

Linear

Imperfections

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Linear Imperfections

■ equation of motion in an accelerator

→ Hills equation

→ sine and cosine like solutions

→ closed orbit

→ sources for closed orbit perturbations

■ dipole perturbations

→ closed orbit response

→ dispersion orbit

→ integer resonances

→ BPMs & dipole correctors

■ quadrupole perturbations

→ one-turn map & tune error

→ beta-beat

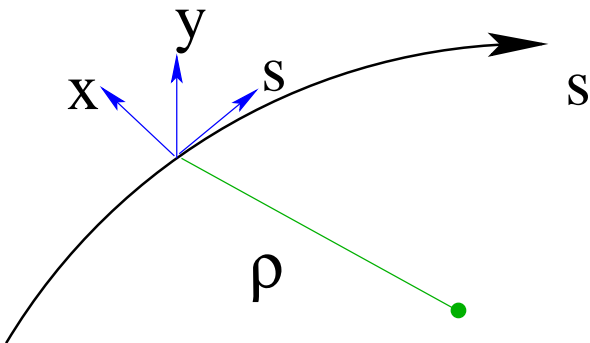
→ half-integer resonances

■ orbit correction

→ local orbit bumps

Variable Definition

Variables in moving coordinate system:



$$\mathbf{x}' = \frac{d}{ds} \mathbf{x}$$

$$\frac{d}{dt} = \frac{ds}{dt} \cdot \frac{d}{ds} \rightarrow \mathbf{x}' = \frac{p_x}{p_0}$$

\swarrow
 \mathbf{v}

Hill's Equation:

$$\frac{d^2 \mathbf{x}}{ds^2} + \mathbf{K}(s) \cdot \mathbf{x} = \mathbf{0}; \quad \mathbf{K}(s) = \mathbf{K}(s + L);$$

$$\mathbf{K}(s) = \begin{cases} 0 & \text{drift} \\ 1/\rho^2 & \text{dipole} \\ 0.3 \cdot \frac{B[\text{T/m}]}{p[\text{GeV}]} & \text{quadrupole} \end{cases}$$

Perturbations:

$$\frac{d^2 \mathbf{x}}{ds^2} + \mathbf{K}(s) \cdot \mathbf{x} = \mathbf{G}(s); \quad \mathbf{G}(s) = \frac{\mathbf{F}(s)_{\text{Lorentz}}}{\mathbf{v} \cdot \mathbf{p}_0}$$

Sinelike and Cosinelike Solutions

■ system of first order linear differential equations:

$$\vec{y} = \begin{pmatrix} x \\ x' \end{pmatrix} \longrightarrow \vec{y}' + \begin{pmatrix} 0 & 1 \\ K & 0 \end{pmatrix} \cdot \vec{y} = 0$$

$$K = \text{const} \longrightarrow$$

$$\vec{Y}_1(s) = \begin{pmatrix} \sin(\sqrt{K} \cdot s) \\ \sqrt{K} \cdot \cos(\sqrt{K} \cdot s) \end{pmatrix} \quad \vec{Y}_2(s) = \begin{pmatrix} \cos(\sqrt{K} \cdot s) \\ -\sqrt{K} \cdot \sin(\sqrt{K} \cdot s) \end{pmatrix}$$

■ initial conditions:

$$\vec{Y}_1(0) = \begin{pmatrix} Y_1 \\ Y_1' \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{Y}_2(0) = \begin{pmatrix} Y_2 \\ Y_2' \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

■ general solution: $\vec{y}(s) = a \cdot \vec{Y}_1(s) + b \cdot \vec{Y}_2(s)$

■ transport map: $\vec{y}(s) = \underline{\underline{M}}(s - s_0) \cdot \vec{y}(s_0)$

with:
$$= \begin{pmatrix} \cos(\sqrt{K} \cdot [s-s_0]) & \sin(\sqrt{K} \cdot [s-s_0]) \\ -\sqrt{K} \cdot \sin(\sqrt{K} \cdot [s-s_0]) & \sqrt{K} \cdot \cos(\sqrt{K} \cdot [s-s_0]) \end{pmatrix}$$

Sinelike and Cosinelike Solutions

Floquet theorem:

$$\vec{Y}_1(s) = \begin{pmatrix} \sqrt{\beta(s)} \cdot \sin(\phi(s) + \phi_0) \\ [\cos(\phi(s) + \phi_0) + \alpha(s) \cdot \sin(\phi(s) + \phi_0)] / \sqrt{\beta(s)} \end{pmatrix}$$

$$\vec{Y}_2(s) = \begin{pmatrix} \sqrt{\beta(s)} \cdot \cos(\phi(s) + \phi_0) \\ -[\sin(\phi(s) + \phi_0) + \alpha(s) \cdot \cos(\phi(s) + \phi_0)] / \sqrt{\beta(s)} \end{pmatrix}$$

$$\beta(s) = \beta(s + L); \quad \phi(s) = \int \frac{1}{\beta} ds; \quad \alpha(s) = -\frac{1}{2} \beta'(s)$$

'sinelike' and 'cosinelike' solutions:

$$\vec{C}(s) = a \cdot \vec{Y}_1(s) + b \cdot \vec{Y}_2(s) \quad \vec{S}(s) = c \cdot \vec{Y}_1(s) + d \cdot \vec{Y}_2(s)$$

$$\text{with: } \vec{C}(s_0) = \begin{pmatrix} C(s_0) \\ C'(s_0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{S}(s_0) = \begin{pmatrix} S(s_0) \\ S'(s_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

→ one can generate a transport matrix in analogy to the case with constant $K(s)$!

Sineline and Cosineline Solutions

█ 'sineline' and 'cosineline' solutions:

$$\vec{S}(s) = \begin{pmatrix} \sqrt{\beta(s)\beta(s_0)} \cdot \sin(\phi(s) + \phi_0) \\ \sqrt{\beta(s_0)} \cdot [\cos(\phi(s) + \phi_0) + \alpha(s) \cdot \sin(\phi(s) + \phi_0)] / \sqrt{\beta(s)} \end{pmatrix}$$

$$\vec{C}(s) = \begin{pmatrix} \sqrt{\beta(s)} \cdot [\cos(\phi(s) + \phi_0) + \alpha(s_0) \cdot \sin(\phi(s) + \phi_0)] / \sqrt{\beta(s_0)} \\ -(1 + \alpha\alpha_0) \cdot [\sin(\phi(s) + \phi_0) + (\alpha_0 - \alpha) \cdot \cos(\phi(s) + \phi_0)] / \sqrt{\beta\beta_0} \end{pmatrix}$$

█ transport map from s_0 to s : $\vec{y}(s) = \underline{M}(s, s_0) \cdot \vec{y}(s_0)$

with: $\underline{M} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix}$

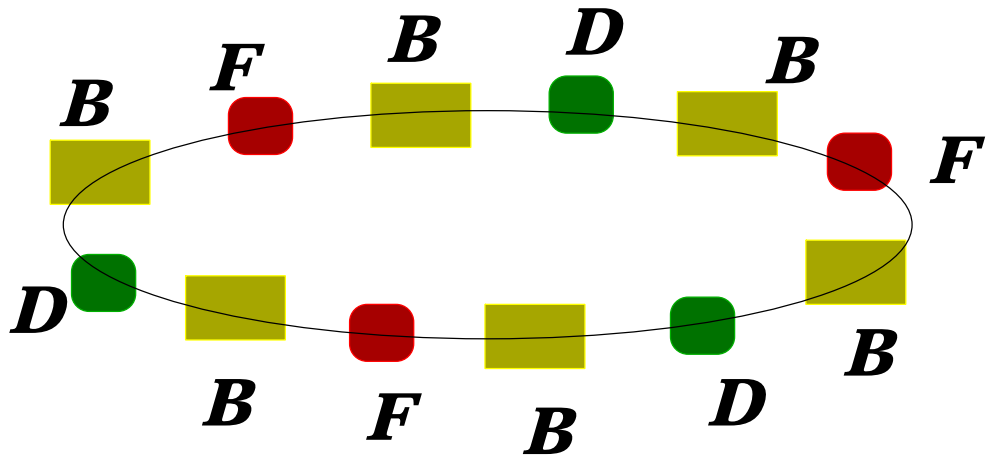
█ transport map for $s = s_0 + L$:

$$\underline{M} = \underline{I} \cdot \cos(2\pi Q) + \underline{J} \cdot \sin(2\pi Q)$$

$$\underline{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \underline{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}; \quad \gamma = [1 + \alpha^2] / \beta$$

Closed Orbit

- particles oscillate around an ideal orbit:



