INSTABILITIES

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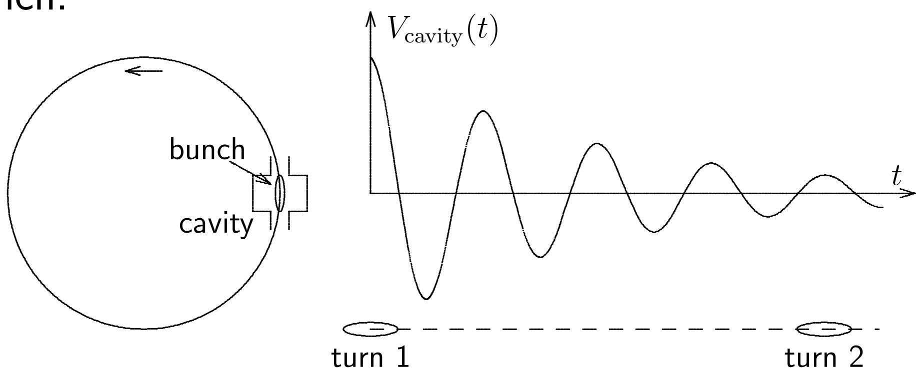
Review of Instabilities Mechanisms
Instabilities I
Instabilities II

Review of Instabilities Mechanisms

OVERVIEW

The motion of a single particle in a storage ring is determined by the external guide fields (dipole and quadrupole magnets, RF-system, etc.), initial conditions and synchrotron radiation. Many particles in a beam may represent a sizable charge and current which act as a source of electromagnetic fields (self fields). They are modified by boundary conditions imposed by the beam surroundings (vacuum chambers, cavities, etc.) and act back on the beam. This can lead to a **frequency shift** (change of the betatron or synchrotron frequency), to an increase of a small disturbance of the beam, i.e. an **instability** or to a **change of the particle distribution**, e.g. bunch lengthening. These phenomena are called **collective effects** being due to a **coherent** or **collective** action of many particles. The role played in this process by the electrical properties of the beam surroundings is expressed by an **impedance**.

As an example we take a bunch in a storage ring going through a cavity where it induces electromagnetic fields which oscillate and slowly decay away. In the next turn the same bunch finds some field left and gets influenced by it. Depending on the phase of the field seen in the next turn a small initial perturbation is increased or decreased leading to an exponentially growing or decaying oscillation of the bunch.



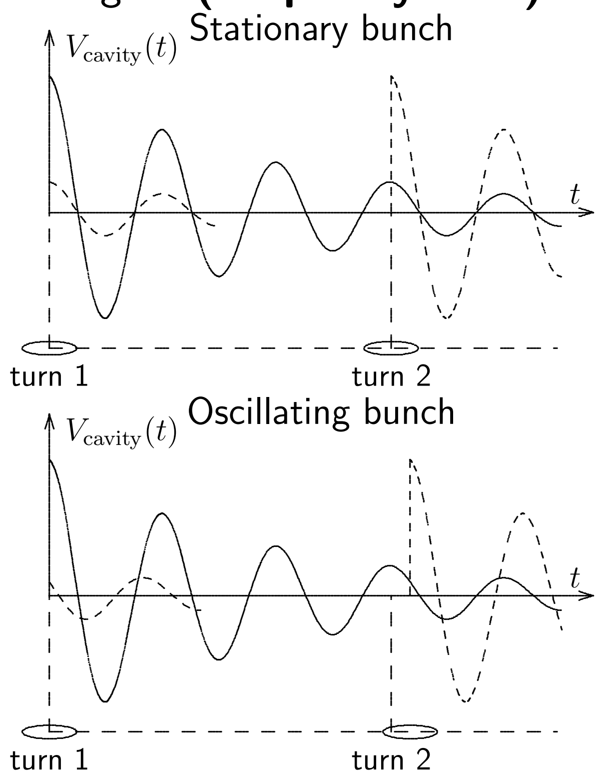
Multi-turn effects

In the example the induced fields have a memory and the instability is built up over many turns. These self fields are often small compared to the guide fields and their effects is treated as a **perturbation** in 3 steps.

a) We determine the stationary particle distribution given by the guide field, initial condition and synchrotron radiation.

b) We consider small disturbances and calculate the fields they create including the boundary conditions (impedance).

