Particle motion in Hamiltonian Formalism II

Or how to derive and solve equations of motion

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Introduction to Accelerator Physics

Everness Hotel, Chavannes de Bogis, Switzerland

September 25th – October 8th, 2021

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Summary of Lecture I

- 2^{nd} order dif. equations of motion from Newton's law (in **configuration space**) can be solved by **transforming** them to pairs of 1^{st} order dif. equations (in **phase space**)
- Natural appearance of **invariant** of motion ("**energy**")
- Non-linear oscillators have **frequencies** which **depend** on the **invariant** (or "**amplitude**")
- Connected invariant of motion to system's **Hamiltonian** (derived through **Lagrangian**)
- Shown that through the **Hamiltonian** , the **equations** of **motions** can be **derived**
- **Poisson bracket** operators are helpful for discovering integrals of motion

Canonical transformations

Canonical Transformations

- Find a **function** for transforming the Hamiltonian from variable $(\mathbf{q}, \mathbf{p})$ to $(\mathbf{Q}, \mathbf{P})$, so system becomes **simpler** to study
- Transformation should be **canonical** (or **symplectic**), so that **Hamiltonian** properties (**phase-space volume**) are preserved

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- These "mixed variable" **generating** functions are derived by

$$F_1(\mathbf{q},\mathbf{Q}) : p_i = \frac{\partial F_1}{\partial q_i},\ \ P_i = -\frac{\partial F_1}{\partial Q_i} \qquad F_3(\mathbf{Q},\mathbf{p}) : q_i = -\frac{\partial F_3}{\partial p_i},\ \ P_i = -\frac{\partial F_3}{\partial Q_i}$$

$$F_2(\mathbf{q},\mathbf{P}) : p_i = \frac{\partial F_2}{\partial q_i},\ \ Q_i = \frac{\partial F_2}{\partial P_i} \qquad F_4(\mathbf{p},\mathbf{P}) : q_i = -\frac{\partial F_4}{\partial p_i},\ \ Q_i = \frac{\partial F_4}{\partial P_i}$$

- A general **non-autonomous Hamiltonian** is transformed to

$$H(\mathbf{Q},\mathbf{P},t) = H(\mathbf{q},\mathbf{p},t) + \frac{\partial F_j}{\partial t}\ ,\quad j = 1,2,3,4$$

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- One generating function can be constructed by the other through **Legendre transformations**, e.g.

$$F_2(\mathbf{q},\mathbf{P}) = F_1(\mathbf{q},\mathbf{Q}) - \mathbf{Q}\cdot\mathbf{P} \;,\quad F_3(\mathbf{Q},\mathbf{p}) = F_1(\mathbf{q},\mathbf{Q}) - \mathbf{q}\cdot\mathbf{p} \;,\quad \ldots$$

with the inner product define as $\mathbf{q}\cdot\mathbf{p} = \sum_i q_i p_i$

Preservation of Phase Volume

- A fundamental property of canonical transformations is the **preservation** of **phase space volume**
- This **volume** preservation in phase space can be represented in the **old** and **new variables** as

$$\int \prod_{i=1}^{n} dp_i dq_i = \int \prod_{i=1}^{n} dP_i dQ_i$$

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$$\prod_{i=1}^{n} dp_i dq_i = \frac{\partial(P_1, \ldots, P_n, Q_1, \ldots, Q_n)}{\partial(p_1, \ldots, p_n, q_1, \ldots, q_n)} \prod_{i=1}^{n} dP_i dQ_i$$

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- These two relationships imply that the **Jacobian** of a **canonical transformation** should have **determinant** equal to 1

$$\left|\frac{\partial(P_1, \ldots, P_n, Q_1, \ldots, Q_n)}{\partial(p_1, \ldots, p_n, q_1, \ldots, q_n)}\right| = \left|\frac{\partial(p_1, \ldots, p_n, q_1, \ldots, q_n)}{\partial(P_1, \ldots, P_n, Q_1, \ldots, Q_n)}\right| = 1$$

Examples of transformations

- The transformation $Q = -p \ , \quad P = q$, which **interchanges conjugate variables** is area preserving, as the Jacobian is

$$\frac{\partial(P,Q)}{\partial(p,q)} = \begin{vmatrix} \frac{\partial P}{\partial p} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial Q}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

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- On the other hand, the transformation from **Cartesian to polar** coordinates $q = P\cos Q \ , \quad p = P\sin Q$ is not, since

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- There are actually **"polar" coordinates** that are **canonical**, given by $q = -\sqrt{2P}\cos Q \,, \quad p = \sqrt{2P}\sin Q$ for which

$$\frac{\partial(q,p)}{\partial(Q,P)} = \begin{vmatrix} \sqrt{2P}\sin Q & \sqrt{2P}\cos Q \\ -\frac{\cos Q}{\sqrt{2P}} & \frac{\sin Q}{\sqrt{2P}} \end{vmatrix} = 1$$

The Relativistic Hamiltonian for electromagnetic fields

Single-particle relativistic Hamiltonian

- Neglecting self fields and radiation, motion can be described by a “single-particle” Hamiltonian

$$H(\mathbf{x},\mathbf{p},t) = c\sqrt{\left(\mathbf{p}-\frac{e}{c}\mathbf{A}(\mathbf{x},t)\right)^2 + m^2c^2} + e\Phi(\mathbf{x},t)$$