- additional dipole fields perturb the orbit:

error in dipole field

energy error

$$\alpha = \frac{l}{\rho} = \frac{q \cdot B \cdot l}{p + \Delta p} \approx \left(1 - \frac{\Delta p}{p} \right) \cdot \frac{q \cdot B \cdot l}{p}$$

offset in quadrupole field

$$B_x = g \cdot y$$

$$B_x = g \cdot \tilde{y}$$

$$B_y = g \cdot x$$

$$x = x_0 + \tilde{x} \rightarrow B_y = g \cdot x_0 + g \cdot \tilde{x}$$

dipole component ↑

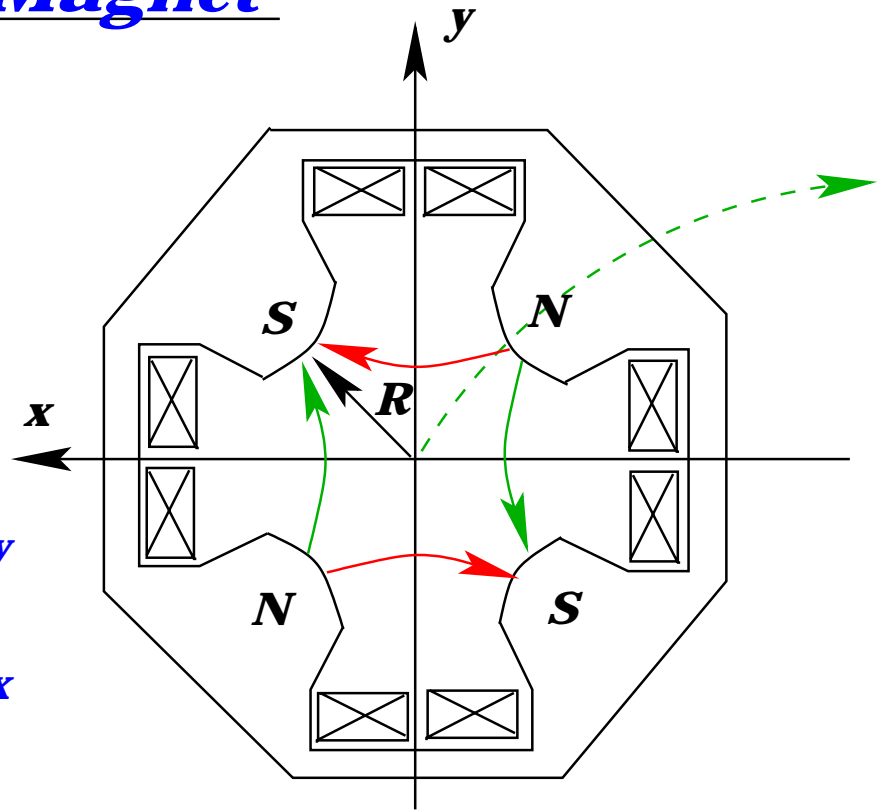
● Quadrupole Magnet

$$B_x = g \cdot y$$

$$B_y = g \cdot x$$

$$F_x = -q \cdot v \cdot B_y$$

$$F_y = q \cdot v \cdot B_x$$



$$\frac{d^2 \mathbf{x}}{ds^2} + \mathbf{K}(s) \cdot \mathbf{x} = \mathbf{G}(s); \quad \mathbf{G}(s) = \frac{\mathbf{F}(s)_{\text{Lorentz}}}{\mathbf{v} \cdot \mathbf{p}_0}$$

● normalized fields:

→ dipole: $k_0(s) = 0.3 \cdot \frac{B_0 [\text{T}]}{p_0 [\text{GeV}]}$

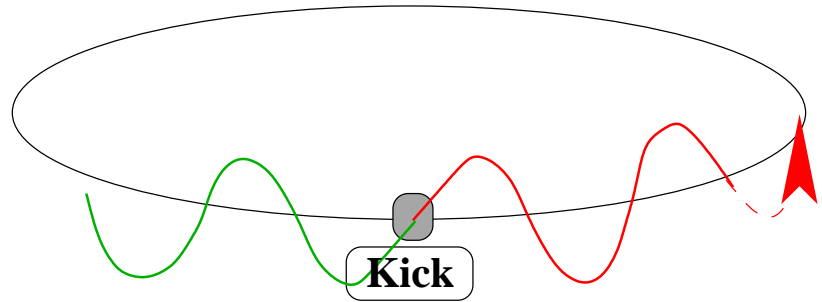
quadrupole: $k_1(s) = 0.3 \cdot \frac{g_0 [\text{T/m}]}{p_0 [\text{GeV}]}$

quadrupole misalignment: $\Delta k_0(s) = 0.3 \cdot \frac{g [\text{T/m}]}{p [\text{GeV}]} \cdot \mathbf{x}_0$

Dipole Error and Orbit Stability

● Q: number of β -oscillations per turn

■ $Q = N$

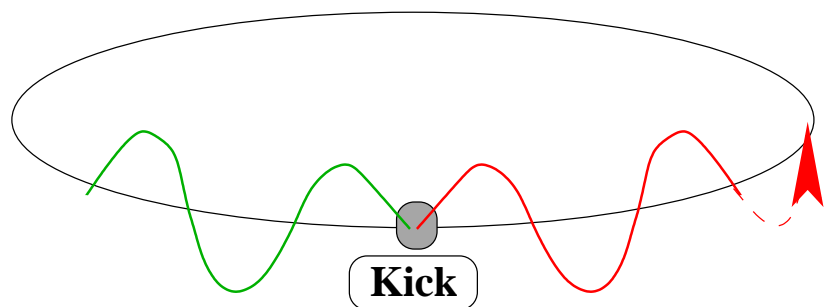


→ *the perturbation adds up*

→ *amplitude growth and particle loss*

↗ *watch out for integer tunes!*

■ $Q = N + 0.5$



→ *the perturbation cancels after each turn*

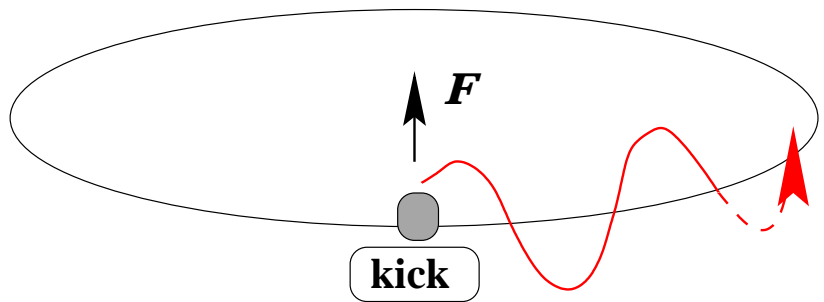
Quadrupole Error and Orbit Stability

● Quadrupole Error:

→ *orbit kick proportional to
beam offset in quadrupole*

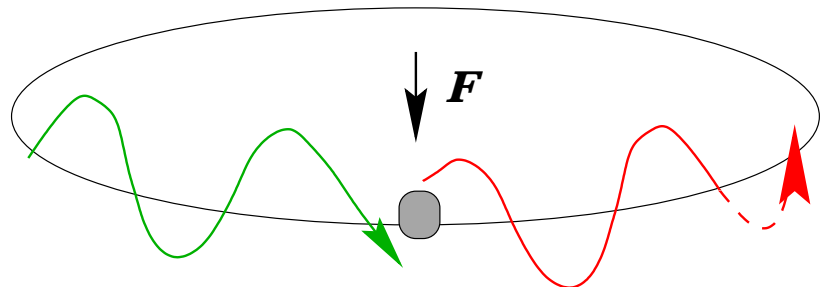
■ $Q = N + 0.5$

1. Turn: $x > 0$



→ *amplitude increase*

2. Turn: $x < 0$



→ *amplitude increase*

↘ *watch out for half integer tunes!*

Sources for Orbit Errors

● *Quadrupole offset:*

■ *alignment* *+/- 0.1 mm*

■ *ground motion*

■ *slow drift*

■ *civilisation*

■ *moon*

■ *seasons*

■ *civil engineering*

● *Error in dipole strength*

■ *power supplies*

■ *calibration*

● *Energy error of particles*

■ *injection energy (RF off)*

■ *RF frequency*

■ *momentum distribution*

Example Quadrupole Alignment in LEP

Transversal tilt dispersion of the 3278 dipoles

$$\sigma = \pm 0.34 \text{ mrd}$$

Vertical dispersion of the 784 quadrupoles
(with respect to the smoothing polynomial)

