The CERN Accelerator School

CAS – Introduction to Accelerator Physics

Collective effects

Part I: Multiparticle systems

Context and outline

Collective effects in the physics of particle accelerators and beam dynamics is yet another very important topic, as it determines the **ultimate performance of many machines**. These effects become increasingly important as the beam intensities are pushed towards the limits.

In terms of the physics, they are challenging to deal with due to **their self-consistent nature** – instead of having to handle the particle dynamics within a fixed environment, the particle distribution actually affects and changes the environment, which in turn impacts the particle distribution and so on.

We will have a look at some important collective effects mostly phenomenologically, trying to give an intuitive picture and showing some real world examples. The main topics that we will discuss are:

- Multiparticle dynamics
  - Moving from single particles to multiparticle systems
- Space charge effects
  - Direct and indirect and space charge
- Wake fields
  - Longitudinal and transverse wake fields and impedances
- Instabilities
  - Coupled bunch and single bunch instabilities in the transverse and the longitudinal planes

We will briefly revise the single particle representation and dynamics and then move **to multi-particle systems** and **their representation**. We will discuss some features of **multi-particle dynamics in absence of collective** effects such as decoherence and filamentation.

Finally, we will look **at direct space charge** as a first real collective effects.

- Part I: Multi-particle effects – direct space charge
  - Multi-particle systems and their representation
  - Incoherent and coherent motion
  - Space charge

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- Part I: Multi-particle effects – direct space charge
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Single particle dynamics

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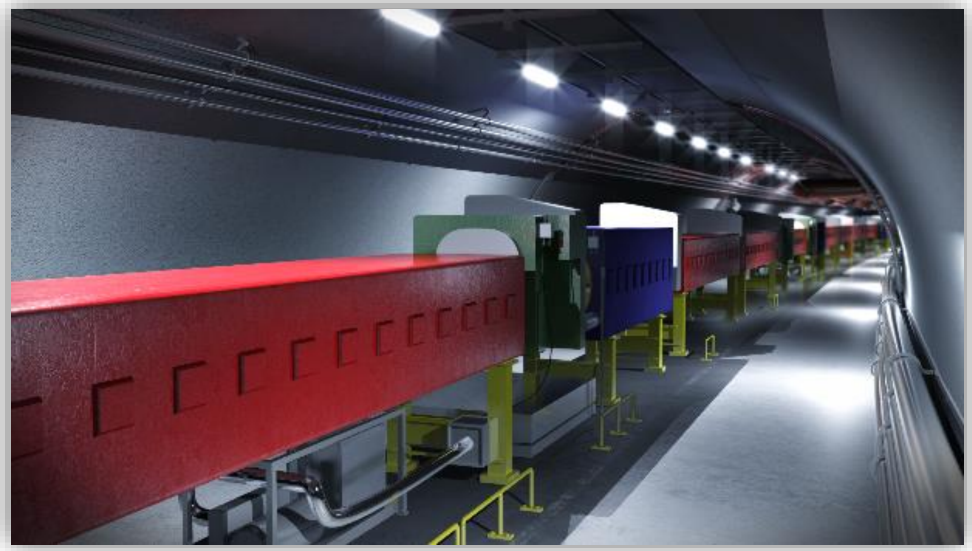
- We have already seen and learned about single particle dynamics…

- We can say that single particle dynamics treats the interaction of **individual particles with external force fields** generated by machine elements:

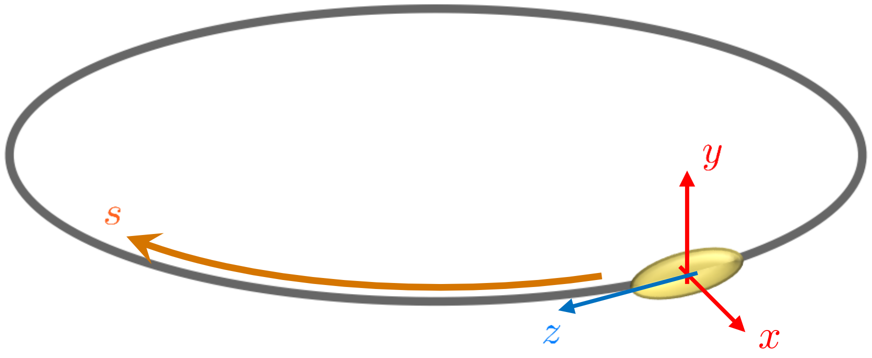
  - Slow magnets to generate guiding fields
  - Fast magnets for injection and extraction
  - RF cavities for bunching and shaping of longitudinal phase space
  - …



- Characteristics of single particle dynamics:

  - External force fields
  - Independent of any given phase space configuration of multi-particle ensembles!

Single particle dynamics – reminder coordinates



- Accelerator coordinates:
  - **Position along the accelerator**

    $$s$$

  - **Accelerator circumference**

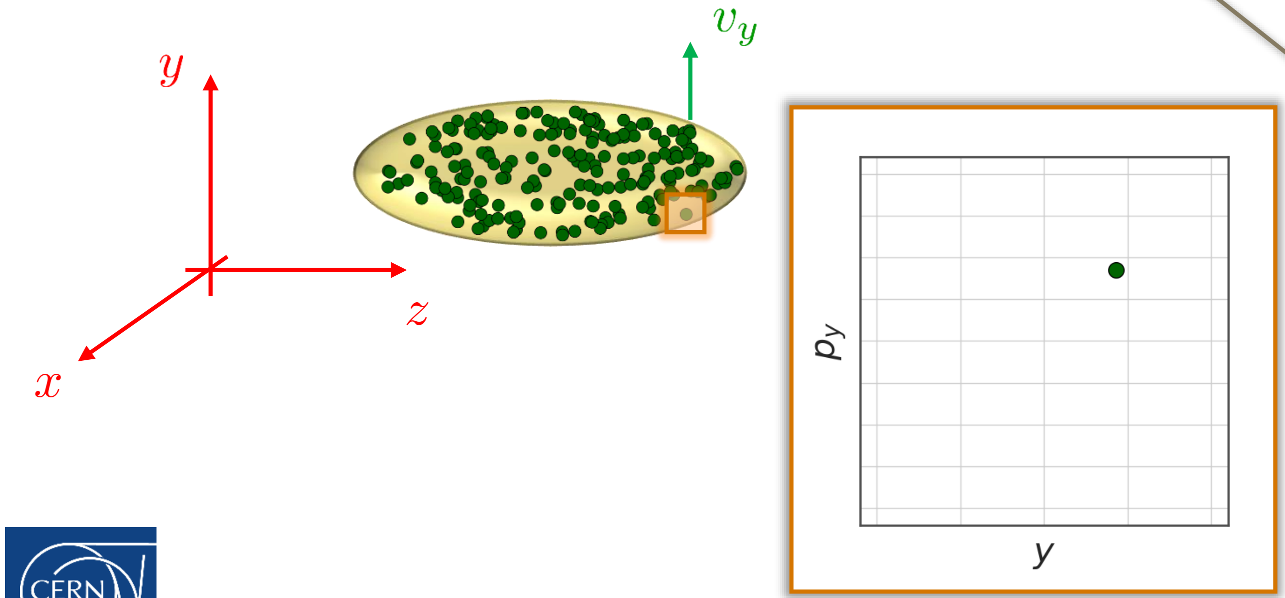
    $$C$$

- Bunch coordinates
  - **Position transverse with respect to orbit**

    $$(x, y)$$

  - **Position longitudinal with respect to reference (synchronous) particle**

    $$z$$

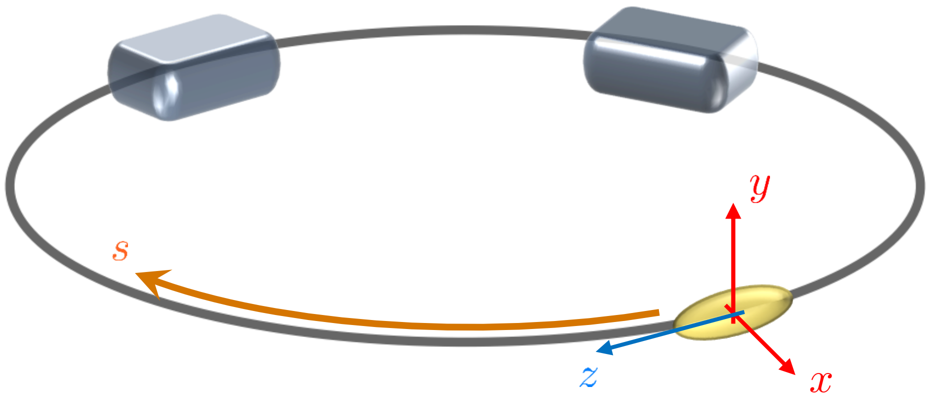


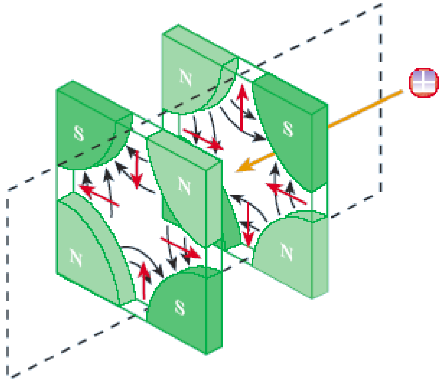
- Phase space coordinates
  - **Representation of particles as unique state in phase space**