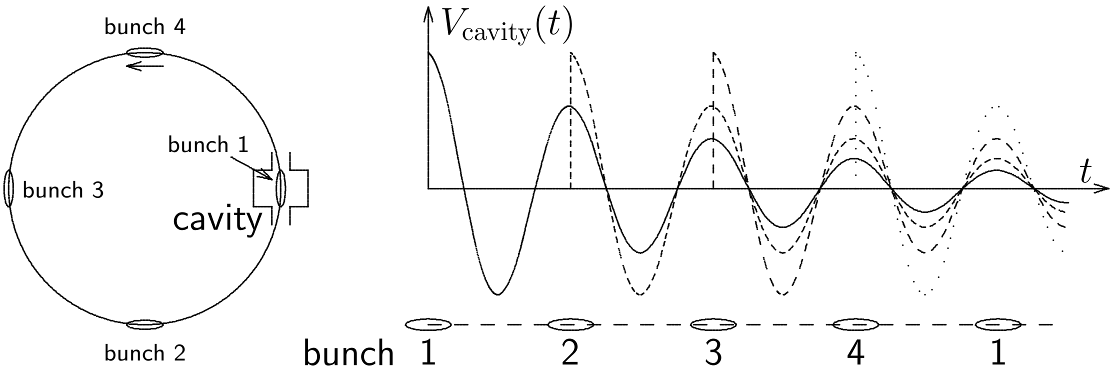
c) We calculate the effect of these fields to see if the initial disturbance is increased **(instability)** or decreased **(damping)** or the oscillation mode changed **(frequency shift)**.



As disturbances we consider orthogonal (independent) oscillation modes and investigate the stability of each. This works for weak interactions which don't alter the nature of the modes but determine only their exponential growth over many turns. Multi-turn effects are driven by narrow frequency band impedances with memory.

Multi-bunch effects

With many circulating bunches their individual oscillations can be coupled by an impedance with a shorter memory bridging just the bunch spacing instead of the revolution time. Multi-turn and multi-bunch instabilities have the same qualitative properties and are called multi-traversal effects.



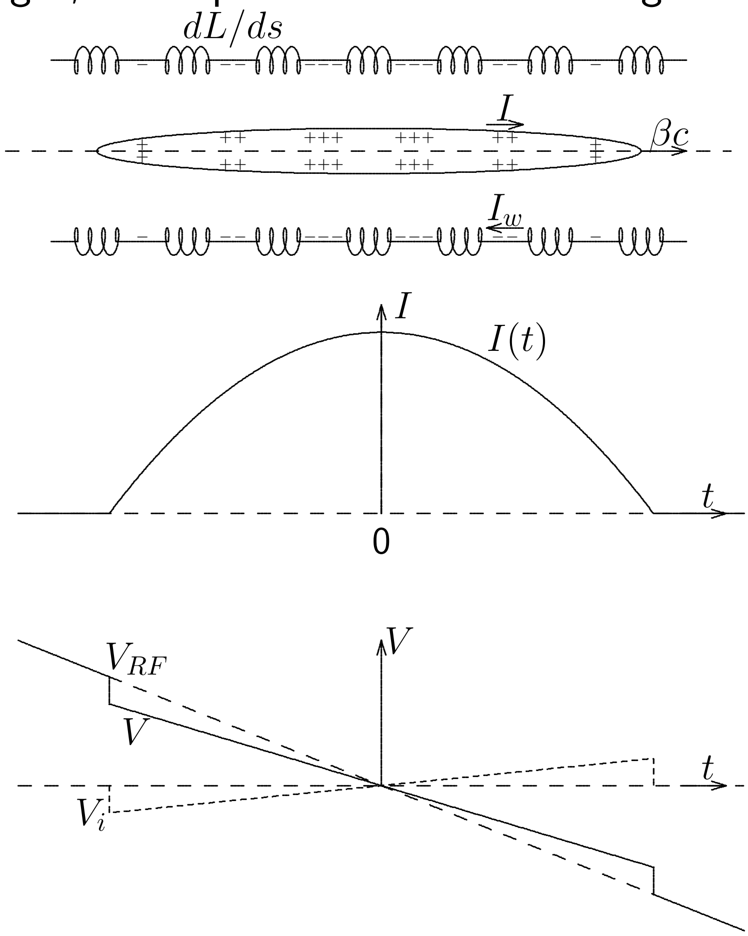
Cures: damp cavity modes, feed-back system.