  - $\mathbf{x} = (x, y, z)$ Cartesian positions
  - $\mathbf{p} = (p_x, p_y, p_z)$ conjugate momenta
  - $\mathbf{A} = (A_x, A_y, A_z)$ magnetic vector potential
  - Φ electric scalar potential

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- The ordinary kinetic momentum vector is written

$$\mathbf{P} = \gamma m \mathbf{v} = \mathbf{p} - \frac{e}{c}\mathbf{A}$$

with $\mathbf{v}$ the velocity vector and $\gamma = (1 - v^2/c^2)^{-1/2}$ the relativistic factor

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- It is generally a **3 degrees of freedom** one plus time (i.e., **4 degrees of freedom**)
- The Hamiltonian represents the **total energy**

$$H \equiv E = \gamma mc^2 + e\Phi$$

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- The **total kinetic momentum** is

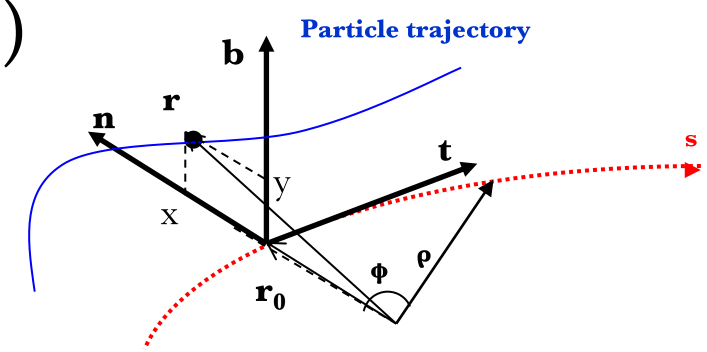
$$P = \left(\frac{H^2}{c^2} - m^2c^2\right)^{1/2}$$

- Using **Hamilton's equations**

$$(\dot{\mathbf{x}},\dot{\mathbf{p}}) = [(\mathbf{x},\mathbf{p}),H]$$

it can be shown that motion is governed by **Lorentz equations**

From Cartesian to "curved" coordinates



- It is useful (especially for **rings**) to transform the Cartesian coordinate system to the **Frenet-Serret system** moving to a closed curve, with path length s
- The **position coordinates** in the two systems are connected by $\mathbf{r} = \mathbf{r_0}(s) + X\mathbf{n}(s) + Y\mathbf{b}(s) = x\mathbf{u_x} + y\mathbf{u_y} + z\mathbf{u_z}$

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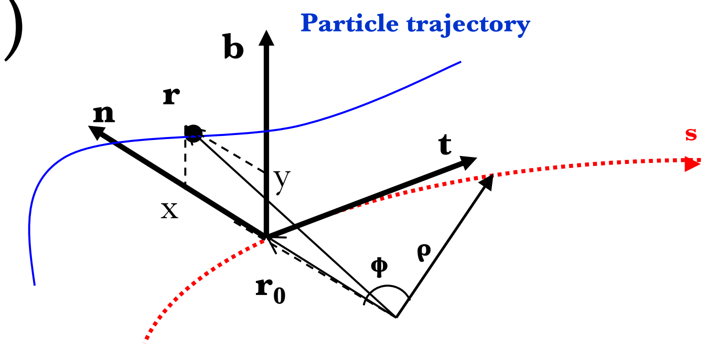

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- The **Frenet-Serret unit vectors** and their derivatives are defined as $(\mathbf{t}, \mathbf{n}, \mathbf{b}) = (\frac{d}{ds}\mathbf{r_0}(s), -\rho(s)\frac{d^2}{ds^2}\mathbf{r_0}(s), \mathbf{t}\times\mathbf{n})$

$$\frac{d}{ds}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\rho(s)} & 0 \\ \frac{1}{\rho(s)} & 0 & -\tau(s) \\ 0 & 0 & \tau(s) \end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix}$$

with $\rho(s)$ the **radius of curvature** and $\tau(s)$ the **torsion** which vanishes in case of planar motion

From Cartesian to "curved" variables

- We are seeking a canonical transformation between

$$(\mathbf{q}, \mathbf{p}) \quad \mapsto \quad (\mathbf{Q}, \mathbf{P}) \ \text{ or}$$
$$(x, y, z, p_x, p_y, p_z) \quad \mapsto \quad (X, Y, s, P_x, P_y, P_s)$$

- The **generating** function is

$$(\mathbf{q}, \mathbf{P}) = -\left(\frac{\partial F_3(\mathbf{p}, \mathbf{Q})}{\partial \mathbf{p}}, \frac{\partial F_3(\mathbf{p}, \mathbf{Q})}{\partial \mathbf{Q}}\right)$$

- By using the **relationship** for the **positions,**

$$\mathbf{r} = \mathbf{r_0}(s) + X\mathbf{n}(s) + Y\mathbf{b}(s) = x\mathbf{u_x} + y\mathbf{u_y} + z\mathbf{u_z}$$

the generating function is

$$F_3(\mathbf{p}, \mathbf{Q}) = -\mathbf{p} \cdot \mathbf{r}$$

From Cartesian to "curved" variables

- For planar motion, the momenta are

$$\mathbf{P} = (P_X, P_Y, P_s) = \mathbf{p}\cdot(\frac{\partial F_3}{\partial X}, \frac{\partial F_3}{\partial Y}, \frac{\partial F_3}{\partial s}) = \mathbf{p}\cdot(\mathbf{n}, \mathbf{b}, (1+\frac{X}{\rho})\mathbf{t})$$