$$\sigma = \pm 0.65 \text{ mm}$$

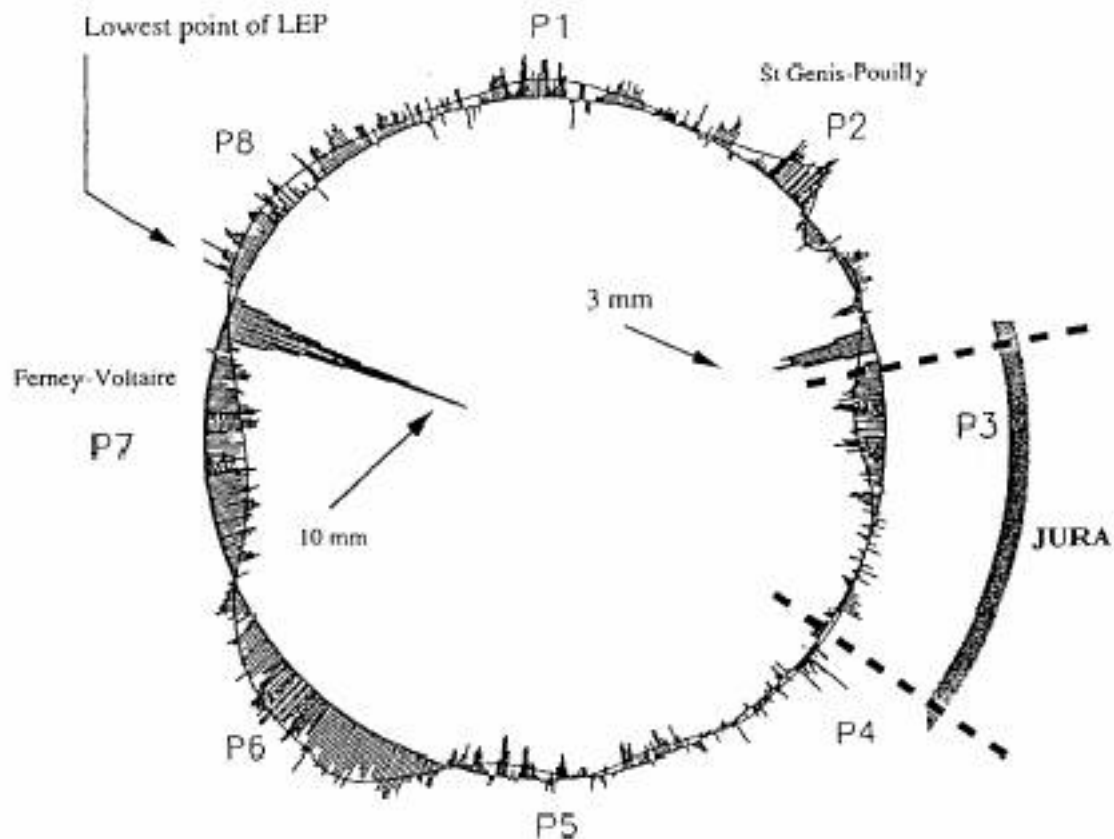


Figure 1 : observed status, end 1992

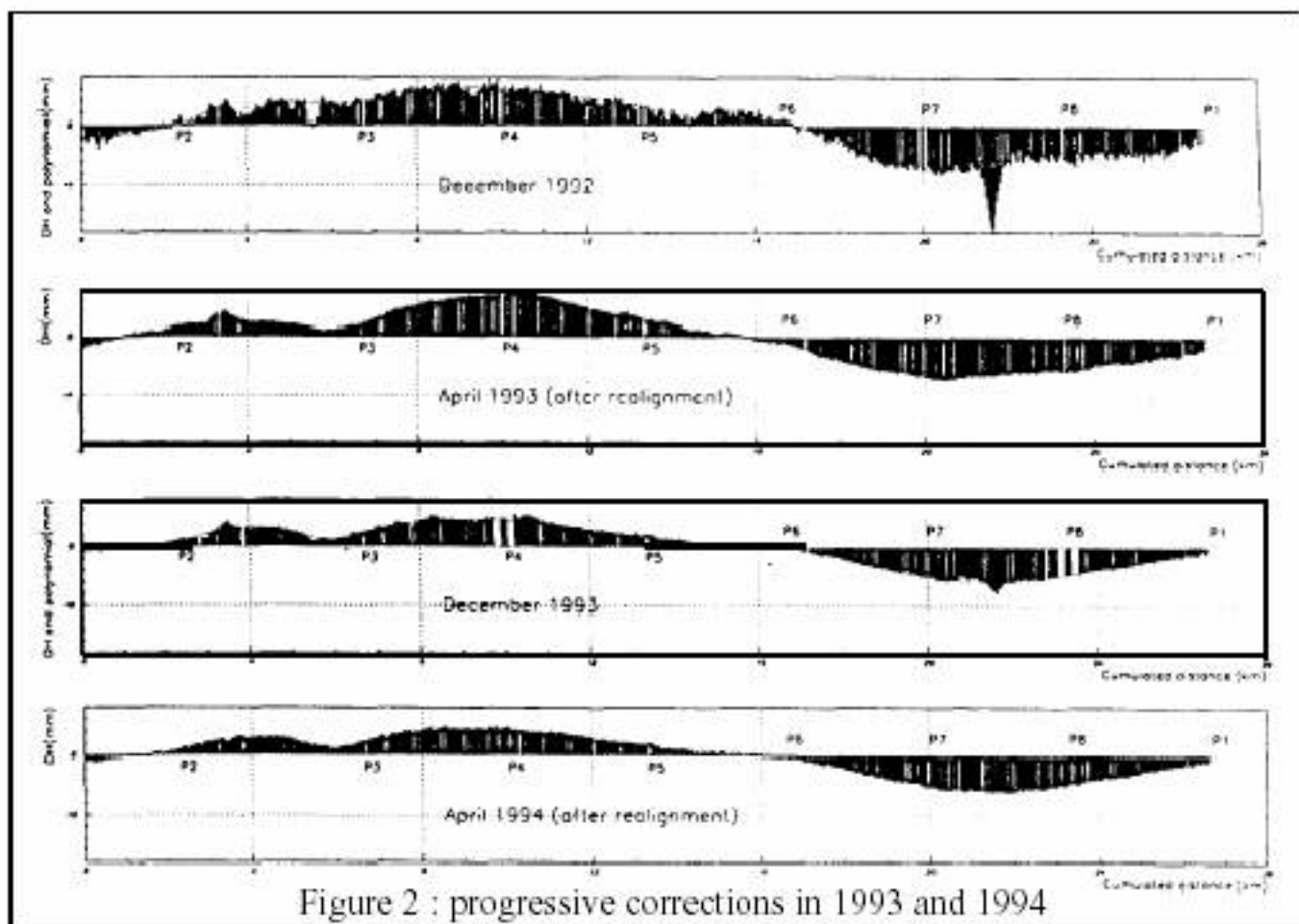


Figure 2 : progressive corrections in 1993 and 1994

Problems Generated by Orbit Errors

● injection errors:

■ *aperture* → *beam losses*

■ *filamentation* → *beam size*

● closed orbit errors:

■ *x-y coupling*

■ *aperture*

■ *energy error*

■ *field imperfections*

■ *dispersion* → *beam size at IP*

■ *beam separation*

Aim:

$\Delta x, \Delta y < 4 \text{ mm}$

rms < 0.5 mm

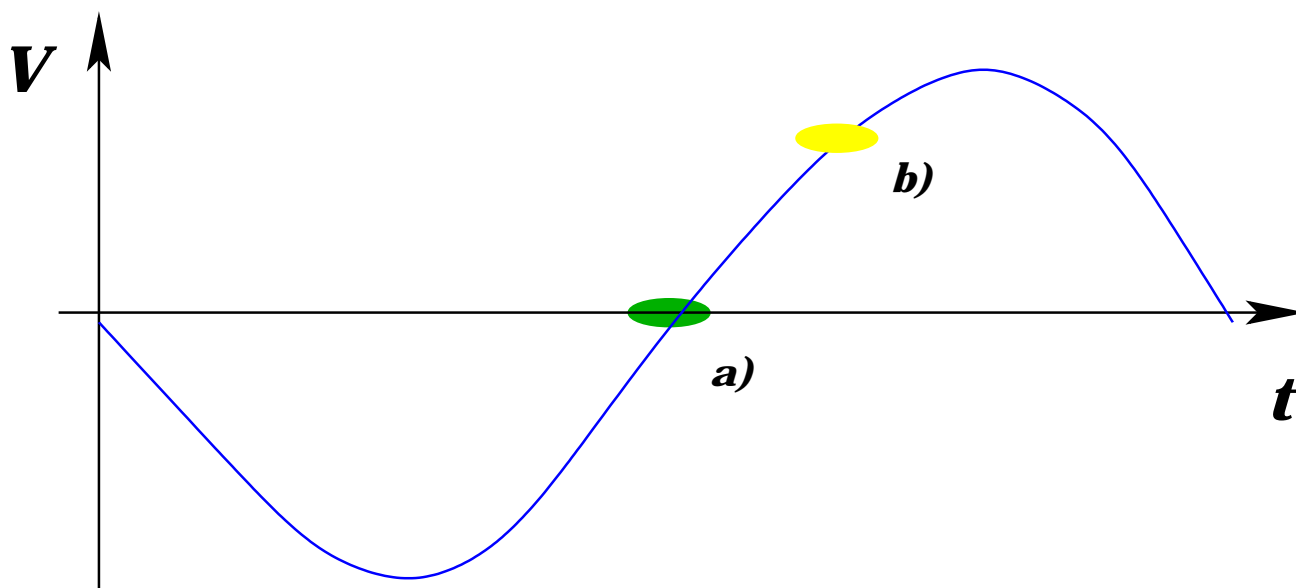


beam monitors and orbit correctors

Synchrotron:

→ the orbit determines the particle energy!

