Introduction to Transverse Beam Optics

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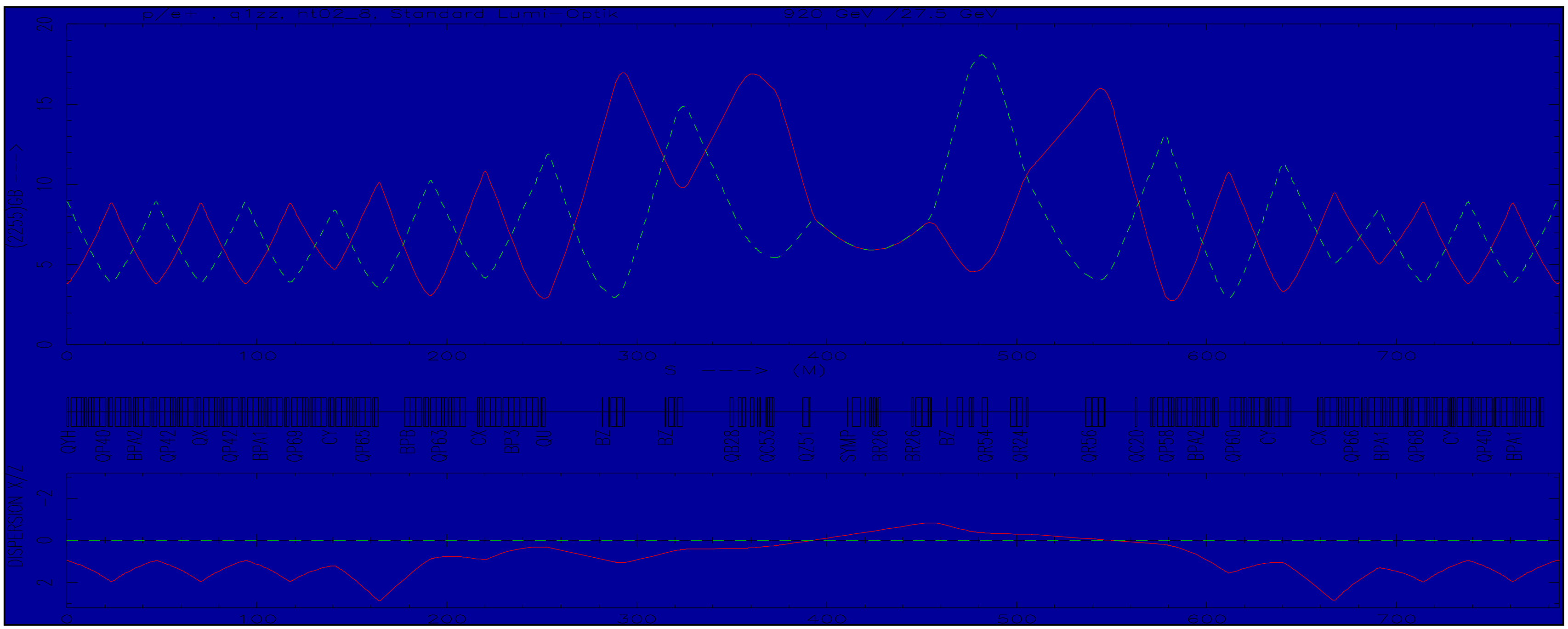
Part II: Periodic Soluion, the Beta Function

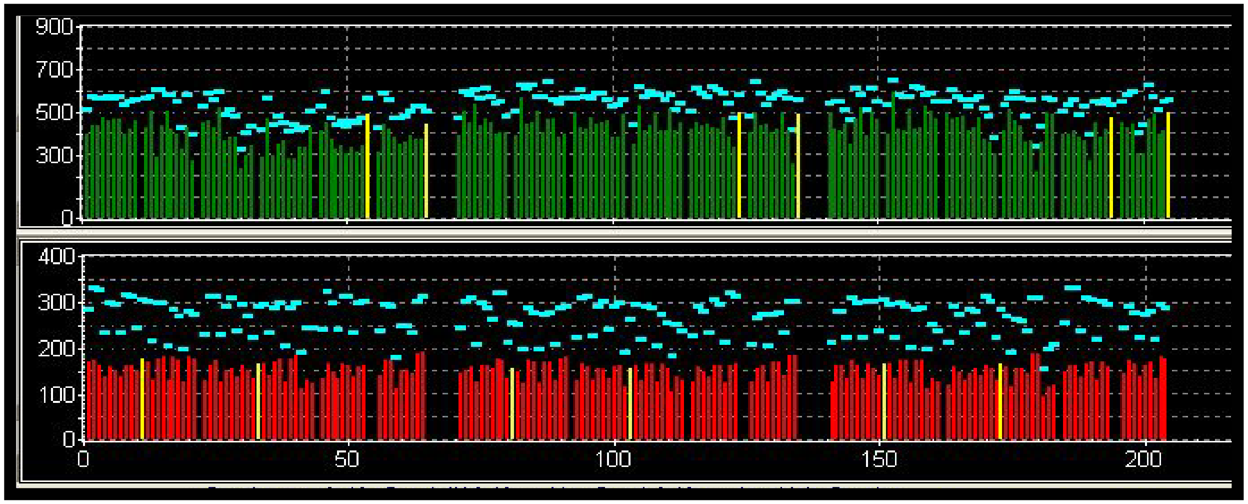


Lattice and Beam Optics of a typical high energy storage ring

I.) the Beta Function

Beam parameters of a typical high energy ring:

Ip = 100 mA
Ie= 50 mA



Example: HERA Bunch pattern

number of particles per bunch:

$$N_p = \frac{100mA}{180} * \frac{\tau_{rev}}{e} = \frac{100 * 10^{-3}}{180} * \frac{Cb}{s} * \frac{21 * 10^{-6}}{1.6 * 10^{-19}} * \frac{s}{Cb}$$

$$\underline{N_p = 7.3 * 10^{10}}$$

... question: do we really have to calculate some 10^{10} single particle trajectories ?

Equation of motion:
Hill´s equation

Example: particle motion with periodic coefficient

consider for the moment: $\Delta p/p=0$
$1/\rho = 0$

equation of motion: $$x''(s) - k(s)x(s) = 0$$

**** restoring force $\neq$ const,***
k(s) = depending on the position s
k(s) = periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.