- Taking into account that the **vector potential** is also transformed in the same way

$$(A_X, A_Y, A_s) = \mathbf{A}\cdot(\mathbf{n}, \mathbf{b}, (1+\frac{X}{\rho})\mathbf{t})$$

the **new Hamiltonian** is given by

$$\mathcal{H}(\mathbf{Q},\mathbf{P},t) = c\sqrt{(P_X - \frac{e}{c}A_X)^2 + (P_Y - \frac{e}{c}A_Y)^2 + \frac{(P_s - \frac{e}{c}A_s)^2}{(1+\frac{X}{\rho(s)})^2} + m^2c^2} + e\Phi$$

Changing of the independent variable

- It is more convenient to use the **path length** s , instead of the **time** as **independent variable**
- The Hamiltonian can be considered as having **4 degrees of freedom**, where the **4th "position"** is **time** and its conjugate momentum is $P_t = -\mathcal{H}$

Changing of the independent variable

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- In the same way, the new Hamiltonian with the path length as the independent variable is just $P_s = -\tilde{\mathcal{H}}(X, Y, t, P_X, P_Y, P_t, s)$ with

$$\tilde{\mathcal{H}} = -\frac{e}{c}A_s - \left(1 + \frac{X}{\rho(s)}\right)\sqrt{(\frac{P_t + e\Phi}{c})^2 - m^2c^2 - (P_x - \frac{e}{c}A_X)^2 - (P_Y - \frac{e}{c}A_Y)^2}$$

- It can be proved that this is indeed a **canonical transformation**
- Note the existence of the **reference orbit for zero vector potential**, for which $(X, Y, P_X, P_Y, P_s) = (0, 0, 0, 0, P_0)$

Neglecting electric fields

- Due to the fact that **longitudinal** (synchrotron) motion is **much slower** than the **transverse** (betatron) one, the electric field can be set to **zero** and the Hamiltonian is written as

$$\tilde{\mathcal{H}} = -\frac{e}{c}A_s - \left(1+\frac{X}{\rho(s)}\right)\sqrt{\underbrace{(\frac{\mathcal{H}}{c})^2 - m^2c^2}_{P^2} - (P_x - \frac{e}{c}A_X)^2 - (P_Y - \frac{e}{c}A_Y)^2}$$

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- The Hamiltonian is then written as

$$\tilde{\mathcal{H}} = -\frac{e}{c}A_s - \left(1+\frac{X}{\rho(s)}\right)\sqrt{(P^2 - (P_x - \frac{e}{c}A_X)^2 - (P_Y - \frac{e}{c}A_Y)^2}$$

- If **static** magnetic fields are considered, the time dependence is also dropped, and the system is having **2 degrees of freedom + "time"** (path length)

Momentum rescaling

- Due to the fact that **total momentum** is **much larger** than the transverse ones, another transformation may be considered, where the transverse momenta are rescaled

$$
\begin{aligned}
(\mathbf{Q}, \mathbf{P}) &\mapsto (\bar{\mathbf{q}}, \bar{\mathbf{p}}) \quad \text{or} \\
(X, Y, t, P_X, P_Y, P_t) &\mapsto (\bar{x}, \bar{y}, \bar{t}, \bar{p}_x, \bar{p}_y, \bar{p}_t) = (X, Y, -c\, t, \frac{P_X}{P_0}, \frac{P_Y}{P_0}, -\frac{P_t}{P_0 c})
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- The new variables are indeed canonical if the Hamiltonian is also rescaled and written as

$$\bar{\mathcal{H}}(\bar{x},\bar{y},\bar{t},\bar{p}_x,\bar{p}_y,\bar{p}_t) = \frac{\tilde{\mathcal{H}}}{P_0} = -e\bar{A}_s - \left(1+\frac{\bar{x}}{\rho(s)}\right)\sqrt{\bar{p}_t^2 - \frac{m^2c^2}{P_0} - (\bar{p}_x - e\bar{A}_x)^2 - (\bar{p}_y - e\bar{A}_y)^2}$$

with $$(\bar{A}_x,\bar{A}_y,\bar{A}_z) = \frac{1}{P_0\,c}(A_x,A_y,A_s)$$

and $$\frac{m^2c^2}{P_0} = \frac{1}{\beta_0^2\gamma_0^2}$$

Moving the reference frame

- Along the reference trajectory $\bar{p}_{t0} = \frac{1}{\beta_0}$ and
$$\frac{d\bar{t}}{ds}\Big|_{P=P_0} = \frac{\partial \bar{H}}{\partial \bar{p}_t}\Big|_{P=P_0} = -\bar{p}_{t0} = -\frac{1}{\beta_0}$$
- It is thus useful to **move** the **reference frame** to the **reference trajectory** for which another canonical transformation is performed

$$(\bar{\mathbf{q}}, \bar{\mathbf{p}}) \mapsto (\hat{\mathbf{q}}, \hat{\mathbf{p}}) \text{ or}$$

$$(\bar{x}, \bar{y}, \bar{t}, \bar{p}_x, \bar{p}_y, \bar{p}_t) \mapsto (\hat{x}, \hat{y}, \hat{t}, \hat{p}_x, \hat{p}_y, \hat{p}_t) = (\hat{x}, \hat{y}, \bar{t} + \frac{s - s_0}{\beta_0}, \hat{p}_x, \hat{p}_y, \bar{p}_t - \frac{1}{\beta_0})$$

