Electromagnetism

Christopher R Prior

Fellow and Tutor in Mathematics

Trinity College, Oxford

ASTeC Intense Beams Group

Rutherford Appleton Laboratory

Contents

- Maxwell's equations and Lorentz Force Law
- Motion of a charged particle under constant Electromagnetic fields
- Relativistic transformations of fields
- Electromagnetic waves
- Waves in a uniform conducting guide
    - Simple example TE_{01} mode
    - Propagation constant, cut-off frequency
    - Group velocity, phase velocity
    - Illustrations

TRINITY
COLLEGE • OXFORD

Reading

- J.D. Jackson: *Classical Electrodynamics*
- H.D. Young and R.A. Freedman: *University Physics (with Modern Physics)*
- P.C. Clemmow: *Electromagnetic Theory*
- *Feynmann Lectures on Physics*
- W.K.H. Panofsky and M.N. Phillips: *Classical Electricity and Magnetism*
- G.L. Pollack and D.R. Stump: *Electromagnetism*

What is electromagnetism?

- The study of Maxwell’s equations, devised in 1863 to represent the relationships between electric and magnetic fields in the presence of electric charges and currents, whether steady or rapidly fluctuating, in a vacuum or in matter.

- The equations represent one of the most elegant and concise way to describe the fundamentals of electricity and magnetism. They pull together in a consistent way earlier results known from the work of Gauss, Faraday, Ampère, Biot, Savart and others.

- Remarkably, Maxwell’s equations are perfectly consistent with the transformations of special relativity.

TRINITY COLLEGE • OXFORD

Maxwell's Equations

- Relate Electric and Magnetic fields generated by charge and current distributions.

E = electric field

D = electric displacement

H = magnetic field

B = magnetic flux density

ρ= charge density

j = current density

μ_0 (permeability of free space) = $4\pi\ 10^{-7}$

ε_0 (permittivity of free space) = $8.854\ 10^{-12}$

c (speed of light) = $2.99792458\ 10^{8}$ m/s

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \wedge \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

In vacuum $\quad \vec{D} = \varepsilon_0 \vec{E}, \quad \vec{B} = \mu_0 \vec{H}, \quad \varepsilon_0 \mu_0 c^2 = 1$

TRINITY
COLLEGE • OXFORD

Maxwell's 1st Equation

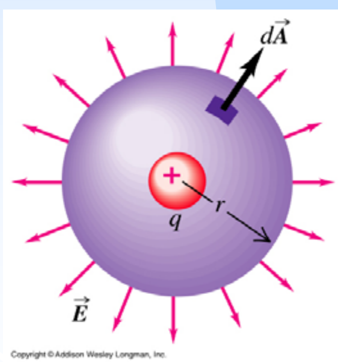
$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

Equivalent to Gauss' Flux Theorem:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \Leftrightarrow \quad \iint_S \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \iiint_V \rho \, dV = \frac{Q}{\varepsilon_0}$$

The flux of electric field out of a closed region is proportional to the total electric charge Q enclosed within the surface.

A point charge *q* generates an electric field



$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^3} \vec{r}$$

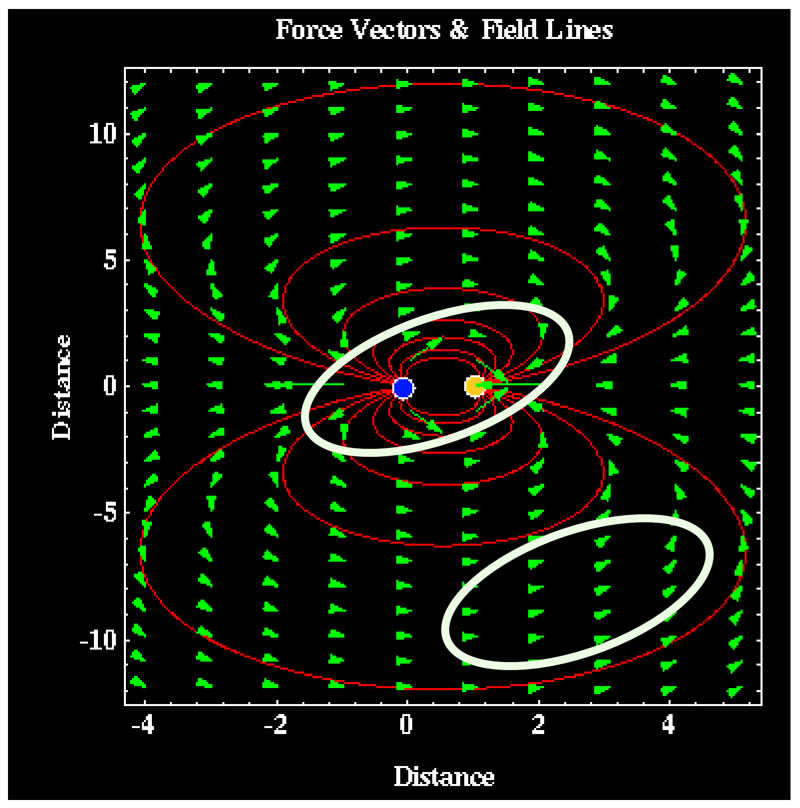
$$\iint_{sphere} \vec{E} \cdot d\vec{S} = \frac{q}{4\pi\varepsilon_0} \iint_{sphere} \frac{dS}{r^2} = \frac{q}{\varepsilon_0}$$

TRINITY
COLLEGE • OXFORD

Area integral gives a measure of the net charge enclosed; divergence of the electric field gives the density of the sources.

Maxwell's 2nd Equation

$$\boxed{\nabla \cdot \vec{B} = 0}$$



$$\nabla \cdot \vec{B} = 0 \quad \Leftrightarrow \quad \oiint \vec{B} \cdot d\vec{S} = 0$$

Gauss' law for magnetism:

The net magnetic flux out of any closed surface is zero. Surround a magnetic dipole with a closed surface. The magnetic flux directed inward towards the south pole will equal the flux outward from the north pole.

If there were a magnetic monopole source, this would give a non-zero integral.

