-Z_i(p\omega_0)\sin(p\omega_0)\right.\\
&+ \frac{\omega_0\hat{\tau}}{2}\left((p+Q_s)Z_i((p+Q_s)\omega_0)\cos((p+Q_s)\omega_0 t)\right.\\
&\qquad \left.\left.+(p-Q_s)Z_i((p-Q_s)\omega_0)\cos((p-Q_s)\omega_0 t)\right)\right].
\end{aligned}
$$

Splitting the trigonometric functions and using τ and $\dot{\tau}$

$$
\begin{aligned}
V_{Ki}(t) &= 2\sum_{\omega>0}^{\infty} I_p\left[-Z_i(p\omega_0)\sin(p\omega_0)\right.\\
&+ \frac{\omega_0\hat{\tau}}{2}\left((p+Q_s)Z_i((p+Q_s)\omega_0)\left(\cos(p\omega_0 t)\tau+\sin(p\omega_0)\frac{\dot{\tau}}{\omega_s}\right)\right.\\
&\qquad \left.\left.+(p-Q_s)Z_r((p-Q_s)\omega_0)\left(\cos(p\omega_0 t)\tau-\sin(p\omega_0 t)\frac{\dot{\tau}}{\omega_s}\right)\right)\right].
\end{aligned}
$$

In the following we assume a narrow band impedance $\delta\omega < \omega_0$ spanning only one revolution harmonic p with its two sidebands.



Average voltage seen by a bunch during the traversal k

The average voltage seen by the N_b particles during a turn k is the convolution of the **single traversal** bunch current $I(t,k)$ with the oscillating voltage excited over many turns in the cavity.

$$\langle V_r \rangle = \frac{1}{eN_b}\int_{-\infty}^{\infty} I(t,k)V_K(t)dt = \frac{1}{I_0 T}\int_{-\infty}^{\infty} I(t,k)V_K(t)dt.$$

We assume small excursion $\tau = \tau_k$ and take only first order in $\tau, \dot{\tau}$

$$I(t-kT_0-\tau) \approx I(t) - \tau\frac{dI}{dt}.$$

Since $I(t)$ is even its derivative is odd. We meet the integrals

$$\int_{-\infty}^{\infty} I(t)\cos(p\omega t)dt = \sqrt{2\pi}\tilde{I}(p\omega_0) = \frac{2\pi}{\omega_0}I_p$$

$$\int \frac{dI}{dt}\sin(p\omega_0 t)dt = -p\omega_0\int I(t)\cos(p\omega_0 t)dt = -p\omega_0\frac{2\pi}{\omega_0}I_p$$

$$\int I(t)\sin(p\omega_0 t)dt = 0\ ,\ \int\frac{dI}{dt}\cos(p\omega_0 t) = 0.$$

The average voltage due to the resistive impedance is

$$\langle V_r \rangle = \frac{2I_p^2}{I_0}\left[Z_r(p\omega_0) - \frac{p\omega_0\dot{\tau}}{2\omega_s}\left(Z_r((p+Q_s)\omega_0) - Z_r((p-Q_s)\omega_0)\right)\right].$$

First term, independent of τ, is energy loss of stationary bunch. The others are proportional to $\dot{\tau} = \eta_c\Delta E/E$ and can give instabilities.
For the reactive impedance we have

$$\langle V_i \rangle = \frac{2I_p^2 p\omega_0\tau}{I_0}\left[-Z_i(p\omega_0) + \frac{1}{2}\left(Z_i((p+Q_s)\omega_0) + Z_i((p-Q_s)\omega_0)\right)\right]$$

which is proportional to τ and creates a synchrotron frequency change.
We abbreviate for later applications

$$Z_{pr}^0 = Z_r(p\omega_0)\ ,\ Z_{pr}^+ = Z_r((p+Q_s)\omega_0)\ ,\ Z_{pr}^- = Z_r((p-Q_s)\omega_0)$$