Single traversal effects

Strong self-fields from broad band impedances change the stationary distribution and modify oscillation modes which are no longer independent. A self consistent solutions is difficult to obtain. The most common such effect is **bunch lengthening**. Small vacuum chamber aperture changes represent at low frequencies an inductive impedance ωL in which the bunch current $I(t)$ induces a voltage

$$V_i(t) = -L\frac{dI}{dt}.$$

It is added to the external RF-voltage, reduces its slope and increases the bunch length, called potential well bunch lengthening.



Cure: smooth chamber.

Longitudinal and transverse effects

Longitudinal effects involving synchrotron (energy, phase) oscillations and longitudinal impedances. They contain longitudinal instabilities, shift of synchrotron frequencies and bunch lengthening.

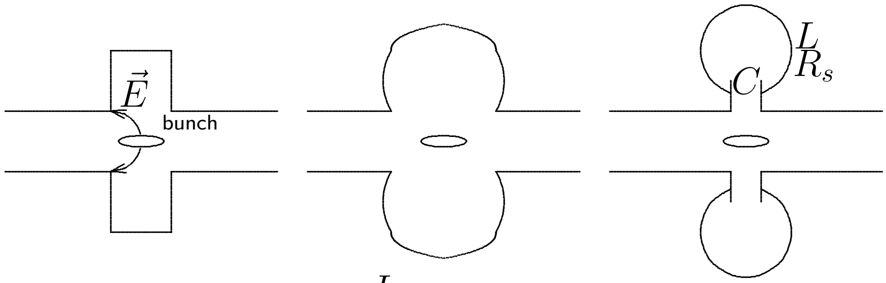
transverse effects involve betatron oscillations and transverse impedances. They contain transverse instabilities and betatron frequency shifts.

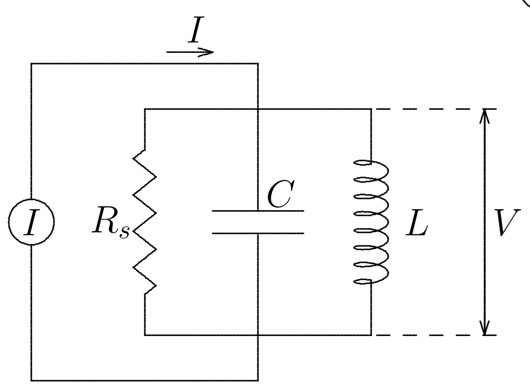
In both cases the longitudinal particle distribution (bunch length) is important since it can be "resolved" by the impedance while the transverse distribution is usually not resolved and does not affect the instability.

The most important longitudinal single traversal effects are synchrotron frequency shifts and bunch lengthening. In the transverse case the effect of the chromaticity is important which can lead to head-tail instabilities.

LONGITUDINAL IMPEDANCES

Longitudinal impedance and wake potential of a resonator



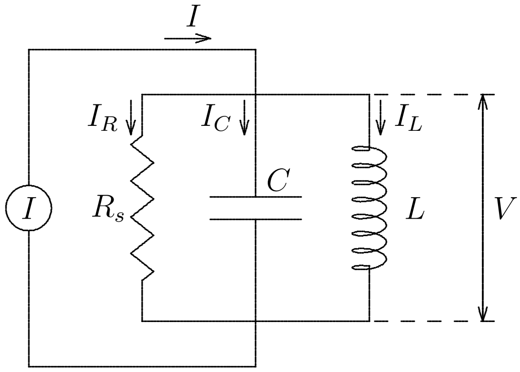


Cavities have narrow band oscillation modes which can drive coupled bunch instabilities. Each resembles an **RCL - circuit** and can, in good approximation, be treated as such. This circuit has a shunt impedance R_s, an inductance L and a capacity C. In a real cavity these parameters cannot easily be separated and we use others which can be measured directly: The **resonance frequency** ω_r, the **quality factor** Q and the **damping rate** α:

$$\omega_r = \frac{1}{\sqrt{LC}}\ ,\ \ Q = R_s\sqrt{\frac{C}{L}} = \frac{R_s}{L\omega_r} = R_s C\omega_r\ ,\ \ \alpha = \frac{\omega_r}{2Q}$$

$$L = \frac{R_s}{Q\omega_r}\ ,\ C = \frac{Q}{\omega_r R_s}.$$

Driving this circuit with a current I gives the voltages and currents across the elements



$$\begin{aligned} V_R &= I_R R_s \\ V_C &= \frac{1}{C}\int I_C dt \\ V_L &= L\frac{dI_L}{dt} \end{aligned}$$

$$V_R = V_C = V_L = V \ , \quad I_R + I_C + I_L = I$$

Differentiating with respect to t gives

$$\dot{I} = \dot{I}_R + \dot{I}_C + \dot{I}_L = \frac{\dot{V}}{R_s} + C\ddot{V} + \frac{V}{L}$$