■ assume: $L >$ design orbit



→ energy increase

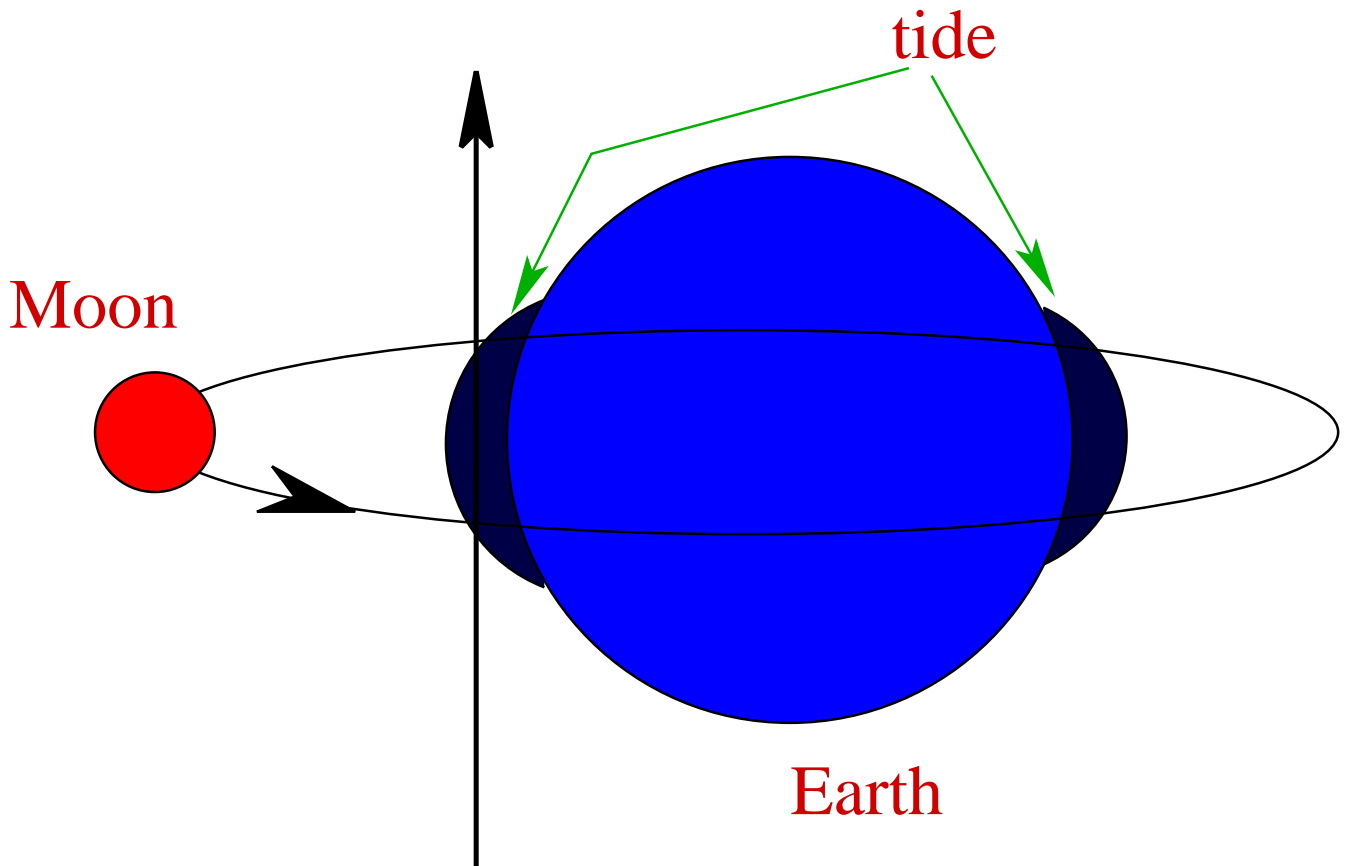
Equilibrium:

$$f_{RF} = h \cdot f_{rev}$$

$$f_{rev} = \frac{1}{2 \cdot \pi} \cdot \frac{q}{m \cdot \gamma} \cdot B$$

→ E depends on orbit and magnetic field!

■ *tidal motion of the earth:*



■ *orbit and beam energy modulation:*

$$f_{mod} = 24 \text{ h}; 12 \text{ h}$$

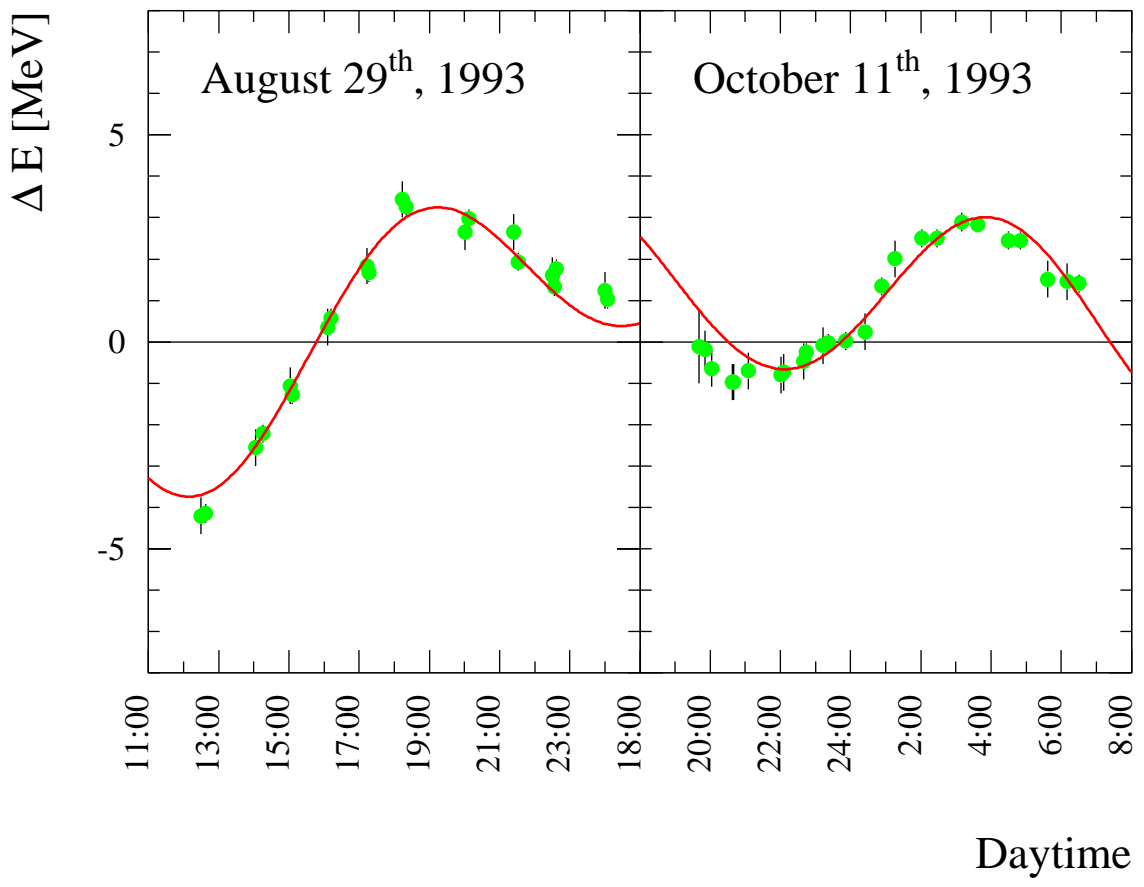
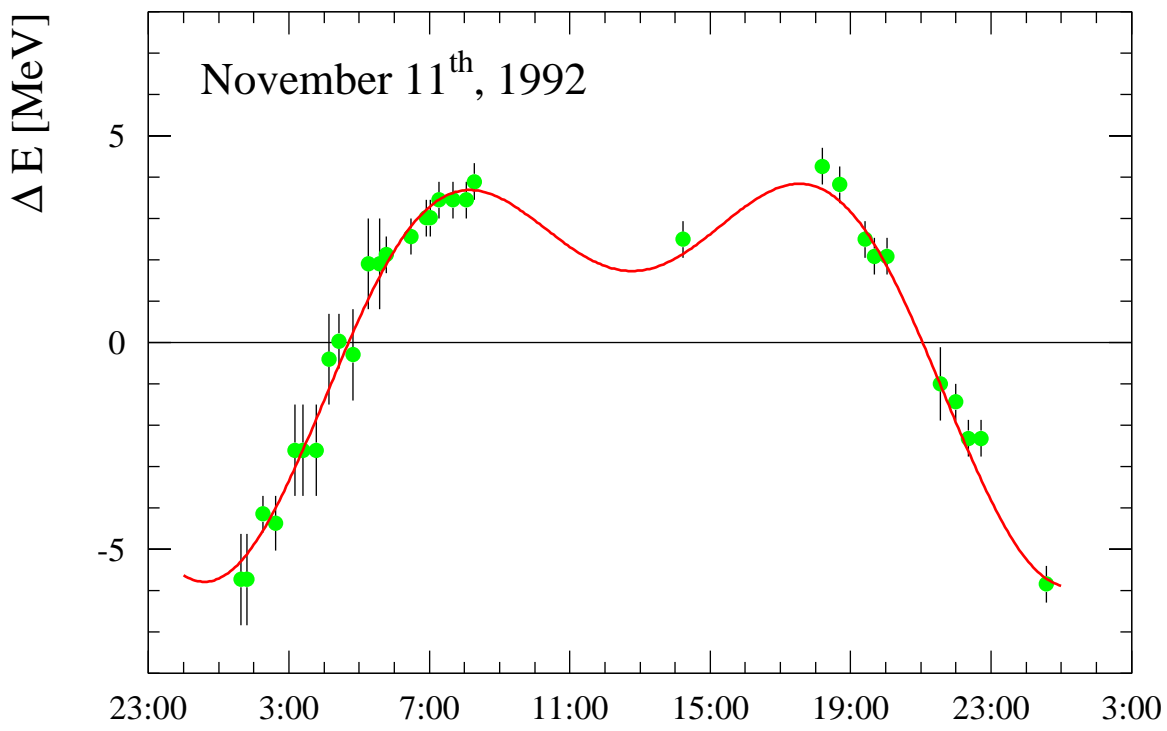
→ $\Delta E \approx 10 \text{ MeV}$