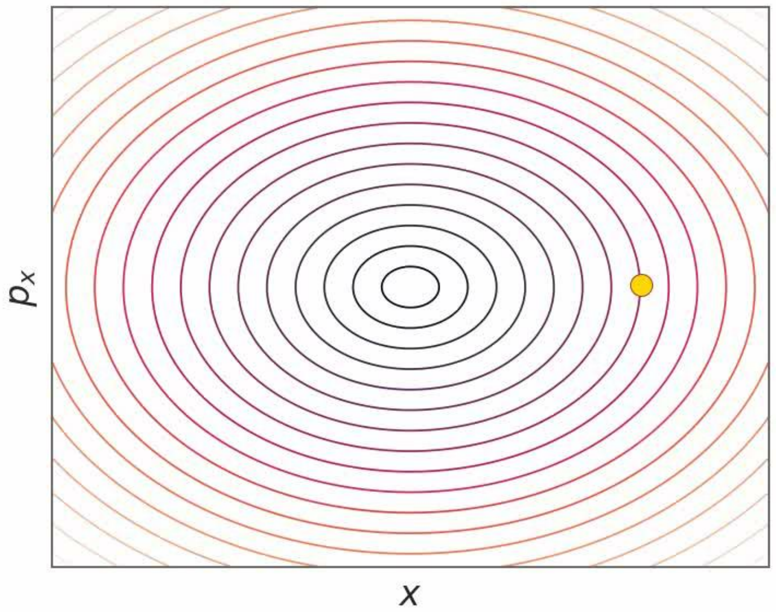
$$\left[\begin{pmatrix} x \\ p_x \end{pmatrix}, \begin{pmatrix} y \\ p_y \end{pmatrix}, \begin{pmatrix} z \\ p_z \end{pmatrix}\right] \in \Gamma \cong \mathbb{R}^6$$

Single particle dynamics – transverse

- Characteristics of single particle dynamics:
  - External force fields
  - Independent of any given multi-particle system phase space configuration!





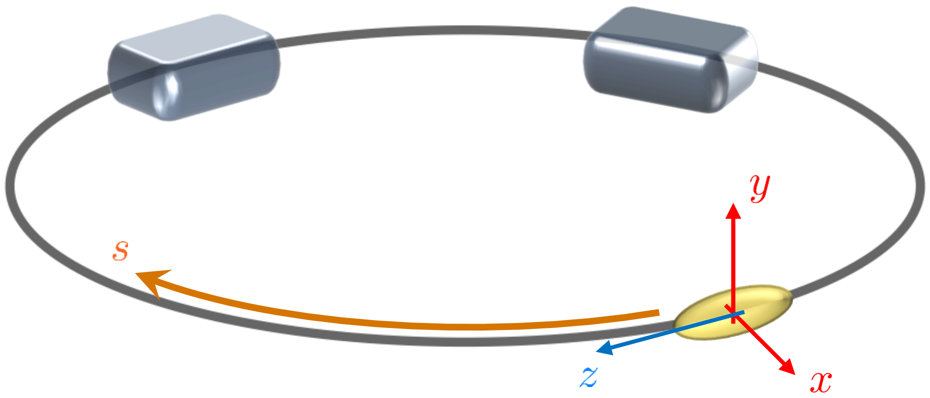


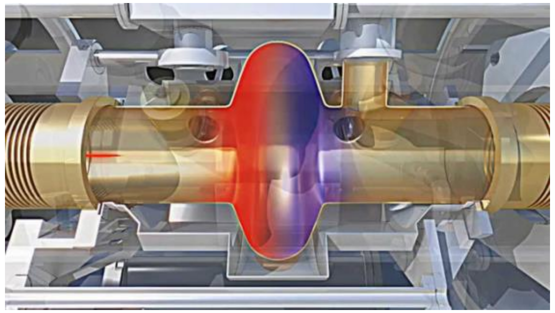
- Examples of single particle dynamics:
  - Transverse focusing → Hill's equation and **betatron motion**

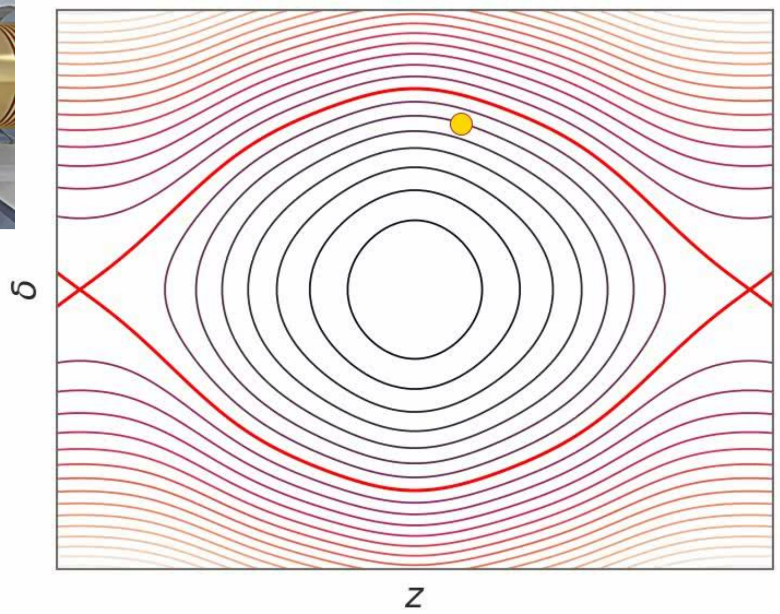
$x'' - K^2(s)\,x^2 = 0$ where $K(s) = K(s+C)$, with the solution $\begin{cases} x = \sqrt{2J\beta_x(s)}\cos\left[\psi(s)\right] \\ x' = -\sqrt{\dfrac{2J}{\beta_x(s)}}\left(\sin\left[\psi(s)\right] - \dfrac{\beta_x(s)'}{2}\cos\left[\psi(s)\right]\right) \end{cases}$

Single particle dynamics – longitudinal

- Characteristics of single particle dynamics:
  - External force fields
  - Independent of any given multi-particle system phase space configuration!







- Examples of single particle dynamics:
  - Longitudinal focusing → Pendulum equation and **synchrotron motion**

$$z' = -\eta\,\delta$$

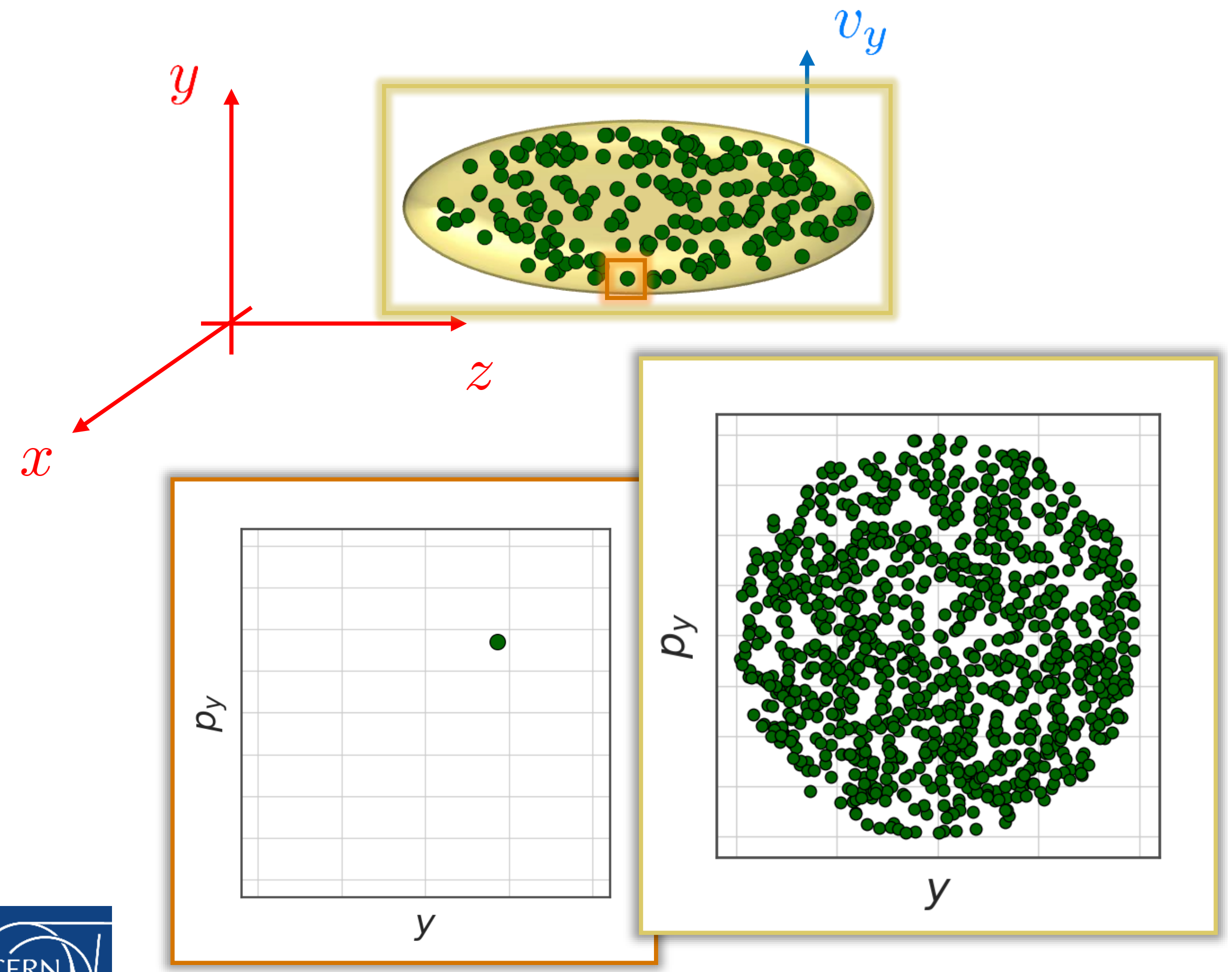
$$\delta' = \frac{e\,V_{\mathrm{RF}}}{m\gamma\beta^2c^2\,C}\sin\left(\frac{2\pi h}{C}\,z\right)$$