Ansatz: $$x(s) = A\,u(s)\cos\{\psi(s)+\phi\}$$

A, Φ = integration constants determined by initial conditions

$$x'(s) = A\,u'(s)\cos\{\psi(s)+\phi\} - A\,u(s)\sin\{\psi(s)+\phi\}\,\psi'(s)$$

$$x''(s) = A\left\{u''(s) - u(s)\psi'^2(s)\right\} cos\left\{\psi(s)+\phi\right\} -$$
$$- A\left\{2u'(s)\psi'(s) + u(s)\psi''(s)\right\} sin\left\{\psi(s)+\phi\right\}$$

insert x(s) and x´(s) into Hill´s equation:

$$A\left\{u''(s) - u(s)\psi'^2(s) - ku(s)\right\} cos\left\{\psi(s)+\phi\right\} -$$
$$- A\left\{2u'(s)\psi'(s) + u(s)\psi''(s)\right\} sin\left\{\psi(s)+\phi\right\} = 0$$

we get two conditions:

$$\Rightarrow \quad u''(s) - u(s)\psi'^2(s) - ku(s) = 0 \qquad (i)$$

$$\Rightarrow \quad 2u'(s)\psi'(s) + u(s)\psi''(s) = 0 \qquad (ii)$$

from (ii) we obtain:

$$2\frac{u'(s)}{u(s)} + \frac{\psi''(s)}{\psi'(s)} = 0 \qquad \Rightarrow \qquad \psi(s) = \int_0^s \frac{d\tilde{s}}{u^2(\tilde{s})}$$

... the phase of the oscillation is given by its amplitude

The Betafunction

inserting into (i):

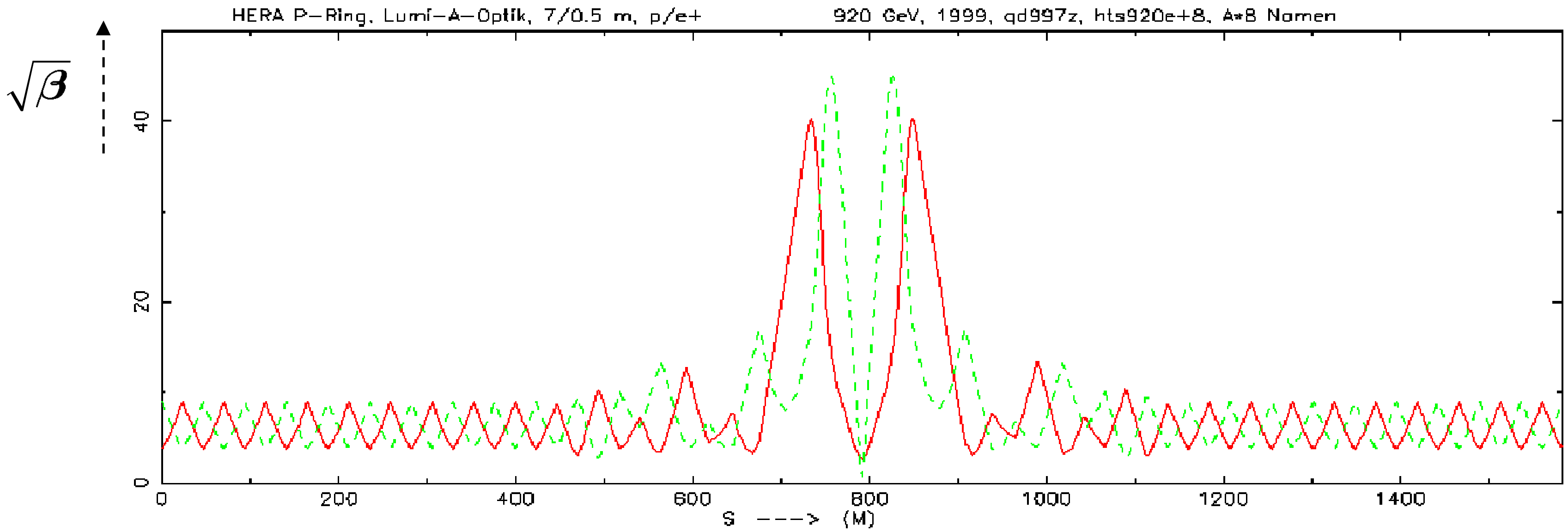
$$u''(s) - \frac{1}{u^3(s)} - k(s)u(s) = 0 \qquad (iii)$$

following tradition we define instead of u(s) ...

$$\beta(s) := u^2(s) \qquad A = \sqrt{\varepsilon}$$

and get for the particle trajectory

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s) + \phi\} \qquad \textbf{where} \qquad \psi(s) = \int_0^s \frac{d\tilde{s}}{\beta(\tilde{s})}$$



The Betafunction

** β is uniquely determined by the equation*

$$u''(s) - \frac{1}{u^3(s)} - k(s)u(s) = 0 \qquad (iii)$$

** equation (iii) cannot be solved analytically*
… but numerically if needed

** by definition: $\beta > 0$*

** β represents the focusing properties but – unlike k(s) – it depends on the total configuration of the ring.*

** β is a periodic function: $\beta(s+C_0) = \beta(s)$*

** β defines at any position s the amplitude of the transverse particle oscillation*

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$$

... allright, but what is ε

II.) Phase Space Ellipse and Liouville´s Theorem

... may I introduce you to Mr. Liouville:

„under the influence of conservative forces, the particle density in phase space is constant.“

Joseph Liouville, 1809-1882

general solution of the particle trajectory:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\} \qquad (i)$$

$$x'(s) = \sqrt{\varepsilon}\,\frac{1}{2}\,\frac{1}{\sqrt{\beta(s)}}\,\beta'(s)\cos\{\psi(s)+\phi\} - \sqrt{\varepsilon}\sqrt{\beta(s)}\,\sin\{\psi(s)+\phi\}\,\psi'(s)$$

the phase $\Psi(s)$ is determined by $\beta(s)$, namely $\Psi'(s) = 1/\beta(s)$ and defining the new variable $\alpha(s) = -\beta'(s)/2$ we get

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\} + \sin\{\psi(s)+\phi\}\right]$$

using (i) we can replace the cosine term in this expression

$$\cos\{\psi(s)+\phi\} = \frac{x(s)}{\sqrt{\varepsilon}\sqrt{\beta(s)}}$$

$$\Rightarrow x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left\{\frac{\alpha(s)x(s)}{\sqrt{\varepsilon\beta(s)}} + sin\{\psi(s)+\phi\}\right\}$$

$$\Rightarrow sin\{\psi(s)+\phi\} = -\frac{\sqrt{\beta(s)}x'(s)}{\sqrt{\varepsilon}} - \frac{\alpha(s)x(s)}{\sqrt{\beta(s)}\sqrt{\varepsilon}}$$

remember from school: $sin^2x + cos^2x = 1$

$$\Rightarrow \left\{\frac{\sqrt{\beta(s)}x'(s)}{\sqrt{\varepsilon}} + \frac{\alpha(s)x(s)}{\sqrt{\beta(s)}\sqrt{\varepsilon}}\right\}^2 + \left(\frac{x(s)}{\sqrt{\varepsilon\beta(s)}}\right)^2 = 1$$