Gauss' law for magnetism is then a statement that

There are no magnetic monopoles

TRINITY COLLEGE • OXFORD

Maxwell's 3rd Equation

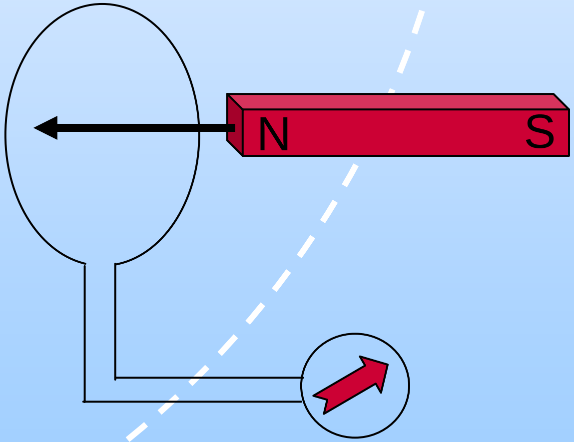
$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Equivalent to Faraday's Law of Induction:

$$\iint_S \nabla \wedge \vec{E} \cdot d\vec{S} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

(for a fixed circuit *C*) $\Leftrightarrow$ $\oint_C E \cdot d\vec{l} = -\frac{d}{dt}\iint_S \vec{B} \cdot d\vec{S} = -\frac{d\Phi}{dt}$

The electromotive force round a circuit $\varepsilon = \oint \vec{E} \cdot d\vec{l}$ is proportional to the rate of change of flux of magnetic field, $\Phi = \oint \vec{B} \cdot d\vec{l}$ through the circuit.



Faraday's Law is the basis for electric generators. It also forms the basis for inductors and transformers.

TRINITY
COLLEGE • OXFORD

Maxwell's 4th Equation

$$\nabla \wedge \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

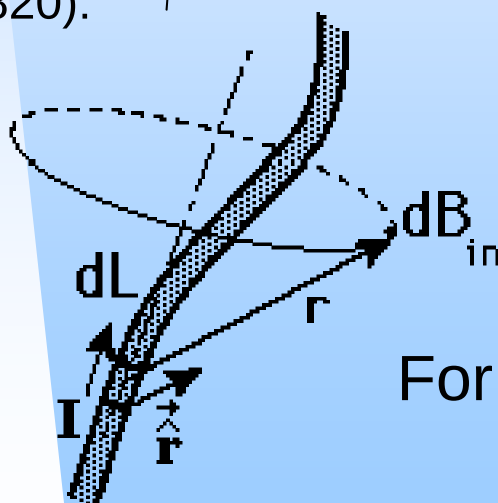
Ampère

Biot

Originates from Ampère's (Circuital) Law : $\nabla \wedge \vec{B} = \mu_0 \vec{j}$

$$\oint_C \vec{B} \cdot d\vec{l} = \iint_S \nabla \wedge \vec{B} \cdot d\vec{S} = \mu_0 \iint_S \vec{j} \cdot d\vec{S} = \mu_0 I$$

Satisfied by the field for a steady line current (Biot-Savart Law, 1820):



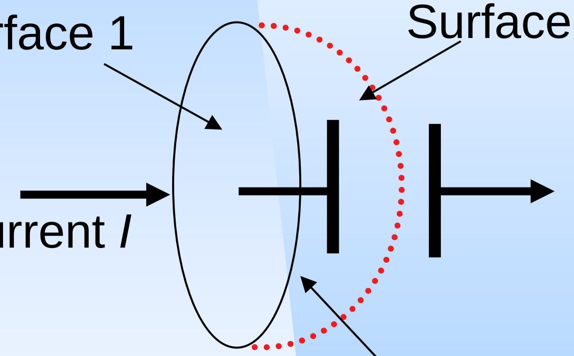
$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l} \wedge \vec{r}}{r^3}$$

For a straight line current $B_\theta = \dfrac{\mu_0 I}{2\pi r}$

TRINITY COLLEGE • OXFORD

Need for displacement current

- Faraday: vary B-field, generate E-field
- Maxwell: varying E-field should then produce a B-field, but not covered by Ampère's Law.



- Apply Ampère to surface 1 (flat disk): line integral of ***B*** = $\mu_0 I$
- Applied to surface 2, line integral is zero since no current penetrates the deformed surface.
- In capacitor, $E = \dfrac{Q}{\varepsilon_0 A}$, so $I = \dfrac{dQ}{dt} = \varepsilon_0 A \dfrac{dE}{dt}$
- Displacement current density is $\vec{j}_d = \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}$

$$\nabla \wedge \vec{B} = \mu_0 \left(\vec{j} + \vec{j}_d \right) = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

TRINITY COLLEGE • OXFORD

Consistency with charge conservation

- **Charge conservation:** Total current flowing out of a region equals the rate of decrease of charge within the volume.

$$\oiint \vec{j} \cdot d\vec{S} = -\frac{d}{dt} \iiint \rho \, dV$$

$$\Leftrightarrow \quad \iiint \nabla \cdot \vec{j} \, dV = -\iiint \frac{\partial \rho}{\partial t} dV$$

$$\Leftrightarrow \quad \nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$$

- **From Maxwell's equations:** Take divergence of (modified) Ampère's equation

$$\nabla \cdot \nabla \wedge \overline{B} = \mu_0 \nabla \cdot \vec{j} + \frac{1}{c^2} \frac{\partial}{\partial t} \left(\nabla \cdot \vec{E} \right)$$

$$\Rightarrow \quad 0 = \mu_0 \nabla \cdot \vec{j} + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \left(\frac{\rho}{\varepsilon_0} \right)$$

$$\Rightarrow \quad 0 = \nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t}$$

Charge conservation is implicit in Maxwell's Equations

Maxwell's Equations in Vacuo

- In vacuum

$$\vec{D} = \varepsilon_0 \vec{E}, \quad \vec{B} = \mu_0 \vec{H}, \quad \varepsilon_0 \mu_0 = \frac{1}{c^2}$$

- Source-free equations:

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \wedge \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

- Source equations

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \wedge \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}$$

- Equivalent integral forms (sometimes useful for simple geometries)