- V_{RF}: RF voltage
- $h = \frac{\omega_{\mathrm{RF}}}{\omega_0}$: harmonic number
- ω_0: Revolution frequency
- C: circumference

Multi-particle dynamics – state representation (theory)

- Representation of a **single particle state**:

$$(\vec{q}, \vec{p})_1 = (x_1, p_{x1}, y_1, p_{y1}, z_1, p_{z1})$$



- Representation of a **multi-particle state**:

$$\begin{aligned}(\vec{q}, \vec{p})_N = (&x_1, p_{x1}, y_1, p_{y1}, z_1, p_{z1},\\ &x_2, p_{x2}, y_2, p_{y2}, z_1, p_{z2},\\ &\ldots\ldots\\ &x_N, p_{xN}, y_1, p_{yN}, z_N, p_{zN})\end{aligned}$$

- The multi-particle state is conveniently represented by a **probability density function Ψ** – which neglecting correlations can be reduced (BBGKY) to the single **particle distribution function**

The probability P (at any time t) to find a given particle at state $(\vec{q}, \vec{p})$:

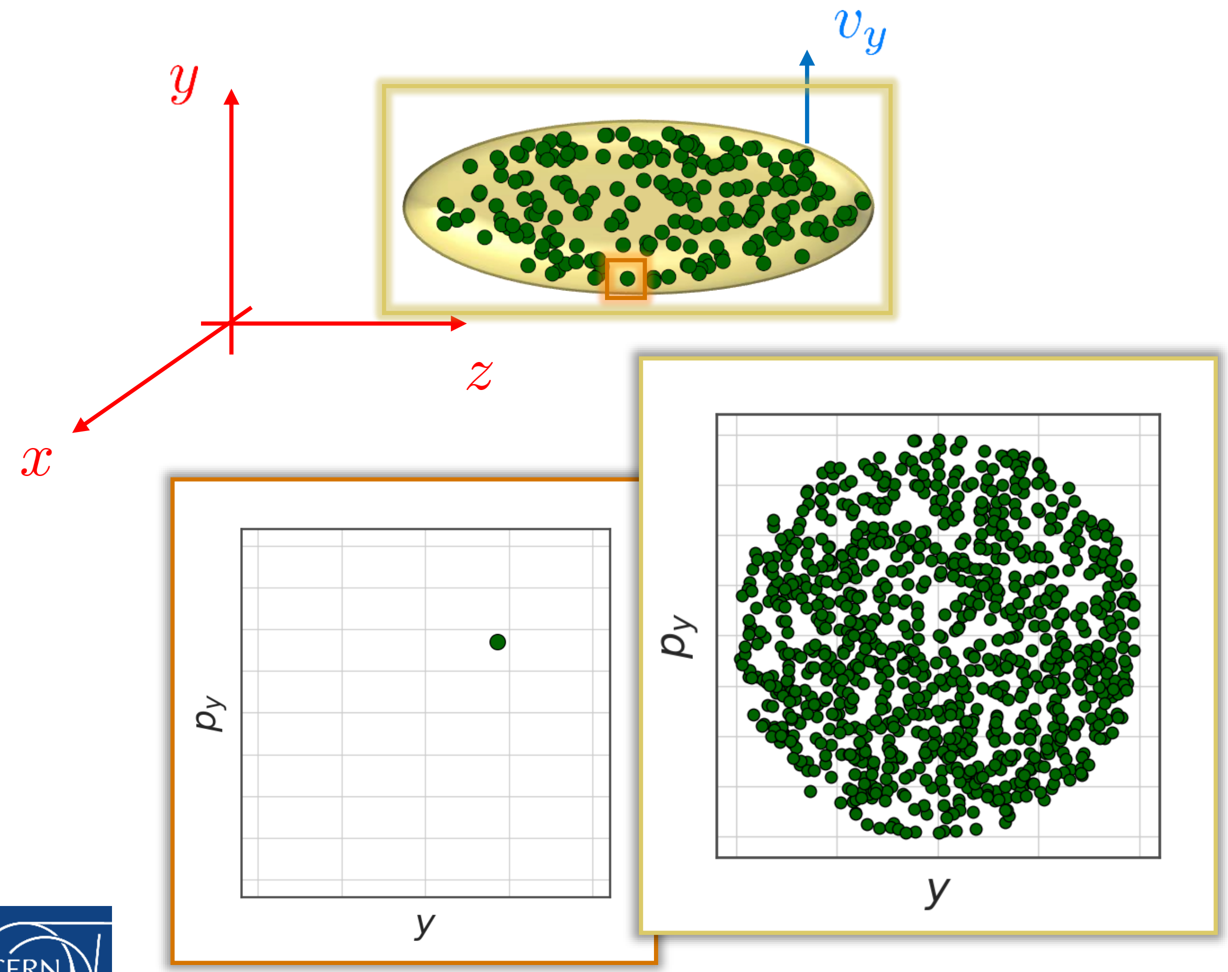
$$P\Big|_{(\vec{q},\vec{p});t} = \frac{1}{N}\,\psi(\vec{q}, \vec{p}, t)$$

Normalization: $1 = \frac{1}{N}\int \psi(\vec{q}, \vec{p}, t)\, d\vec{q}d\vec{p}$

Multi-particle dynamics – state representation (numerics)

- Representation of a **single particle state**:

$$(\vec{q}, \vec{p})_1 = (x_1, p_{x1}, y_1, p_{y1}, z_1, p_{z1})$$




```
In [6]: df = pd.DataFrame(bunch.get_coords_n_momenta_dict())
        df
```

Out[6]:

	dp	x	xp	y	yp	z
0	0.001590	0.000566	-2.285393e-05	-0.001980	4.283152e-06	0.353427
1	0.001978	0.000370	1.954404e-05	-0.000359	5.543904e-05	0.159670
2	0.003492	-0.000829	-2.773707e-05	0.000291	6.627340e-05	-0.251489
3	0.002195	-0.001668	-2.317633e-05	0.001878	-1.870926e-05	-0.038597
4	0.000572	0.000990	5.493907e-05	0.000152	-1.951051e-05	0.492968
5	-0.000418	0.001088	4.778027e-05	0.003320	-7.716856e-06	0.415582
6	-0.000114	-0.000194	1.065400e-05	0.001798	-4.984276e-07	-0.349064
7	0.001100	-0.001257	-6.873217e-05	-0.002374	5.657645e-06	-0.023157
8	0.002706	0.005351	-1.867898e-07	-0.000765	3.012523e-05	-0.291095
9	0.003508	0.000499	1.865768e-05	-0.001032	-5.363820e-05	0.211726
10	-0.001711	-0.003168	4.372560e-05	-0.001933	-2.151020e-05	-0.145358
11	-0.002150	-0.000565	-1.853825e-05	-0.003895	-6.192450e-06	0.072499
12	0.002059	0.003453	-3.808703e-05	0.000118	3.179588e-05	-0.001816
13	0.002709	0.000241	-3.457535e-05	0.000474	5.057865e-05	-0.005464
14	-0.001593	0.000711	-1.667091e-05	-0.002523	-3.804168e-05	[illegible]
15	-0.000830	-0.000393	-7.473946e-05	[illegible]	[illegible]	[illegible]
16	-0.001743	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Multi-particle dynamics – state representation

We have seen three ways of representing multi-particle states:

- **Graphically →** best for intuition
- **Theoretically →** **very good for analytical calculations to gain theoretical insights and, e.g., scaling laws**
- **Numerically →** **very good to deploy in numerical simulations to study generic and realistic cases**

Let's continue with just few more words on the theoretical description of multi-particle states before moving back to more practical examples.

