# Dynamics of Non-Linear Beams with Space-Charge

Chris Prior
STFC/RAL and Trinity College Oxford

## Basic equations

$$
\begin{aligned}
x'' + k_x(s)x - \frac{q}{m_0\gamma^3\beta^2c^2}\, E_x(x,y,z,s) &= 0 \qquad (1)\\
y'' + k_y(s)y - \frac{q}{m_0\gamma^3\beta^2c^2}\, E_y(x,y,z,s) &= 0 \qquad (2)\\
z'' + k_z(s)z - \frac{q}{m_0\gamma^3\beta^2c^2}\, E_z(x,y,z,s) &= 0 \qquad (3)
\end{aligned}
$$

Particle coordinates $(x, y, z)$ with respect to frame whose motion is given by $s$

Factor $1/\gamma^2$ from electrostatic-magnetostatic effects

Other factor $\gamma$ from relativistic mass $m = m_0\gamma$

Space-charge field $\mathbf{E}$ from Maxwell's equation:

$$\nabla \cdot \mathbf{E} = \frac{q}{\epsilon_0}\, n(x,y,z,s) \qquad (4)$$

where $n(x, y, z, s)$ is the number density of the beam distribution.

$n(x, y, z, s)$ given by the particle density $f(x, y, z, x', y', z', s)$ in six-dimensional phase space, which must satisfy the Vlasov equation

$$\frac{\partial f}{\partial s} + (\mathbf{x}' \cdot \nabla) f - (\mathbf{k} - \frac{q}{m_0 \gamma^3 \beta^2 c^2} \mathbf{E}) \cdot \nabla_{\mathbf{x}'} f = 0, \tag{5}$$

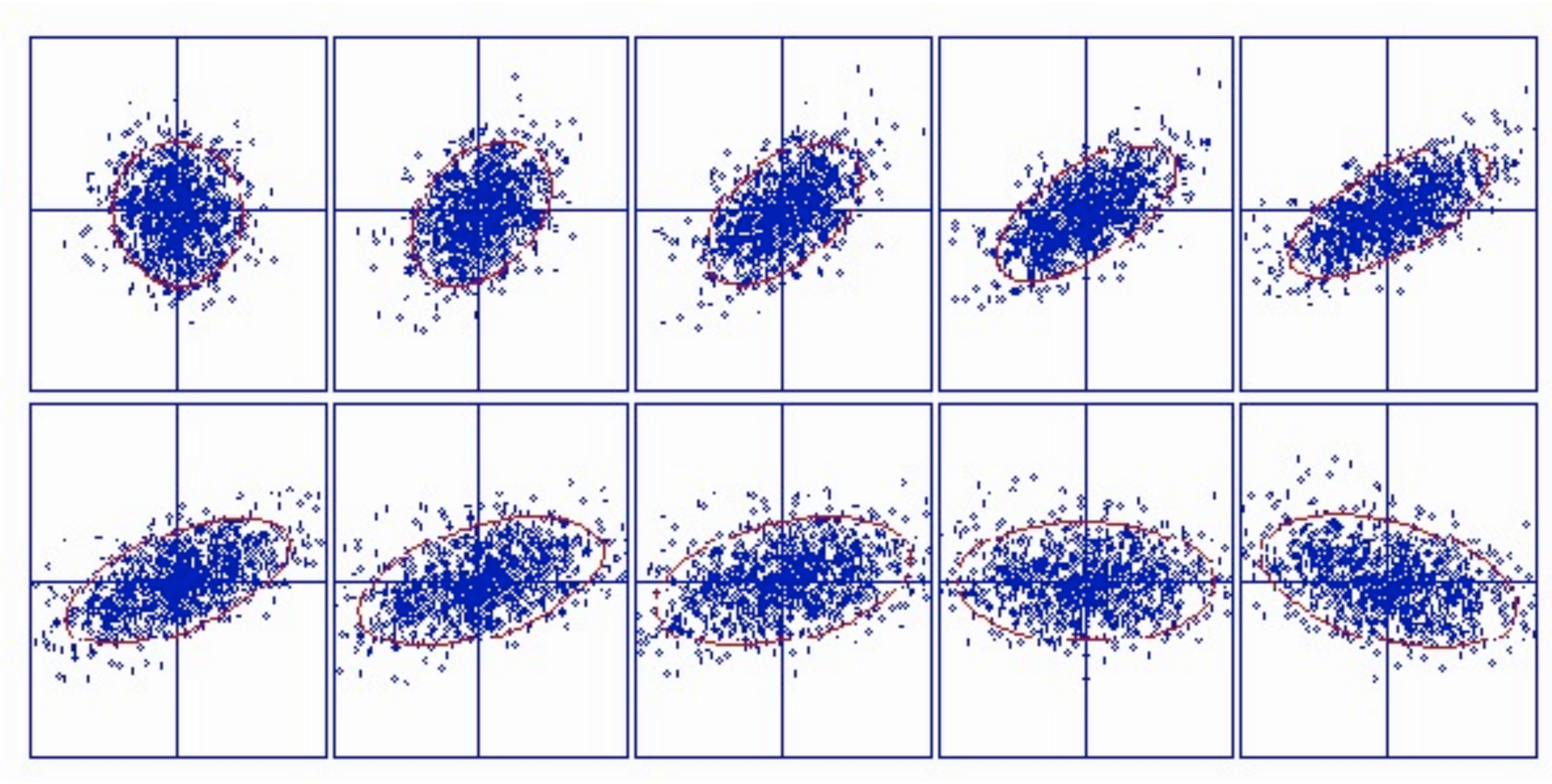
through

$$n = \iiint f(x, y, z, x', y', z', s) \, \mathrm{d}x' \, \mathrm{d}y' \, \mathrm{d}z'. \tag{6}$$

The total number of particles in the beam is

$$N = \iiint n(x, y, z, s) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z. \tag{7}$$

This is a complete set of seven coupled equations in which the distribution determines the forces, which determine the motion, which determines the distribution, and so on.



How do we study a very non-linear beam?

# RMS Properties of Beams

For non-linear beams, use the moments of the distribution and consider the behaviour of r.m.s quantities.

Define the average value of a quantity $g(x, y, z, x', y', z', s)$ by

$$< g >= \frac{1}{N} \int \dots \int \; gf \, \mathrm{d}x \dots \mathrm{d}z'$$

The *r.m.s. envelope* is

$$\tilde{x} = \sqrt{< x^2 >}$$

and the *r.m.s. emittance* is

$$\tilde{\epsilon}_x = \left[ < x^2 >< {x'}^2 > - < xx' >^2 \right]^{\frac{1}{2}}$$

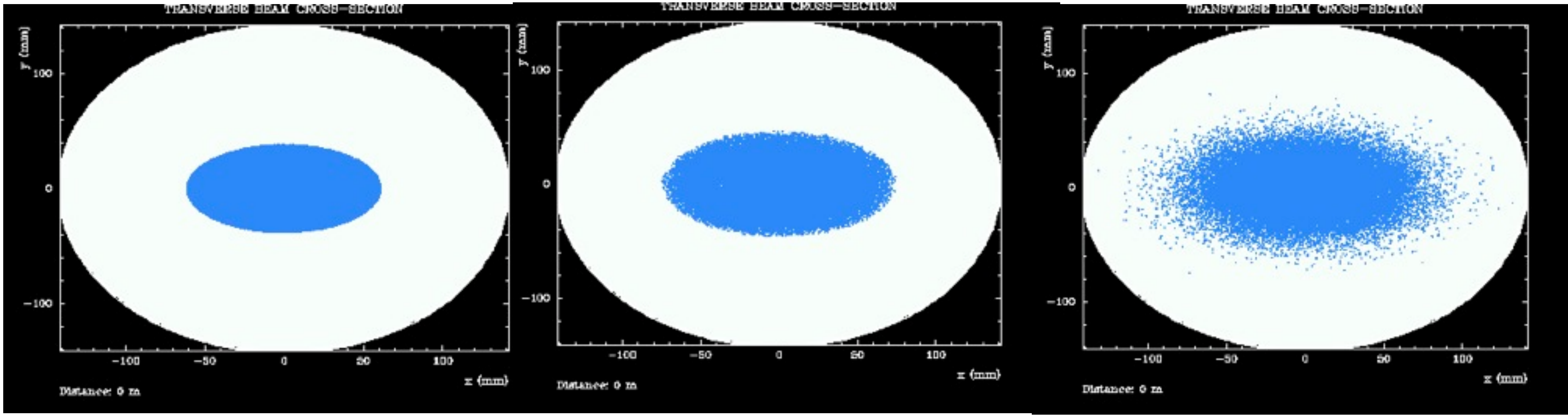
Note: invariant under rotations of phase-space coordinates