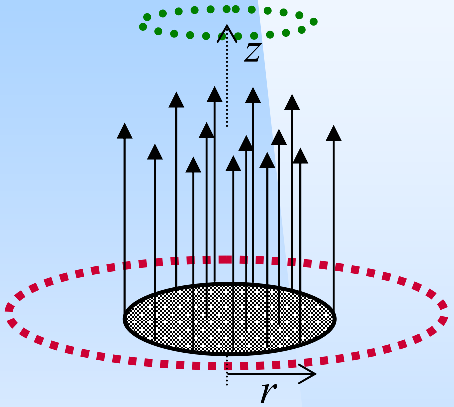
$$\oint\!\!\!\!\oint \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \iiint \rho \, dV$$

$$\oint\!\!\!\!\oint \vec{B} \cdot d\vec{S} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint \vec{B} \cdot d\vec{S} = -\frac{d\Phi}{dt}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \iint \vec{j} \cdot d\vec{S} + \frac{1}{c^2}\frac{d}{dt} \iint \vec{E} \cdot d\vec{S}$$

Example: Calculate E from B



$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\iint \vec{B}\cdot dS$$

$$B_z = \begin{cases} B_0 \cos\omega t & r < r_0 \\ 0 & r > r_0 \end{cases}$$

$$r < r_0 \quad 2\pi r E_\theta = -\pi r^2 B_0 \omega \cos\omega t$$

$$\Rightarrow \quad E_\theta = -\frac{B_0 \omega r}{2}\cos\omega t$$

$$r > r_0 \quad 2\pi r E_\theta = -\pi r_0^2 B_0 \omega \cos\omega t$$

$$\Rightarrow \quad E_\theta = -\frac{\omega r_0^2 B_0}{2r}\cos\omega t$$

Also from $\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\nabla \wedge \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{dt}$ then gives current density necessary to sustain the fields

TRINITY
COLLEGE • OXFORD

Lorentz force law

- Supplement to Maxwell's equations, gives force on a charged particle moving in an electromagnetic field:

$$\vec{f} = q\left(\vec{E} + \vec{v} \wedge \vec{B}\right)$$

- For continuous distributions, have a force density $\vec{f}_d = \rho\vec{E} + \vec{j} \wedge \vec{B}$
- Relativistic equation of motion
  - 4-vector form: $$F = \frac{dP}{d\tau} \Rightarrow \gamma\left(\frac{\vec{v}\cdot\vec{f}}{c}, \vec{f}\right) = \gamma\left(\frac{1}{c}\frac{dE}{dt}, \frac{d\vec{p}}{dt}\right)$$
  - 3-vector component:

$$\boxed{\frac{d}{dt}\left(m_0\gamma\vec{v}\right) = \vec{f} = q\left(\vec{E} + \vec{v} \wedge \vec{B}\right)}$$

TRINITY COLLEGE • OXFORD

Motion of charged particles in constant electromagnetic fields

$$\boxed{\frac{d}{dt}\left(m_0\gamma\vec{v}\right)=\vec{f}=q\left(\vec{E}+\vec{v}\wedge\vec{B}\right)}$$

- Constant E-field gives uniform acceleration in straight line
- Solution of $\frac{d}{dt}\left(\gamma\vec{v}\right)=\frac{q}{m_0}E$

$$x=\frac{m_0c^2}{qE}\left[\sqrt{1+\left(\frac{qEt}{m_0c}\right)^2}-1\right]$$

$$\approx\frac{1}{2}\frac{qE}{m_0}t^2 \quad \text{for } qE<<m_0c$$

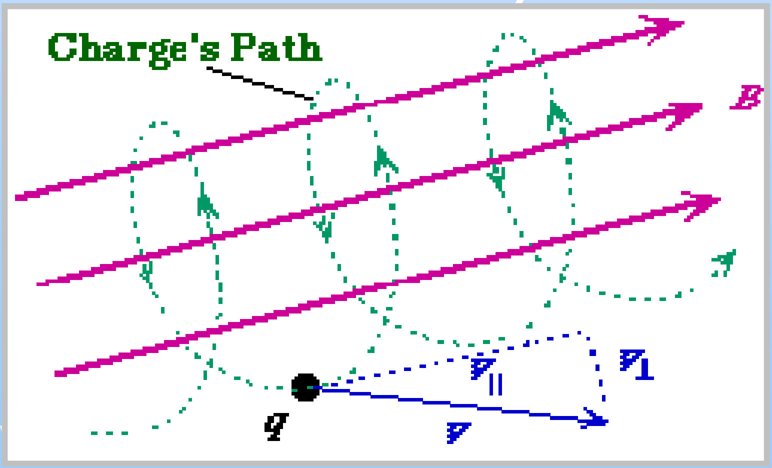
- Energy gain $=qEx$

- Constant magnetic field gives uniform spiral about B with constant energy.

$$\frac{d\vec{v}}{dt}=\frac{q}{m_0\gamma}\vec{v}\wedge\vec{B}$$

$$\vec{v}_{//}=\text{constant}$$

$$\left|\vec{x}_{\perp}\right|=\text{constant}$$



Relativistic Transformations of E and B

- According to observer O in frame F, particle has velocity v, fields are $\vec{E}$ and $\vec{B}$ and Lorentz force is $$\vec{f} = q\left(\vec{E} + \vec{v} \wedge \vec{B}\right)$$
- In Frame F′, particle is at rest and force is $$\vec{f}' = q'\vec{E}'$$
- Assume measurements give same charge and force, so $$q = q' \quad \text{and} \quad \vec{E}' = E + \vec{v} \wedge \vec{B}$$
- Point charge q at rest in F: $$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}, \quad \vec{B} = 0$$
- See a current in F′, giving a field $$\vec{B} = -\frac{\mu_0 q}{4\pi}\frac{\vec{v} \wedge \vec{r}}{r^3} = -\frac{1}{c^2}\vec{v} \wedge \vec{E}$$
- Suggests $$\vec{B}' = \vec{B} - \frac{1}{c^2}\vec{v} \wedge \vec{E}$$

Rough idea

Exact:

$$\boxed{\begin{aligned} \vec{E}'_\perp &= \gamma\left(E_\perp + \vec{v} \wedge \vec{B}\right), \quad \vec{E}'_{//} = \vec{E}_{//} \\ \vec{B}'_\perp &= \gamma\left(B_\perp - \frac{\vec{v} \wedge \vec{E}}{c^2}\right), \quad \vec{B}'_{//} = \vec{B}_{//} \end{aligned}}$$