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- The mixed variable generating function is

$(\hat{\mathbf{q}}, \bar{\mathbf{p}}) = (\frac{\partial F_2(\bar{\mathbf{q}}, \hat{\mathbf{p}})}{\partial \hat{\mathbf{p}}}, \frac{\partial F_2(\bar{\mathbf{q}}, \hat{\mathbf{p}})}{\partial \bar{\mathbf{q}}})$ providing

$$F_2(\bar{\mathbf{q}}, \hat{\mathbf{p}}) = \bar{x}\hat{p}_x + \bar{y}\hat{p}_y + (\bar{t} + \frac{s - s_0}{\beta_0})(\hat{p}_t + \frac{1}{\beta_0})$$

- The Hamiltonian is then

$$\hat{\mathcal{H}}(\hat{x}, \hat{y}, \hat{t}, \hat{p}_x, \hat{p}_y, \hat{p}_t) = \frac{1}{\beta_0}(\frac{1}{\beta_0} + \hat{p}_t) - e\hat{A}_s - \left(1 + \frac{\hat{x}}{\rho(s)}\right)\sqrt{(\hat{p}_t + \frac{1}{\beta_0})^2 - \frac{1}{\beta_0^2\gamma_0^2} - (\hat{p}_x - e\hat{A}_x)^2 - (\hat{p}_y - e\bar{A}_y)^2}$$

Relativistic and transverse field approximations

- First note that $\hat{p}_t = \bar{p}_t - \frac{1}{\beta_0} = \bar{p}_t - \bar{p}_{t0} = \frac{P_t - P_0}{P_0} \equiv \delta$ and $l = \hat{t}$
- In the **ultra-relativistic limit** $\beta_0 \to 1$, $\frac{1}{\beta_0^2\gamma^2} \to 0$ and the Hamiltonian is written as

$$\mathcal{H}(x,y,l,p_x,p_y,\delta) = (1+\delta) - e\hat{A}_s - \left(1+\frac{x}{\rho(s)}\right)\sqrt{(1+\delta)^2 - (p_x - e\hat{A}_x)^2 - (p_y - e\hat{A}_y)^2}$$

where the "hats" are dropped for simplicity

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  where the "hats" are dropped for simplicity
- If we consider **only transverse field** components, the **vector potential** has **only** a **longitudinal** component and the Hamiltonian is written as

$$\mathcal{H}(x,y,l,p_x,p_y,\delta) = (1+\delta) - e\hat{A}_s - \left(1+\frac{x}{\rho(s)}\right)\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}$$

- Note that the Hamiltonian is non-linear even in the absence of any field component (i.e. for a drift)!

- Summary of canonical transformations and approximations
  - From Cartesian to Frenet-Serret (rotating) coordinate system (bending in the horizontal plane)
  - Changing the independent variable from time to the path length s
  - Electric field set to zero, as longitudinal (synchrotron) motion is much slower then transverse (betatron) one
  - Consider static and transverse magnetic fields
  - Rescale the momentum and move the origin to the periodic orbit
  - For the ultra-relativistic limit $\beta_0 \to 1\ ,\quad \frac{1}{\beta_0^2\gamma^2} \to 0$ the Hamiltonian becomes

$$\mathcal{H}(x, y, l, p_x, p_y, \delta) = (1+\delta) - e\hat{A}_s - \left(1 + \frac{x}{\rho(s)}\right)\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}$$

with $\frac{P_t - P_0}{P_0} \equiv \delta$

High-energy, large ring approximation

- It is useful for study purposes (especially for finding an "integrable" version of the Hamiltonian) to make an extra approximation
- For this, **transverse momenta** (rescaled to the reference momentum) are considered to be **much smaller** than **1**, i.e. the square root can be expanded.
- Considering also the large machine approximation $x << \rho$, (dropping cubic terms), the Hamiltonian is simplified to

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2(1+\delta)} - \frac{x(1+\delta)}{\rho(s)} - e\hat{A}_s$$

- This expansion may **not** be **a good idea**, especially for **low energy**, **small** size **rings**

Linear magnetic fields

- Assume a simple case of **linear transverse magnetic fields,**

$$\begin{aligned} B_x &= \quad b_1(s)y \\ B_y &= -b_0(s) + b_1(s)x \end{aligned},$$

  - main bending field $\quad -B_0 \equiv b_0(s) = \frac{P_0 c}{e\rho(s)} \; [\mathrm{T}]$
  - normalized quadrupole gradient $\quad K(s) = b_1(s)\frac{e}{cP_0} = \frac{b_1(s)}{B\rho} \; [1/\mathrm{m}^2]$
  - magnetic rigidity $\quad B\rho = \dfrac{P_0 c}{e} \; [\mathrm{T}\cdot\mathrm{m}]$

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  - magnetic rigidity $\quad B\rho = \dfrac{P_0 c}{e} \ [\mathrm{T} \cdot \mathrm{m}]$

- The vector potential has only a longitudinal component which in curvilinear coordinates is

$$B_x = -\frac{1}{1+\frac{x}{\rho(s)}} \frac{\partial A_s}{\partial y} \ , \quad B_y = \frac{1}{1+\frac{x}{\rho(s)}} \frac{\partial A_s}{\partial x}$$