Using $L = R_s/(\omega_r Q)$ and $C = Q/(\omega_r R_s)$ gives differential equation

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I}$$

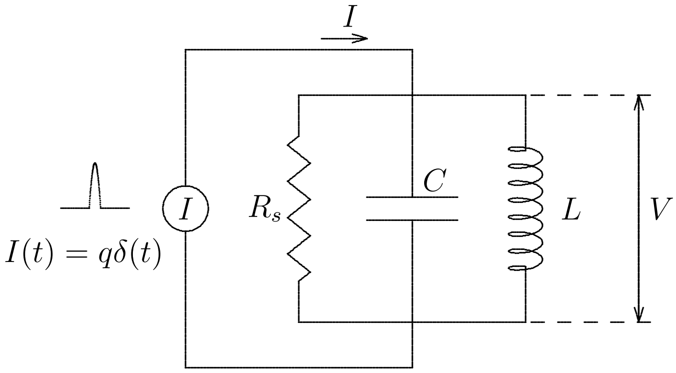
The solution of the homogeneous equation represents a damped oscillation

$$V(t) = \hat{V}\mathrm{e}^{-\alpha \mathrm{t}} \cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\mathrm{t} + \phi\right)$$

$$V(t) = \mathrm{e}^{-\alpha \mathrm{t}}\left(\mathrm{A}\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\mathrm{t}\right) + \mathrm{B}\sin\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\mathrm{t}\right)\right)$$

Green (wake) function

We calculate the response of RCL circuit (cavity mode) to a delta function pulse (very short bunch) $I(t) = q\delta(t)$



$$\omega_r = \frac{1}{\sqrt{LC}}$$
$$Q = R_s\sqrt{\frac{C}{L}}$$
$$\alpha = \frac{\omega_r}{2Q}$$

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I}$$

The charge q will charge up the capacity to a voltage

$$V(0^+) = \frac{q}{C} = \frac{\omega_r R_s}{Q}q \text{ using } C = \frac{Q}{\omega_r R_s}$$

Energy stored in capacitor equals energy lost by charge

$$U = \frac{q^2}{2C} = \frac{\omega_r R_s}{2Q}q^2 = \frac{V(0^+)}{2}q = k_{pm}q^2$$

where we introduced the **parasitic mode loss factor**

$$k_{pm} = \frac{\omega_r R_s}{2Q} \quad \text{measured usually in [V/pC]}$$

The charged capacitor C will now discharge first through the resistor R_s and then also through the inductance L

$$\dot{V}(0^+) = -\frac{\dot{q}}{C} = -\frac{I_R}{C} = -\frac{1}{C}\frac{V(0^+)}{R_s} = -\frac{\omega_r^2 R_s}{Q^2}q = -\frac{2\omega_r k_{pm}}{Q}q$$

The resonance circuit has now the initial conditions

$$V(0^+) = 2k_{pm}q \;\text{ and }\; \dot{V}(0^+) = -\frac{2\omega_r k_{pm}}{Q}q$$

The solution of the homogeneous differential equation is

$$V(t) = \mathrm{e}^{-\alpha \mathrm{t}}\left(\mathrm{A}\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right) + \mathrm{B}\sin\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right)\right)$$

$$\begin{aligned}\dot{V}(t) = \mathrm{e}^{-\alpha \mathrm{t}} \quad & \left(\left(-A\alpha + B\omega_r\sqrt{1-\frac{1}{4Q^2}}\right)\cos\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}\,t\right)\right. \\ & \left. - \left(B\alpha + A\omega_r\sqrt{1-\frac{1}{4Q^2}}\right)\sin\left(\omega_r\sqrt{1-\frac{1}{4Q^2}}\,t\right)\right).\end{aligned}$$