Multi-particle dynamics – state representation

A multi-particle ensemble is characterized by its **macroscopic statistical properties**, i.e.:

$$N = \int \psi\left(\vec{q}, \vec{p}\right)\, d\vec{q}d\vec{p}$$

$$\langle x \rangle = \frac{1}{N} \int x \cdot \psi\left(\vec{q}, \vec{p}\right)\, d\vec{q}d\vec{p}$$

$$\sigma_x^2 = \frac{1}{N} \int \left(x - \langle x \rangle\right)^2 \cdot \psi\left(\vec{q}, \vec{p}\right)\, d\vec{q}d\vec{p}$$

with similar definitions for $\langle y \rangle, \sigma_y, \langle z \rangle, \sigma_z$

One of the most important parameters to characterize the quality of a particle bunch is the **(normalized) statistical emittance** – it is a measure of the particle bunch size in phase space.

$$\varepsilon_x^{n,\mathrm{rms}} = \sqrt{\sigma_x^2 \sigma_{p_x}^2 - \sigma_x \sigma_{p_x}}$$

Multi-particle dynamics – Hamilton equations of motion

Miraculously, the motion of particles in a particle accelerator happens according to the **Hamilton equations of motion**:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}\,, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

→ sometimes also more compactly written as

$$\frac{d}{dt}\vec{x} = J\,\nabla_{\vec{x}} H\,, \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

This has important consequences on the possible degrees of freedom for the motion of any particle ensemble in phase space – or also the evolution of the emittance of a particle bunch…

Remark: multi-particle dynamics – Liouville's theorem

… in an accelerator environment, the **multi-particle system moves in phase space like an incompressible fluid**

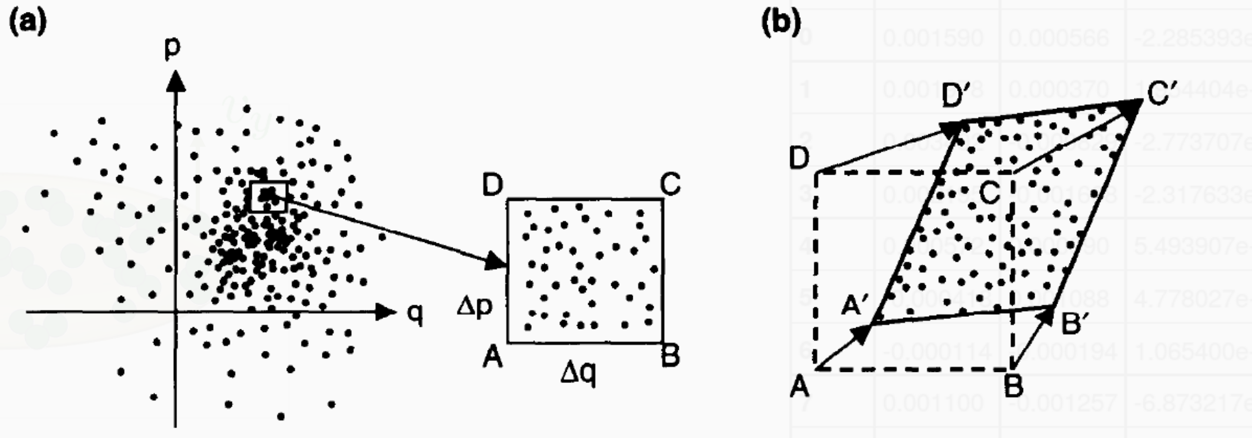


Figure 6.3. (a) Phase space distribution of particles at time t. A rectangular box $ABCD$ with area $\Delta q\,\Delta p$ is drawn and magnified. (b) At a later time, $t + dt$, the box moves and deforms into a parellelogram with the same area as $ABCD$. All particles inside the box move with the box.

Liouville's Theorem*:

$$\frac{\partial \Psi}{\partial t} + \sum_i^N \left(\frac{\partial \Psi}{\partial q_i} \frac{\partial q_i}{\partial t} + \frac{\partial \Psi}{\partial p_i} \frac{\partial p_i}{\partial t} \right) = 0$$

i.e., the occupied phase space density remains constant.

(*) Follows from just two physics assumptions:

1. Conservation of number of systems
2. Hamilton equations of motion

See, e.g.,
https://cds.cern.ch/record/572817?ln=de or
http://www.damtp.cam.ac.uk/user/tong/dynamics/four.pdf

Remark: multi-particle dynamics – Liouville's theorem

- Re[presentation of a multi-particle ensemble in] ph[ase space]

… in an accelerator environment, the **multi-particle system moves in phase space like an incompressible fluid**

From Liouville's theorem, we can immediately write down the **Vlasov equation** which often forms the basis of any analytical computations of collective effects:

$$\frac{\partial \Psi}{\partial t} = \{H, \Psi\} = \{H_{\text{ext}} + H_{\text{coll}}, \Psi_0 + \Psi_{\text{pert}}\}$$

- H: Hamiltonian
- Ψ: Single particle probablity density function
- $\{\cdot,\cdot\}$: Poisson bracket

If we know these **magic functions H** (and we in fact mostly do), most of the collective behavior of the beam can be calculated (see, e.g., N. Mounet "Direct Vlasov Solvers" at the Numerical Methods for Analysis, Design and Modelling of Particle Accelerators, Thessaloniki (Greece)).

Liouville's Theorem*:

$$\frac{\partial \Psi}{\partial t} + \sum_i^N \left(\frac{\partial \Psi}{\partial q_i} \frac{\partial q_i}{\partial t} + \frac{\partial \Psi}{\partial p_i} \frac{\partial p_i}{\partial t} \right) = 0$$

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https://cds.cern.ch/record/572817?ln=de or
http://www.damtp.cam.ac.uk/user/tong/dynamics/four.pdf

We have very briefly reviewed single particle representation and dynamics in a particle accelerator. We have then introduced the concept of **multi-particle systems** and their **representation in phase space**.

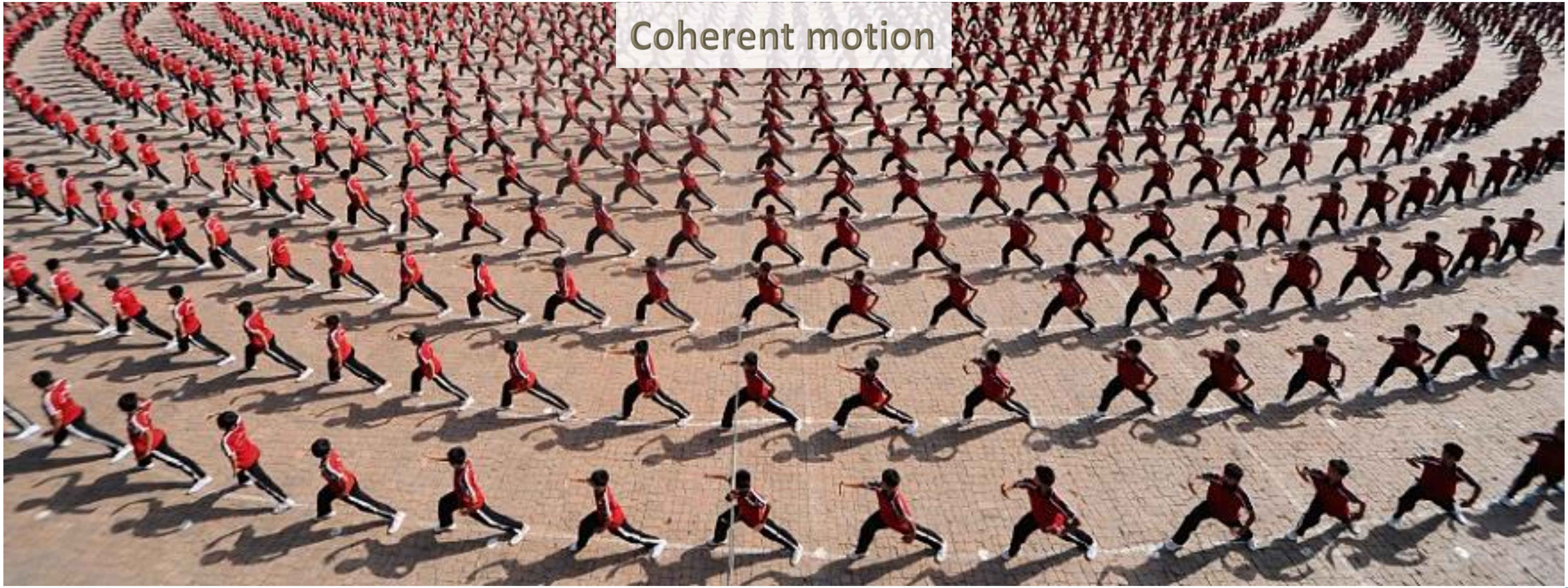
Two very fundamental and important points are the **definition of the statistical emittance** as a measure of the particle bunch quality and **Liouville's theorem** which describes the motion of a multi-particle system in phase space.

Let's now discuss some **peculiarities of multi-particle system dynamics** – in particular, we will discuss incoherent and coherent motion. We will **not yet be touching collective effects**.