If the beam projection in $x$-$x'$ phase space is a uniform filling of the tilted ellipse

$$\hat{\gamma}_x x^2 + 2\hat{\alpha}_x xx' + \hat{\beta}_x {x'}^2 \leqslant \epsilon \qquad (\hat{\beta}_x \hat{\gamma}_x - \hat{\alpha}_x^2 = 1),$$

then $\tilde{x} = \frac{1}{2}\sqrt{\epsilon \hat{\beta}},$ and $\tilde{\epsilon} = \frac{1}{4}\epsilon$

Science & Technology Facilities Council




Beams are *equivalent* if they have the same first and second moments

Note:

$$\begin{aligned}
\tilde{\epsilon}^2 &= \langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2 = \langle x'^2\rangle\left\{\langle x^2\rangle - \frac{\langle xx'\rangle^2}{\langle x'^2\rangle}\right\} \\
&= \langle x'^2\rangle\left\{\langle x^2\rangle - 2\frac{\langle xx'\rangle^2}{\langle x'^2\rangle} + \frac{\langle xx'\rangle^2\langle x'^2\rangle}{\langle x'^2\rangle^2}\right\} \\
&= \langle x'^2\rangle\left\langle x^2 - 2xx'\frac{\langle xx'\rangle}{\langle x'^2\rangle} + x'^2\left(\frac{\langle xx'\rangle}{\langle x'^2\rangle}\right)^2\right\rangle \\
&= \langle x'^2\rangle\left\langle\left(x - x'\frac{\langle xx'\rangle}{\langle x'^2\rangle}\right)^2\right\rangle
\end{aligned}$$

confirmed positive, valid definition

Define RMS Twiss parameters to identify the RMS emittance ellipse

$$a = \sqrt{\epsilon\beta} \quad \Longrightarrow \quad \beta_{rms} = \frac{\tilde{x}^2}{\tilde{\epsilon}} = \frac{\langle x^2 \rangle}{\tilde{\epsilon}}$$

$$a' = -\hat{\alpha}\sqrt{\frac{\epsilon}{\hat{\beta}}} \Longrightarrow aa' = -\hat{\alpha}\epsilon \quad \Longrightarrow \quad \alpha_{rms} = -\frac{\langle xx' \rangle}{\tilde{\epsilon}}$$

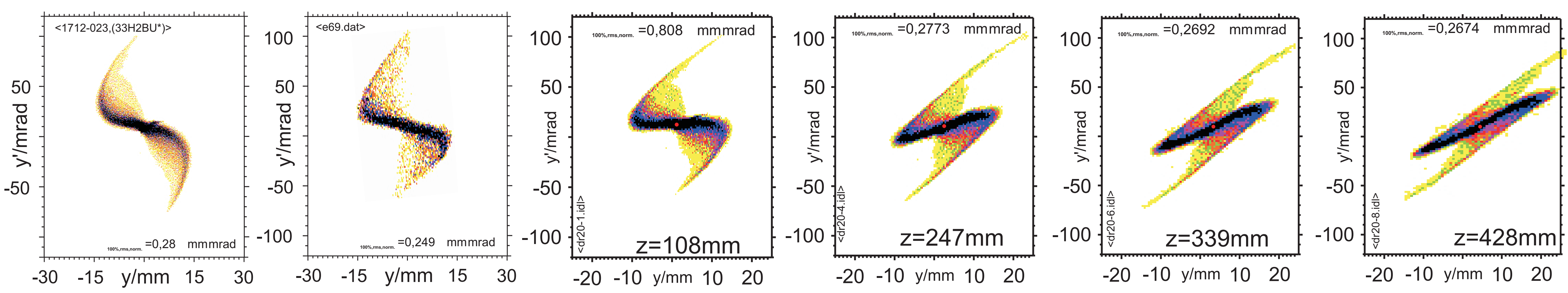
$$\hat{\gamma} = \frac{1+\hat{\alpha}^2}{\hat{\beta}} \quad \Longrightarrow \quad \gamma_{rms} = \frac{1+\alpha_{rms}^2}{\beta_{rms}} = \frac{\langle x'^2 \rangle}{\tilde{\epsilon}}$$

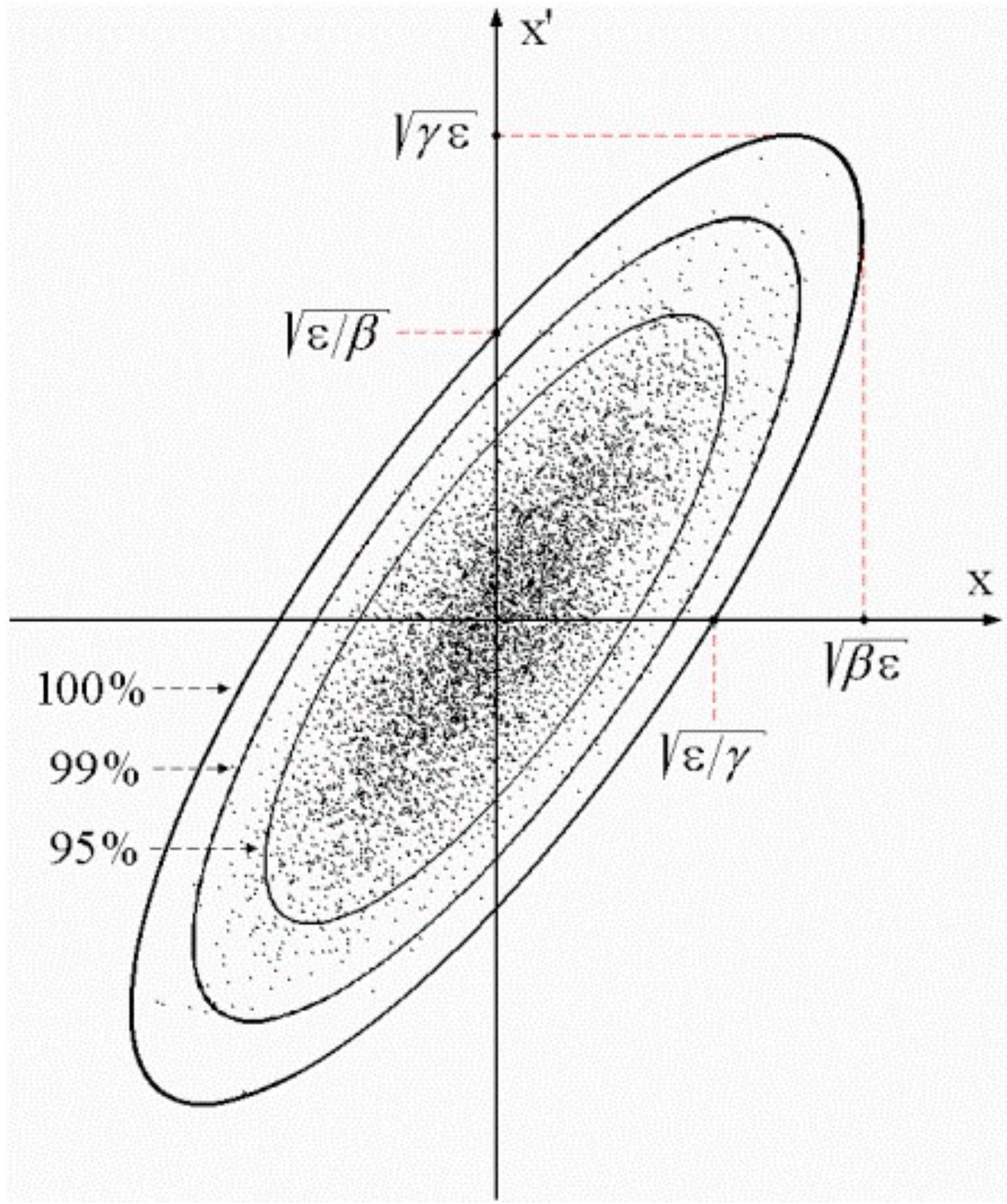
$\alpha_{rms}$ and $\beta_{rms}$ give orientation and aspect ratio of an "emittance ellipse".