The initial conditions give

$$A = 2k_{pm}q \text{ and } -A\alpha + B\omega_r\sqrt{1-\frac{1}{4Q^2}} = -\frac{2\omega_r k_{pm}}{Q}q.$$

The resonator voltage at t, excited by a $I(t) = q\delta t$ at $t = 0$ is

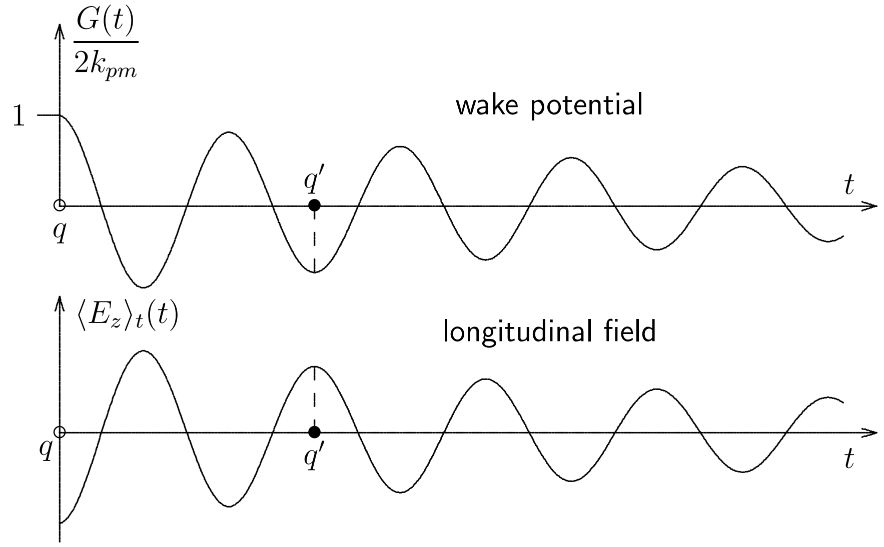
$$V(t) = 2qk_{pm}\mathrm{e}^{-\alpha \mathrm{t}}\left(\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right) - \frac{\sin\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right)}{2\mathrm{Q}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}}\right)$$

This voltage induced by charge q at $t = 0$ is seen by a second point charge q' traversing the cavity at t and loosing or gaining an energy $U = q'V(t)$. This energy gain/loss per unit source and probe charges is called **Green or wake function** $G(t)$. For our resonator (cavity resonance):

$$G(t) = 2k_{pm}\mathrm{e}^{-\alpha \mathrm{t}}\left(\cos\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right) - \frac{\sin\left(\omega_\mathrm{r}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}\,\mathrm{t}\right)}{2\mathrm{Q}\sqrt{1-\frac{1}{4\mathrm{Q}^2}}}\right)$$

for $Q \gg 1$ this simplifies to the damped oscillation

$$G(t) \approx 2k_{pm}\mathrm{e}^{-\alpha \mathrm{t}}\cos(\omega_\mathrm{r}\mathrm{t}) \ , \ \mathrm{k_{pm}} = \frac{\omega_\mathrm{r}\mathrm{R_s}}{2\mathrm{Q}} \ , \ \alpha = \frac{\omega_\mathrm{r}}{2\mathrm{Q}}$$

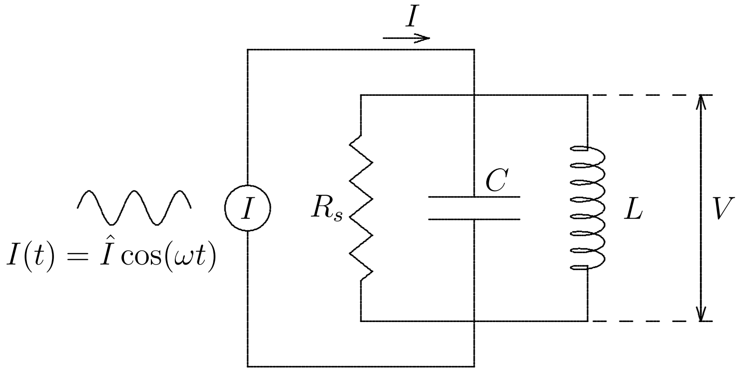


The wake potential is related to the longitudinal field E_z by a field integral over the object length. Since the field changes this integration has follow a particle going with the speed of light through the object taking the momentary field value

$$V = -\int_{z_1}^{z_2} E_z(z,t)dz = -f_t\int_{z_1}^{z_2} E_z(z)dz = -\langle E_z\rangle_t \Delta z.$$

with the transit time factor f_t correcting the instantaneous integral over z. We use a wake potential being positive where the particle loses energy consistent with the sign used for resistors.