- The previous expressions can be integrated to give

$$A_s(x,y,s) = \frac{P_0 c}{e} \left[-\frac{x}{\rho(s)} - \left(\frac{1}{\rho(s)^2} + K(s) \right) \frac{x^2}{2} + K(s) \frac{y^2}{2} \right] = P_0 c \, \hat{A}_s(x,y,s)$$

The integrable Hamiltonian

- The Hamiltonian for linear fields can be finally written as

$$\mathcal{H} = \frac{p_x^2+p_y^2}{2(1+\delta)} - \frac{x\delta}{\rho(s)} + \frac{x^2}{2\rho(s)^2} + \frac{K(s)}{2}(x^2 - y^2)$$

- Hamilton's equation are

$$\frac{dx}{ds} = \frac{p_x}{1+\delta}, \quad \frac{dp_x}{ds} = \frac{\delta}{\rho(s)} - \left(\frac{1}{\rho^2(s)} + K(s)\right)x$$

$$\frac{dy}{ds} = \frac{p_y}{1+\delta}, \quad \frac{dp_y}{ds} = K(s)y$$

and they can be written as two second order uncoupled differential equations, i.e. **Hill's equations**

$$x'' + \frac{1}{1+\delta}\overbrace{\left(\frac{1}{\rho(s)^2} + K(s)\right)}^{K_x} x = \frac{\delta}{\rho(s)}$$

$$y'' - \frac{1}{1+\delta}\underbrace{K(s)}_{K_y} y = 0$$

with the usual solution for $\delta = 0$ and $u = x, y$

$$u(s) = \sqrt{\epsilon\beta(s)}\cos(\psi(s) + \psi_0)$$

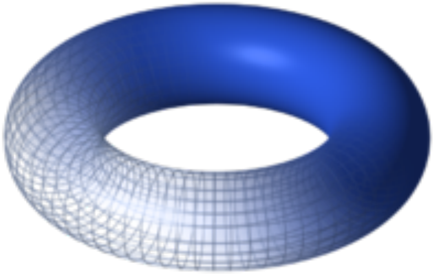
$$u'(s) = \sqrt{\frac{\epsilon}{\beta(s)}}\left(\sin(\psi(s) + \psi_0) + \alpha(s)\cos(\psi(s) + \psi_0)\right)$$

Action-angle variables

- There is a canonical transformation to some **optimal set** of variables which can simplify the phase-space motion
- This set of variables are the **action-angle** variables
- The action vector is defined as the integral $\mathbf{J} = \oint \mathbf{p} d\mathbf{q}$ over closed paths in phase space.
- An **integrable Hamiltonian** is written as a function of only the actions, i.e. $H_0 = H_0(\mathbf{J})$. Hamilton's equations give

$$\dot{\phi}_i = \frac{\partial H_0(\mathbf{J})}{\partial J_i} = \omega_i(\mathbf{J}) \Rightarrow \phi_i = \omega_i(\mathbf{J})t + \phi_{i0}$$

$$\dot{J}_i = -\frac{\partial H_0(\mathbf{J})}{\partial \phi_i} = 0 \Rightarrow J_i = \text{const.}$$



i.e. the **actions are integrals of motion** and the **angles** are **evolving linearly with time**, with **constant frequencies** which depend on the actions

- The actions define the surface of an **invariant torus**, topologically equivalent to the product of n circles

Accelerator Hamiltonian in action-angle variables

- Considering **on-momentum** motion, the Hamiltonian can be written as

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2} + \frac{K_x(s)x^2 - K_y(s)y^2}{2}$$

- The generating function from the original to action angle variables is

$$F_1(x, y, \phi_x, \phi_y; s) = -\frac{x^2}{2\beta_x(s)}\left[\tan\phi_x(s) + a_x(s)\right] - \frac{y^2}{2\beta_y(s)}\left[\tan\phi_y(s) + a_y(s)\right]$$

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- The **old variables** with respect to **actions and angles** are

$$u(s) = \sqrt{2\beta_u(s)J_u}\cos\phi_u(s)\ ,\quad p_u(s) = -\sqrt{\frac{2J_u}{\beta_u(s)}}\left(\sin\phi_u(s) + \alpha_u(s)\cos\phi_u(s)\right)$$

and the Hamiltonian takes the form

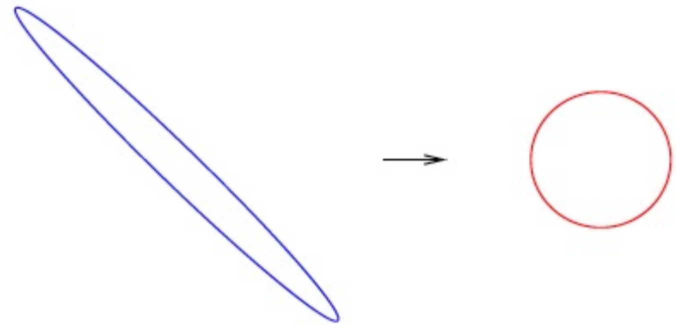
$$\mathcal{H}_0(J_x, J_y, s) = \frac{J_x}{\beta_x(s)} + \frac{J_y}{\beta_y(s)}$$