Single particle emittance: $\gamma_{rms}x'^2 + 2\alpha_{rms}xx' + \beta_{rms}x^2$

100% emittance is maximum for all particles

Statistics then used to find the more meaningful 90% emittance





x'
$\sqrt{\gamma\varepsilon}$
$\sqrt{\varepsilon/\beta}$
x
$\sqrt{\beta\varepsilon}$
$\sqrt{\varepsilon/\gamma}$
100%
99%
95%

How important is RMS emittance? How does it evolve? What causes it to change?

$$x'' = -k_x(s)x + \frac{q}{m_0\gamma^3\beta^2c^2}E_x$$

Calculations:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} < x^2 > &= 2 < xx' > \\
\frac{\mathrm{d}}{\mathrm{d}s} < xx' > &= < x'^2 > + < xx'' > \\
&= < x'^2 > -k_x(s) < x^2 > + \frac{q}{m_0\gamma^3\beta^2c^2} < xE_x > \\
\frac{\mathrm{d}}{\mathrm{d}s} < x'^2 > &= 2 < x'x'' > \\
&= -2k_x(s) < xx' > + \frac{2q}{m_0\gamma^3\beta^2c^2} < x'E_x > .
\end{aligned}$$

Science & Technology Facilities Council

# Evolution of RMS Emittance under Space-Charge

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\tilde{\epsilon}^2 &= \frac{\mathrm{d}}{\mathrm{d}s}\left[\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right] \\
&= 2\langle xx'\rangle\langle x'^2\rangle + 2\langle x^2\rangle\left\{-k_x(s)\langle xx'\rangle + \frac{q}{m_0\gamma^3\beta^2c^2}\langle x'E_x\rangle\right\} \\
&\quad -2\langle xx'\rangle\left\{\langle x'^2\rangle - k_x(s)\langle x^2\rangle + \frac{q}{m_0\gamma^3\beta^2c^2}\langle xE_x\rangle\right\} \\
&= \frac{2q}{m_0\gamma^3\beta^2c^2}\left[< x^2 >< x'E_x > - < xx' >< xE_x >\right]
\end{aligned}$$

RMS emittance

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}s}\tilde{\epsilon}^2 = \frac{2q}{m_0\gamma^3\beta^2c^2}\left[< x^2 >< x'E_x > - < xx' >< xE_x >\right]}$$

RMS emittance will be constant under linear space-charge forces $E_x \propto x$.

Science & Technology Facilities Council

# Evolution of RMS Envelope under Space-Charge

$$
\begin{aligned}
2\tilde{x}\frac{\mathrm{d}\tilde{x}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\tilde{x}^2 &= 2\langle xx'\rangle \\
\Longrightarrow \qquad \tilde{x}\frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}s^2} + \left(\frac{\mathrm{d}\tilde{x}}{\mathrm{d}s}\right)^2 &= \langle x'^2\rangle - k_x(s)\langle x^2\rangle + \frac{q}{m_0\gamma^3\beta^2c^2}\langle xE_x\rangle \\
&= \langle x'^2\rangle - k_x(s)\tilde{x}^2 + \frac{q}{m_0\gamma^3\beta^2c^2}\langle xE_x\rangle \\
\Longrightarrow \qquad \tilde{x}\left(\frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}s^2} + k_x(s)\tilde{x}\right) &= \langle x'^2\rangle - \frac{\langle xx'\rangle^2}{\tilde{x}^2} + \frac{q}{m_0\gamma^3\beta^2c^2}\langle xE_x\rangle \\
&= \frac{\tilde{\epsilon}^2}{\tilde{x}^2} + \frac{q}{m_0\gamma^3\beta^2c^2}\langle xE_x\rangle
\end{aligned}
$$

RMS envelope equation

$$
\boxed{\frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}s^2} + k_x(s)\tilde{x} - \frac{\tilde{\epsilon}^2}{\tilde{x}^3} - \frac{q}{m_0\gamma^3\beta^2c^2}\frac{\langle xE_x\rangle}{\tilde{x}} = 0}
$$

Science & Technology Facilities Council

# Special 2D Distributions

(a) Kapchinskij-Vladimirskij (KV) distribution

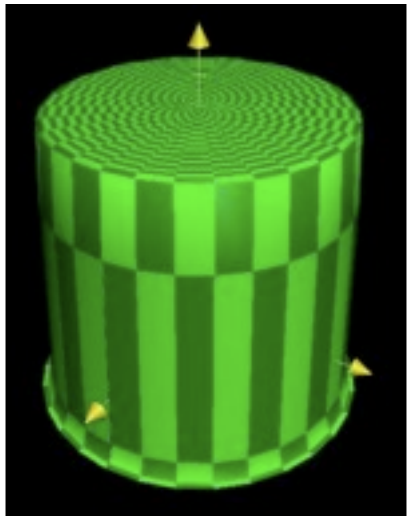
(b) Non-stationary Waterbag distribution

(c) Non-stationary Parabolic distribution

(d) Non-stationary Gaussian distribution

# (a) KV Distribution

Particles uniformly populate the surface of a hyper-ellipsoid in 4D phase-space



$$f(x,y,x',y') = \frac{N}{\pi^2 ab\epsilon_x\epsilon_y}\delta\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{a^2x'^2}{\epsilon_x^2}+\frac{b^2y'^2}{\epsilon_y^2}-1\right\}$$

where $\delta$ is the Dirac delta-function.

Real-space number density is

$$n(x,y) = \iint_{\text{all space}} f(x,y,x',y')\,\mathrm{d}x'\,\mathrm{d}y' = \frac{N}{\pi ab}, \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2} \leqslant 1.$$

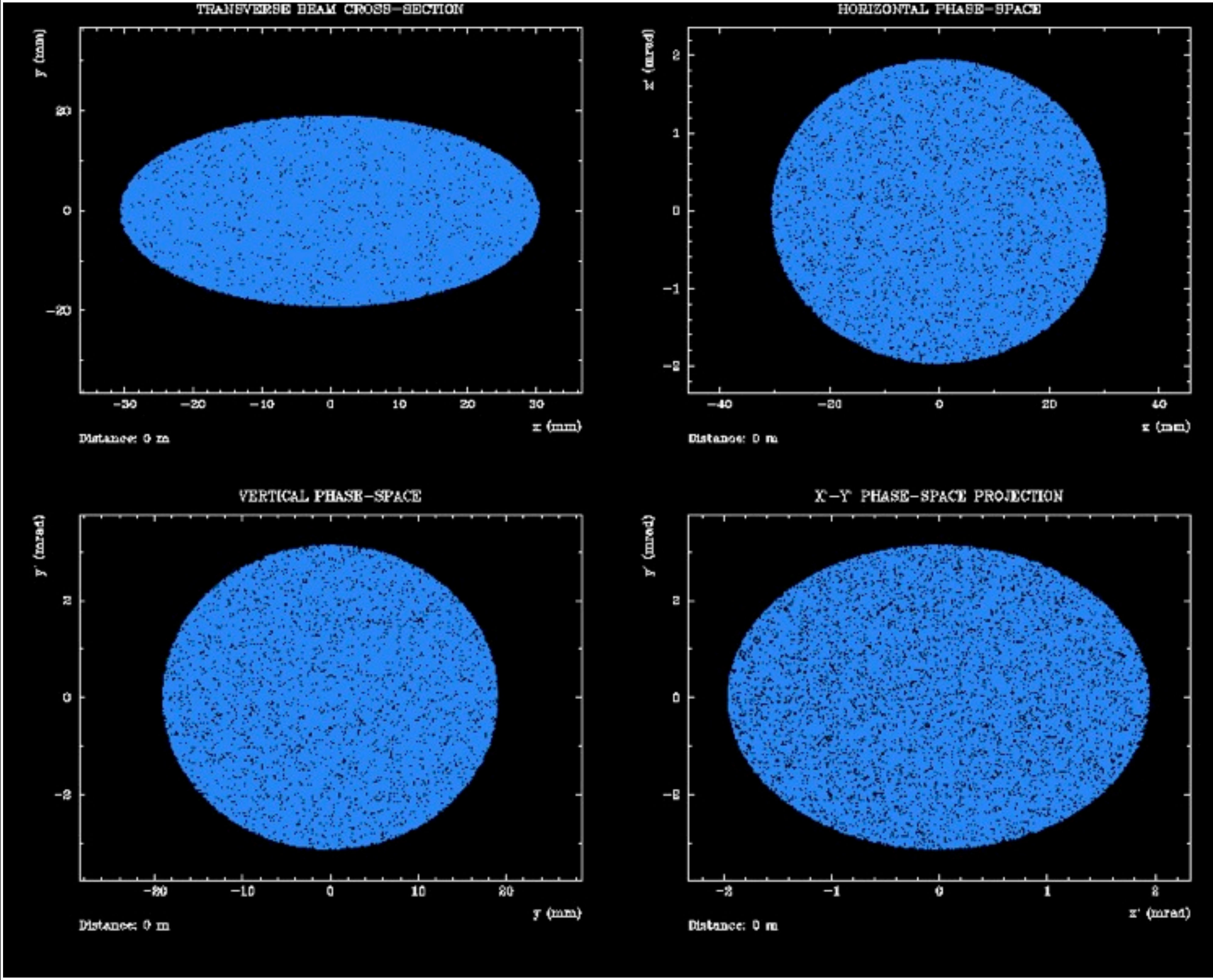
$\Longrightarrow$ a uniform elliptical beam.