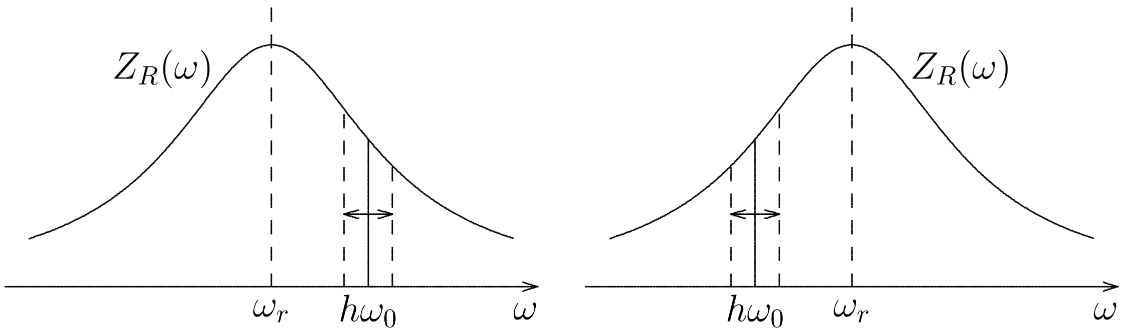
$$Z_{pi}^0 = Z_i(p\omega_0)\ ,\ Z_{pi}^+ = Z_i((p+Q_s)\omega_0)\ ,\ Z_{pi}^- = Z_i((p-Q_s)\omega_0)$$

$$\langle V \rangle = \frac{2I_p^2}{I_0}\left[Z_{pr}^0 - \frac{p\omega_0\dot{\tau}}{2\omega_s}\left(Z_{pr}^+ - Z_{pr}^-\right)\right] + \frac{2I_p^2 p\omega_0\tau}{I_0}\left[-Z_{pi}^0 + \frac{1}{2}\left(Z_{pi}^+ + Z_{pi}^-\right)\right]$$

ROBINSON INSTABILITY

Qualitative treatment

The most important longitudinal instability is based on the interaction between the bunch and a cavity with memory, called **Robinson** instability. As a qualitative treatment we consider a single circulating bunch interaction with a cavity of resonance frequency ω_r and impedance $Z(\omega)$ of which we consider only the resistive part Z_r.



The revolution frequency ω_0 depends on energy deviation ΔE

$$\frac{\Delta\omega_0}{\omega_0} = -\eta_c \frac{\Delta E}{E}.$$

While the bunch is executing a coherent dipole mode oscillation $\epsilon(t) = \hat{\epsilon}\cos(\omega_s t)$ its energy and revolution frequency are modulated. **Above transition** ω_0 is **small** when the **energy is high** and ω_0 is **large** when the **energy is small**. If the cavity is tuned to a resonant frequency slightly smaller than the RF frequency $\omega_r < p\omega_0$ the bunch sees a higher impedance and **loses more energy** when it has an **energy excess** and it **loses less energy** when it has a **lack of energy**. This leads to a **damping** of the oscillation. If $\omega_r > p\omega_0$ this is reversed and leads to an **instability**. Below transition energy the dependence of the revolution frequency is reversed which changes the stability criterion.

Quantitative treatment

We consider a narrow-band cavity and one circulating bunch executing a synchrotron oscillation $\tau = \hat{\tau}\cos(\omega_s t)$ and producing sidebands to revolution frequency harmonics which induce an averaged voltage $\langle V\rangle$ and resulting energy loss $U = e\langle V\rangle$ in a turn. We will now include this in the energy gain/loss in the equation of synchrotron oscillation

$$\delta\epsilon = \delta\left(\frac{\Delta E}{E}\right) = \frac{e\hat{V}\sin(\phi_s)}{E} + \frac{h\omega_0 e\hat{V}\cos\phi_s}{E}\tau - \frac{U}{E}.$$

For $\delta\epsilon \ll 1$ we use a smooth approximation

$$\delta\epsilon = \dot{\epsilon}T_0 = \dot{\epsilon}\frac{2\pi}{\omega_0}\ ,\ \dot{\epsilon} = \frac{\omega_0 e\hat{V}\sin\phi_s}{2\pi E} + \frac{\omega_0^2 he\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{U}{E}.$$

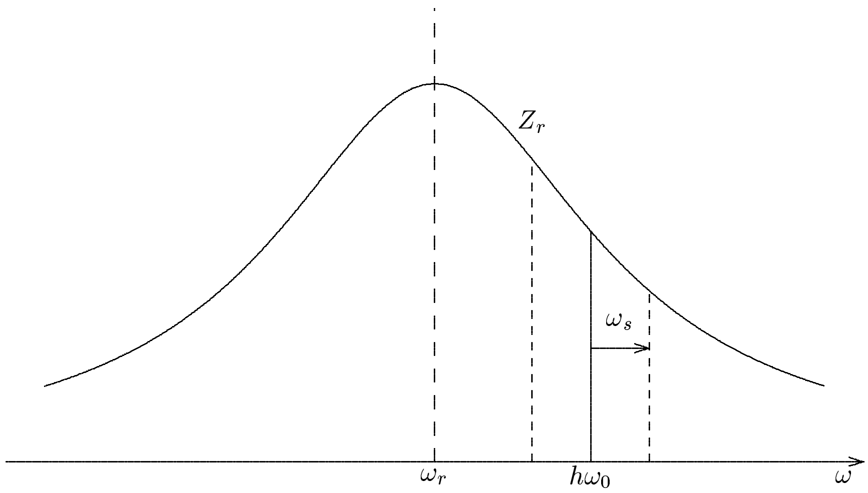
The energy loss U depends on τ and $\dot{\tau}$