$$\varepsilon = \left\{\sqrt{\beta(s)}x'(s) + \frac{\alpha(s)x(s)}{\sqrt{\beta(s)}}\right\}^2 + \frac{x^2(s)}{\beta(s)}$$

$$\varepsilon = \beta(s)x'^2(s) + 2\alpha(s)x(s)x'(s) + x^2(s)\frac{1+\alpha^2(s)}{\beta(s)}$$

we define a new variable: $\gamma(s) = (1+\alpha^2)/\beta$

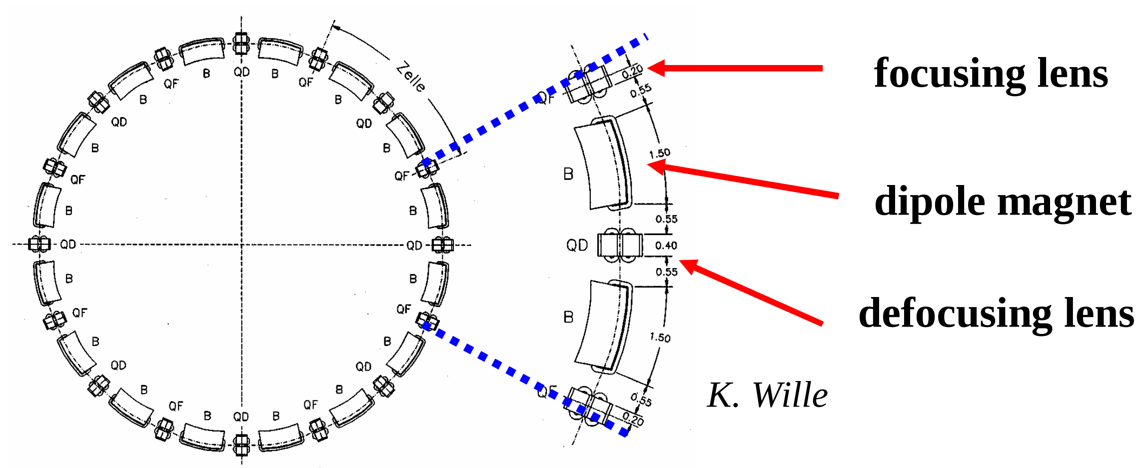
$$\varepsilon = \beta(s)\cdot x'^2(s) + 2\alpha(s)\cdot x(s)x'(s) + \gamma(s)\cdot x^2(s) = const$$

Remember: transformation through a system of lattice elements

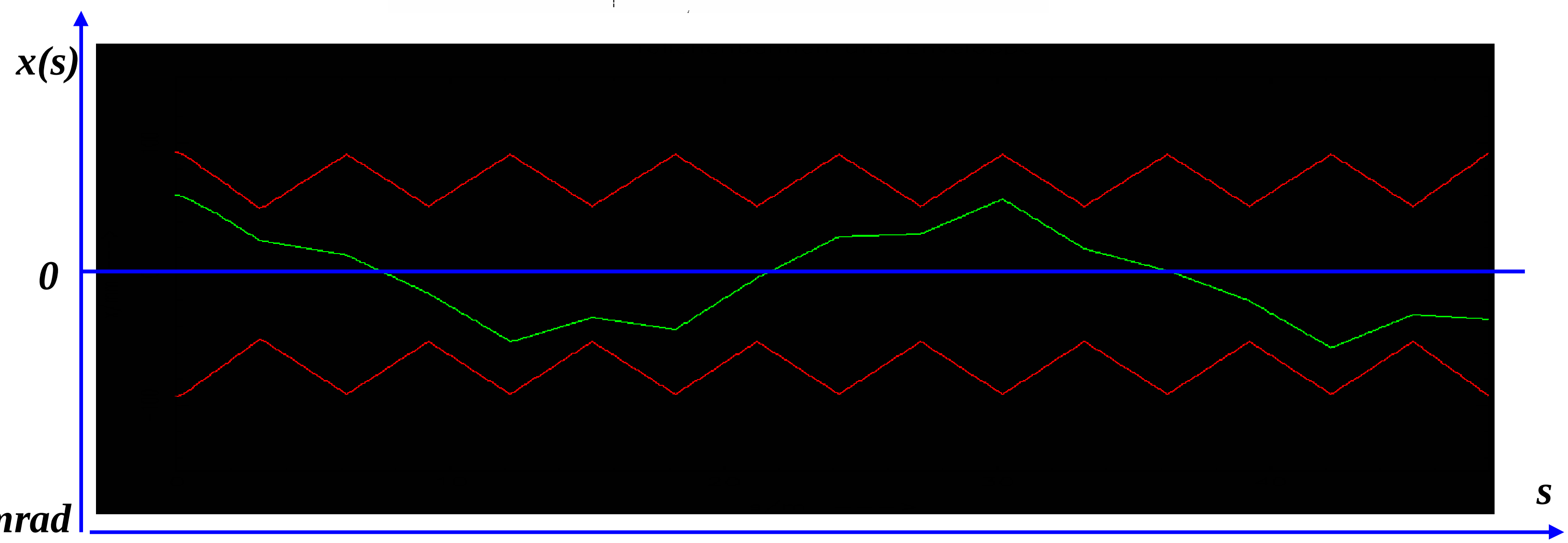
combine the single element solutions by multiplication of the matrices

$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D^*}.....$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s2,s1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