TRINITY
COLLEGE • OXFORD

Electromagnetic waves

- Maxwell's equations predict the existence of electromagnetic waves, later discovered by Hertz.
- No charges, no currents:

$$\nabla \wedge \vec{H} = \frac{\partial \vec{D}}{\partial t} \qquad \nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{D} = 0 \qquad \nabla \cdot \vec{B} = 0$$

$$\begin{aligned}\nabla \wedge \left(\nabla \wedge \vec{E}\right) &= -\nabla \wedge \frac{\partial \vec{B}}{\partial t} \\ &= -\frac{\partial}{\partial t}\left(\nabla \wedge \vec{B}\right) \\ &= -\mu \frac{\partial^2 \vec{D}}{\partial t^2} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}\end{aligned}$$

$$\begin{aligned}\nabla \wedge \left(\nabla \wedge \vec{E}\right) &= \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E} \\ &= -\nabla^2 \vec{E}\end{aligned}$$

3D wave equation :

$$\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

TRINITY COLLEGE • OXFORD

Nature of electromagnetic waves

- A general plane wave with angular frequency ω travelling in the direction of the wave vector $\vec{k}$ has the form

$$\vec{E} = \vec{E}_0 \exp[\, j(\omega t - \vec{k}\cdot\vec{x})] \qquad \vec{B} = \vec{B}_0 \exp[\, j(\omega t - \vec{k}\cdot\vec{x})]$$

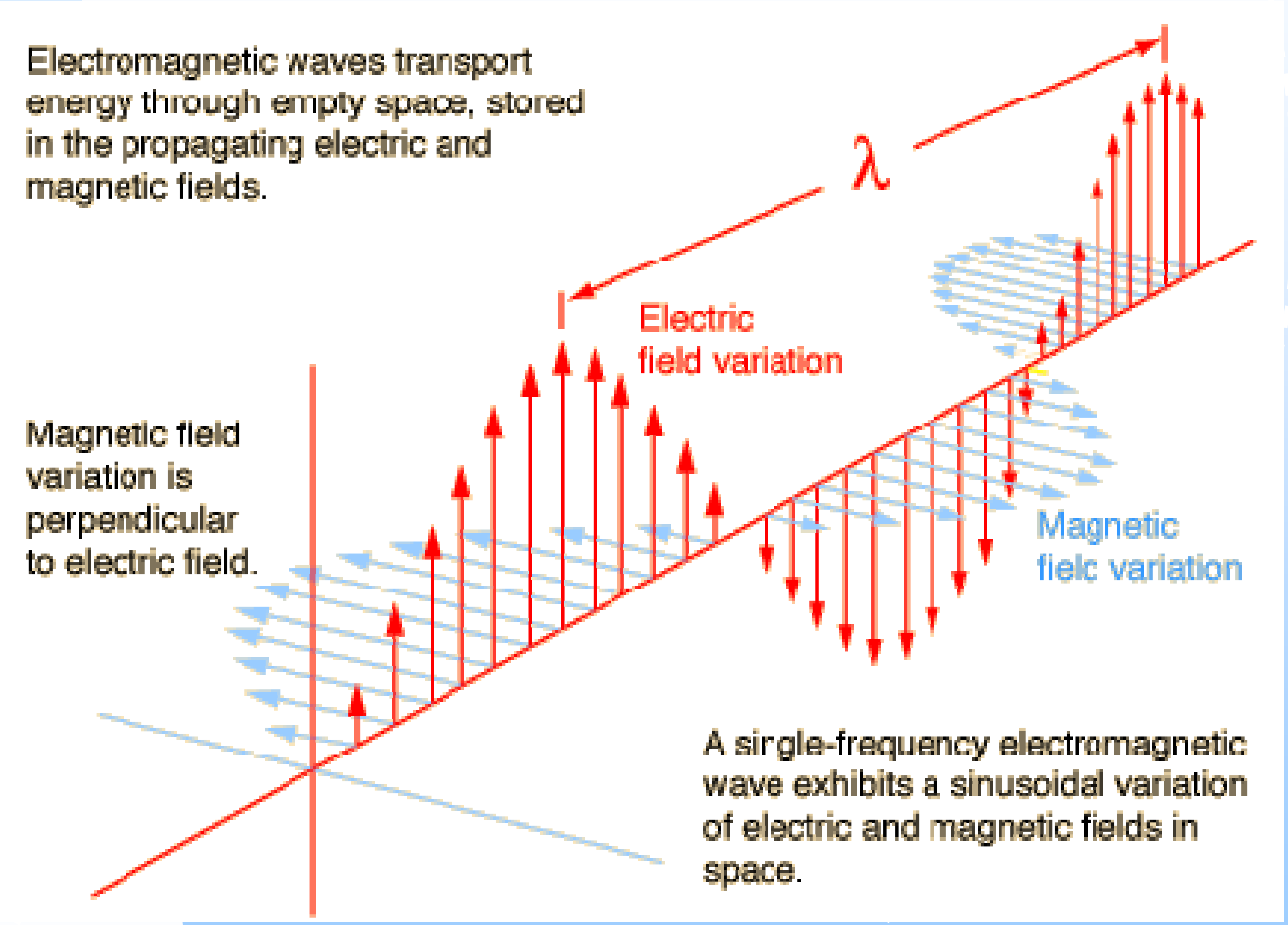
- Phase $\omega t - \vec{k}\cdot\vec{x}$ = 2π × number of waves and so is a Lorentz invariant.
- Apply Maxwell's equations

$$\boxed{\begin{array}{l} \nabla \leftrightarrow -j\vec{k} \\ \dfrac{\partial}{\partial t} \leftrightarrow j\omega \end{array}}$$

$$\boxed{\begin{array}{c} \nabla\cdot\vec{E} = 0 = \nabla\cdot\vec{B} \quad \leftrightarrow \quad \vec{k}\cdot\vec{E} = 0 = \vec{k}\cdot\vec{B} \\ \nabla\wedge\vec{E} = -\dot{\vec{B}} \quad \leftrightarrow \quad \vec{k}\wedge\vec{E} = \omega\vec{B} \end{array}}$$

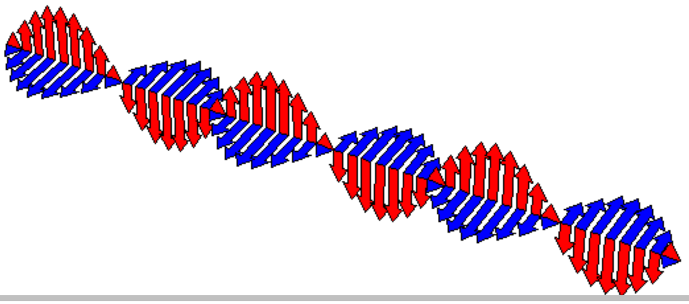
Waves are transverse to the direction of propagation, and $\overleftarrow{E}, \vec{B}$ and $\vec{k}$ are mutually perpendicular