$$\langle V\rangle = \frac{2I_p^2}{I_0}\left[Z_{pr}^0 - \frac{p\omega_0\dot{\tau}}{2\omega_s}\left(Z_{pr}^+ - Z_{pr}^-\right)\right] + \frac{2I_p^2 p\omega_0\tau}{I_0}\left[-Z_{pi}^0 + \frac{1}{2}\left(Z_{pi}^+ + Z_{pi}^-\right)\right]$$

$$\begin{aligned}
\text{giving } \dot{\epsilon} &= \frac{\omega_0 e\hat{V}\sin\phi_s}{2\pi E} - \frac{2I_p^2 Z_{pr}^0}{I_0}\omega_0^2 he\hat{V} \\
&+ \frac{\omega_0^2 he\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{2I_p^2 p\omega_0\tau}{I_0}\left[-Z_{pi}^0 + \frac{1}{2}\left(Z_{pi}^+ + Z_{pi}^-\right)\right] \\
&+ \frac{p\omega_0\dot{\tau}}{2\omega_s}\left(Z_{pr}^+ - Z_{pr}^-\right) \\
\dot{\tau} &= \eta_c\epsilon.
\end{aligned}$$

For equilibrium $\hat{V}\sin\phi_s = \dfrac{2I_p^2 Z_{pr}^0}{I_0}$, using $\omega_{s0}^2 = -\omega_0^2\dfrac{h\eta_c e\hat{V}\cos\phi_s}{2\pi E}$

$$\begin{aligned}
\ddot{\tau} &+ \frac{pI_p^2\omega_{s0}}{2I_0 h\hat{V}\cos\phi_s}(Z_{pr}^+ - Z_{pr}^-)\dot{\tau} \\
&+\omega_{s0}^2\left(1 - \frac{pI_p^2}{2I_0 h\hat{V}\cos\phi_s}(-2Z_{pi}^0 + Z_{pi}^+ + Z_{pi}^-)\right)\tau = 0.
\end{aligned}$$



$$\ddot{\tau} + \frac{pI_p^2\omega_{s0}}{2I_0h\hat{V}\cos\phi_s}(Z_{pr}^+ - Z_{pr}^-)\dot{\tau}$$
$$+\omega_{s0}^2\left(1 - \frac{pI_p^2}{2I_0h\hat{V}\cos\phi_s}(-2Z_{pi}^0 + Z_{pi}^+ + Z_{pi}^-)\right)\tau = 0.$$

This second order equation describes a damped oscillation

$$\epsilon = \hat{\epsilon}\mathrm{e}^{-\alpha_s t}\cos(\omega_s t + \phi)$$

with the damping or growth rate

$$\alpha_s = \frac{\omega_{s0}pI_p^2(Z_{pr}^+ - Z_{pr}^-)}{2I_0h\hat{V}\cos\phi_s}.$$

The growth rate is given by the difference of the resistive impedance at the upper and lower synchrotron sideband. Above transition $\cos\phi_s < 0$ and $\alpha_s > 0$, i.e. stability if $Z_{pr}^- > Z_{pr}^+$ already obtained from qualitative arguments.