$$\approx 0.02\%$$

aim: $\Delta E \lesssim 0.003\%$

→ *requires correction!*

energy modulation due to tidal motion of earth

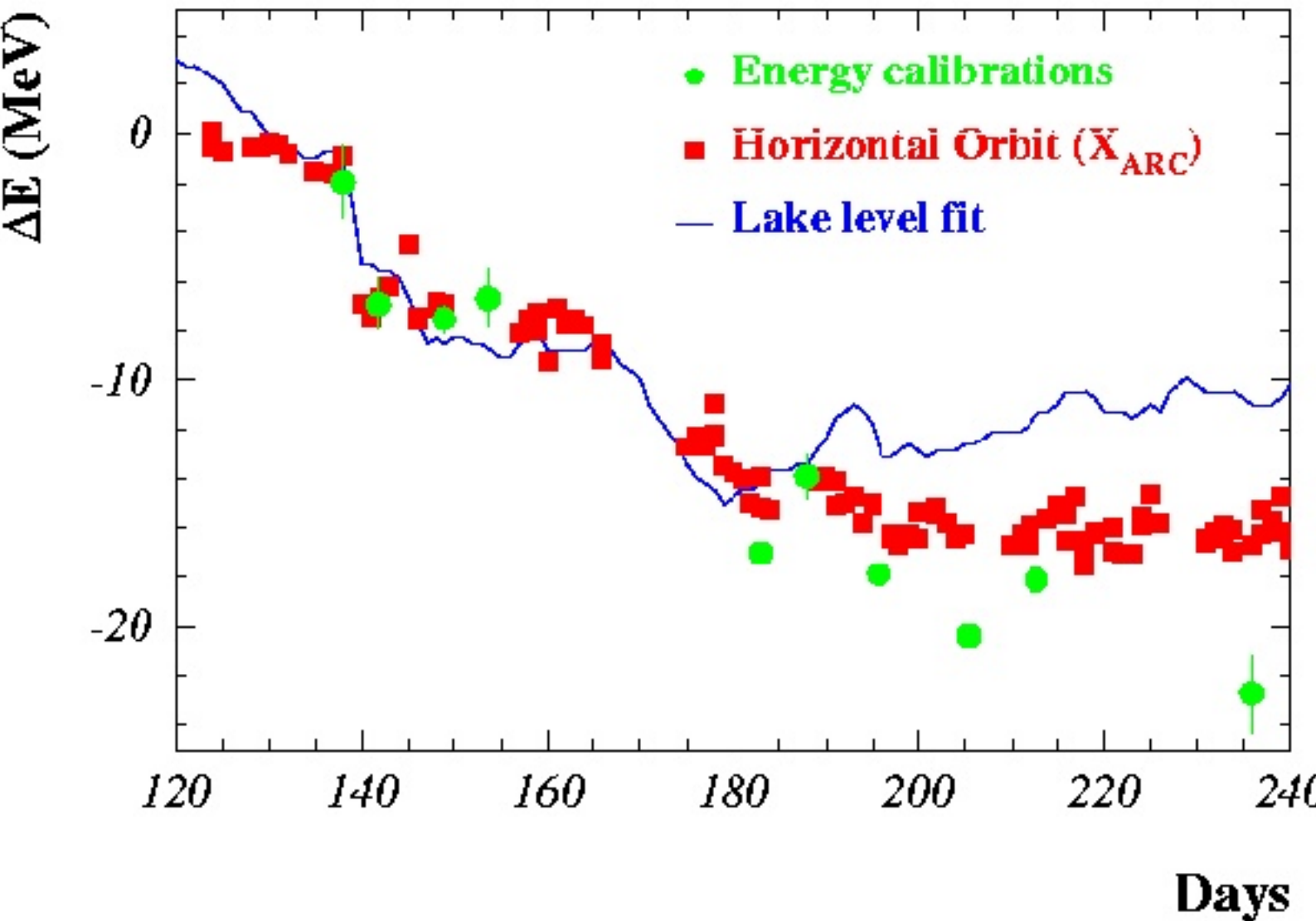


→ $\Delta E \approx 10 \text{ MeV}$

■ energy modulation due to lake level changes

changes in the water level of lake Geneva change the position of the LEP tunnel and thus the quadrupole positions

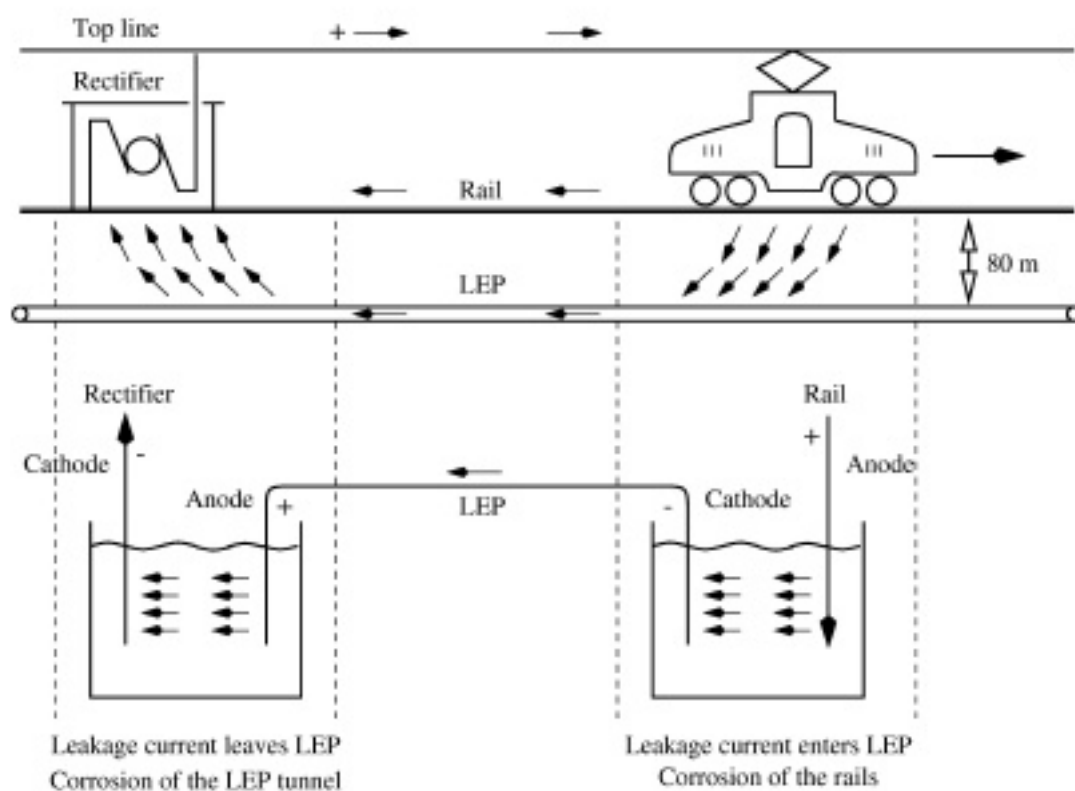
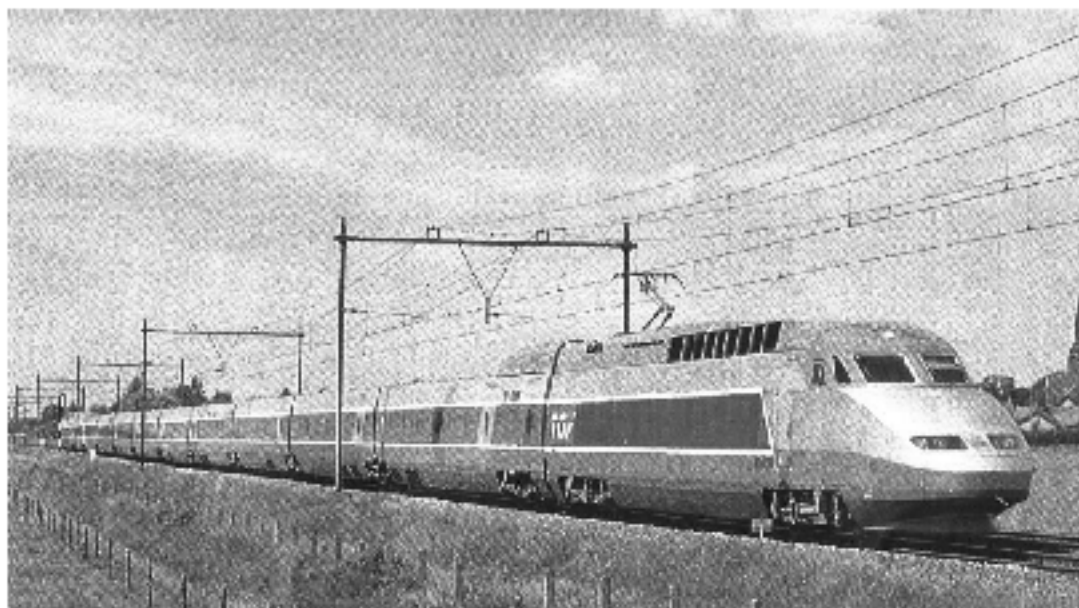
→ orbit and energy perturbations