Plane electromagnetic wave



TRINITY COLLEGE • OXFORD

Plane Electromagnetic Waves



$$\nabla \wedge \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \quad \leftrightarrow \quad \vec{k} \wedge \vec{B} = -\frac{\omega}{c^2}\vec{E}$$

Combined with $\vec{k} \wedge \vec{E} = \omega \vec{B}$

deduce that $\dfrac{|\vec{E}|}{|\vec{B}|} = \dfrac{\omega}{k} = \dfrac{kc^2}{\omega}$

$\Rightarrow$ speed of wave in vacuum is $\dfrac{\omega}{|\vec{k}|} = c$

Wavelength $\lambda = \dfrac{2\pi}{|\vec{\mathrm{k}}|}$

Frequency $\nu = \dfrac{\omega}{2\pi}$

The fact that $\omega t - \vec{k}\cdot\vec{x}$ is an invariant tells us that

$$\Lambda = \left(\frac{\omega}{c}, \vec{k}\right)$$

is a Lorentz 4-vector, the 4-Frequency vector. Deduce frequency transforms as

$$\omega' = \gamma\left(\omega - \vec{v}\cdot\vec{k}\right) = \omega\sqrt{\frac{c-v}{c+v}}$$

TRINITY COLLEGE • OXFORD

Waves in a conducting medium

- For a medium of conductivity σ, $\boxed{\vec{j} = \sigma \vec{E}}$
- Modified Maxwell: $\nabla \wedge \vec{H} = \vec{j} + \varepsilon \dot{\vec{E}} = \sigma \vec{E} + \varepsilon \dot{\vec{E}}$
- Put $\vec{E} = \vec{E}_0 \exp[j(\omega t - \vec{k} \cdot \vec{x})]$ $\vec{B} = \vec{B}_0 \exp[j(\omega t - \vec{k} \cdot \vec{x})]$

$$-j\vec{k} \wedge \vec{H} = \underbrace{\sigma \vec{E}}_{\text{conduction current}} + \underbrace{j\omega\varepsilon \vec{E}}_{\text{displacement current}}$$

Dissipation factor

$$D = \frac{\sigma}{\omega \varepsilon}$$

$$\text{Copper}: \quad \sigma = 5.8 \times 10^{7}, \varepsilon = \varepsilon_0 \quad \Rightarrow \quad D = 10^{12}$$

$$\text{Teflon}: \quad \sigma = 3 \times 10^{-8}, \varepsilon = 2.1\varepsilon_0 \quad \Rightarrow \quad D = 2.57 \times 10^{-4}$$

TRINITY COLLEGE • OXFORD

Attenuation in a Good Conductor

$$-j\vec{k}\wedge\vec{H}=\sigma\vec{E}+j\omega\varepsilon\vec{E}$$

Combine with $\nabla\wedge\vec{E}=-\dot{\vec{B}} \quad\Rightarrow\quad \vec{k}\wedge\vec{E}=\omega\mu\vec{H}$

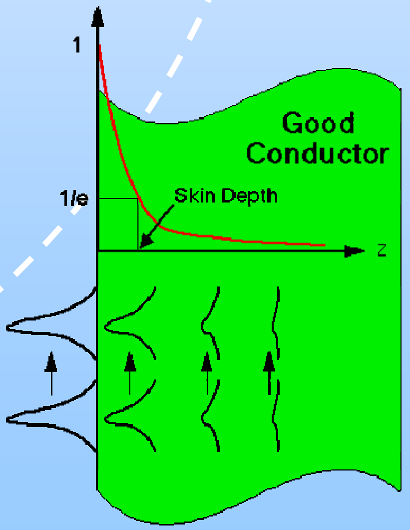
$$\Rightarrow\quad \vec{k}\wedge\left(\vec{k}\wedge\vec{E}\right)=\omega\mu\,\vec{k}\wedge\vec{H}=-\omega\mu\left(-j\sigma+\omega\varepsilon\right)\vec{E}$$

$$\Rightarrow\quad k^2=\omega\mu\left(-j\sigma+\omega\varepsilon\right)$$

For a good conductor D >> 1, $\sigma \gg \omega\varepsilon \qquad k^2\approx -j\omega\mu\sigma \quad\Rightarrow\quad k\approx\sqrt{\frac{\omega\mu\sigma}{2}}(1-j)$

Wave form is $\exp\left[j\left(\omega t-\frac{x}{\delta}\right)\right]\exp\left(-\frac{x}{\delta}\right)$

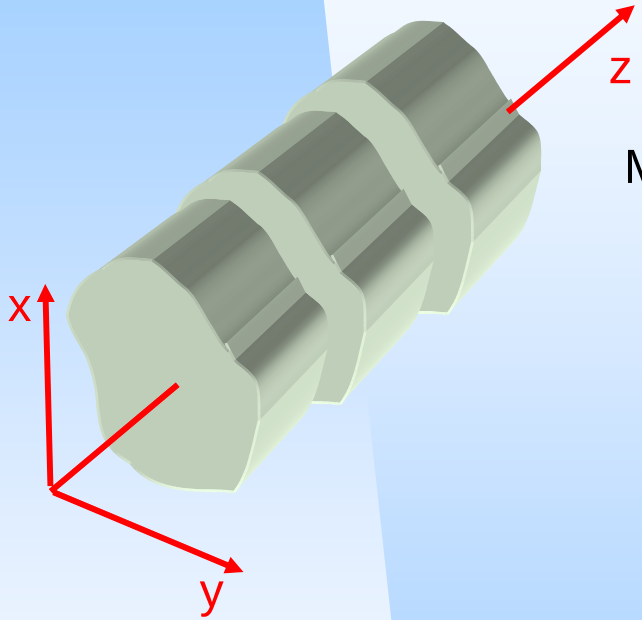
where $\delta=\sqrt{\frac{2}{\omega\mu\sigma}}$ is the skin - depth



[copper.mov] [water.mov]