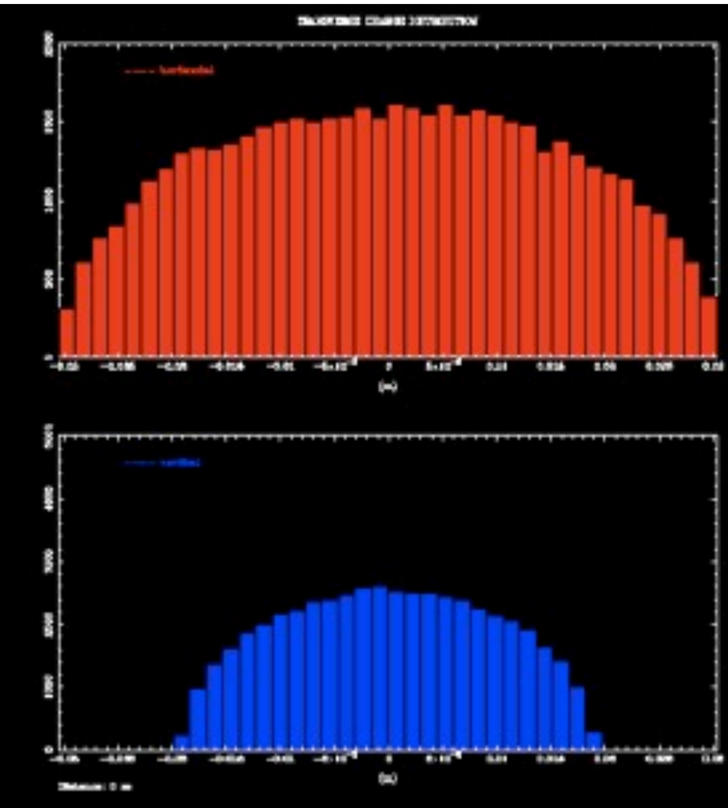
$$\langle x^2\rangle = \tfrac{1}{4}a^2, \quad \langle x'^2\rangle = \frac{\epsilon_x^2}{4a^2} \quad \langle xx'\rangle = 0,$$

$$\Longrightarrow \qquad \tilde{x} = \tfrac{1}{2}a, \quad \tilde{y} = \tfrac{1}{2}b, \quad \tilde{\epsilon_x} = \tfrac{1}{4}\epsilon_x, \quad \tilde{\epsilon_y} = \tfrac{1}{4}\epsilon_y.$$

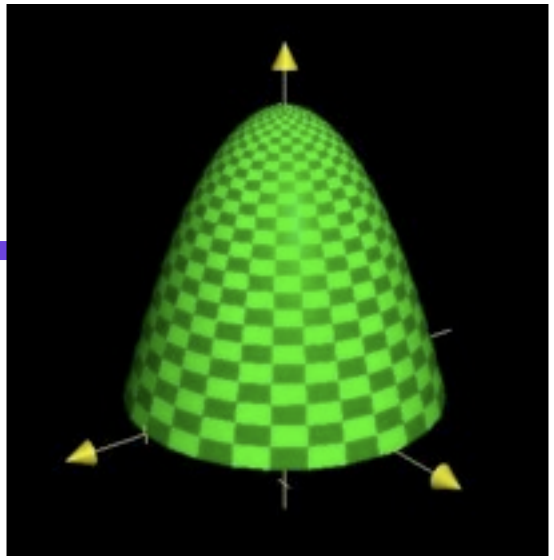
Space-charge forces linear $\Longrightarrow$ RMS emittances, $\tilde{\epsilon}_x, \tilde{\epsilon}_y$ are constant.

Distribution of particles is preserved in a linear focusing system $\rightarrow$ stationary


TRANSVERSE BEAM CROSS-SECTION
y (mm)
x (mm)
Distance: 0 m
HORIZONTAL PHASE-SPACE
x' (mrad)
x (mm)
Distance: 0 m
VERTICAL PHASE-SPACE
y' (mrad)
y (mm)
Distance: 0 m
X'-Y' PHASE-SPACE PROJECTION
y' (mrad)
x' (mrad)
Distance: 0 m



# (b) Waterbag Distribution



Particles uniformly fill the 4D hyper-ellispoid in phase-space:

$$f(x, y, x', y') = \frac{2N}{\pi^2 a^4}, \qquad x^2 + y^2 + x'^2 + y'^2 \leqslant a^2$$

if we assume a round beam with equal emittances and use normalised variables.

Real-space density is $\quad n(x, y) = \frac{2N}{\pi a^2}\left(1 - \frac{r^2}{a^2}\right), \qquad r^2 = x^2 + y^2 \leqslant a^2$

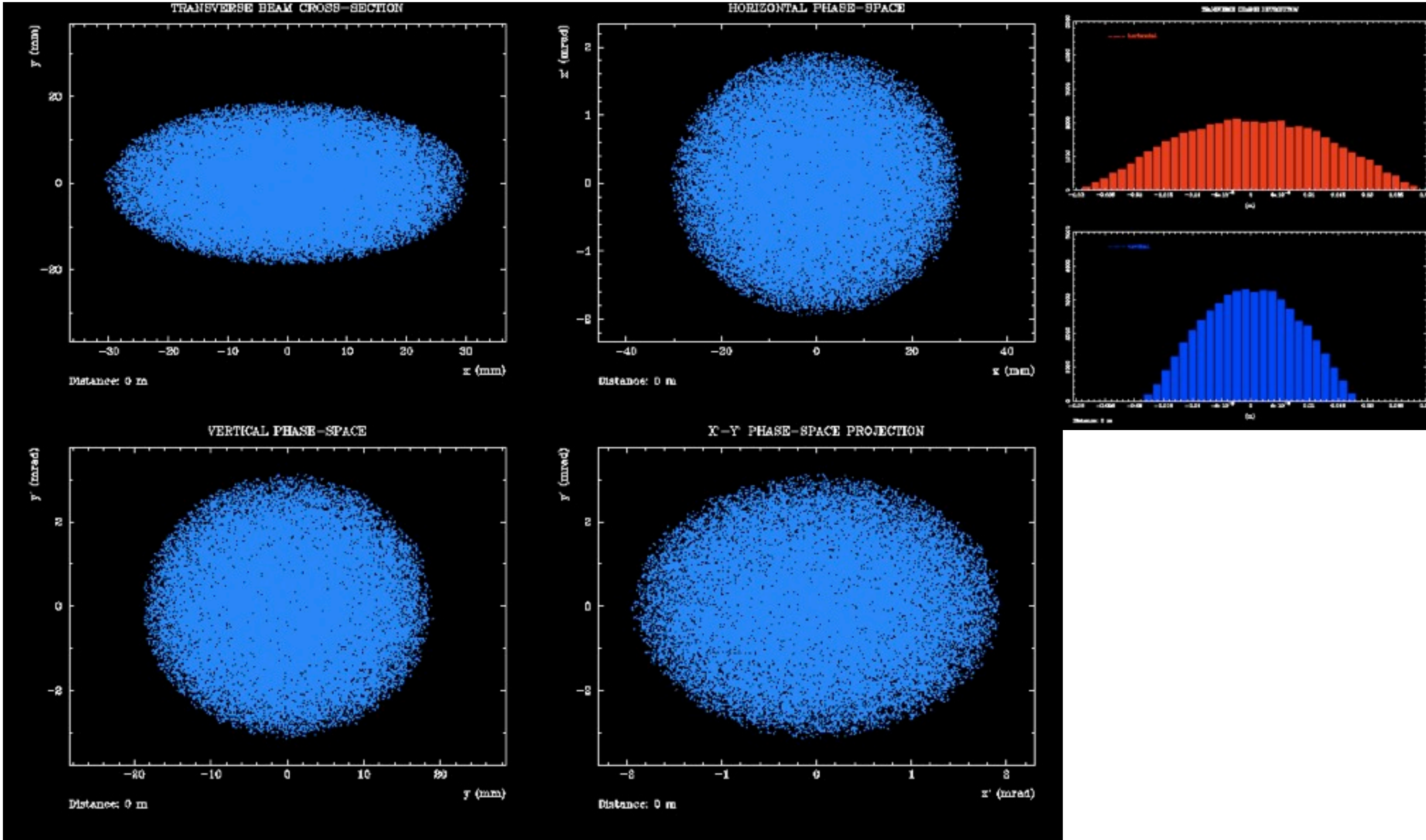
Space-charge fields are

$$E_r = \frac{q}{2\pi\epsilon_0 r}\int_0^r n(r) 2\pi r \,\mathrm{d}r = \frac{Nq}{2\pi\epsilon_0 r}\left[1 - \left(1 - \frac{r^2}{a^2}\right)^2\right], \qquad r \leqslant a$$

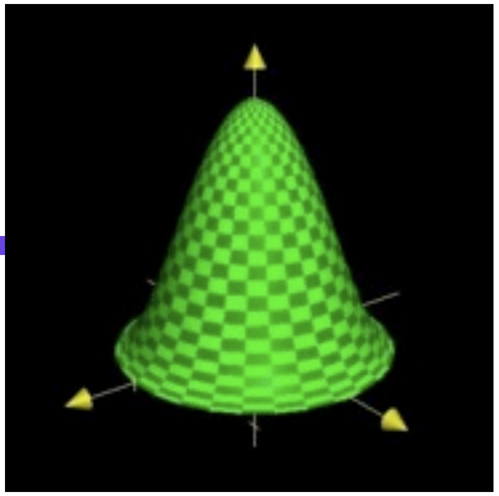
Then $\quad \langle x^2 \rangle = \frac{1}{6}a^2 = \langle x'^2 \rangle, \quad \langle xx' \rangle = 0 \quad \Longrightarrow \quad \tilde{\epsilon}_x = \frac{1}{6}a^2 = \frac{1}{6}\epsilon.$

Space-charge fields are non-linear, so RMS emittance is not constant and the initial distribution will change with time.
This 2D-waterbag distribution is not stationary.