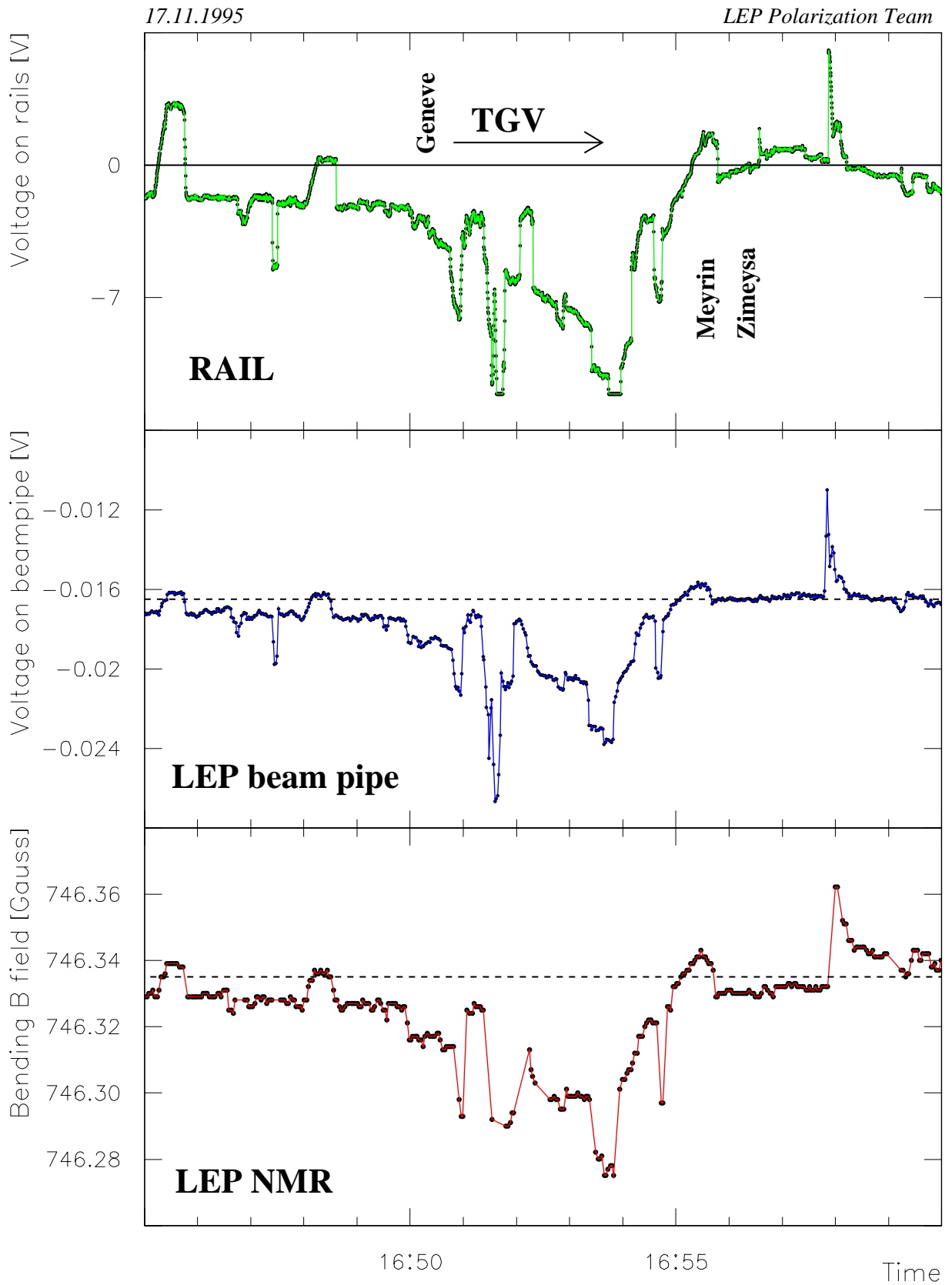
→ $\Delta E \approx 20 \text{ MeV}$

energy modulation due current perturbations in the main dipole magnets

TGV line between Geneva and Bellegarde



■ correlation of NMR dipole field measurements
with the voltage on the TGV train tracks

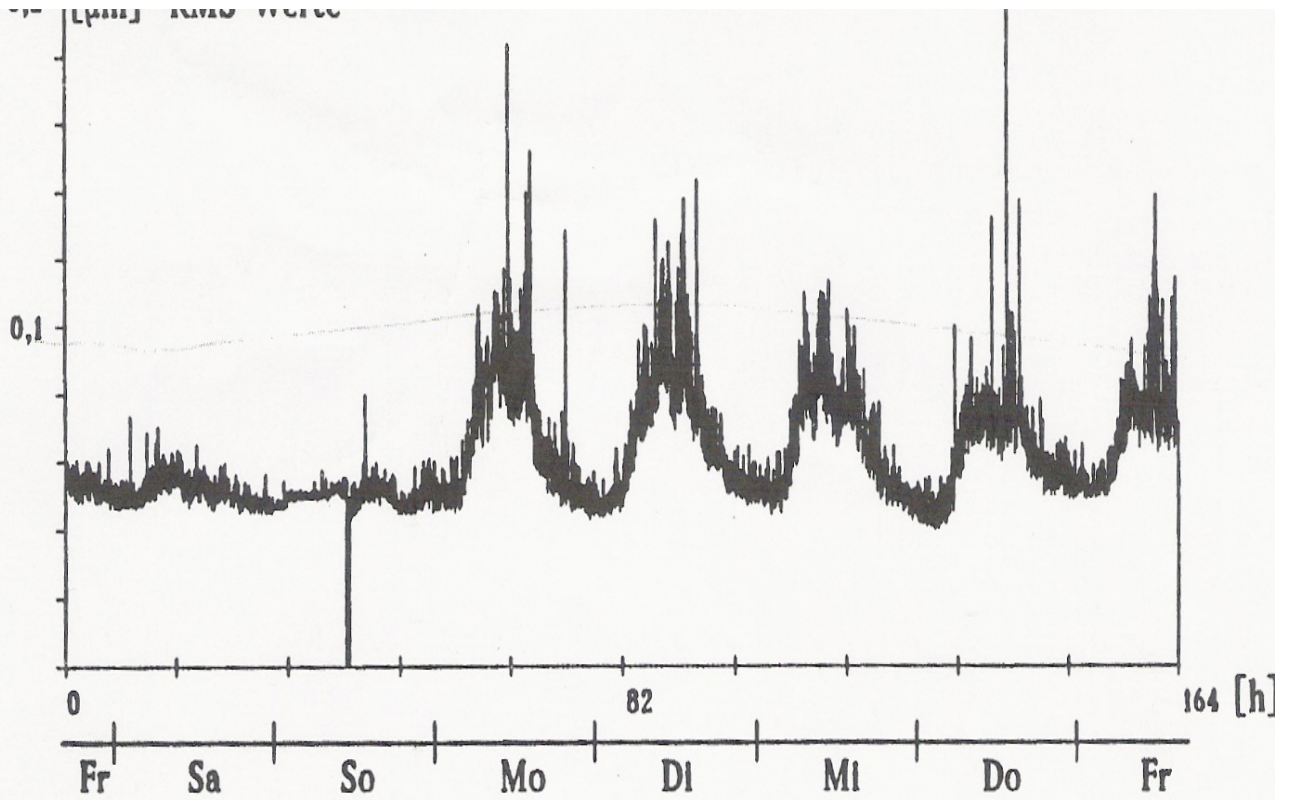


$\Delta E \approx 5 \text{ MeV}$ for LEP operation at 45 GeV

■ ground motion due to human activity

quadrupole motion in HERA-p (DESY Hamburg)