TRINITY COLLEGE • OXFORD

Maxwell's Equations in a uniform perfectly conducting guide



Hollow metallic cylinder with perfectly conducting boundary surfaces

Maxwell's equations with time dependence *exp(jωt)* are:

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -j\omega\mu\vec{H}$$

$$\nabla \wedge \vec{H} = \frac{\partial \vec{D}}{\partial t} = j\omega\varepsilon\vec{E}$$

$$\Rightarrow \quad \nabla^2\vec{E} = \nabla\left(\nabla\cdot\vec{E}\right) - \nabla\wedge\left(\nabla\wedge\vec{E}\right) = j\omega\mu\nabla\wedge\vec{H} = -\omega^2\varepsilon\mu\vec{E}$$

$$\left(\nabla^2 + \omega^2\mu\varepsilon\right)\begin{Bmatrix}\vec{E}\\ \vec{H}\end{Bmatrix} = 0$$

Assume $\vec{E}(x,y,z,t) = \vec{E}(x,y)e^{(j\omega t-\gamma z)}$

$\vec{H}(x,y,z,t) = \vec{H}(x,y)e^{(j\omega t-\gamma z)}$

γ is the propagation constant

Then $\left[\nabla_t^2 + (\omega^2\varepsilon\mu + \gamma^2)\right]\begin{Bmatrix}\vec{E}\\ \vec{H}\end{Bmatrix} = 0$

Can solve for the fields completely in terms of E_z and H_z

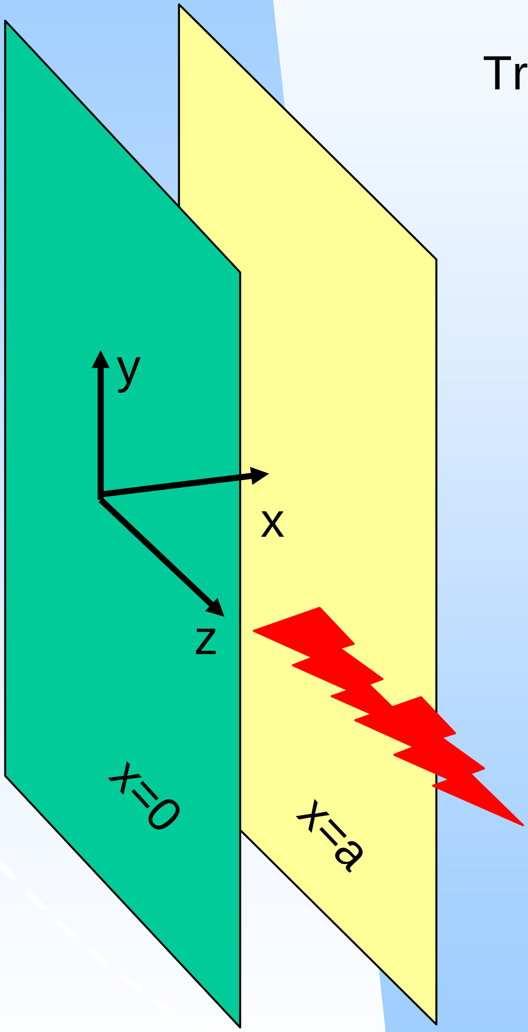
TRINITY COLLEGE • OXFORD

Special cases

- Transverse magnetic (TM modes):
  - H_z=0 everywhere, E_z=0 on cylindrical boundary
- Transverse electric (TE modes):
  - E_z=0 everywhere, $\frac{\partial H_z}{\partial n} = 0$ on cylindrical boundary
- Transverse electromagnetic (TEM modes):
  - E_z=H_z=0 everywhere
  - requires $\gamma^2 + \omega^2 \varepsilon\mu = 0$ or $\gamma = \pm j\omega\sqrt{\varepsilon\mu}$

TRINITY COLLEGE • OXFORD

A simple model with E_z=0



Transport between two infinite parallel conducting plates:

$$\vec{E} = (0,1,0)E(x)\ e^{(j\omega t-\gamma z)} \quad \text{where } E(x) \text{ satisfies}$$

$$\nabla_{\mathrm{t}}^2 E = \frac{d^2E}{dx^2} = -K^2E, \quad K^2 = \omega^2\varepsilon\mu + \gamma^2$$

$$\text{i.e.} \quad E = A\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}Kx$$

To satisfy boundary conditions, $E=0$ on $x=0$ and $x=a$, so

$$E = A\sin Kx, \quad K = K_n = \frac{n\pi}{a}, \quad n \text{ integer}$$

Propagation constant is $\gamma = \sqrt{K_n^2 - \omega^2\varepsilon\mu}$

$$= \frac{n\pi}{a}\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2} \quad \text{where } \omega_c = \frac{K_n}{\sqrt{\varepsilon\mu}}$$

TRINITY COLLEGE • OXFORD

Cut-off frequency, ω_c

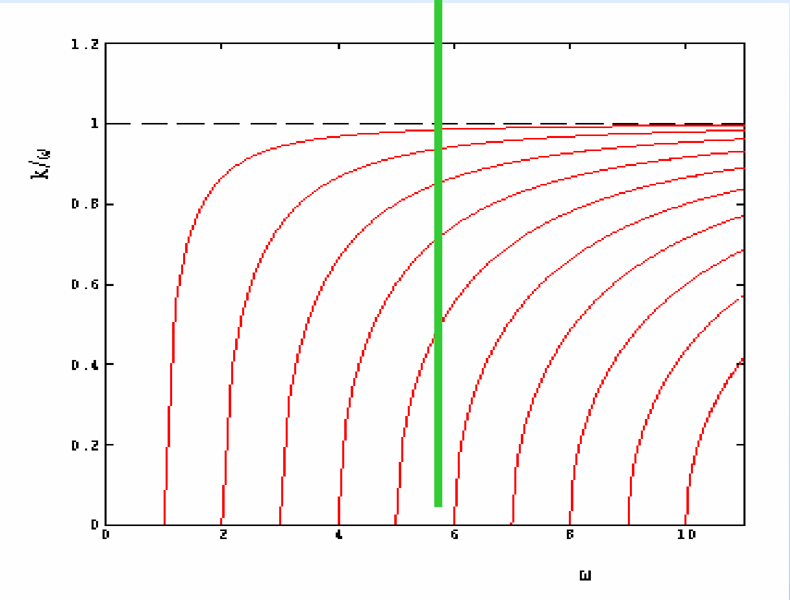
$$\gamma = \frac{n\pi}{a}\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2}, \quad E = A\sin\frac{n\pi x}{a}\, e^{j\omega t-\gamma z}, \quad \omega_c = \frac{n\pi}{a\sqrt{\varepsilon\mu}}$$