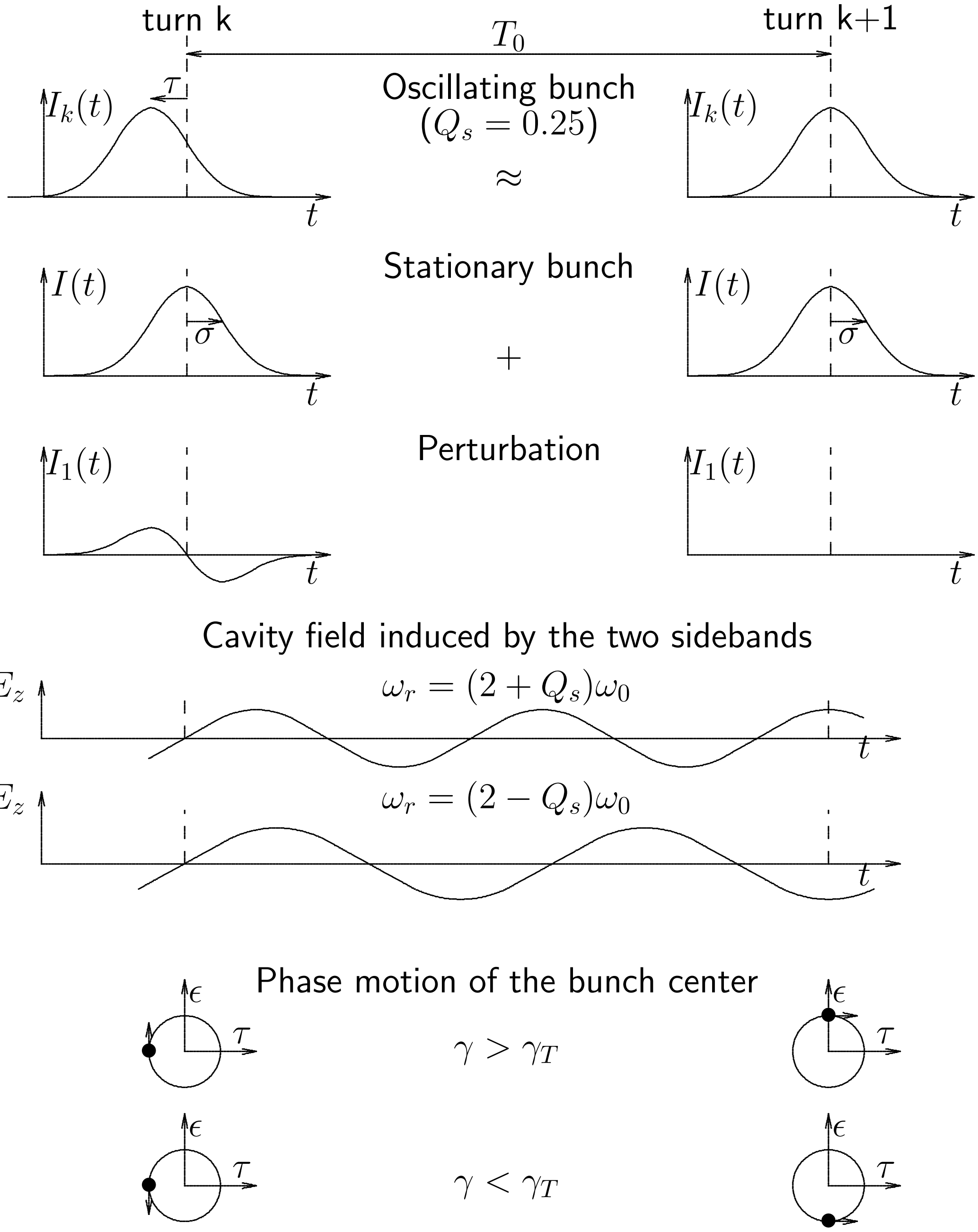
The RF cavity itself has a narrow-band impedance around $h\omega_0$ which can drive an instability. Since the bunch length is much shorter than the RF wavelength we have $I_p = I_h \approx I_0$ so that

$$\alpha_s \approx \frac{\omega_{s0}I_0(Z_{pr}^+ - Z_{pr}^-)}{2I_0\hat{V}\cos\phi_s}.$$

There is also a frequency shift due to the reactive impedance

$$\omega_s^2 = \omega_{s0}^2\left(1 + \frac{pI_p^2Z_{pi}^0}{I_0h\hat{V}\cos\phi_s} - \frac{pI_p^2(Z_{pi}^+ + Z_{pi}^-)}{I_0h\hat{V}\cos\phi_s}\right).$$