- Part I: Multi-particle effects – direct space charge
    - Multi-particle systems and their representation
    - Incoherent and coherent motion
    - Space charge

Incoherent vs. coherent motion

- When considering multi-particle systems we need to **differentiate between incoherent ("microscopic") and coherent ("macroscopic")** motion



Incoherent vs. coherent motion

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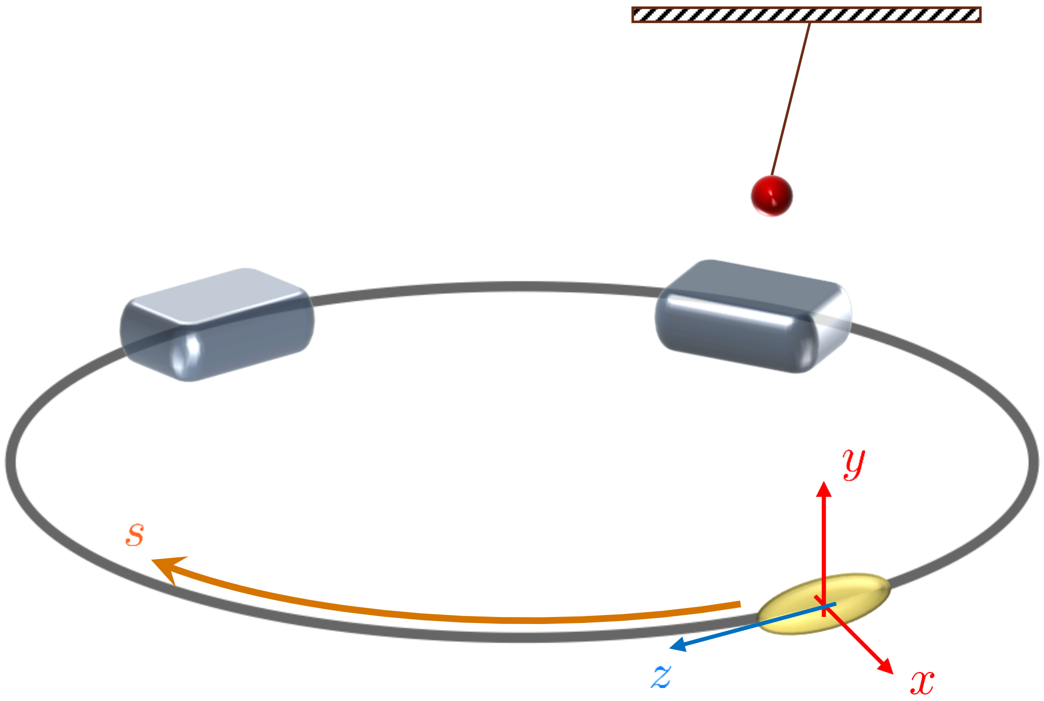
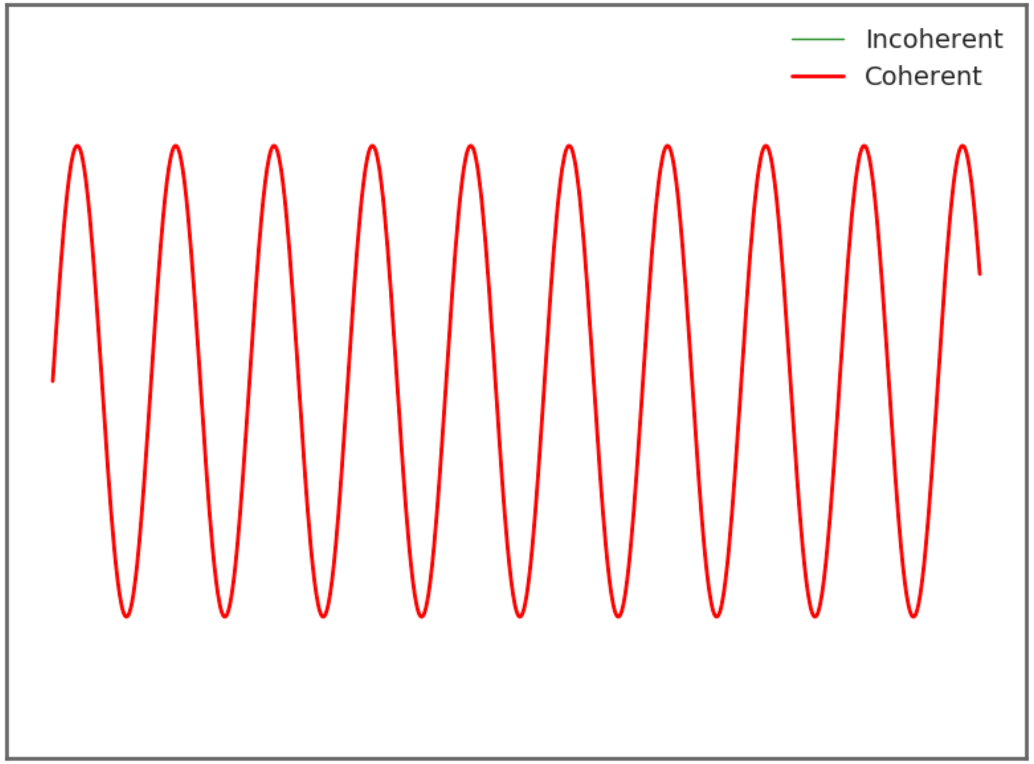
- When considering multi-particle systems we need to **differentiate between incoherent ("microscopic") and coherent ("macroscopic")** motion

Incoherent vs. coherent motion – model

- A single particle
  → incoherent **is identical to** coherent motion



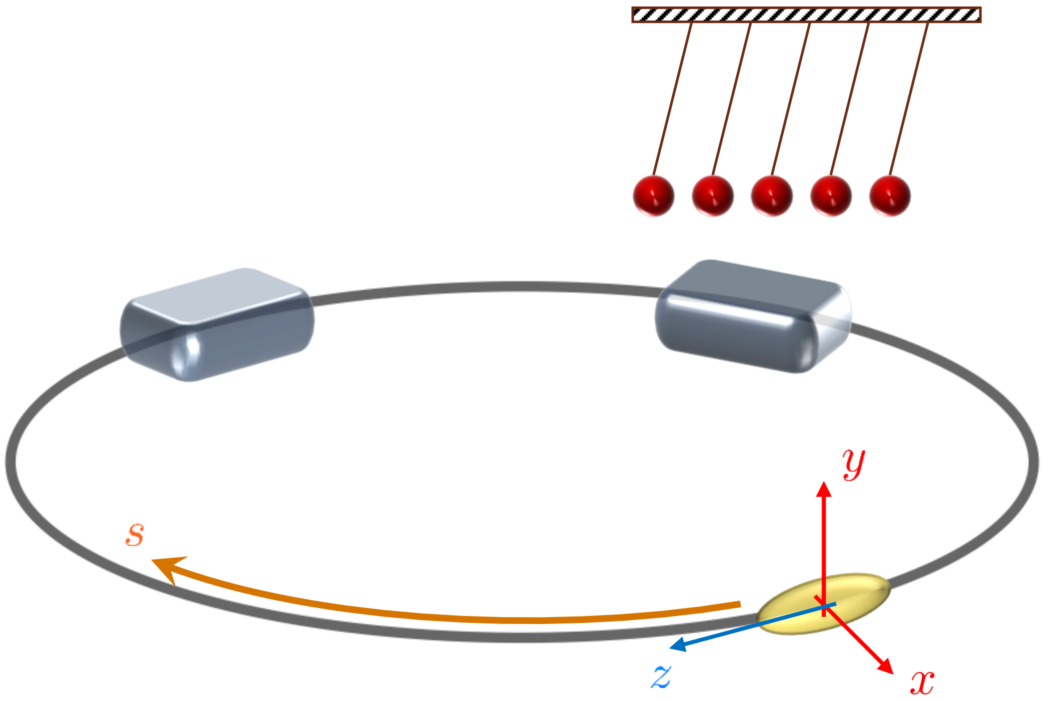


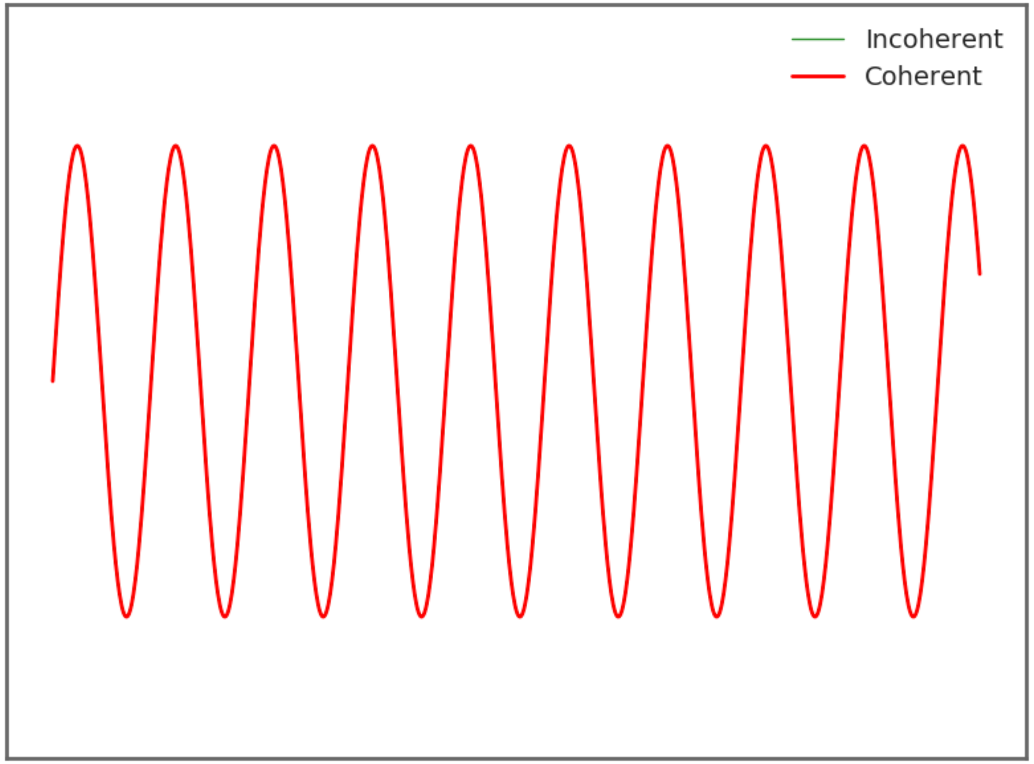
- Coherent – macroscopic quantities:

$$\langle x\rangle = \frac{1}{N}\int x\cdot\psi\left(\vec{q},\vec{p}\right)\,d\vec{q}d\vec{p}\,,\quad \sigma_x^2 = \frac{1}{N}\int\left(x-\langle x\rangle\right)^2\cdot\psi\left(\vec{q},\vec{p}\right)\,d\vec{q}d\vec{p}\,,\quad \varepsilon_x^n = \sqrt{\sigma_x^2\sigma_{p_x}^2 - \sigma_x\sigma_{p_x}}$$

Incoherent vs. coherent motion – model

- Five particles – in phase
  → incoherent **is same as** coherent motion





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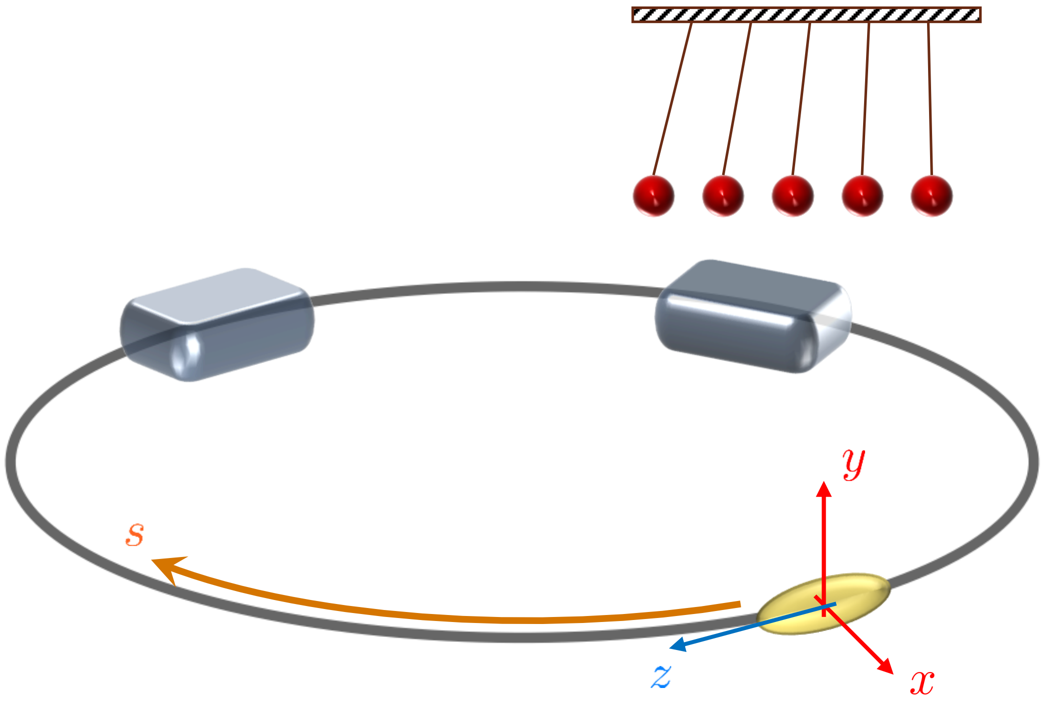
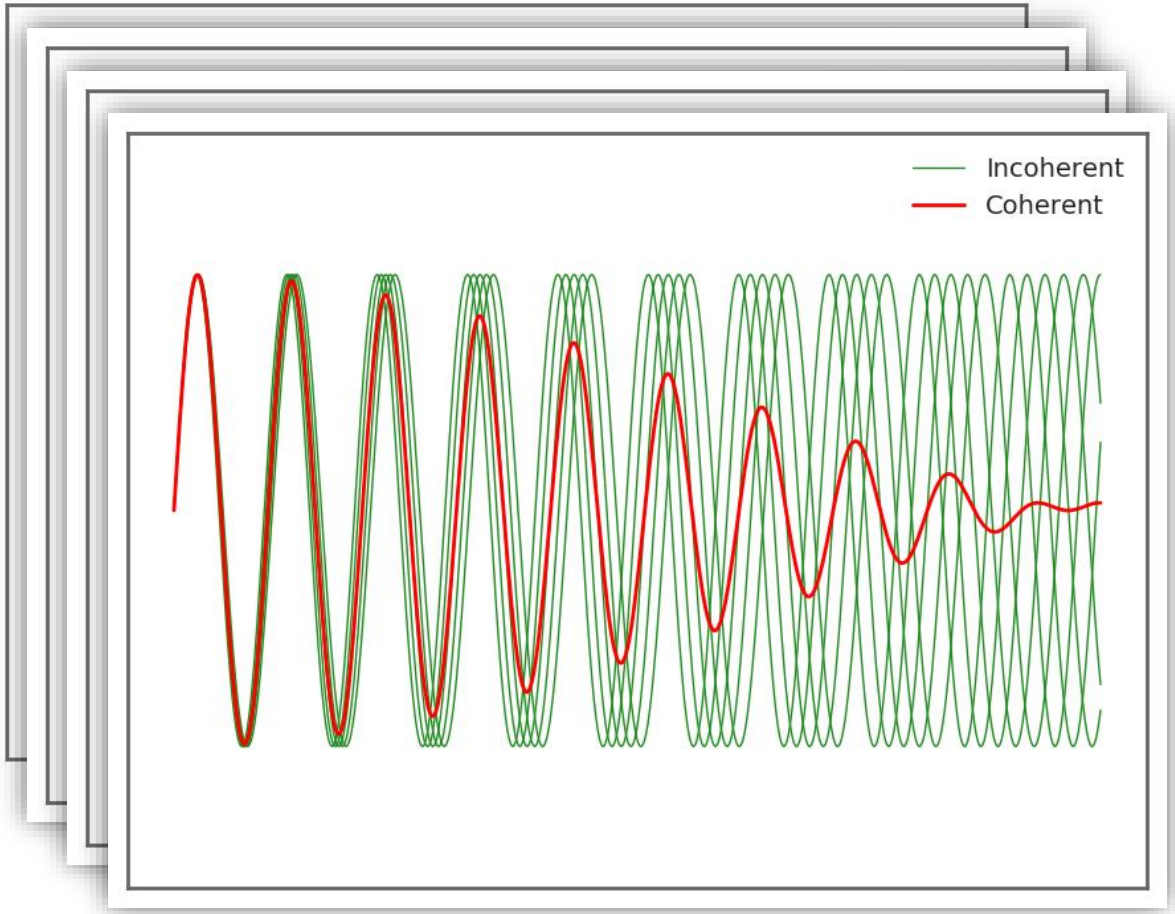
$$\langle \boldsymbol{x} \rangle = \frac{\boldsymbol{1}}{\boldsymbol{N}} \int \boldsymbol{x} \cdot \boldsymbol{\psi} \left(\vec{\boldsymbol{q}}, \vec{\boldsymbol{p}} \right) \, \boldsymbol{d}\vec{\boldsymbol{q}} \boldsymbol{d}\vec{\boldsymbol{p}} \,, \quad \sigma_x^2 = \frac{1}{N} \int \left(x - \langle x \rangle \right)^2 \cdot \psi \left(\vec{q}, \vec{p} \right) \, d\vec{q} d\vec{p} \,, \quad \varepsilon_x^n = \sqrt{\sigma_x^2 \sigma_{p_x}^2 - \sigma_x \sigma_{p_x}}$$