Science & Technology Facilities Council


TRANSVERSE BEAM CROSS-SECTION
y (mm)
x (mm)
Distance: 0 m
HORIZONTAL PHASE-SPACE
x' (mrad)
x (mm)
Distance: 0 m
VERTICAL PHASE-SPACE
y' (mrad)
y (mm)
Distance: 0 m
X'-Y' PHASE-SPACE PROJECTION
y' (mrad)
x' (mrad)
Distance: 0 m

# (c) Parabolic Distribution



Beam fills the 4D hyper-ellipsoid with a parabolic density:

$$f(x, y, x', y') = \frac{6N}{\pi^2 a^4}\left(1 - \frac{\rho^2}{a^2}\right) \qquad \rho^2 = x^2 + y^2 + x'^2 + y'^2 \leqslant a^2.$$

Real-space density is $\quad n(x, y) = \frac{3N}{\pi a^2}\left(1 - \frac{r^2}{a^2}\right)^2, \quad r^2 = x^2 + y^2 \leqslant a^2$

Space-charge fields are

$$E_r = \frac{q}{2\pi\epsilon_0 r}\int_0^r n(r) 2\pi r \,\mathrm{d}r = \frac{Nq}{2\pi\epsilon_0 r}\left[1 - \left(1 - \frac{r^2}{a^2}\right)^3\right], \qquad r \leqslant a$$

Then $\quad \langle x^2 \rangle = \frac{1}{8}a^2 = \langle x'^2 \rangle, \quad \langle xx' \rangle = 0 \quad \Longrightarrow \quad \tilde{\epsilon}_x = \frac{1}{8}a^2 = \frac{1}{6}\epsilon.$

Non-linear forces $\Longrightarrow$ beam evolves, distribution changes with time, non-stationary.

Note alternative form of distributions: $n(x, y) = \frac{m}{\pi ab}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{m-1}$

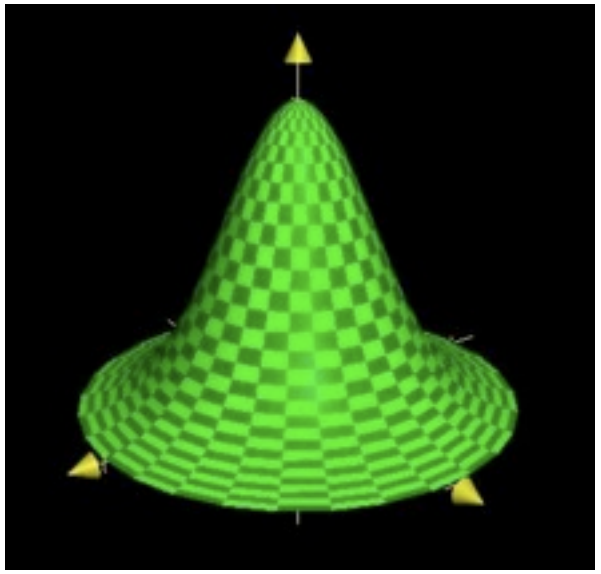
$m = 0$ is KV, $m = 1$ is Waterbag, $m = 2$ is parabolic, etc.

# (d) Gaussian Distribution

Gaussian model distribution, cut off at $n$ standard deviations ($3 \leq n \leq 10$) to avoid unrealistic tails

$$f(x,y,x',y') = \frac{N}{4\pi^2\sigma^4} \exp\left(-\frac{\rho^2}{2\sigma^2}\right), \qquad \rho \leqslant n\sigma.$$

Projection in real-space is also Gaussian:



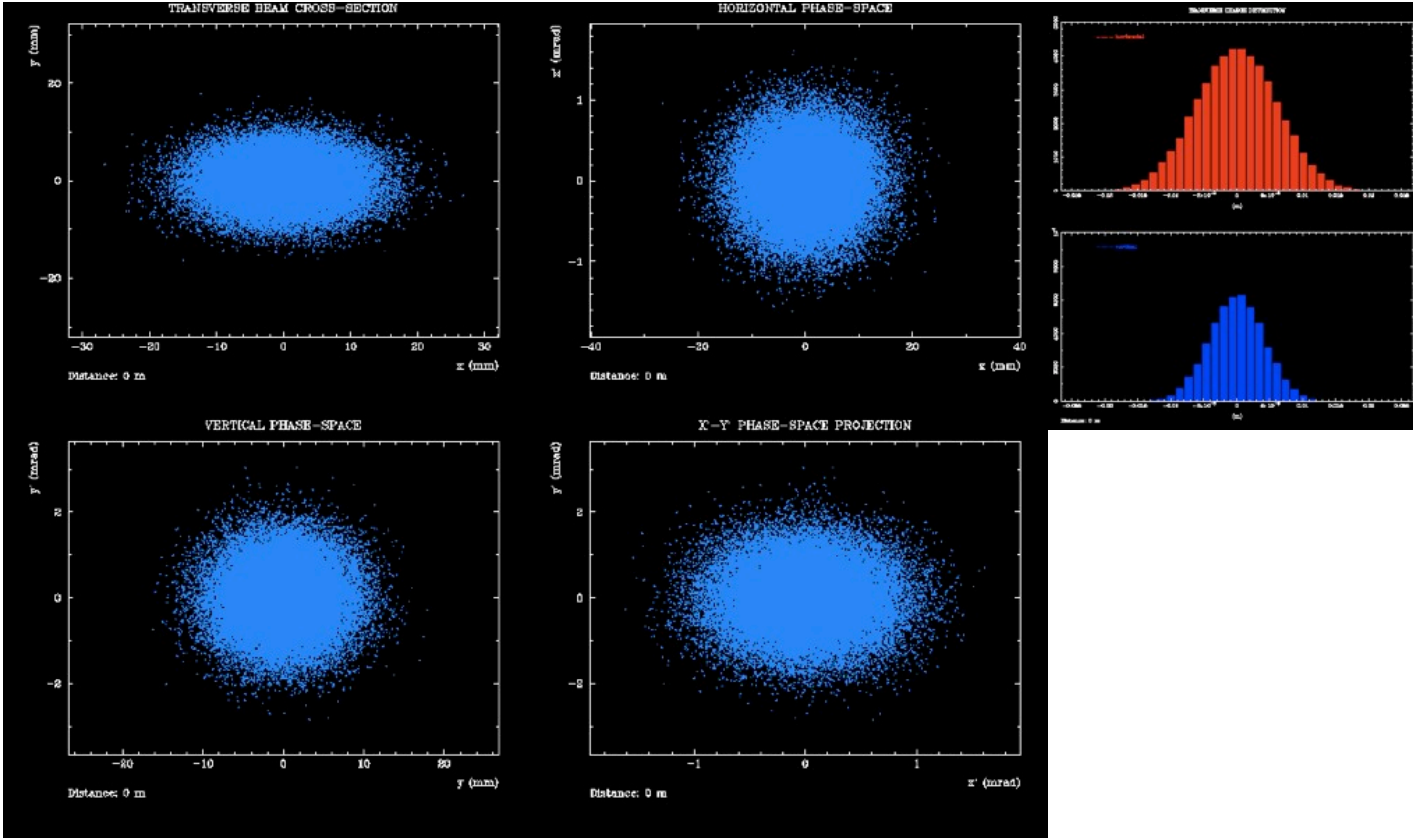
$$n(x,y) = \frac{N}{2\pi\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right), \qquad r \leqslant n\sigma$$

Space-charge fields given by

$$E_r = \frac{Nq}{2\pi\epsilon_0 r}\left[1 - \exp\left(-\frac{r^2}{a^2}\right)\right], \qquad r \leqslant n\sigma$$

RMS quantities: $\quad \langle x^2 \rangle = \sigma^2 = \langle x'^2 \rangle, \quad \langle xx' \rangle = 0 \qquad \Longrightarrow \tilde{\epsilon} = \frac{1}{n^2}\epsilon$

Space-charge fields are non-linear, so distribution is not stationary.