typical values in a strong foc. machine:
$x \approx mm$, $x' \leq mrad$

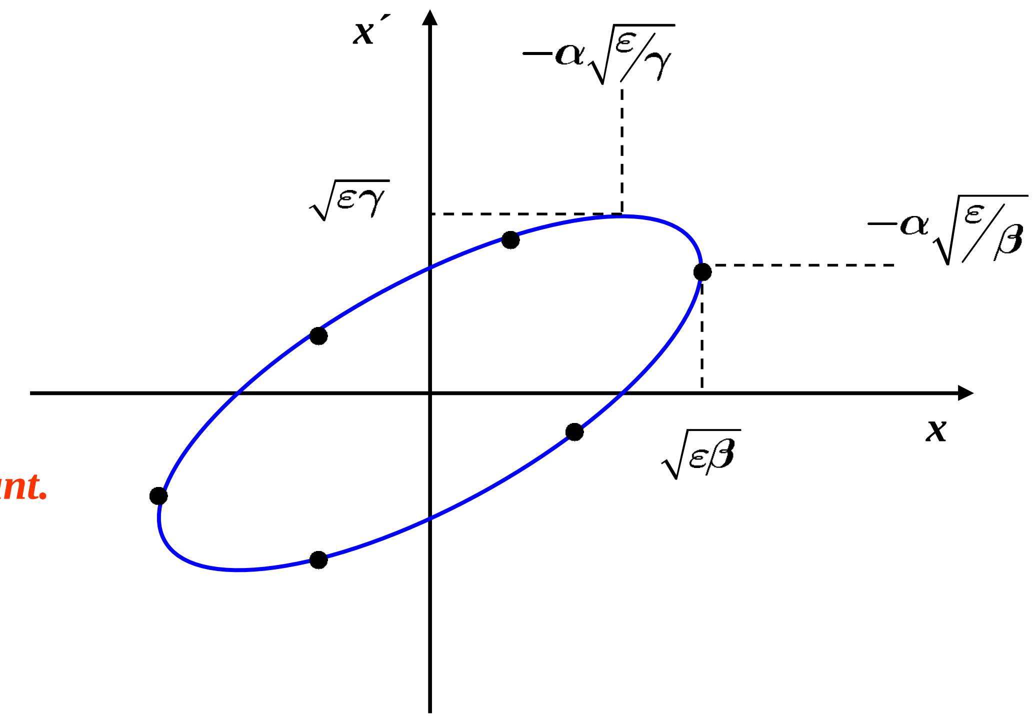


Phase Space Ellipse and Liouville´s Theorem

$$\varepsilon = \beta(s) \cdot x'^2(s) + 2\alpha(s) \cdot x(s)x'(s) + \gamma(s) \cdot x^2(s)$$

parametric representation of an ellipse in the x, x´ phase space,
ε = "Courant Snyder Invariant"

Area of the ellipse: $A = \pi * \varepsilon$

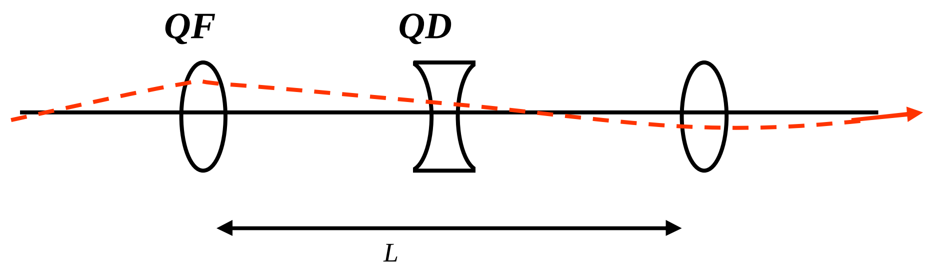


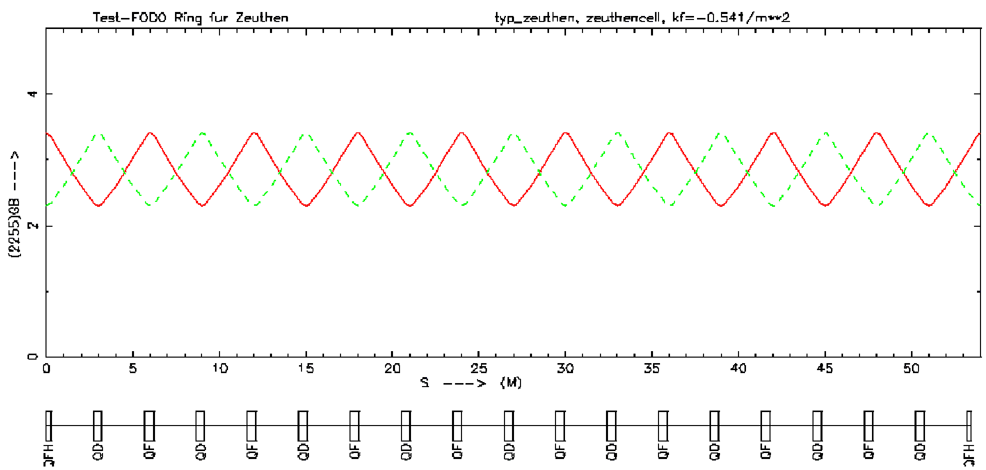
**** Nota bene: as α, β, γ are functions of s the shape of the ellipse will change if we go around the ring, but the area is constant.***

**** if α = 0 the β-function reaches its extreme value and the ellipse is upright in x or x´ direction.***

Example: FODO Structure

regular, periodic pattern of focusing and defocusing quadrupole lenses



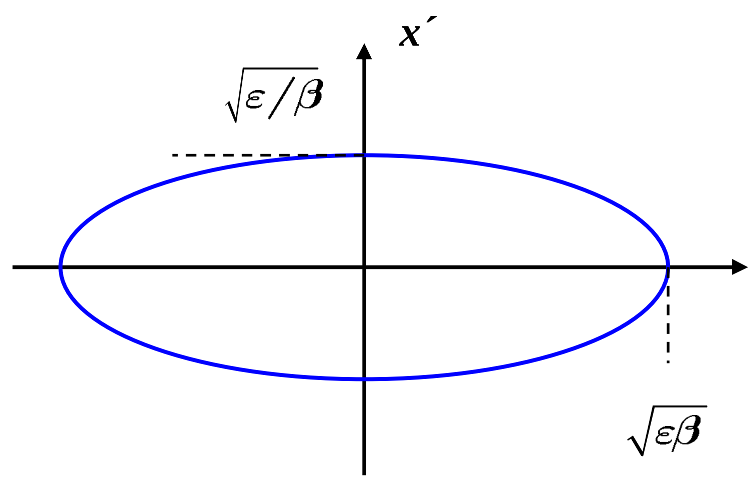


optics calculation:

β_x, β_y in a FODO

$\alpha = -1/2\ \beta' = 0$ in the center of the quadrupole

... in the center of a foc. quadrupole the particle amplitude reaches its maximum



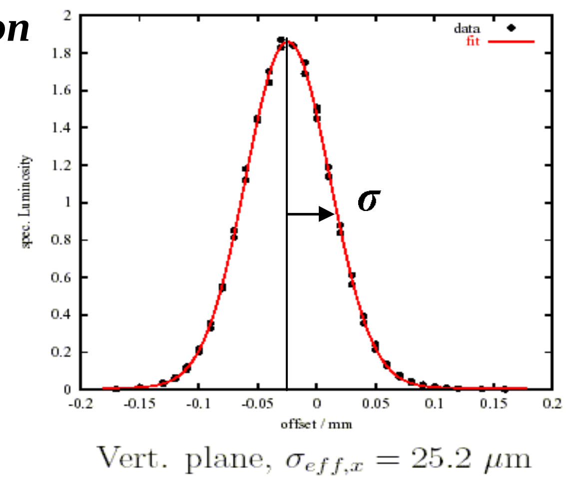
III.) Beam Emittance and Envelope

consider a particle whose ellipse in phase space surrounds all other ellipses
→ due to Mr. Liouville the area of this single particle will forever enclose all others
→ define an emittance of the beam in the sense that

$$Area = \pi\hat{\varepsilon} \quad \Leftrightarrow \quad \sigma_{beam}(s) = \sqrt{\hat{\varepsilon}\beta}$$