- The "time" (longitudinal position) dependence can be eliminated by the transformation to **normalized coordinate**

$$\begin{pmatrix}\mathcal{U}\\ \mathcal{U}'\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{\beta}} & 0\\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta}\end{pmatrix}\begin{pmatrix}u\\ u'\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}\mathcal{U}\\ \mathcal{U}'\end{pmatrix} = \sqrt{2J}\begin{pmatrix}\cos(\nu\phi)\\ \sin(\nu\phi)\end{pmatrix} \text{ with } \nu = \frac{1}{2\pi}\oint\frac{du}{\beta(s)}$$

Linear normal forms

- Make a coordinate transformation so that we get a simpler form of the matrix, i.e. **ellipses** are transformed to circles (simple rotation)



$$\mathcal{M} = \mathcal{A} \circ \mathcal{R} \circ \mathcal{A}^{-1} \quad \text{or :} \quad \mathcal{R} = \mathcal{A}^{-1} \circ \mathcal{M} \circ \mathcal{A}$$

- Using linear algebra, the solution is

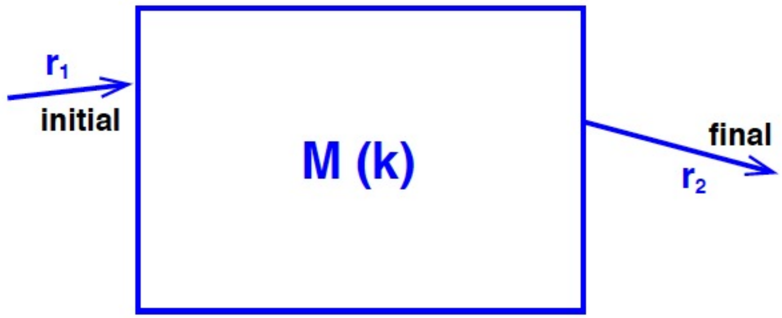
$$\mathcal{A} = \begin{pmatrix} \sqrt{\beta(s_0)} & 0 \\ -\frac{\alpha(s_0)}{\sqrt{\beta(s_0)}} & \frac{1}{\sqrt{\beta(s_0)}} \end{pmatrix} \quad \text{and} \quad \mathcal{R} = \begin{pmatrix} \cos(\mu_x) & \sin(\mu_x) \\ -\sin(\mu_x) & \cos(\mu_x) \end{pmatrix}$$

- This transformation can be extended to a non-linear system (see Advanced course)

Symplectic maps

Maps

- A generalization of the matrix (which can only describe linear systems), is a **map**, which transforms a system from some initial to some final coordinates



- Analyzing the map, will give useful information about the behavior of the system
- There are different ways to build the map:
  - Taylor (Power) maps
  - Lie transformations
  - Truncated Power Series Algebra (TPSA), can generate maps from straight-forward tracking
- Preservation of **symplecticity** is important

Symplectic maps

- Consider two sets of canonical variables $\mathbf{z}$, $\bar{\mathbf{z}}$ which may be even considered as the evolution of the system between two points in phase space
- A transformation from the one to the other set can be constructed through a **map** $\mathcal{M} : \mathbf{z} \mapsto \bar{\mathbf{z}}$
- The **Jacobian matrix** of the map $M = M(\mathbf{z}, t)$ is composed by the elements $M_{ij} \equiv \frac{\partial \bar{z}_i}{\partial z_j}$
- The map is **symplectic** if $M^T J M = J$ where $J = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$
- It can be shown that $\det(M) = 1$
- It can be shown that the variables defined through a symplectic map $[\bar{z}_i, \bar{z}_j] = [z_i, z_j] = J_{ij}$ which is a known relation satisfied by canonical variables
- In other words, symplectic maps **preserve** Poisson brackets

Why symplecticity is important

- **Symplecticity** guarantees that the **transformations** in phase space are **area preserving**
- To understand what deviation from symplecticity produces consider the simple case of the **quadrupole** with the general matrix written as

$$\mathcal{M}_{\rm Q} = \begin{pmatrix} \cos(\sqrt{k}L) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}L) \\ -\sqrt{k}\sin(\sqrt{k}L) & \cos(\sqrt{k}L) \end{pmatrix}$$