RMS



peak to peak

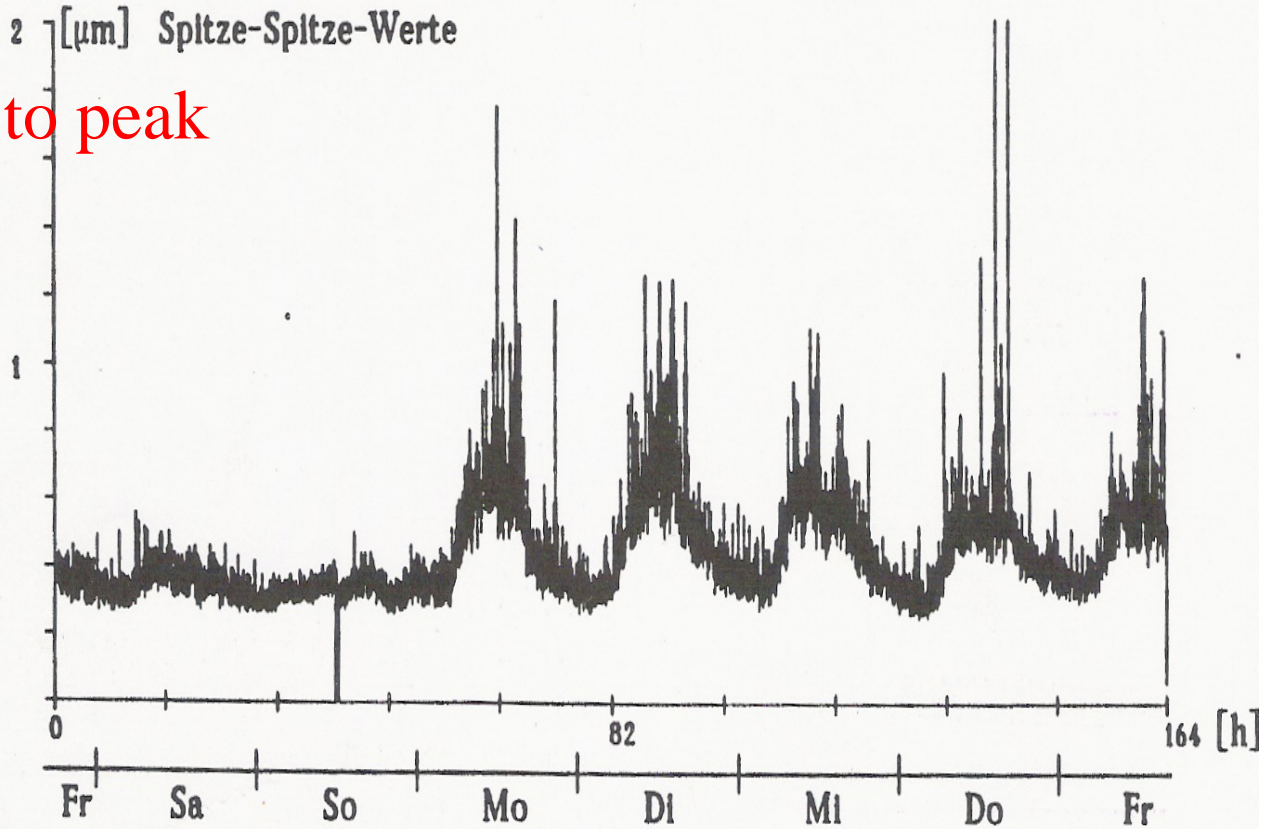


Abb. 3.13 Zeitabhängigkeit der Bodenbewegung
oben RMS-Werte
unten Spitze-Spitze-Werte

Closed Orbit Response

inhomogeneous equation:

$$\frac{d^2 \mathbf{x}}{d s^2} + \mathbf{K}(s) \cdot \mathbf{x} = \mathbf{G}(s); \quad \mathbf{G}(s) = \Delta \mathbf{k}_0 (s)$$

$$\vec{y}' + \begin{pmatrix} 0 & 1 \\ \mathbf{K} & 0 \end{pmatrix} \cdot \vec{y} = \vec{\mathbf{G}}; \quad \vec{\mathbf{G}} = \begin{pmatrix} 0 \\ \mathbf{G} \end{pmatrix}$$

$$\vec{y}(s) = a \cdot \vec{\mathbf{S}}(s) + b \cdot \vec{\mathbf{C}}(s) + \vec{\Psi}(s)$$

we need to find only one solution!

variation of the constant:

$$\vec{\Psi}(s) = c(s) \cdot \vec{\mathbf{S}}(s) + d(s) \cdot \vec{\mathbf{C}}(s)$$

Closed Orbit Response

variation of the constant in matrix form:

$$\vec{\psi}(s) = \underline{\phi}(s) \cdot \vec{u}(s); \quad \text{with}$$

$$\underline{\phi}(s) = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix}$$

substitute into differential equation:

$$\underline{\phi}(s) \cdot \vec{u}'(s) = \vec{G}(s)$$

$$\vec{u}(s) = \int_{s_0}^s \underline{\phi}(t)^{-1} \cdot \vec{G}(t) dt$$

$$\vec{y}(s) = a \cdot \vec{S}(s) + b \cdot \vec{C}(s) + \underline{\phi}(s) \cdot \int_{s_0}^s \underline{\phi}(t)^{-1} \cdot \vec{G}(t) dt$$

Closed Orbit Response

periodic boundary conditions:

$$\vec{y}(s) = a \cdot \vec{S}(s) + b \cdot \vec{C}(s) + \underline{\phi}(s) \cdot \int_{s_0}^s \underline{\phi}(t)^{-1} \cdot \vec{G}(t) dt$$

with

$$\vec{y}(s) = \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}; \quad x(s) = x(s + L); \quad x'(s) = x'(s + L)$$



periodic boundary conditions determine coefficients a and b



$$x(s) = \frac{\sqrt{\beta(s)}}{2 \sin(\pi \cdot Q)} \cdot \int_{s_0}^{s_0 + \text{circ}} \sqrt{\beta(t)} \cdot G(t) \cos[\phi(t) - \phi(s) - \pi Q] dt$$

Closed Orbit Response

Example: particle momentum error

normalized dipole strength: $\mathbf{k}_o(\mathbf{s}) = 0.3 \cdot \frac{B[T]}{p[GeV]}$

$$k_o(s) = \frac{1}{\rho(t)} - \frac{1}{\rho(t)} \cdot \frac{\Delta p}{p_o} \longrightarrow G(t) = \frac{1}{\rho(t)} \cdot \frac{\Delta p}{p_o}$$

$$x(s) = \frac{\sqrt{\beta(s)}}{2 \sin(\pi \cdot Q)} \cdot \oint \sqrt{\beta(t)} \cdot G(t) \cos[|\phi(t) - \phi(s)| - \pi Q] dt$$

$$\longrightarrow x(s) = D(s) \cdot \frac{\Delta p}{p}$$

with

$$D(s) = \frac{\sqrt{\beta(s)}}{2 \sin(\pi \cdot Q)} \cdot \oint \frac{\sqrt{\beta(t)}}{\rho(t)} \cdot \cos[|\phi(t) - \phi(s)| - \pi Q] dt$$



Dispersion Orbit

Orbit Correction

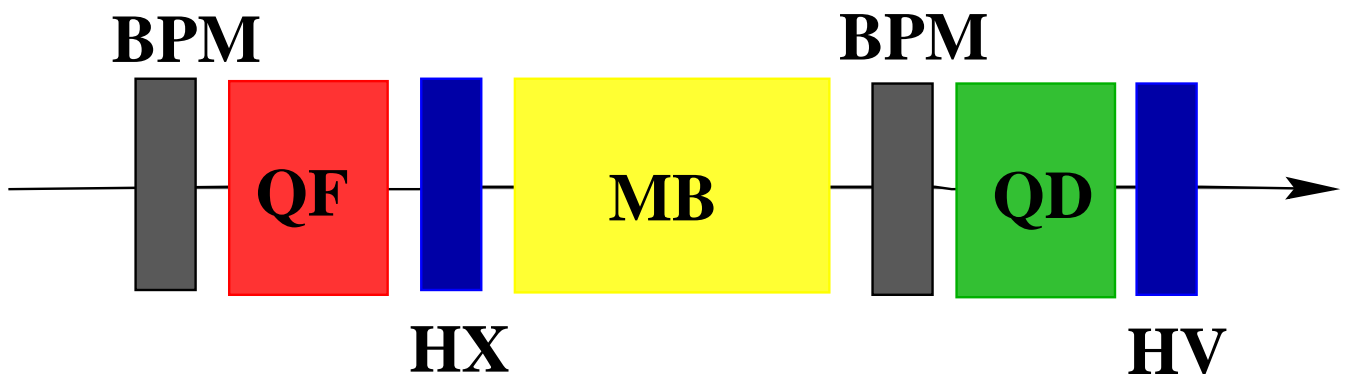
- the orbit error in a storage ring with conventional magnets is dominated by the contributions from the quadrupole alignment errors
- orbit perturbation is proportional to the local β -functions at the location of the dipole error
 - alignment errors at QF cause mainly horizontal orbit errors
 - alignment errors at QD causes mainly vertical orbit errors

Orbit Correction

■ aim at a local correction of the dipole error due to the quadrupole alignment errors

→ place orbit corrector and BPM next to the main quadrupoles

→ horizontal BPM and corrector next to QF
vertical BPM and corrector next to QD



→ orbit in the opposite plane?

relative alignment of BPM and quadrupole?