- $\omega<\omega_c$ gives real solution for γ, so attenuation only. No wave propagates: cut-off modes.
- $\omega>\omega_c$ gives purely imaginary solution for γ, and a wave propagates without attenuation.

$$\gamma = jk, \quad k = \sqrt{\varepsilon\mu}\left(\omega^2-\omega_c^2\right)^{1/2} = \omega\sqrt{\varepsilon\mu}\left(1-\frac{\omega_c^2}{\omega^2}\right)^{1/2}$$

- For a given frequency ω only a finite number of modes can propagate.

$$\omega > \omega_c = \frac{n\pi}{a\sqrt{\varepsilon\mu}} \quad \Rightarrow \quad n < \frac{a\omega}{\pi}\sqrt{\varepsilon\mu}$$



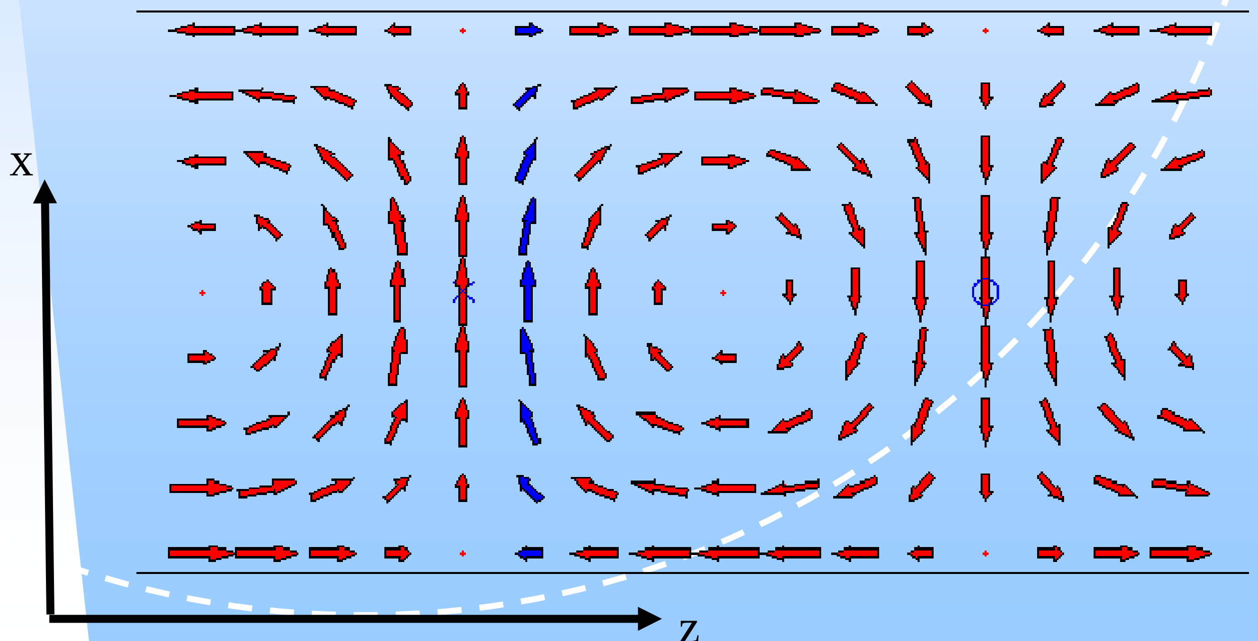
For given frequency, convenient to choose *a* s.t. only *n=1* mode occurs.

TRINITY
COLLEGE • OXFORD

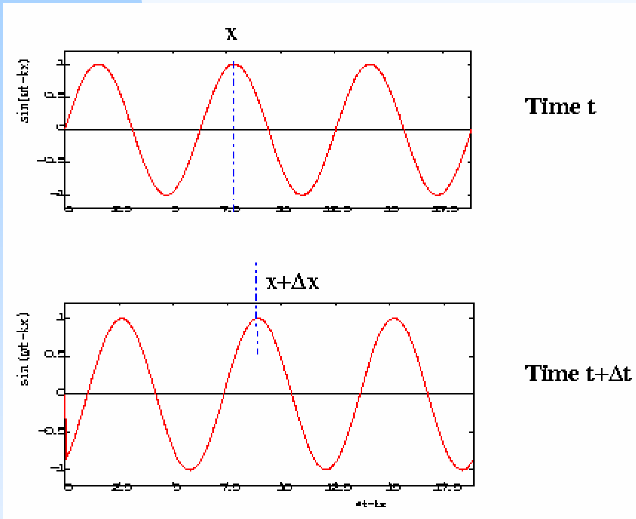
Propagated electromagnetic fields

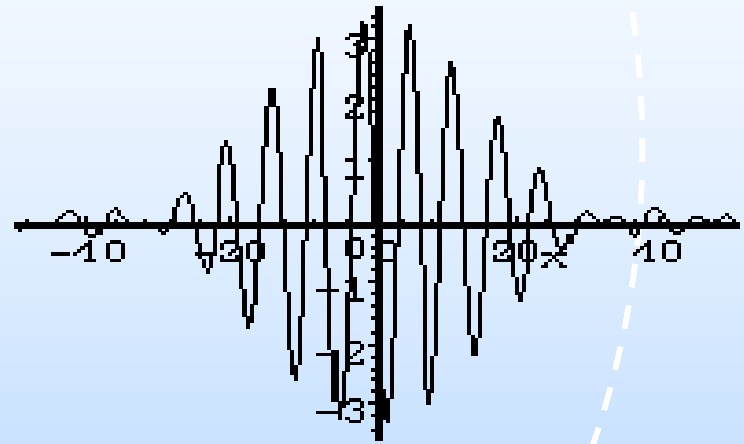
- From $\nabla \wedge \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$, (assuming A is real)

$$\vec{H} = \frac{j}{\omega\mu} \nabla \wedge \vec{E} \quad \Rightarrow \quad \begin{cases} H_x = -\dfrac{Ak}{\omega\mu} \sin\left(\dfrac{n\pi x}{a}\right) \cos(\omega t - kz) \\ H_y = 0 \\ H_z = -\dfrac{A}{\omega\mu} \dfrac{n\pi}{a} \cos\left(\dfrac{n\pi x}{a}\right) \sin(\omega t - kz) \end{cases}$$