TRANSVERSE BEAM CROSS-SECTION
y (mm)
x (mm)
Distance: 0 m
HORIZONTAL PHASE-SPACE
x' (mrad)
x (mm)
Distance: 0 m
VERTICAL PHASE-SPACE
y' (mrad)
y (mm)
Distance: 0 m
X'-Y' PHASE-SPACE PROJECTION
y' (mrad)
x' (mrad)
Distance: 0 m

Recall: For a 2D uniform beam with elliptical cross section $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1$, space-charge forces are linear and given by

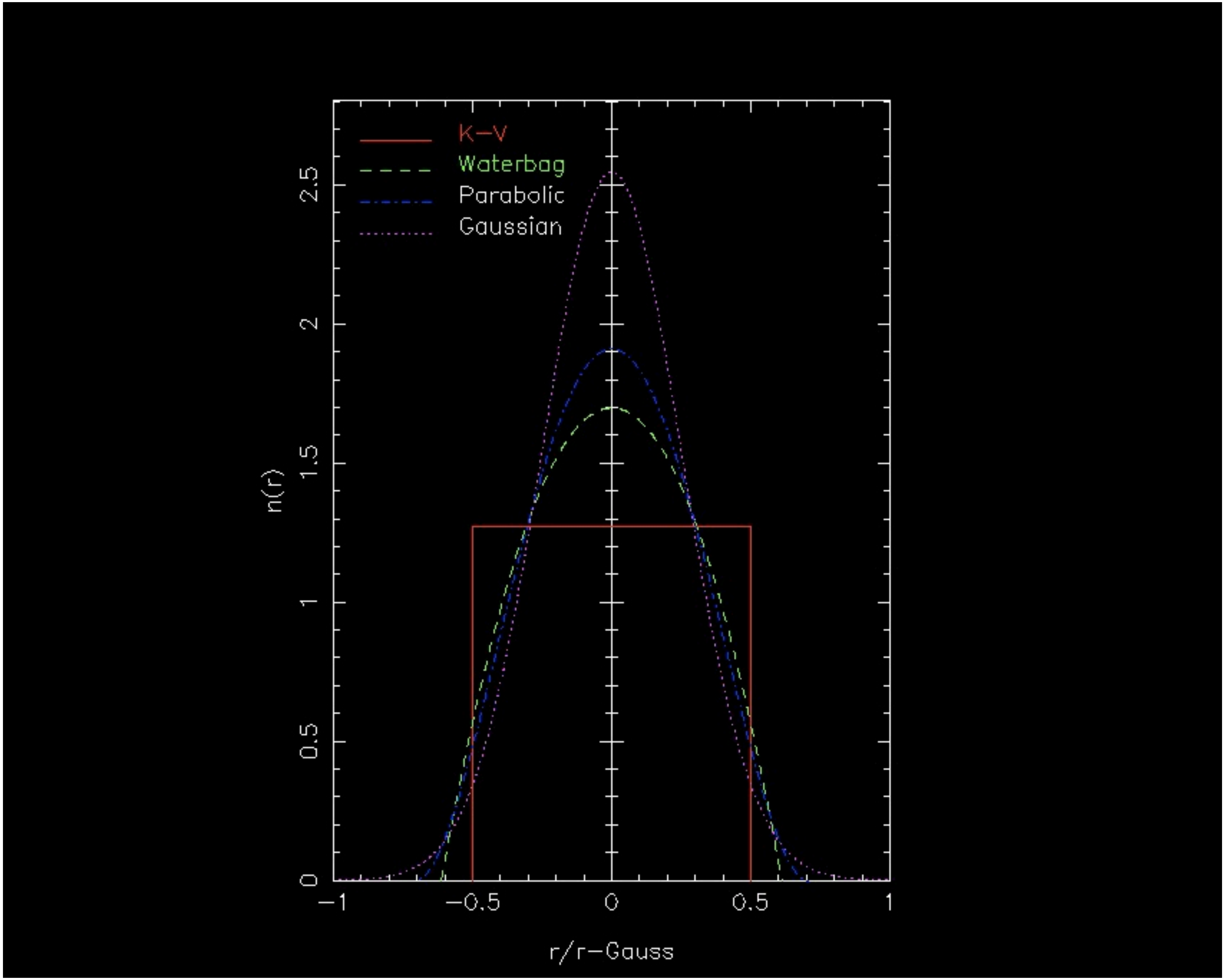
$$\mathbf{E} = \frac{Nq}{\pi\epsilon_0(a+b)}\left(\frac{x}{a}, \frac{y}{b}\right),$$

where $N$ is the number of particles per unit length.

Equations of particle motion and envelope equations are then:

$$\boxed{\begin{aligned} x'' + k_x(s)x - \frac{2K}{a+b}\,\frac{x}{a} &= 0 \\ y'' + k_y(s)y - \frac{2K}{a+b}\,\frac{y}{b} &= 0 \\ a'' + k_x(s)a - \frac{\epsilon_x^2}{a^3} - \frac{2K}{a+b} &= 0 \\ b'' + k_b(s)b - \frac{\epsilon_y^2}{b^3} - \frac{2K}{a+b} &= 0 \end{aligned}}$$

$K = \frac{I}{I_0}\,\frac{2}{(\beta\gamma)^3}$ is the Perveance and $I_0 = \frac{4\pi\epsilon_0 m_0 c^3}{q}$



K-V
Waterbag
Parabolic
Gaussian
n(r)
r/r-Gauss
0
0.5
1
1.5
2
2.5
−1
−0.5
0
0.5
1

# Use of RMS Envelope Equations

Compare KV $$a'' + ka - \frac{\epsilon^2}{a^3} - \frac{K}{a} = 0$$

RMS $$\tilde{x}'' + k\tilde{x} - \frac{\tilde{\epsilon}^2}{\tilde{x}^3} - \frac{q}{m_0\gamma^3\beta^2c^2}\frac{\langle xE_x\rangle}{\tilde{x}} = 0$$

Sacherer showed that for ellipsoidal particle densities of the form

$$n(x,y,z,s) = n\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, s\right)$$

the averages $\langle xE_x\rangle, \langle yE_y\rangle$ etc depend only very weakly on the exact charge distribution.

For 2D axisymmetric beam, radius $R$ and uniform density $n$,

$$E_r = \frac{nqr}{2\epsilon_0} \implies E_x = \frac{nqx}{2\epsilon_0}$$

Then
$$\begin{aligned}\langle xE_x\rangle = \langle yE_y\rangle &= \tfrac{1}{2}\big(\langle xE_x\rangle + \langle yE_y\rangle\big) = \frac{nq}{4\epsilon_0}\langle r^2\rangle \\ &= \frac{nq}{4\epsilon_0} \times \frac{1}{2}R^2 = \frac{Nq}{8\pi\epsilon_0}\end{aligned}$$

Therefore RMS space-charge term is

$$\frac{q}{m_0\gamma^3\beta^2c^2}\frac{\langle xE_x\rangle}{\tilde{x}} = \frac{Nq^2}{8\pi\epsilon_0 m_0\gamma^3\beta^2c^2}\frac{1}{\tilde{x}} = \frac{\frac{1}{4}K}{\tilde{x}}$$

and general RMS envelope equation is

$$\tilde{x}'' + k\tilde{x} - \frac{\epsilon^2}{\tilde{x}^3} - \frac{\frac{1}{4}K}{\tilde{x}} = 0$$

Now, for KV, $\tilde{x} = \frac{1}{2}a,\ \tilde{\epsilon} = \frac{1}{4}\epsilon$

so KV envelope equation is, $a'' + ka - \frac{\epsilon^2}{a^3} - \frac{K}{a} = 0$

$$\Longrightarrow \qquad 2\tilde{x}'' + 2k\tilde{x} - \frac{16\tilde{\epsilon}^2}{8\tilde{x}^3} - \frac{K}{2\tilde{x}} = 0$$

This is the same as the general RMS envelope equation.

So, if we have a channel designed for a KV beam, it will also serve for a non-linear beam with the same RMS beam size.

# Emittance Growth

What causes RMS emittance to evolve?

$$\tilde{\epsilon}^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2$$

$\langle x'^2 \rangle \sim$ Kinetic Energy $\qquad \langle x^2 \rangle \sim$ Potential Energy

What else is present?