LEP Orbit

Horizontal Orbit:

■ *beam offset in quadrupoles:*

→ *Lake Geneva*

→ *moon*

→ *energy error*

Vertical Orbit:

■ *beam offset in quadrupoles*

■ *beam separation*

→ *orbit deflection depends on particle energy*

→ *vertical dispersion [D(s)]*

$$\sigma_y = \sqrt{\varepsilon \cdot \beta_y + \delta_y^2 \cdot D^2}$$

→ *small vertical beam size relies on good orbit*

■ *1994: 13000 vertical orbit corrections in physics*

Quadrupole Gradient Error

one turn map:

can be generated by matrix multiplication:

$$\longrightarrow \vec{z}_{n+1} = \underline{\underline{M}} \cdot \vec{z}_n \quad \vec{z} = \begin{pmatrix} x \\ x' \end{pmatrix}$$

and can be expressed in terms of the C and S solutions

$$\underline{\underline{M}} = \underline{\underline{I}} \cdot \cos(2\pi Q) + \underline{\underline{J}} \cdot \sin(2\pi Q)$$

$$\underline{\underline{I}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \underline{\underline{J}} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}; \quad \gamma = [1 + \alpha^2] / \beta$$

remember: $\cos(2\pi Q) = \frac{1}{2} \text{trace } \underline{\underline{M}}$

\longrightarrow the coefficients of: $\frac{\underline{\underline{M}} - \underline{\underline{I}} \cdot \cos(2\pi Q)}{\sin(2\pi Q)}$

provide the optic functions at s_0

Quadrupole Gradient Error

transfer matrix for single quadrupole:

$$m_0 = \begin{pmatrix} 1 & 0 \\ -k_1 \cdot l & 1 \end{pmatrix}$$

matrix for single quadrupole with error:

$$m = \begin{pmatrix} 1 & 0 \\ -[k_1 + \Delta k_1] \cdot l & 1 \end{pmatrix}$$

one turn matrix with quadrupole error:

$$M = m \cdot m_0^{-1} \cdot M_0$$

trace M



$$\cos(2\pi Q) = \cos(2\pi Q_0) - \frac{1}{2} \beta \cdot \Delta k_1 \cdot l \cdot \sin(2\pi Q_0)$$

Quadrupole Gradient Error

distributed perturbation:

$$\cos(2\pi Q) = \cos(2\pi Q_0) - \frac{\sin(2\pi Q_0)}{2} \cdot \int \beta \cdot \Delta k_1 ds$$

$$\longrightarrow \Delta Q = \frac{1}{4\pi} \cdot \int \beta \cdot \Delta k_1 ds$$

chromaticity:

$$k_1 = \frac{e \cdot g}{p}$$

momentum error $\longrightarrow \Delta k_1 = -k_1 \cdot \frac{\Delta p}{p}$

$$\begin{aligned} \Delta Q &= -\frac{1}{4\pi} \cdot \int \beta \cdot k_1 \cdot ds \cdot \frac{\Delta p}{p} \\ &= \xi \cdot \frac{\Delta p}{p} \end{aligned}$$

β - *Beat*

quadrupole error:

$$\longrightarrow \vec{z}_{n+1} = \underline{M} \cdot \vec{z}_n \quad \underline{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

with

$$\underline{M} = \underline{I} \cdot \cos(2\pi Q) + \underline{J} \cdot \sin(2\pi Q)$$

$$\underline{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \underline{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}; \quad \gamma = [1 + \alpha^2] / \beta$$

$$\longrightarrow \text{calculate: } \frac{m_{12}}{\sin(2\pi Q)}$$

$$\Delta\beta(s) = \frac{\beta(s)}{2 \sin(2\pi Q)} \cdot \int_{s_0}^{s_0 + \text{circ}} \beta(t) \cdot \Delta k(t) \cos[2[\phi(t) - \phi(s)] - 2\pi Q] dt$$



β - beat oscillates with twice the betatron frequency

Local Orbit Bumps I

deflection angle:

$$\theta_i = \int_{\text{dipole}} G_i(t) dt = \frac{0.3 \cdot B_i[\text{T}] \cdot l}{p[\text{GeV}]}$$

trajectory response:

[no periodic boundary conditions]

$$\longrightarrow x(s) = \sqrt{\beta_i \beta(s)} \cdot \theta_i \cdot \sin[\phi(s) - \phi_i]$$

$$\longrightarrow x'(s) = \sqrt{\beta_i / \beta(s)} \cdot \theta_i \cdot \cos[\phi(s) - \phi_i]$$

Local Orbit Bumps II

closed orbit bump:

compensate the trajectory perturbation with

additional corrector kicks further down stream

→ closure of the perturbation within one turn

→ local orbit excursion

→ possibility to correct orbit errors locally

→ closure with one additional corrector magnet

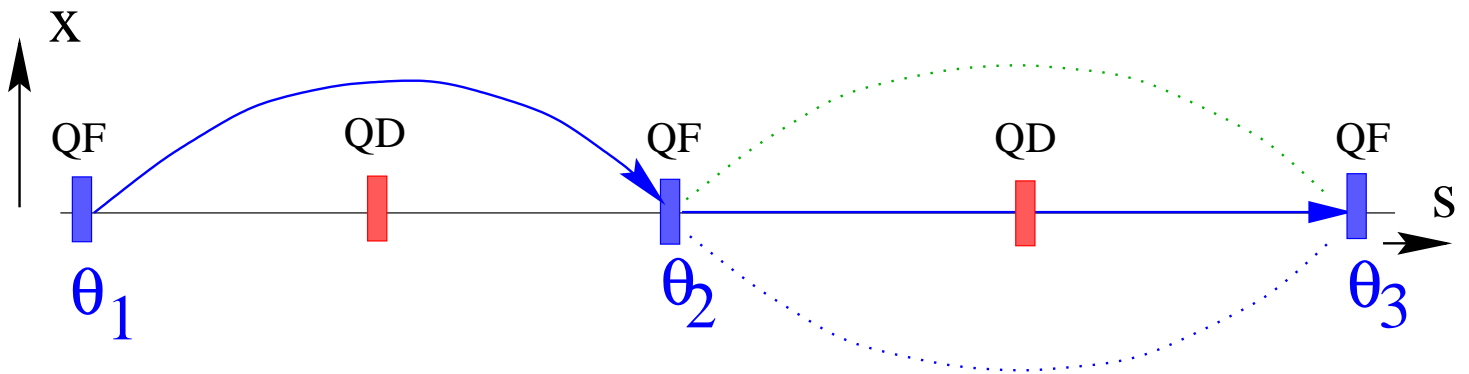
→ π - bump

→ closure with two additional corrector magnets

→ three corrector bump

Local Orbit Bumps III

■ π - bump: (quasi local correction of error)



→
$$\theta_2 = \frac{-\sqrt{\beta_1}}{\sqrt{\beta_2}} \cdot \theta_1$$

■ limits / problems:

→ closure depends on lattice phase advance

→ requires 90° lattice

→ sensitive to lattice errors

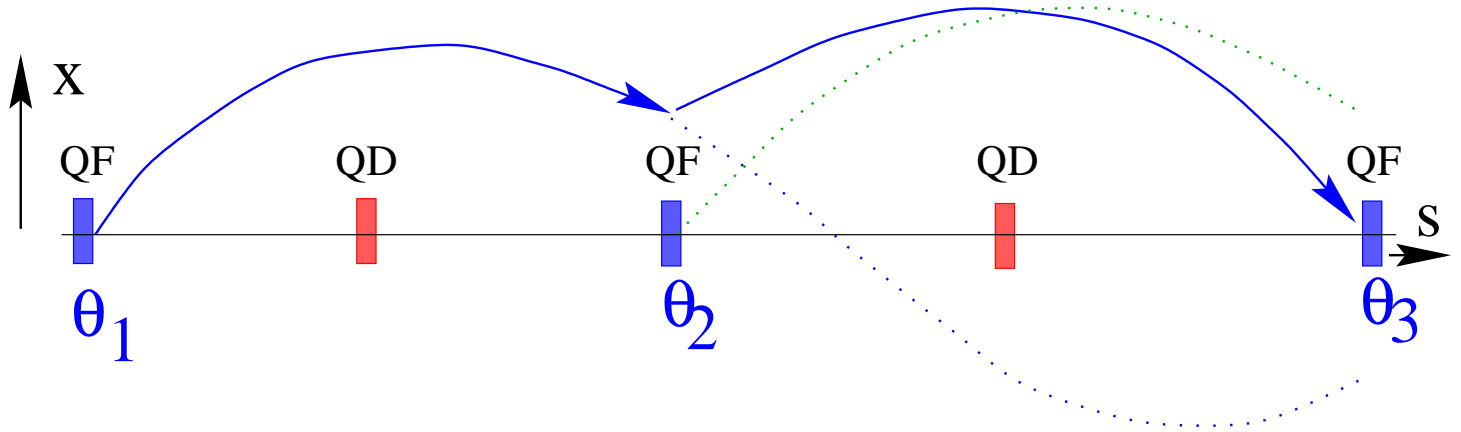
→ requires horizontal BPMs at QF and QD

→ sensitive to BPM errors

→ requires large number of correctors

Local Orbit Bumps IV

3 corrector bump: (quasi local correction of error)



$$\theta_2 = -\frac{\sqrt{\beta_1}}{\sqrt{\beta_2}} \cdot \frac{\sin(\Delta\phi_{3-1})}{\sin(\Delta\phi_{3-2})} \cdot \theta_1$$

$$\theta_3 = \left(\frac{\sin(\Delta\phi_{3-1})}{\tan(\Delta\phi_{3-2})} - \cos(\Delta\phi_{3-1}) \right) \cdot \frac{\sqrt{\beta_1}}{\sqrt{\beta_3}} \cdot \theta_1$$

works for any lattice phase advance

requires only horizontal BPMs at QF

limits / problems:

sensitive to BPM errors

large number of correctors

can not control x'

Summary Linear Imperfections

avoid machine tunes near integer resonances:

- they amplifies the response to dipole field errors
- a closed orbit perturbation propagates with the betatron phase around the storage ring
- discontinuities in the derivative of the closed orbit response at the location of the perturbation

avoid storage ring tunes near half-integer resonances:

- they amplifies the response to quadrupole field errors
- betafunction perturbations propagate with twice the betatron phase advance around the storage ring

integral expressions are mainly used for estimates
numerical programs mainly rely on maps

- closed orbit = fixed point of '1-turn' map
- dispersion = eigenvector of extended '1-turn' map
- tune is given by the trace of the '1-turn' map
- twiss functions are given by the matrix elements