Phase and group velocities





$$\int_{-\infty}^{\infty} A(k)e^{j[\omega(k)t-kx]}dk$$

Plane wave exp *j(ωt-kx)* has constant phase *ωt-kx* at peaks

$$\omega\Delta t - k\Delta x = 0$$

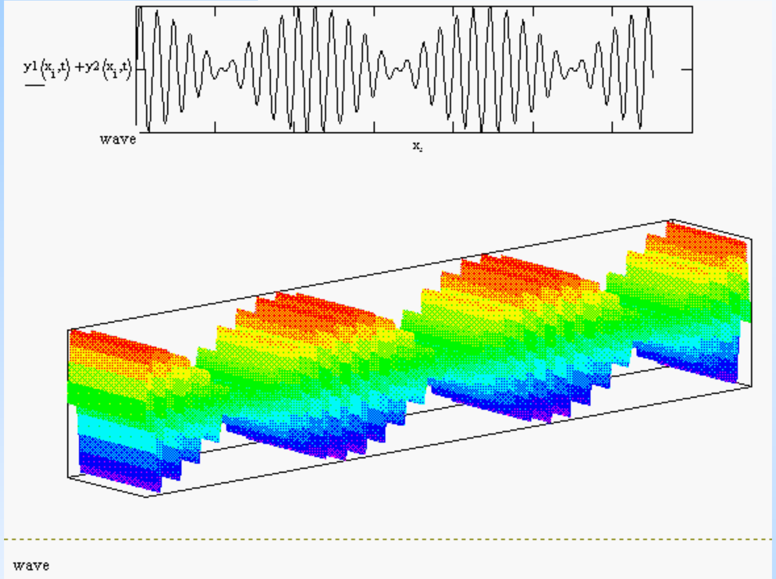
$$\Leftrightarrow \qquad v_p = \frac{\Delta x}{\Delta t} = \frac{\omega}{k}$$

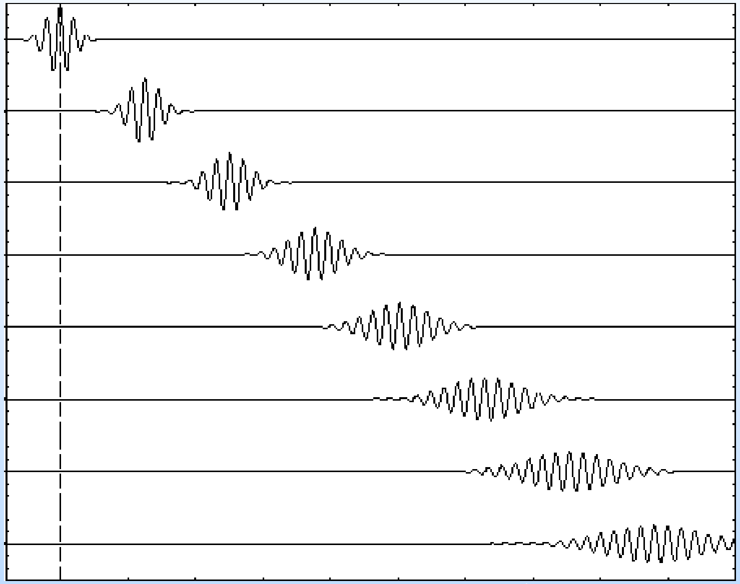
Superposition of plane waves. While shape is relatively undistorted, pulse travels with the group velocity

$$v_g = \frac{d\omega}{dk}$$

TRINITY
COLLEGE • OXFORD

Wave packet structure





- Phase velocities of individual plane waves making up the wave packet are different,
- The wave packet will then disperse with time

Phase and group velocities in the simple wave guide

- Wave number is $k = \sqrt{\varepsilon\mu}\left(\omega^2 - \omega_c^2\right)^{1/2} < \omega\sqrt{\varepsilon\mu}$

  so wavelength in guide $\lambda = \dfrac{2\pi}{k} > \dfrac{2\pi}{\omega\sqrt{\varepsilon\mu}}$, the free-space wavelength

- Phase velocity is $v_p = \dfrac{\omega}{k} > \dfrac{1}{\sqrt{\varepsilon\mu}}$, larger than free-space velocity

- Group velocity is less than infinite space value

$$k^2 = \varepsilon\mu\left(\omega^2 - \omega_c^2\right) \quad \Rightarrow \quad v_g = \frac{d\omega}{dk} = \frac{k}{\omega\varepsilon\mu} < \frac{1}{\sqrt{\varepsilon\mu}}$$

TRINITY COLLEGE • OXFORD

Calculation of wave properties

- If a=3 cm, cut-off frequency of lowest order mode is

$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{2a\sqrt{\varepsilon\mu}} \cong 5\,\mathrm{GHz}$$

- At 7 GHz, only the n=1 mode propagates and

$$k = \sqrt{\varepsilon\mu}\left(\omega^2 - \omega_c^2\right)^{1/2} \approx 103\,\mathrm{m}^{-1}$$

$$\lambda = \frac{2\pi}{k} \approx 6\,\mathrm{cm}$$

$$v_p = \frac{\omega}{k} \approx 4.3\times10^8\,\mathrm{ms}^{-1}$$

$$v_g = \frac{k}{\omega\varepsilon\mu} = 2.1\times10^8\,\mathrm{ms}^{-1}$$

TRINITY COLLEGE • OXFORD

Waveguide animations

- TE1 mode above cut-off ppwg_1-1.mov
- TE1 mode, smaller ω ppwg_1-2.mov
- TE1 mode at cut-off ppwg_1-3.mov
- TE1 mode below cut-off ppwg_1-4.mov
- TE1 mode, variable ω ppwg_1_vf.mov
- TE2 mode above cut-off ppwg_2-1.mov
- TE2 mode, smaller ppwg_2-2.mov
- TE2 mode at cut-off ppwg_2-3.mov
- TE2 mode below cut-off ppwg_2-4.mov

TRINITY
COLLEGE • OXFORD