ε = beam emittance

in practice: transverse particle density in a beam ≈ Gauß distribution



Example: HERA transverse beam profile measured at the interaction point

beam envelope

$$\sigma_{beam}(s) = \sqrt{\varepsilon_B \cdot \beta}$$

$$\varepsilon = \beta(s) \cdot x'^2(s) + 2\alpha(s) \cdot x(s)x'(s) + \gamma(s) \cdot x^2(s)$$

solve for x´ and require $\frac{dx'}{dx} = 0$

beam divergence:

$$x'_{max}(s) = \sqrt{\varepsilon_B}\sqrt{\frac{1+\alpha^2}{\beta}} = \sqrt{\varepsilon_B \gamma}$$

... so sorry $\varepsilon \neq const.$

According to Hamiltonian mechanics:
phase space diagram relates the variables q and p

$$q = position = x$$

$$p = momentum = mc\gamma\beta_x$$

Liouvilles Theorem: $\int p\, dq = const$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta}$$

where $\beta = v/c$

and Liouville tells us:

$$\int p\, dq = const = mc \int \gamma\beta_x\, dx = mc\gamma\beta \int x' dx$$

$$\boxed{\Rightarrow \int x' dx = \frac{const}{\beta\gamma}}$$

the beam emittance shrinks during acceleration $\varepsilon \sim 1/\gamma$

IV.) Transformation of α, β, γ

consider two positions in the storage ring: s_0 , s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} \qquad M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

since ε = const:

$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2 = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 {x_0}^2$$

express x_0, x'_0 as a function of x, x´.

$$\begin{pmatrix} x \\ x' \end{pmatrix}_0 = M^{-1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_s , \qquad M^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix}$$

... remember W = CS´-SC´ = 1

$$\rightarrow x_0 = S'x - Sx' \qquad x_0' = -C'x + Cx'$$

inserting into ε

$$\varepsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$= \beta_0 (Cx' - C'x)^2 + 2\alpha_0 (S'x - Sx')(Cx' - C'x) + \gamma_0 (S'x - Sx')^2$$

sort via x, x´and compare the coefficients to get ....

$$\beta(s) = C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0$$

$$\alpha(s) = -CC'\beta_0 + (SC' + S'C)\alpha_0 - SS'\gamma_0$$

$$\gamma(s) = C'^2\beta_0 - 2S'C'\alpha_0 + S'^2\gamma_0$$

in matrix notation:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

!

1.) this expression is important

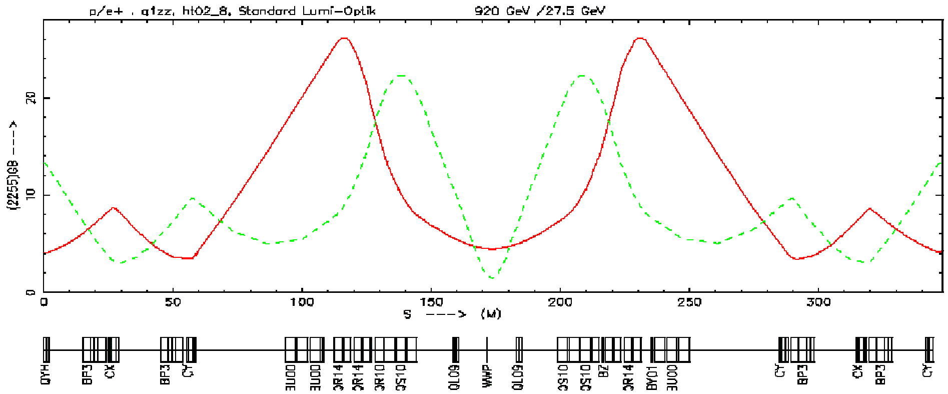
2.) given the twiss parameters α, β, γ at any point in the lattice we can transform them and calculate their values at any other point in the ring.

3.) the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to calculate single particle trajectories.

4.) go back to point 1.)

Example: symmetric Drift space

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$



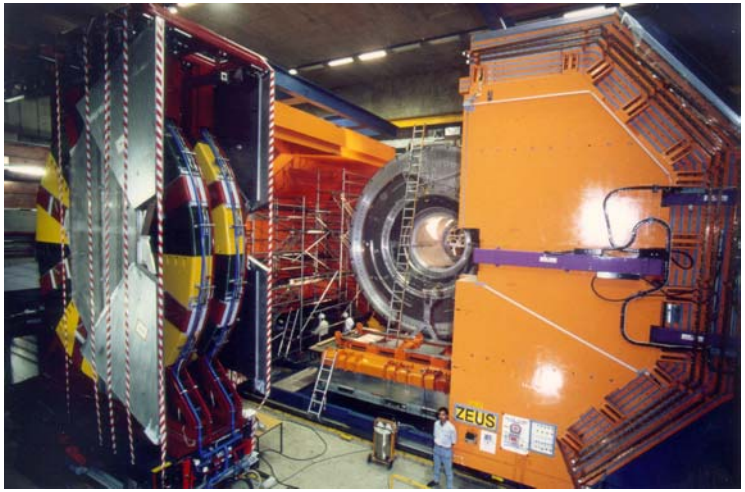
transformation of the Twiss parameters in a drift:

$$\beta(l) = \beta_0 - 2\alpha_0 l + \gamma_0 l^2, \qquad \alpha(l) = \alpha_0 - \gamma_0 l, \qquad \gamma(l) = \gamma_0$$

start at a symmetry point

$$\alpha_0 = 0, \qquad \gamma_0 = \frac{1}{\beta_0}$$

$$\boxed{\beta(l) = \beta_0 + \frac{l^2}{\beta_0}}$$



Example: Zeus detector at HERA

* *in a (symmetric) drift space the β function grows quadratically with the distance l to the starting point*
* *small beam sizes lead to a fast increase of the beam envelope in the drift*
* *collision points in a lattice need special care*

V.) Transfer Matrix M ... *yes we had that topic already*

general solution of Hill´s equation

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\,\cos\{\psi(s)+\phi\}$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\} + \sin\{\psi(s)+\phi\}\right]$$

remember the trigonometrical gymnastics: sin(a+b)=sin(a) cos(b)+cos(a) sin(b)
cos(a+b)=cos(a) cos(b) - sin(a)sin(b)

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta_s}\left(\cos\psi_s\cos\phi - \sin\psi_s\sin\phi\right)$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}}\left[\alpha_s\cos\psi_s\cos\phi - \alpha_s\sin\psi_s\sin\phi + \sin\psi_s\cos\phi + \cos\psi_s\sin\phi\right]$$

define the starting point: $s(0) = s_0$, $x(0) = x_0$, $x´(0) = x´_0$, $\alpha(0) = \alpha_0$, $\beta(0) = \beta_0$, $\Psi(0) = 0$

$$\cos\phi = \frac{x_0}{\sqrt{\varepsilon\beta_0}} \quad , \qquad \sin\phi = -\frac{1}{\sqrt{\varepsilon}}\left(x'_0\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right)$$

inserting above ...