Impedance



We assume now a **harmonic** excitation of the circuit with a current of the form $I = \hat{I}\cos(\omega t)$

The differential equation of the harmonic excitation is

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I} = -\frac{\omega_r R_s}{Q}\hat{I}\omega\sin(\omega t)$$

The solution of the homogeneous equation is a damped oscillation which disappears after some time. We are left with the particular solution of the form $V(t) = A\cos(\omega t) + B\sin(\omega t)$. Inserting this into the differential equation and separating cosine and sine terms gives

$$-(\omega^2 - \omega_r^2)A + \frac{\omega_r\omega}{Q}B = 0 \text{ and } (\omega^2 - \omega_r^2)B + \frac{\omega_r\omega}{Q}A = \frac{\omega_r\omega R_s}{Q}\hat{I}$$

The voltage induced by the harmonic excitation of the resonator becomes

$$V(t) = \hat{I}R_s\frac{\cos(\omega t) + Q\frac{\omega^2-\omega_r^2}{\omega_r\omega}\sin(\omega t)}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}$$

$$V(t) = \hat{I} R_s \frac{\cos(\omega t) + Q\frac{\omega^2-\omega_r^2}{\omega_r \omega}\sin(\omega t)}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}$$

The voltage has a cosine term being **in phase** with the exciting current. It can absorb energy and is called **resistive** term.
The sine term of the voltage is **out of phase** with the exciting current and does not absorb energy, it is called **reactive**.
The ratio between the voltage and current is the **impedance**. It is a **function of frequency** ω and has a resistive part $Z_r(\omega)$ and a reactive part $Z_i(\omega)$

$$Z_r(\omega) = R_s \frac{1}{1 + Q^2\left(\frac{\omega_r^2-\omega^2}{\omega_r\omega}\right)^2} \quad , \quad Z_i(\omega) = -R_s \frac{Q\frac{\omega^2-\omega_r^2}{\omega_r\omega}}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega_r\omega}\right)^2}$$

The resistive part of the impedance is always positive, the reactive is positive below the resonant frequency $\omega < \omega_r$ and negative above of it $\omega > \omega_r$.

Complex notation

We have used a harmonic excitation of the form

$$I(t) = \hat{I}\cos(\omega t) = \hat{I}\frac{\mathrm{e}^{j\omega t} + \mathrm{e}^{-j\omega t}}{2} \ \text{ with } \ 0 \leq \omega \leq \infty$$

It is often more convenient to use a complex notation

$$I(t) = \hat{I}\mathrm{e}^{j\omega t} \ \text{ with } \ -\infty \leq \omega \leq \infty$$

giving more compact expressions. Using the differential equation

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I}$$

with $I(t) = \hat{I}\exp(j\omega t)$ and seeking a solution of the form $V(t) = V_0\exp(j\omega t)$, where V_0 is in general complex, one gets

$$\left(-\omega^2\mathrm{e}^{j\omega t} + j\frac{\omega_r\omega}{Q}\mathrm{e}^{j\omega t} + \omega_r^2\mathrm{e}^{j\omega t}\right)V_0 = j\frac{\omega_r\omega R_s}{Q}\hat{I}\mathrm{e}^{j\omega t}$$