Incoherent vs. coherent motion – model

- Five particles – in phase but detuned
  → **decoherence** of coherent motion





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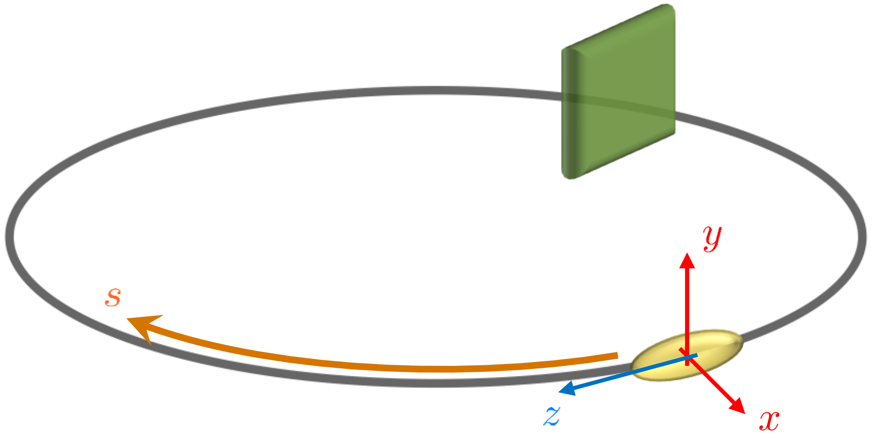
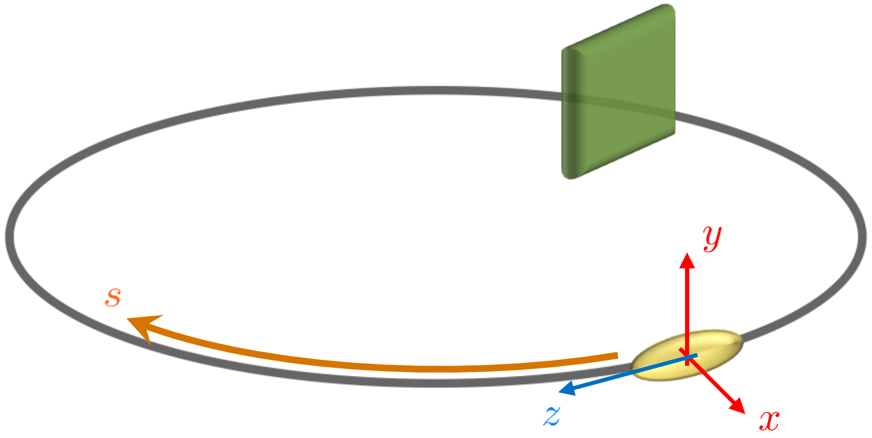
Incoherent vs. coherent motion – simulation

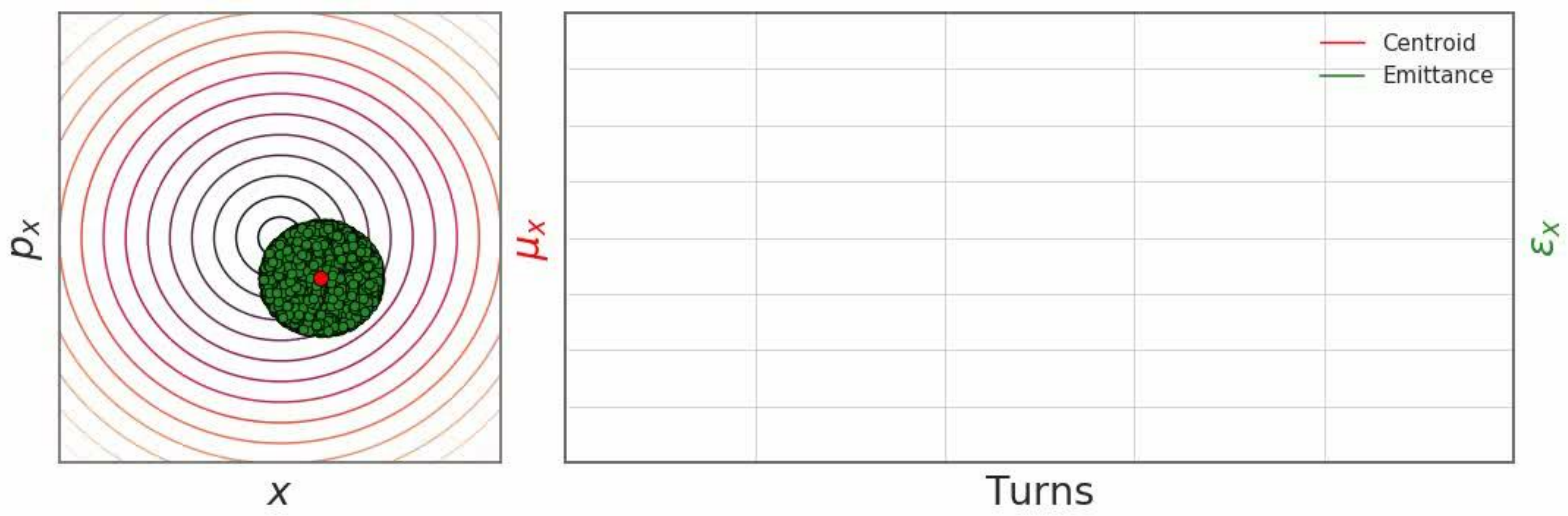


- Let's observe a system of five hundred particles:
  - Although each and every individual particle performs more or less large oscillations around the orbit, a coherent motion is hardly visible – for a matched injection.

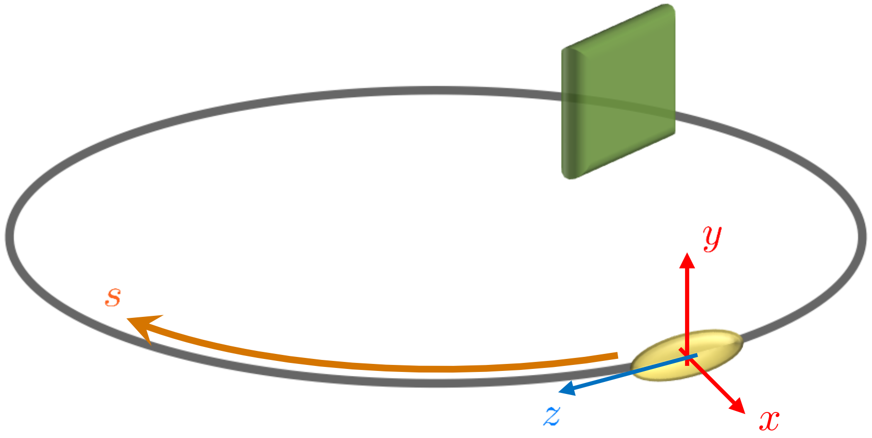
Incoherent vs. coherent motion – simulation



- Let's observe a system of five hundred particles:
  - For an offset injection, all particles move **around the orbit at a constant tune** and, at the same time, a clear and **strong coherent motion** is observed.



Incoherent vs. coherent motion – simulation



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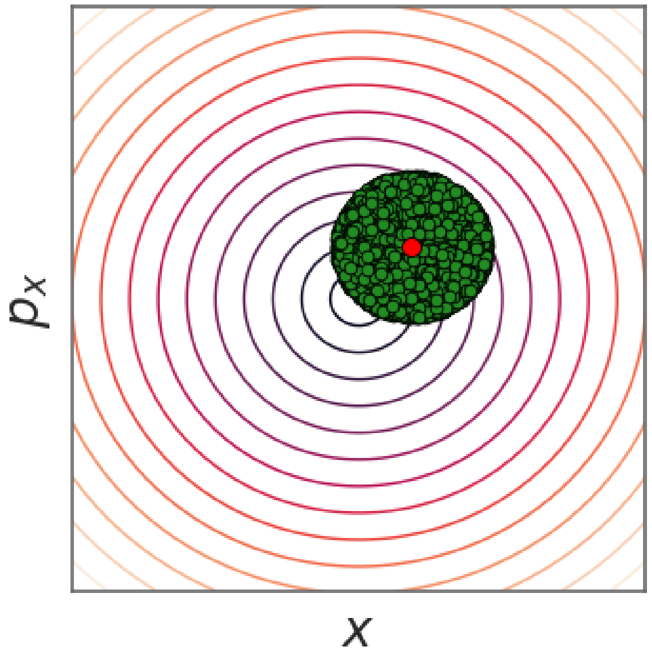