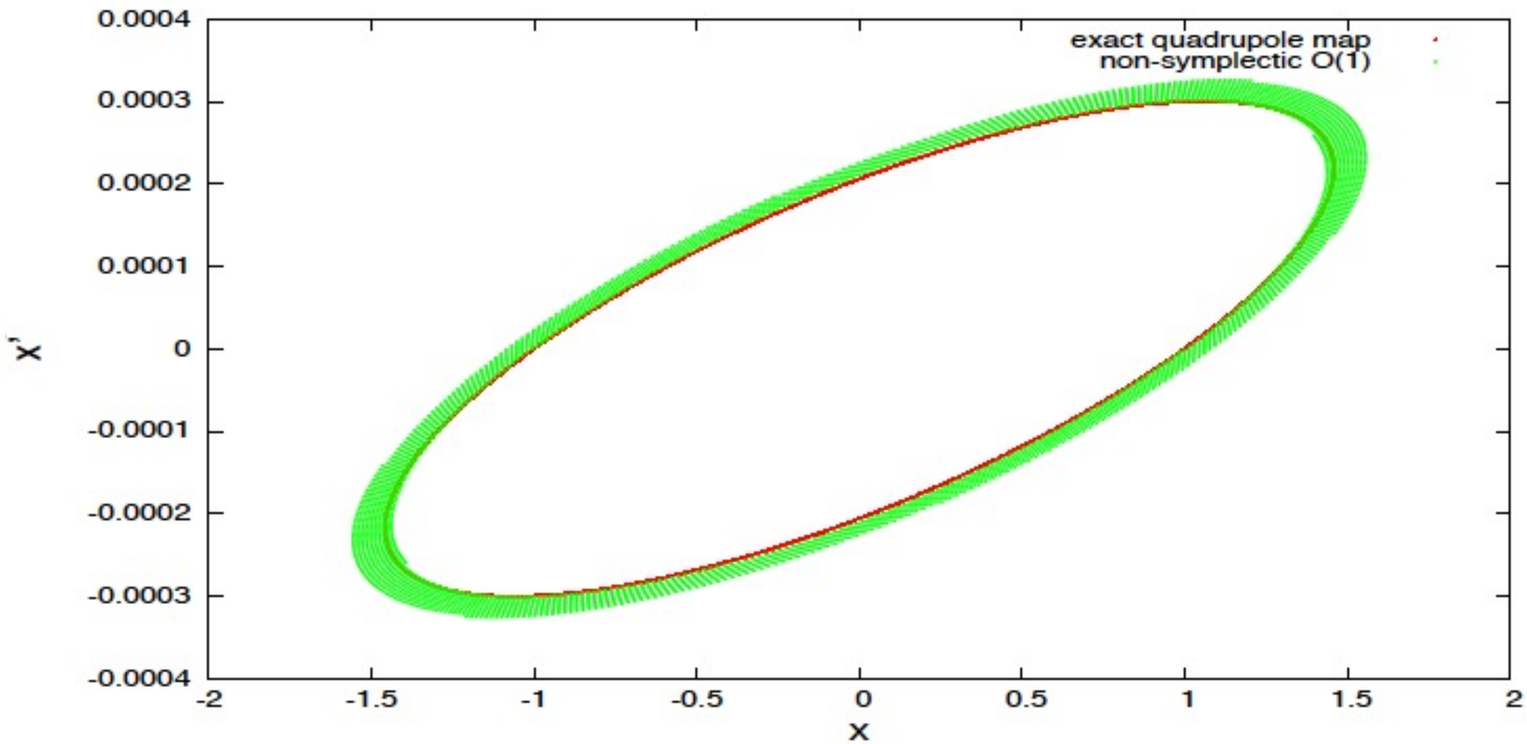
- Take the Taylor expansion for small lengths, up to first order

$$\mathcal{M}_{\rm Q} = \begin{pmatrix} 1 & L \\ -kL & 1 \end{pmatrix} + O(L^2)$$

- This is indeed **not symplectic** as the determinant of the matrix is equal to $1 + kL^2$, i.e. there is a deviation from symplecticity at 2nd order in the quadrupole length

Phase portrait for non-symplectic matrix

- The iterated non-symplectic matrix does not provide the well-know elliptic trajectory in phase space
- Although the trajectory is very close to the original one, it **spirals outwards towards infinity**



Lie formalism

- The Poisson bracket properties satisfy what is mathematically called a **Lie** algebra
- They can be represented by (Lie) operators of the form

$$:f:g = [f,g] \quad \text{and} :f:^2 g = [f,[f,g]] \quad \text{etc.}$$

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- For a Hamiltonian system $H(\mathbf{z},t)$ there is a **formal solution** of the equations of motion $\frac{d\mathbf{z}}{dt} = [H,\mathbf{z}] =: H:\mathbf{z}$ written as $\mathbf{z}(t) = \sum_{k=0}^{\infty} \frac{t^k :H:^k}{k!}\mathbf{z}_0 = e^{t:H:}\mathbf{z}_0$ with a symplectic map $\mathcal{M} = e^{:H:}$

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- The 1-turn accelerator map can be represented by the composition of the maps of each element $\mathcal{M} = e^{:f_2:}\, e^{:f_3:}\, e^{:f_4:} \ldots$ where f_i (called the generator) is the Hamiltonian for each element, a polynomial of degree m in the variables $z_1, \ldots, z_n$

General non-linear Accelerator Hamiltonian

- Considering the **general expression** of the the **longitudinal component** of the **vector potential** is (see appendix)
  - In curvilinear coordinates (curved elements)

$$A_s = (1+\frac{x}{\rho(s)})B_0 \Re e \sum_{n=0}^{\infty} \frac{b_n + i a_n}{n+1}(x+iy)^{n+1}$$

  - In Cartesian coordinates $A_s = B_0 \Re e \sum_{n=0}^{\infty} \frac{b_n + i a_n}{n+1}(x+iy)^{n+1}$

  with the **multipole coefficients** being written as

$$a_n = \frac{1}{B_0\, n!} \frac{\partial^n B_x}{\partial x^n}\Big|_{x=y=0} \quad \text{and} \quad b_n = \frac{1}{B_0\, n!} \frac{\partial^n B_y}{\partial x^n}\Big|_{x=y=0}$$