Qualitative understanding

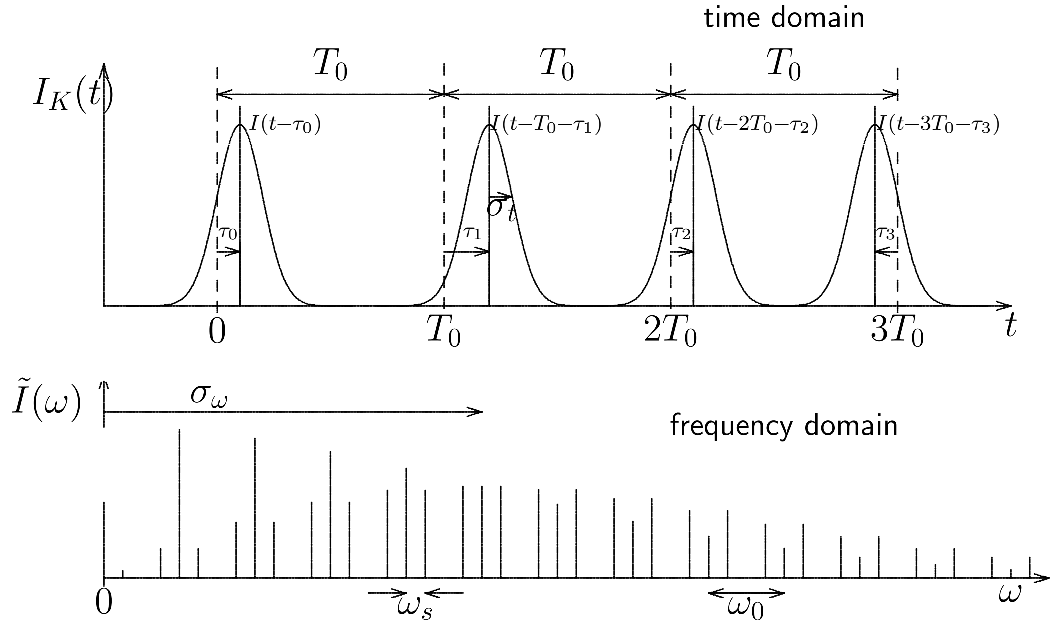


The growth rate is given by the side-band impedance difference

$$\alpha_s = \frac{\omega_{s0} p I_p^2 (Z_{pr}^+ - Z_{pr}^-)}{2 I_0 h \hat{V} \cos \phi_s}.$$

This can be understood qualitatively taking as example $p = 2$ and $Q_s = \omega_s/\omega_0 = 0.25$. At $\gamma > \gamma_T$ the voltage induced by the upper sideband enhances the oscillation, the one from the lower sideband reduces it. Below transition the situation is reversed.

More accurate treatment



$$I_K(t) = \sum_{-\infty}^{\infty} I(t - kT_0 - \tau_k) \ , \ \tau_k = \hat{\tau}\cos(2\pi Q_s k) \approx \tau_k = \hat{\tau}\cos(\omega_s t)$$

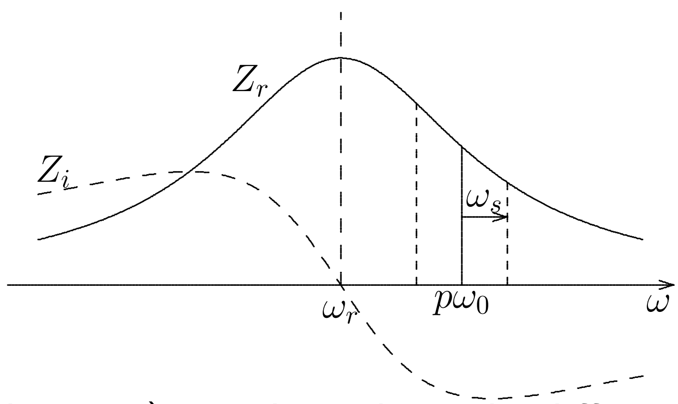
The bunch position is a function of turns k rather than time t giving current components

$$\omega_{p\pm} = \omega_0(p \pm Q_s), \quad I_{p\pm} = \frac{\omega_0}{\sqrt{2\pi}}\tilde{I}(\omega_{p\pm}), \quad \tilde{I}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} I(t)\mathrm{e}^{-\omega t}dt.$$

This gives a slight modification of the growth rate of synchrotron oscillations

$$\tau = \hat{\tau}\mathrm{e}^{-\Delta\omega_{si}t}\cos\left((\omega_s + \Delta\omega_{sr})t\right)$$

$$\frac{1}{\tau_s} = \omega_s \frac{\Big((p+Q_s)I_{p+}^2 Z_r(\omega_{p+}) - (p-Q_s)I_{p-}^2 Z_r(\omega_{p-})\Big)}{2hI_0\hat{V}\cos\phi_s}$$