$$x(s) = \sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\psi_s + \alpha_0 \sin\psi_s\right\} x_0 + \left\{\sqrt{\beta_s\beta_0}\,\sin\psi_s\right\} x_0'$$

$$x'(s) = \frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s\right\} x_0 + \sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\psi_s - \alpha_s \sin\psi_s\right\} x_0'$$

which can be expressed ... for convenience ... in matrix form

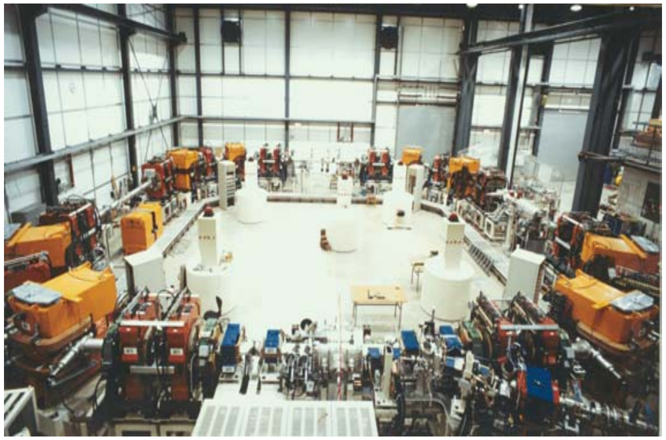
$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

*

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\Delta\psi + \alpha_0 \sin\Delta\psi\right) & \sqrt{\beta_s\beta_0}\,\sin\Delta\psi \\ \frac{(\alpha_0-\alpha_s)\cos\Delta\psi - (1+\alpha_0\alpha_s)\sin\Delta\psi}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\Delta\psi - \alpha_s \sin\Delta\psi\right) \end{pmatrix}$$

* ***we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.***
* ***and nothing but the α β γ at these positions.***
* ***... !***

Question: ***what will happen, if you do not make too many mistakes and your particle performs one complete turn ?***



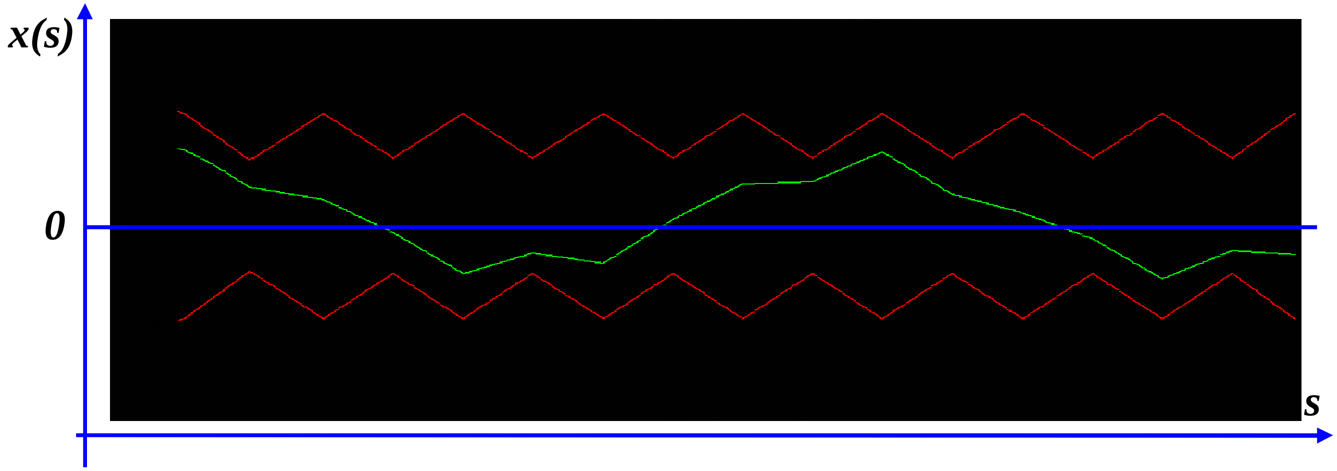
$$\beta(s+L) = \beta(s)$$
$$\alpha(s+L) = \alpha(s)$$
$$\gamma(s+L) = \gamma(s)$$

... and refer to a symmetry point:
$s=s_0, \ \alpha_0 = 0$

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos\psi_{turn} & \beta_0 \sin\psi_{turn} \\ \frac{-1}{\beta_0}\sin\psi_{turn} & \cos\psi_{turn} \end{pmatrix}$$

Definition: ***phase advance of the particle oscillation per revolution in units of 2π is called tune***

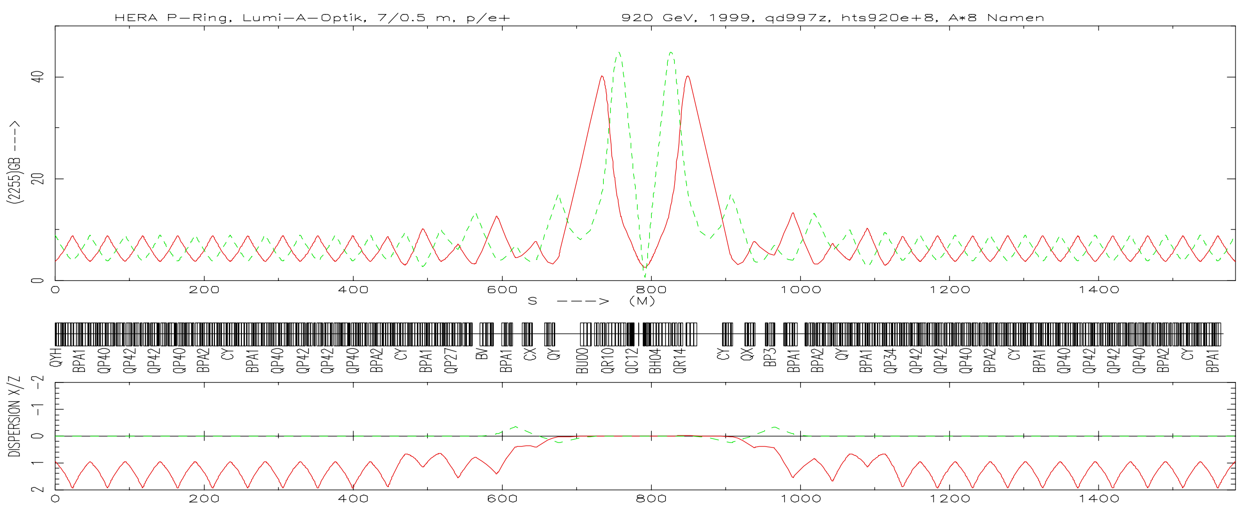
$$Q = \frac{\Delta\psi_{1\,turn}}{2\pi}$$