Science & Technology Facilities Council

# Field Energy of a Beam

Self-field energy is

$$U = \tfrac{1}{2}\int \rho\phi \,\mathrm{d}V = \tfrac{1}{2}\int q\phi n \,\mathrm{d}V = \tfrac{1}{2}Nq\langle\phi\rangle$$

where $\rho = qn$ is the charge density and $\phi$ is the electrostatic potential, $\mathbf{E} = -\nabla\phi$

For the four special distributions, take a circular boundary of radius $R$ where $\phi = 0$.

$$\text{e.g. for a KV beam, } E_{r<a} = \frac{Nq}{2\pi\epsilon_0 a^2} r, \quad E_{r>a} = \frac{Nq}{2\pi\epsilon_0 r}$$

$$\Longrightarrow \quad \phi_{r<a} = \frac{Nq}{4\pi\epsilon_0}\left\{1 + 2\ln\frac{R}{a} - \frac{r^2}{a^2}\right\}.$$

Therefore

$$\begin{aligned} U_{KV} &= \tfrac{1}{2}Nq\int_0^a \phi_{r<a}(r)\,2\pi r\,\mathrm{d}r \\ &= \frac{N^2q^2}{4\pi\epsilon_0}\left(\frac{1}{4} + \ln\frac{R}{a}\right) \qquad (\text{and } a = 2\tilde{x} = 2\sqrt{\langle x^2\rangle}) \end{aligned}$$

$$
\begin{aligned}
U_{KV} &= u_0 \left\{ \ln \frac{R}{\langle x^2 \rangle} + \frac{1}{4} - \ln 2 \right\} &&= u_0 \left\{ \ln \frac{R}{\langle x^2 \rangle} - 0.4431 \right\} && \\
U_{WB} &= u_0 \left\{ \ln \frac{R}{\langle x^2 \rangle} + \frac{11}{24} - \frac{1}{2} \ln 6 \right\} &&= u_0 \left\{ \ln \frac{R}{\langle x^2 \rangle} - 0.4375 \right\} && \Delta U_{WB} = 0.0056\, u_0 \\
U_{PA} &= u_0 \left\{ \ln \frac{R}{\langle x^2 \rangle} + \frac{73}{120} - \frac{1}{2} \ln 8 \right\} &&= u_0 \left\{ \ln \frac{R}{\langle x^2 \rangle} - 0.4314 \right\} && \Delta U_{PA} = 0.0118\, u_0 \\
U_{GA} &= u_0 \left\{ \ln \frac{R}{\langle x^2 \rangle} + \frac{C}{2} - \ln 2 \right\} &&= u_0 \left\{ \ln \frac{R}{\langle x^2 \rangle} - 0.4046 \right\} && \Delta U_{GA} = 0.0386\, u_0
\end{aligned}
$$

Here $C$ is Euler's constant (0.577215665), $u_0 = \dfrac{N^2 q^2}{4\pi\epsilon_0} = \dfrac{I^2}{4\pi\epsilon_0 \beta^2 c^2}$ and $\Delta$ is the difference from the uniform KV model for equivalent beams (same $\langle x^2 \rangle$).

Deduce that, for equivalent beams, the uniform KV distribution has the smallest field energy.

# Energy Conservation

Kinetic energy of a single particle in the beam frame is $\frac{1}{2}m_0\gamma\beta^2c^2x'^2$

Potential energy from external forces is $\frac{1}{2}m_0\gamma\beta^2c^2k_xx^2$

$$T = \frac{1}{2}\int nm_0\gamma\beta^2c^2\sum x'^2\,\mathrm{d}V = \frac{1}{2}Nm_0\gamma\beta^2c^2\sum\langle x'^2\rangle$$

$$V = \frac{1}{2}\int nm_0\gamma\beta^2c^2\sum k_xx^2\,\mathrm{d}V = \frac{1}{2}Nm_0\gamma\beta^2c^2\sum k_x(s)\langle x^2\rangle$$

Calculations give **Energy conservation law:**

$$\boxed{T + V + \frac{1}{\gamma^2}U = \text{constant}}$$

For a continuous distribution with axisymmetry,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}s}\tilde{\epsilon}_x^2 = -\frac{1}{2}K\tilde{x}^2\frac{\mathrm{d}}{\mathrm{d}s}\frac{\Delta U}{u_0}}$$

where $\Delta U$ is the non-linear field energy and $K$ is the perveance

$$K = \frac{I}{I_0}\frac{2}{(\beta\gamma)^3} = \frac{Nq^2}{2\pi\epsilon_0 m_0\gamma^3\beta^2c^2}.$$

Solve in conjunction with RMS envelope equation

$$\boxed{\frac{\mathrm{d}^2}{\mathrm{d}s^2}\tilde{x} + k_x(s)\tilde{x} - \frac{\tilde{\epsilon}_x^2}{\tilde{x}^3} - \frac{K}{4\tilde{x}} = 0.}$$

An initially non-uniform beam in a focusing channel will release some or all of its non-linear field energy, which will be converted via kinetic and potential energies into RMS emittance as the beam evolves towards a stationary distribution.

# Emittance Growth

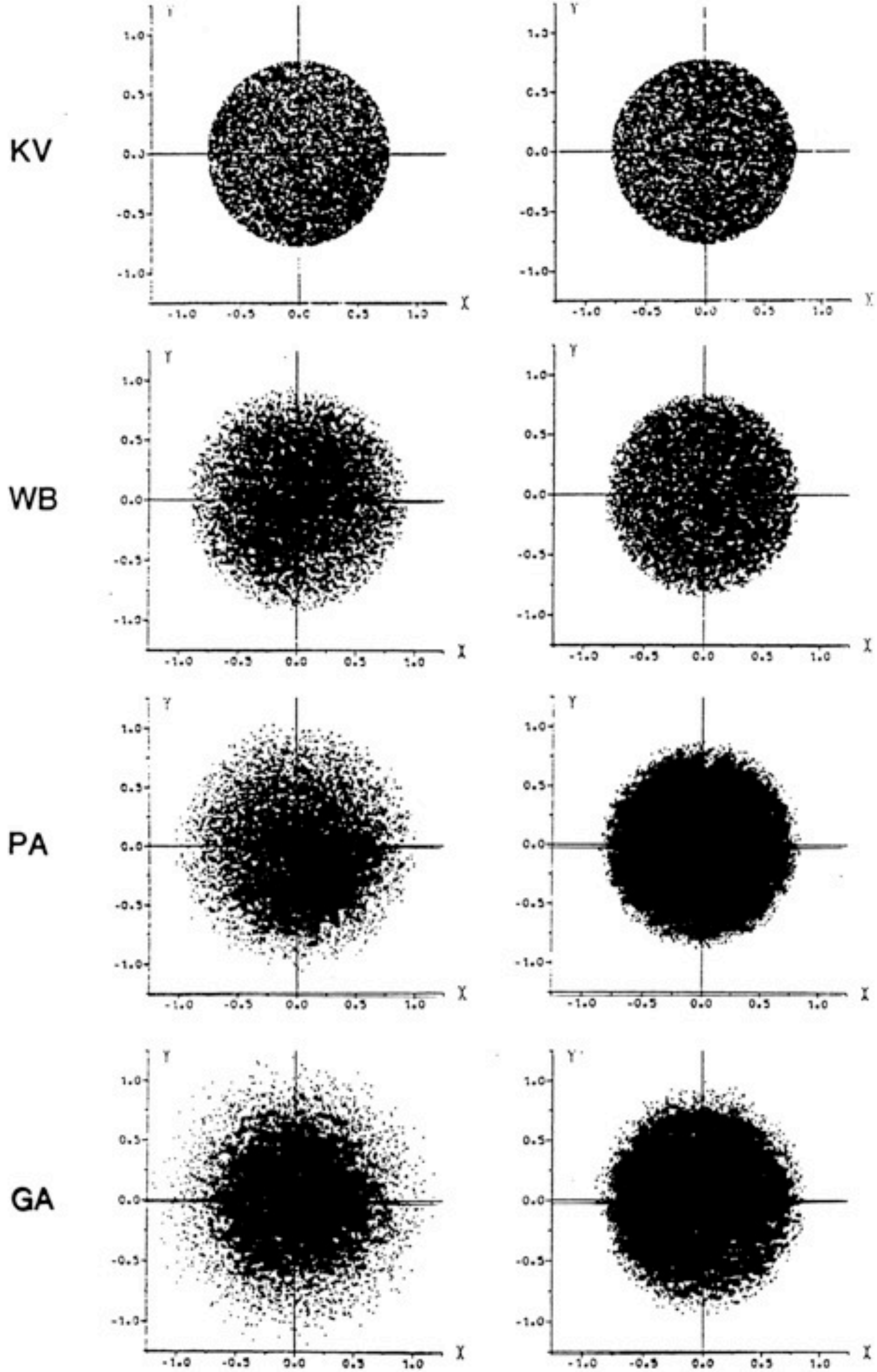
**Example**

- Focusing channel with $\sigma_0 = 60^\circ$ and $\sigma = 15^\circ$
- High levels of space charge, so the RMS beam size is approximately constant
- Assuming beam evolves to a minimum field energy state $\Delta U_{\text{final}} = 0$

then theory predicts emittance growth of:

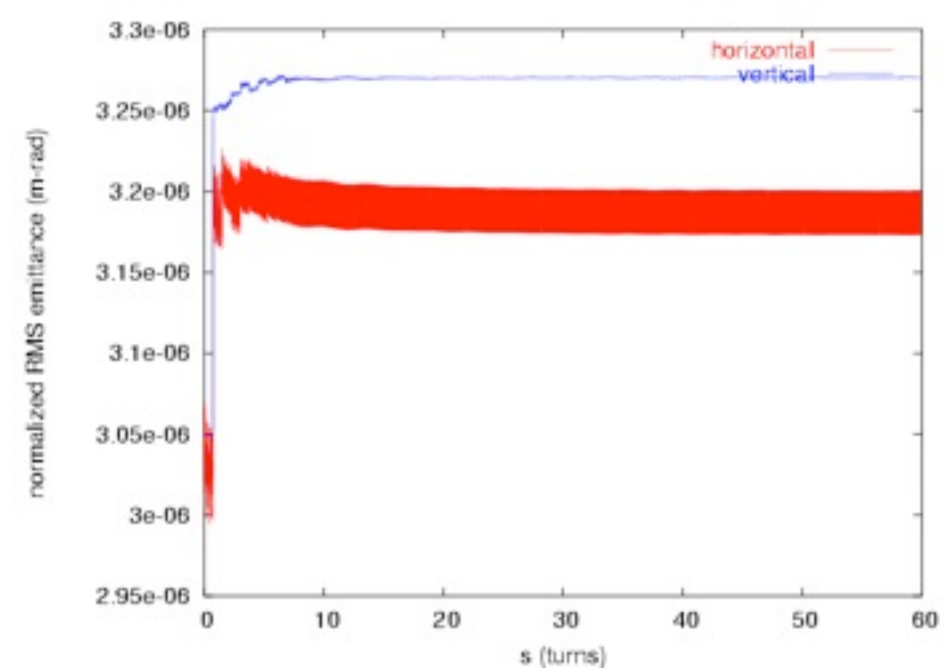
| **Distribution** | $\Delta U / u_0$ | **Emittance Increase** |
|---|---|---|
| Waterbag | 0.0056 | 8% |
| Parabolic | 0.0118 | 16% |
| Gaussian | 0.0386 | 47% |

Science & Technology Facilities Council




# Evolution of particle densities for 2D transverse distributions, showing tendency towards a stationary (KV) state

Simulations of emittance increase accord with theory. The increase takes place very rapidly, generally within one quarter of a plasma oscillation.



# Causes of Emittance Growth

- Non-linear self-forces arising from non-stationary beam profiles
- Non-linear applied forces
- Chromatic aberrations
- Beam mis-match causing oscillation of RMS radius
- Beam off-centering causing coherent oscillations about the central orbit
- Misalignment of magnets
- Coulomb scattering between particles
- Instabilities
- Non-linear coupling between transverse and longitudinal motion
- External statistical fluctuations (e.g. rf noise)

Science & Technology Facilities Council

# Quantifying Beam Halo

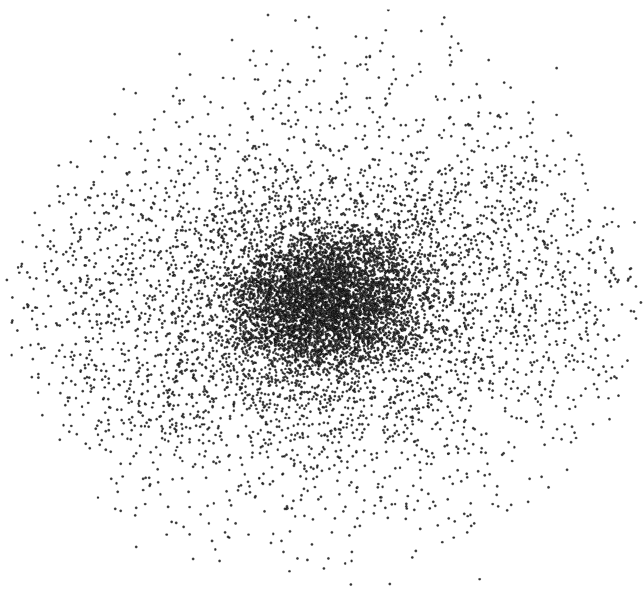
**Kurtosis** – an idea from statistics to measure tails of distributions, adapted for beams in accelerators. *T. Wangler (LANL)*

$$I_2 = \langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2$$ ⇠ RMS emittance

$$I_4 = \langle x^4\rangle\langle x'^4\rangle + 3\langle x^2x'^2\rangle^2 - 4\langle xx'^3\rangle\langle x^3x'\rangle$$ ⇠ 4th moments

$$H = \frac{1}{2I_2}\sqrt{3I_4} - 2$$

$$= \frac{\sqrt{3\langle x^4\rangle\langle x'^4\rangle + 9\langle x^2x'^2\rangle^2 - 12\langle xx'^3\rangle\langle x^3x'\rangle}}{2\langle x^2\rangle\langle x'^2\rangle - 2\langle xx'\rangle^2} - 2.$$



Elliptical symmetry in phase space

$$\Longrightarrow \boxed{H = \begin{cases} 0 & \text{for the KV distribution} \\ 1 & \text{for the Gaussian distribution.} \end{cases}}$$

H is called the halo parameter

Multi-particle simulations show that significant halo in the 2D phase-space projection corresponds to $\boxed{H > 1}$.

An alternative approach is to use the *spatial profile parameter*:

$$\boxed{h = \frac{\langle x^4 \rangle}{\langle x^2 \rangle^2} - 2}$$

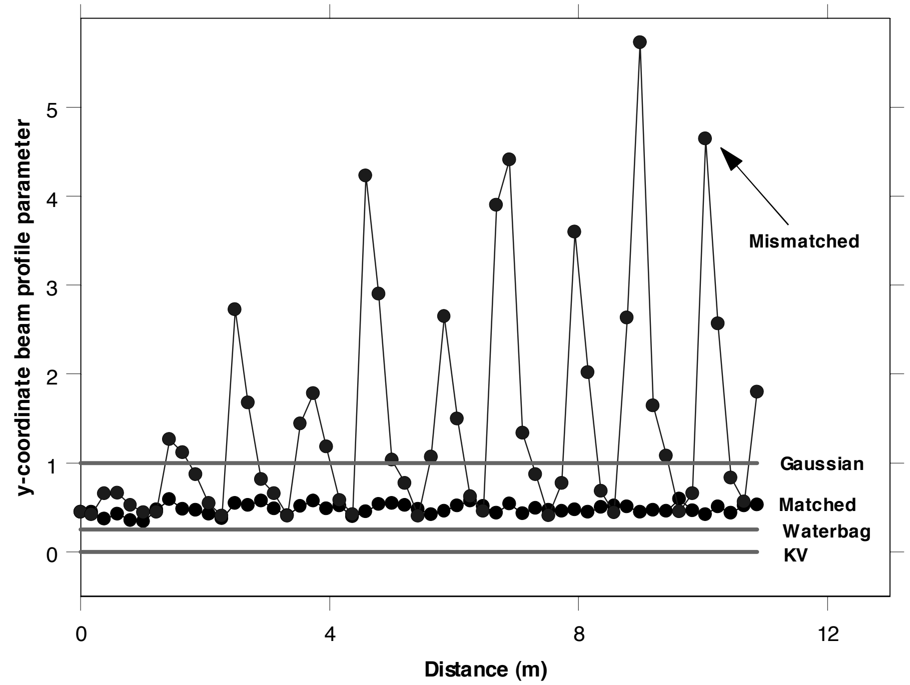
For beams with elliptical symmetry and densities

$$\rho(x, x') = f\left(\beta x'^2 + 2\alpha x x' + \gamma x^2\right),$$

direct calculation shows $H = h$.

But not true for more general distributions

Simulations show that halo can "hide" in phase space and is not observed in some spatial projections.




Results from a beam halo experiment at LANL Beam-profile parameter from computer simulation at drift-space locations along the beamline for the matched and mismatched beams. Values for uniform (KV), Waterbag and Gaussian beams are shown. The excursions above the Gaussian level indicate a large halo

Science & Technology Facilities Council