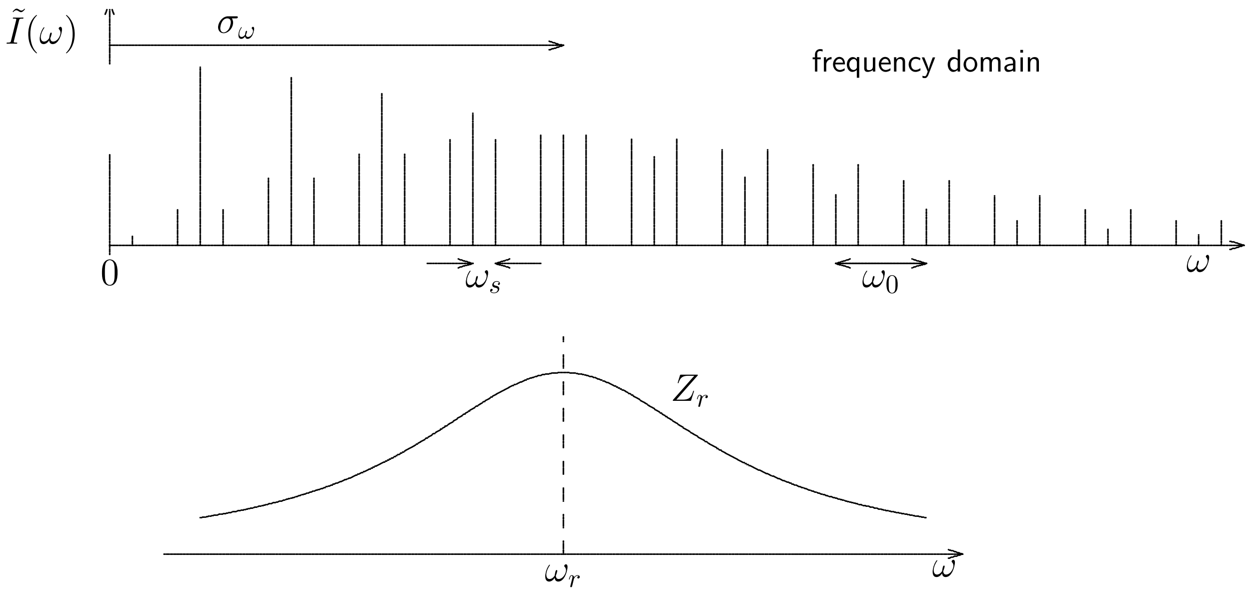


The growth (or damping) rate depends on the difference in impedance between the upper and lower sideband.