„veni vidi vici ...“ *.... or in english ... „we got it !“*

$$M(s) = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos 2\pi Q + \alpha_s \sin 2\pi Q & \beta_s \sin 2\pi Q \\ \dfrac{-(1+\alpha_s^2)}{\beta_s} \sin 2\pi Q & \cos 2\pi Q - \alpha_s \sin 2\pi Q \end{pmatrix}$$

** determine the matrices of the single lattice elements*
** calculate the matrix product to get the one turn matrix (at a given starting position s)*
** get all the information about the lattice functions – at that position s*
** ... for any periodic structure: storage ring, substructure*



VI.) Stability Criterion

... following Courant, Snyder: Annals of physics 3, 1958

Transfer Matrix for 1 turn: $M(s+L) = M(s)$

and for N turns: $M(s+N\cdot L) = (M(s))^N$

stable motion: all elements of M have to remain bounded after N turns.
→ eigenvalues of M have to remain bounded.

$$M\begin{pmatrix} x \\ x' \end{pmatrix} = \lambda \begin{pmatrix} x \\ x' \end{pmatrix}$$

→ solved by determinant equation $det(M-\lambda I) = 0$

write formally: $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\lambda^2 - \lambda(a+d) + \underbrace{(ad-bc)}_{det\ M\ =\ 1} = 0$$

introduce a new parameter: $\cos\mu = \frac{1}{2} trace M = \frac{1}{2}(a+d)$

Nota bene: μ = real if $\left|\frac{1}{2} trace M\right| < 1$

solution of the determinant equation: $\lambda_{1/2} = \cos\mu \pm i \sin\mu$

The „Twiss“ Parametrisation of M

$$M = I\cos\mu + J\sin\mu \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\alpha = \frac{a-d}{2\sin\mu}, \qquad \beta = \frac{b}{\sin\mu}, \qquad \gamma = \frac{-c}{\sin\mu}$$

moreover, as det (M) = 1 $\quad \gamma = \dfrac{1+\alpha^2}{\beta}$

Now consider again N turns:

$$M^N = (I\cos\mu + J\sin\mu)^N = I\cos N\mu + J\sin N\mu$$

de Moivre´s formula

** The elements of M remain bounded if the parameter μ is real*
** Stability criterion for periodic structures:*

$|\cos\mu| < 1$
$|trace\,(M)| < 2$

Twiss Parametrisation: α, β, γ

... you know already that if ...

$$M = I\cos\mu + J\sin\mu \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

so if M is given by the lattice elements ...

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = M_1 \cdot M_2 \cdot M_3 \cdot M_4 \dots$$

we get:

$$\beta = \frac{S}{\sin\mu} \qquad \alpha = \frac{C - S'}{2\sin\mu}$$

$$\gamma = \frac{-C'}{\sin\mu} \qquad \cos\mu = \frac{1}{2}(C + S')$$

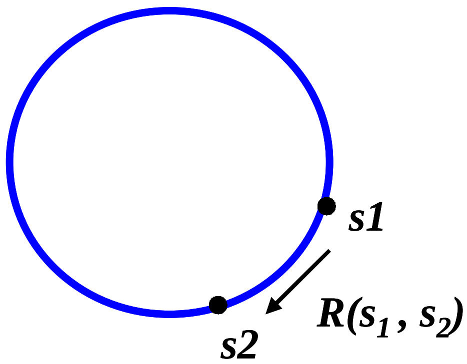
** very usefull for exact and for back on the envelope calculations*

** it has nothing to do with Prof. Twiss (...says Prof. Twiss)*

still about the trace ...

** The transfer matrix for one complete revolution is a function of the position s*

$$M = \begin{pmatrix} \cos\mu + \alpha(s)\sin\mu & \beta(s)\sin\mu \\ -\gamma(s)\sin\mu & \cos\mu - \alpha(s)\sin\mu \end{pmatrix}$$



** The trace of M however does not !*

the transformation from s1 → s2 + 1 turn can be expressed in two ways:

$$R(s_1, s_2 + C) = M_{S2} \cdot R(s_1, s_2)$$

$$R(s_1, s_2 + C) = R(s_1, s_2) \cdot M_{S1}$$

M denotes the 1 turn matrix

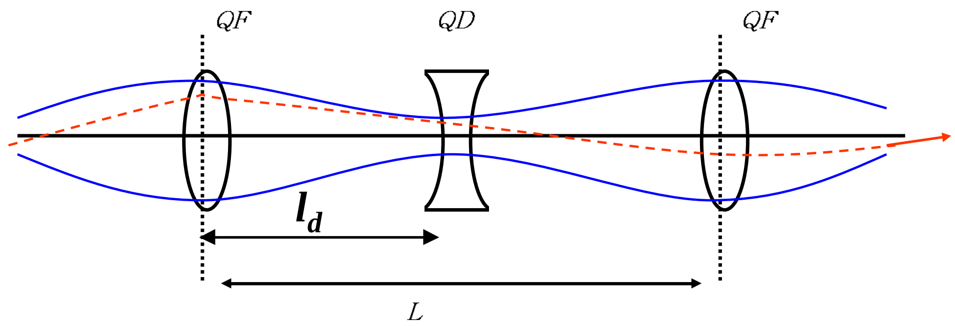
multiply both eq. from rhs by $R^{-1}(s_1,s_2)$

$$M_{s2} = R(s_1, s_2) \cdot M_{s1} \cdot R^{-1}(s_1, s_2)$$

Ms1 and Ms1 are related by a similarity transformation → the trace is unchanged

→ μ does not depend on s

VII.) Example: FoDo Structure



FoDo: ***regular structure of Focusing and Defocusing quadrupole lenses with „nothing" in between.***
Definition: „nothing" = anything that can be neglected to first order: drifts, bending magnets, high energy physics detectors etc.