and for the impedance which is defined as the ration V/I

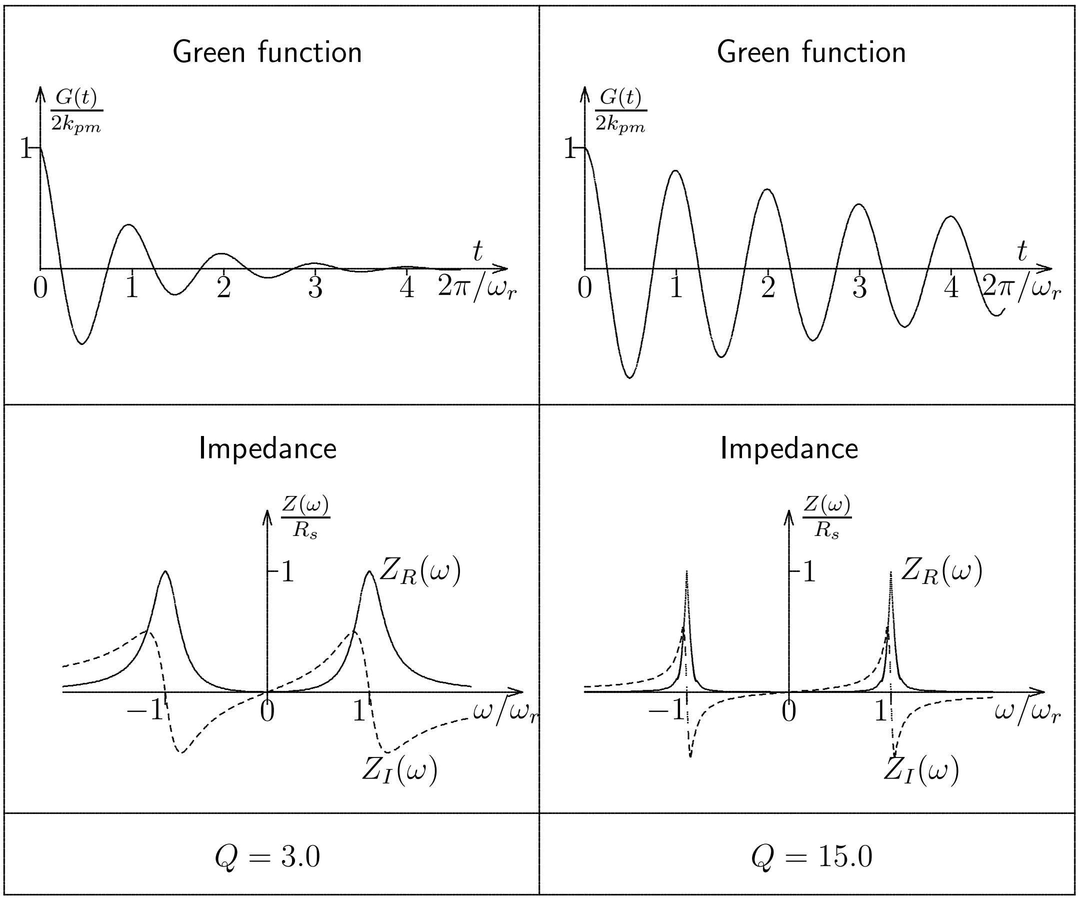
$$Z(\omega) = \frac{V_0}{\hat{I}} = \frac{R_s}{1 + jQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)} = R_s\frac{1 - jQ\frac{\omega^2-\omega_r^2}{\omega\omega_r}}{1 + Q^2\left(\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)^2} = Z_r + jZ_i$$

For $Q \gg 1$ the impedance is only large for $\omega \approx \omega_r$ or for $|\omega - \omega_r|/\omega_r = |\Delta\omega|/\omega_r \ll 1$ and can be simplified

$$Z(\omega) \approx R_s\frac{1 - j2Q\frac{\Delta\omega}{\omega_r}}{1 + 4Q^2\left(\frac{\Delta\omega}{\omega_r}\right)^2}$$

Caution: sometimes $I(t) = \hat{I}\mathrm{e}^{-i\omega t}$ instead of $I(t) = \hat{I}\mathrm{e}^{j\omega t}$ is used, this reverses the sign $Z_i(\omega)$.

Green function and impedance of a resonator



The resonator impedance has some specific properties:

$$\text{at}\,\omega = \omega_r \;\rightarrow\; Z_r(\omega_r)\ \text{has a maximum},\ \ Z_i(\omega_r) = 0$$

$$0 < \omega < \omega_r \;\rightarrow\; Z_i(\omega) > 0\ \text{(inductive)}$$

$$\omega > \omega_r \;\rightarrow\; Z_i(\omega) < 0\ \text{(capacitive)}$$

and any impedance or wake potential has the general properties

$$Z_r(\omega) = Z_r(-\omega)\ \ ,\ \ Z_i(\omega) = -Z_i(-\omega)$$

$$Z(\omega) = \int_{-\infty}^{\infty} G(t)\mathrm{e}^{-j\omega t}dt\ \ Z(\omega) \propto \text{Fourier transform of}\, G(t)$$

$$\text{for}\, t < 0 \;\rightarrow\; G(t) = 0,$$

no fields before particle arrives.