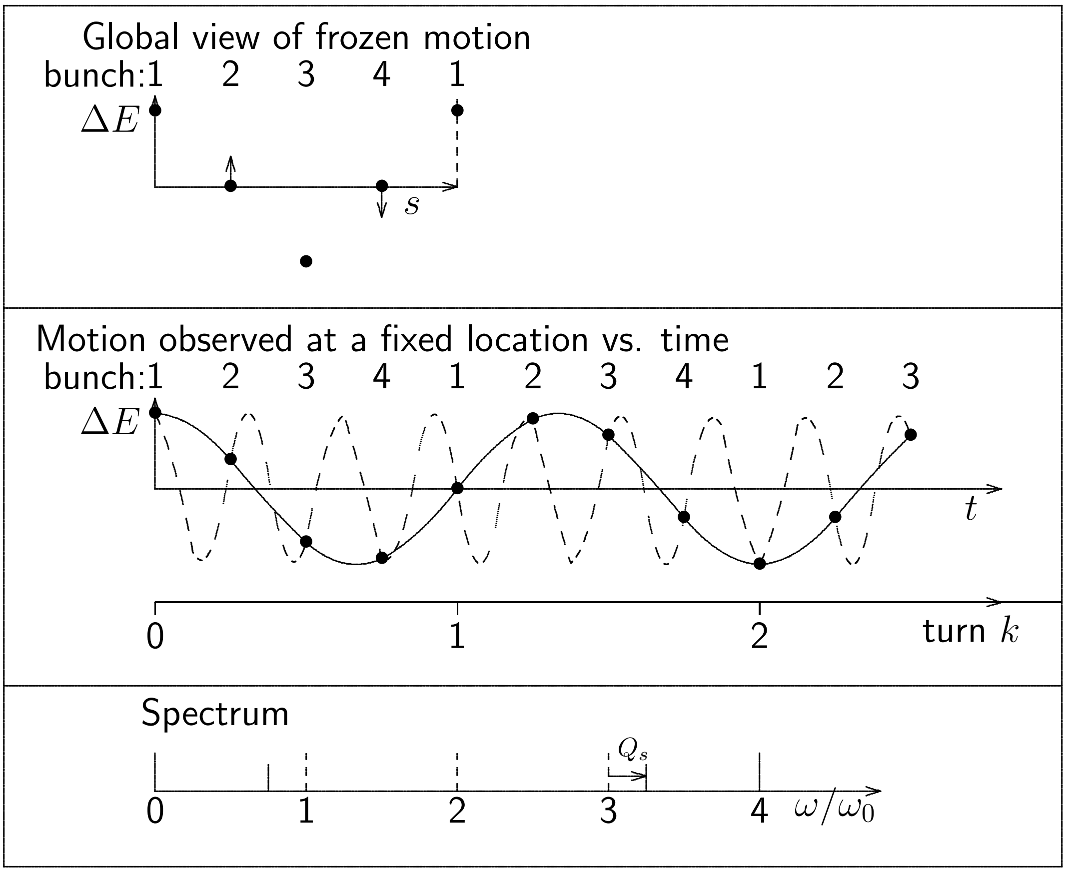
General impedance

For an impedance of larger band-width we have to take the impedance at the sidebands of **all** revolution harmonics $p\omega_0$

$$\frac{1}{\tau_s} = \frac{\omega_s}{2hI_0\hat{V}\cos\phi_s}\sum_p \Big((p+Q_s)I_{p+}^2 Z_r(\omega_{p+}) - (p-Q_s)I_{p-}^2 Z_r(\omega_{p-})\Big)$$



Many bunches



M equidistant bunches have M independent modes of coupled bunch oscillations labeled $0 \le n \le M-1$ related to oscillation phase difference $\Delta\phi$ of adjacent bunches $n = \Delta\phi/(2\pi M)$. Each mode n has one pair of sidebands in each frequency range of $M\omega_0$

$$\omega_{p\pm} = \omega_0(pM \pm (n + Q_s))$$

The example shows $M = 4$, $n = 1$ and $Q_s = \omega_s/\omega_0 = 0.25$ The growth rate of each mode n is given by a sum over the impedance differences of each sideband pair.

$$\frac{1}{\tau_s} = \frac{\omega_s}{2hI_0\hat{V}\cos\phi_s}\sum_p \Big((p+Q_s)I_{p+}^2 Z_r(\omega_{p+}) - (p-Q_s)I_{p-}^2 Z_r(\omega_{p-})\Big)$$

Bunch shape oscillations

In addition to the rigid dipole modes ($m = 1$) there are bunch shape oscillations, quadrupole mode ($m = 2$), sextupole mode ($m = 3$), etc. with the frequencies

$$\omega_{p\pm} = \omega_0(pM \pm (n + mQ_s)).$$

BUNCH LENGTHENING

A ring impedance consists often of many resonances with frequencies ω_r, shunt impedance R_s and quality factors Q. At low frequencies, $\omega < \omega_r$, their impedances are mainly inductive

$$Z(\omega) = R_s \frac{1 - jQ\frac{\omega^2-\omega_r^2}{\omega\omega_r}}{1+\left(Q\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)^2} \approx j\frac{R_s\omega}{Q\omega_r} + ...$$

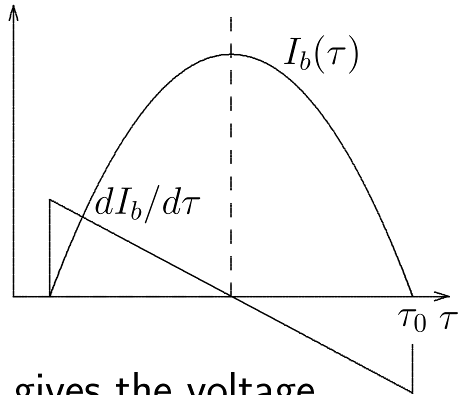
The sum impedance at low frequencies of all these resonances divided by the mode number $n = \omega/\omega_0$ is called

$$\left|\frac{Z}{n}\right|_0 = \sum_k \frac{R_{sk}\omega_0}{Q_k\omega_{rk}} = L\omega_0.$$

with L being the inductance. A bunch with current $I_b(t)$ induces a voltage $V_i = -LdI_b/dt$ which is added to the RF-voltage

$$V(t) = \hat{V}\sin(h\omega_0 t) - L\frac{dI_b}{dt}.$$

Developing around t_s, using $\tau = t - t_s$, $\phi_s = h\omega_0 t_s$ and using a parabolic bunch