Transfer Matrices:

$$M_{half\ cell} = M_{QD/2} \cdot M_{ld} \cdot M_{QF/2}$$

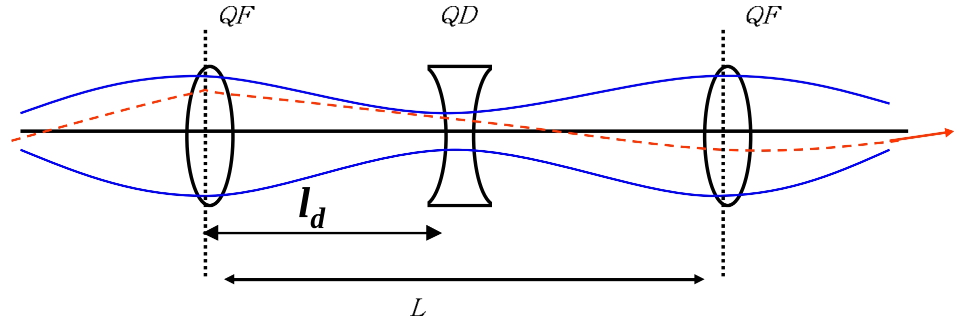
$$M_{half\ cell} = \begin{pmatrix} 1 & 0 \\ 1/\tilde{f} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & l_d \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1/\tilde{f} & 1 \end{pmatrix}$$

we put:
$$l_d = L / 2$$
$$\tilde{f} = 2f$$

$$M_{half\ cell} = \begin{pmatrix} 1 - l_d/\tilde{f} & l_d \\ -l_d/\tilde{f}^2 & 1 + l_d/\tilde{f} \end{pmatrix}$$

... for second half cell set $\tilde{f} \rightarrow -\tilde{f}$

$$M = \begin{pmatrix} 1 - \frac{2l_d^2}{\tilde{f}^2} & 2l_d(1 + \frac{l_d}{\tilde{f}}) \\ 2(\frac{l_d^2}{\tilde{f}^3} - \frac{l_d}{\tilde{f}^2}) & 1 - \frac{2l_d^2}{\tilde{f}^2} \end{pmatrix}$$



compare to the Twiss Parametrisation:

$$M = \begin{pmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix}$$

1.) we get at the starting point (middle of quad): $\alpha = 0$

2.) Phase advance of the cell:

$$|trace(M)| = |2\cos\mu| = 2 - \frac{4l_d^2}{\tilde{f}^2} \quad \rightarrow \quad |\cos\mu| = 1 - \frac{2l_d^2}{\tilde{f}^2}$$

after a good beer you will remember that:

$$\cos\mu = 1 - 2\sin^2\frac{\mu}{2}$$

2.) Phase advance of the cell:

$$\boxed{\left|\sin\frac{\mu}{2}\right| = \frac{L}{4f}}$$

the phase advance of the particle oscillation is given by the cell length L and the focal length f of the quadrupole lenses

FoDo structure in the SPS

3.) Stability Criterion for the FoDo:

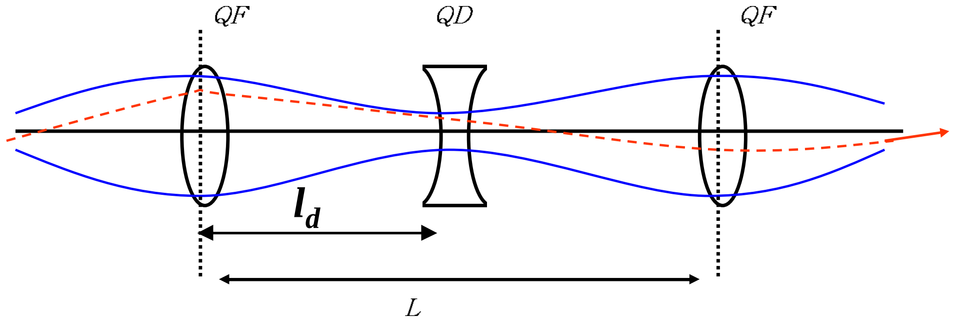
$$|trace(M)| = \left|2 - \frac{4l_d^2}{\tilde{f}^2}\right| < 2 \qquad \boxed{\Rightarrow \quad f > \frac{L_{cell}}{4}}$$

4.) Example: HERA Proton Ring

$L = 47m$ $k = 0.032\ m^{-2}$ $l_{quad} = 1.9m$
$f = 16.4\ m$ $\mu = 90°$

$$\frac{L_{cell}}{4} = 11.75\ m$$

VIII.) Twiss Parameters in a FoDo



assume the transformation from the foc. quad to the defocusing one

→ same procedure as every year, James ...

matrix of the half cell

$$M_{half\ cell} = \begin{pmatrix} 1 - l_d/\tilde{f} & l_d \\ -l_d/\tilde{f}^2 & 1 + l_d/\tilde{f} \end{pmatrix}$$

compare to the matrix in Twiss form

$$M = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\Delta\psi + \alpha_1 \sin\Delta\psi\right) & \sqrt{\beta_2\beta_1}\,\sin\Delta\psi \\ \frac{(\alpha_1 - \alpha_2)\cos\Delta\psi - (1 + \alpha_1\alpha_2)\sin\Delta\psi}{\sqrt{\beta_2\beta_1}} & \sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\Delta\psi - \alpha_2 \sin\Delta\psi\right) \end{pmatrix}$$

$$\hat{\beta} = \frac{L(1 + \sin\frac{\mu}{2})}{\sin\mu} \qquad \breve{\beta} = \frac{L(1 - \sin\frac{\mu}{2})}{\sin\mu}$$

IX.) Résumé:

general solution of the equation of motion $$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s) + \phi\}$$

phase advance $$\psi(s_{1\to 2}) = \int_{s1}^{s2} \frac{1}{\beta(\tilde{s})}\, d\tilde{s}$$

Courant Snyder Invariant $$\varepsilon = \beta(s)\cdot x'^2(s) + 2\alpha(s)\cdot x(s)x'(s) + \gamma(s)\cdot x^2(s)$$

Beam Dimensions: *beam size* $\sigma = \sqrt{\varepsilon\beta(s)}$ *beam divergence* $\sigma' = \sqrt{\varepsilon\gamma(s)}$

Transfer of Twissparameters
$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_{S2} = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + CS' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_{S1}$$

If $\alpha_0 = 0$:

Beta function in a symmetric drift: $$\beta(l) = \beta_0 + \frac{l^2}{\beta_0}$$

Interpretation of β

$$\left.\begin{aligned} \sigma &= \sqrt{\varepsilon\beta} \\ \sigma' &= \sqrt{\varepsilon / \beta} \end{aligned}\right\} \qquad \beta = \frac{\sigma}{\sigma'}$$

Transfer matrix as a function of the Twiss parameters:

$$M = \begin{pmatrix} \sqrt{\dfrac{\beta_s}{\beta_0}}\left(\cos\Delta\psi + \alpha_0 \sin\Delta\psi\right) & \sqrt{\beta_s\beta_0}\,\sin\Delta\psi \\ \dfrac{(\alpha_0 - \alpha_s)\cos\Delta\psi - (1 + \alpha_0\alpha_s)\sin\Delta\psi}{\sqrt{\beta_s\beta_0}} & \sqrt{\dfrac{\beta_0}{\beta s}}\left(\cos\Delta\psi - \alpha_s \sin\Delta\psi\right) \end{pmatrix}$$

Tune: $$Q = \frac{1}{2\pi}\oint \frac{1}{\beta(s)}\,ds \approx \frac{1}{2\pi}\frac{2\pi\bar{R}}{\bar{\beta}}$$

$$Q \approx \frac{\bar{R}}{\bar{\beta}}$$