- The **general non-linear Hamiltonian** can be written as

$$\mathcal{H}(x,y,p_x,p_y,s) = \mathcal{H}_0(x,y,p_x,p_y,s) + \sum_{k_x,k_y} h_{k_x,k_y}(s) x^{k_x} y^{k_y}$$

with the **periodic functions** $h_{k_x,k_y}(s) = h_{k_x,k_y}(s+C)$

Magnetic element Hamiltonians

- Dipole:

$$H = \frac{x\delta}{\rho} + \frac{x^2}{2\rho^2} + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

- Quadrupole:

$$H = \frac{1}{2}k_1(x^2 - y^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

- Sextupole:

$$H = \frac{1}{3}k_2(x^3 - 3xy^2) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

- Octupole:

$$H = \frac{1}{4}k_3(x^4 - 6x^2y^2 + y^4) + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

Map for quadrupole

- Consider the 1D quadrupole Hamiltonian

$$H = \frac{1}{2}(k_1 x^2 + p^2)$$

- For a quadrupole of length L, the map is written as

$$e^{\frac{L}{2}:(k_1 x^2 + p^2):}$$

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- Its application to the transverse variables is

$$e^{-\frac{L}{2}:(k_1 x^2+p^2):} x = \sum_{n=0}^{\infty}\left(\frac{(-k_1 L^2)^n}{(2n)!} x + L\frac{(-k_1 L^2)^n}{(2n+1)!} p\right)$$

$$e^{-\frac{L}{2}:(k_1 x^2+p^2):} p = \sum_{n=0}^{\infty}\left(\frac{(-k_1 L^2)^n}{(2n)!} p - \sqrt{k_1}\frac{(-k_1 L^2)^n}{(2n+1)!} p\right)$$

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$$e^{-\frac{L}{2}:(k_1 x^2 + p^2):} p = \sum_{n=0}^{\infty} \left(\frac{(-k_1 L^2)^n}{(2n)!} p - \sqrt{k_1} \frac{(-k_1 L^2)^n}{(2n+1)!} p \right)$$

- This finally provides the usual quadrupole matrix

$$e^{-\frac{L}{2}:(k_1 x^2 + p^2):} x = \cos(\sqrt{k_1} L) x + \frac{1}{\sqrt{k_1}} \sin(\sqrt{k_1} L) p$$

$$e^{-\frac{L}{2}:(k_1 x^2 + p^2):} p = -\sqrt{k_1} \sin(\sqrt{k_1} L) x + \cos(\sqrt{k_1} L) p$$

Appendix

Magnetic multipole expansion

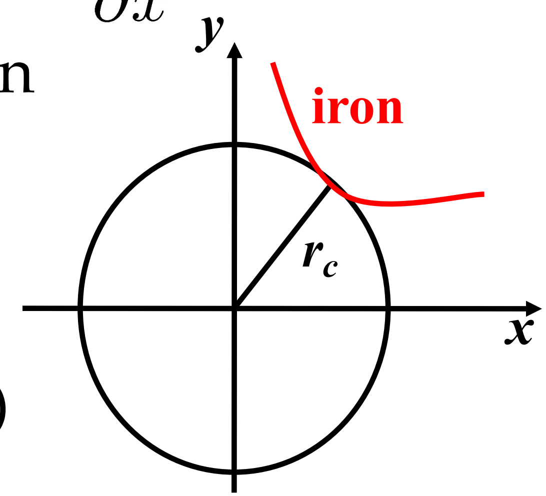
- From Gauss law of magnetostatics, a vector potential exist

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \exists \mathbf{A} : \quad \mathbf{B} = \nabla \times \mathbf{A}$$

- Assuming transverse 2D field, vector potential has only one component A_s. The Ampere's law in vacuum (inside the beam pipe) $\nabla \times \mathbf{B} = 0 \quad \rightarrow \quad \exists V : \quad \mathbf{B} = -\nabla V$
- Using the previous equations, the relations between field components and potentials are

$$B_x = -\frac{\partial V}{\partial x} = \frac{\partial A_s}{\partial y}, \quad B_y = -\frac{\partial V}{\partial y} = -\frac{\partial A_s}{\partial x}$$

i.e. Riemann conditions of an analytic function



Exists complex potential of $z = x + iy$ with power series expansion convergent in a circle with radius $|z| = r_c$ (distance from iron yoke)

$$\mathcal{A}(x+iy) = A_s(x,y) + iV(x,y) = \sum_{n=1}^{\infty} \kappa_n z^n = \sum_{n=1}^{\infty} (\lambda_n + i\mu_n)(x+iy)^n$$

Multipole expansion II

- From the complex potential we can derive the fields

$$B_y + iB_x = -\frac{\partial}{\partial x}(A_s(x,y) + iV(x,y)) = -\sum_{n=1}^{\infty} n(\lambda_n + i\mu_n)(x+iy)^{n-1}$$

- Setting $b_n = -n\lambda_n \ , \quad a_n = n\mu_n$

$$B_y + iB_x = \sum_{n=1}^{\infty}(b_n - ia_n)(x+iy)^{n-1}$$

- Define normalized coefficients

$$b'_n = \frac{b_n}{10^{-4}B_0} r_0^{n-1} \ , \ a'_n = \frac{a_n}{10^{-4}B_0} r_0^{n-1}$$

on a reference radius r_0, 10^{-4} of the main field to get

$$B_y + iB_x = 10^{-4}B_0 \sum_{n=1}^{\infty}(b'_n - ia'_n)\left(\frac{x+iy}{r_0}\right)^{n-1}$$

- **Note:** $n' = n - 1$ is the US convention