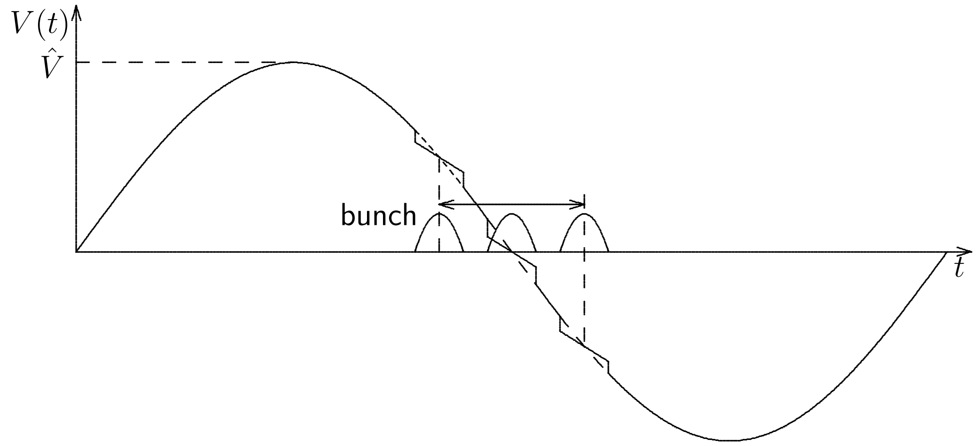
Average current I_0
$I_b(\tau) = \hat{I}(1 - \tau^2/\tau_0^2)$
$dI_b/d\tau = 3\pi I_0\tau/\tau_0^3$

gives the voltage

$$V = \hat{V}\sin\phi_s + \hat{V}\cos\phi_s h\omega_0\tau\left(1 + \frac{3\pi|Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\tau_0)^3}\right).$$

and synchrotron frequency shift for the particles in the bunch

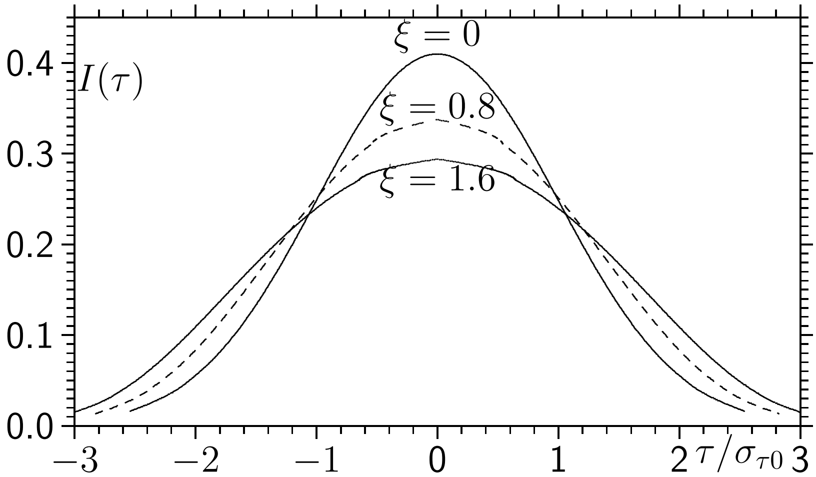
$$\frac{\Delta\omega_s}{\omega_s} \approx \frac{3\pi|Z/n|_0 I_0}{2h\hat{V}\cos\phi_s(\omega_0\tau_0)^3}$$



Only the incoherent frequency of individual particles changed (reduced for $\gamma > \gamma_T$, increased for $\gamma < \gamma_T$). The coherent dipole (rigid bunch) is not affected. This separates the coherent synchrotron frequency from the incoherent distribution and leads to a loss of Landau damping. The reduction of longitudinal focusing increases the bunch length given by a 4th order equation for protons with constant phase space area

$$\left(\frac{\tau_0}{\tau_{00}}\right)^4 + \frac{3\pi |Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\tau_{00})^3}\left(\frac{\tau_0}{\tau_{00}}\right) - 1 = 0$$

and a 3rd order one for electrons having fixed energy spread. The assumed parabolic bunch current is the projection of an elliptic phase space distribution. In this case the bunch form is not changed just its length increased. This is more complicated for other distribution like for the Gaussian shown in the figure.



$$\xi = \frac{\sqrt{2\pi}h^2 I_0 |Z/n|_0}{\hat{V}\cos\phi(h\omega_0\sigma_{\tau 0})^3}$$