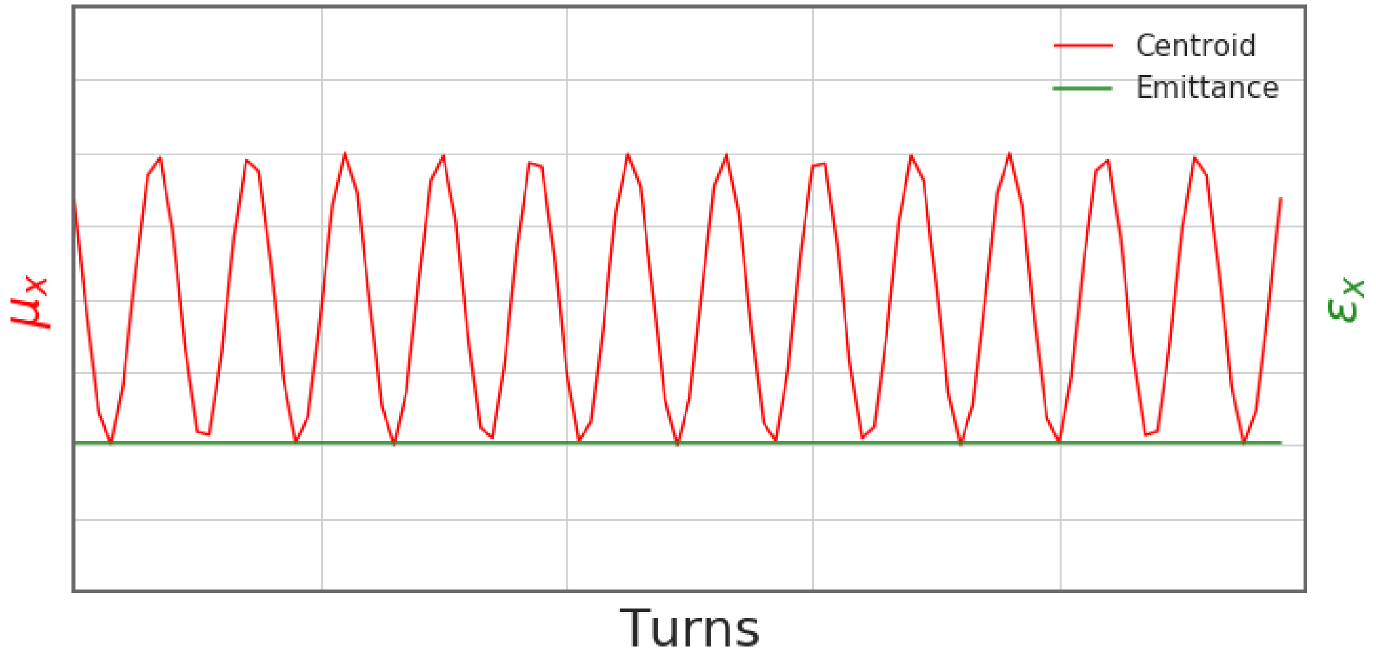

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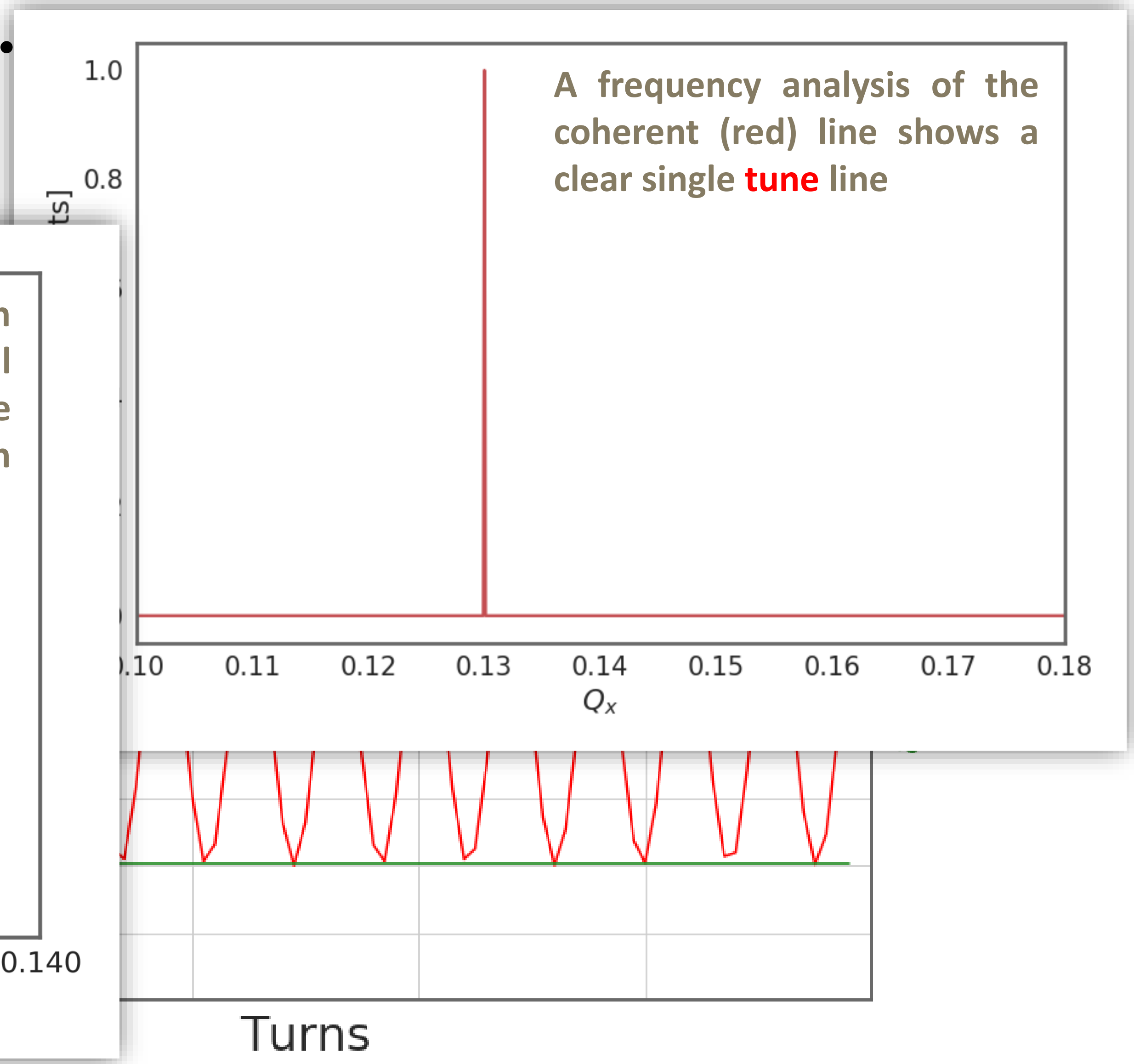
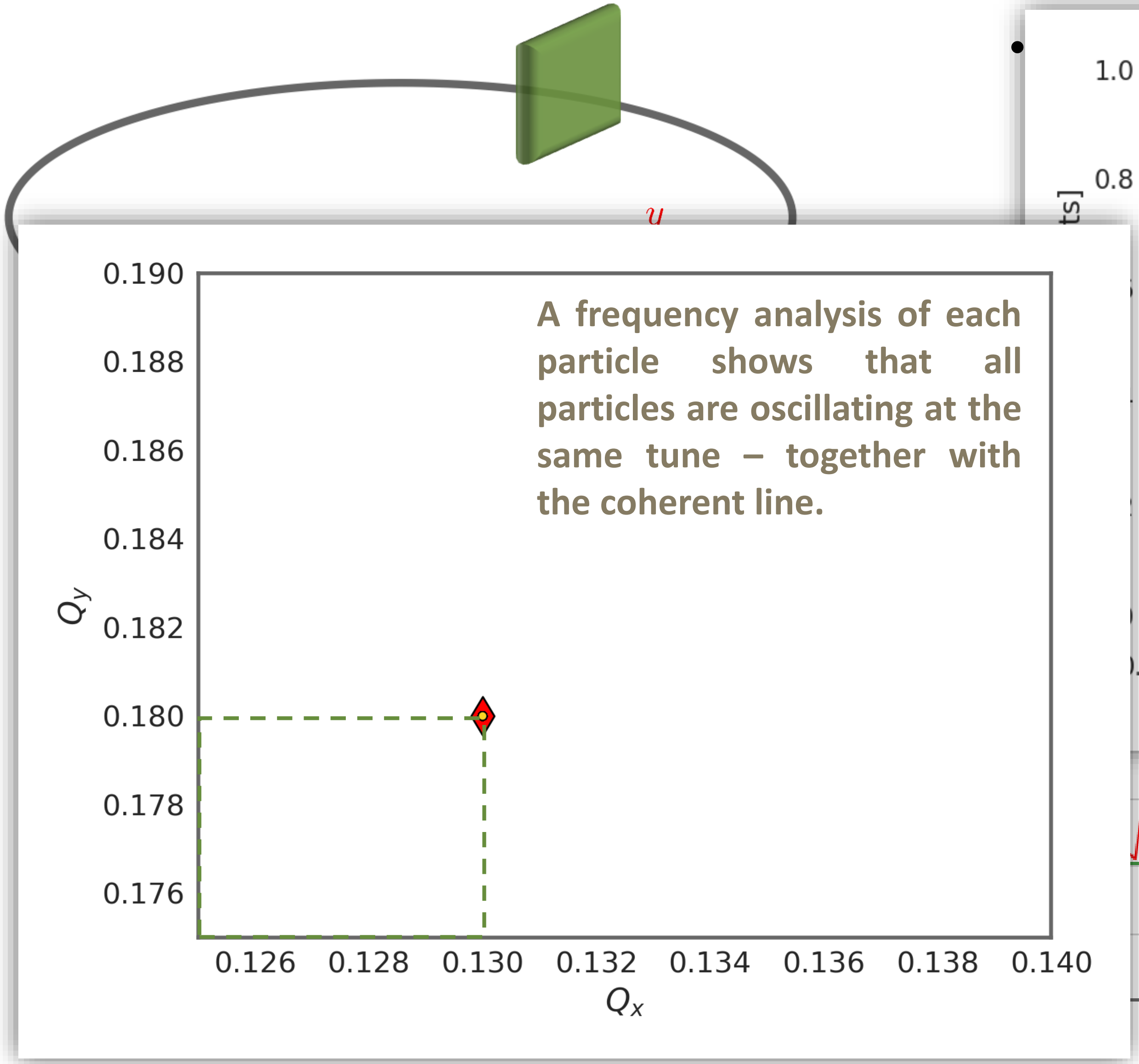
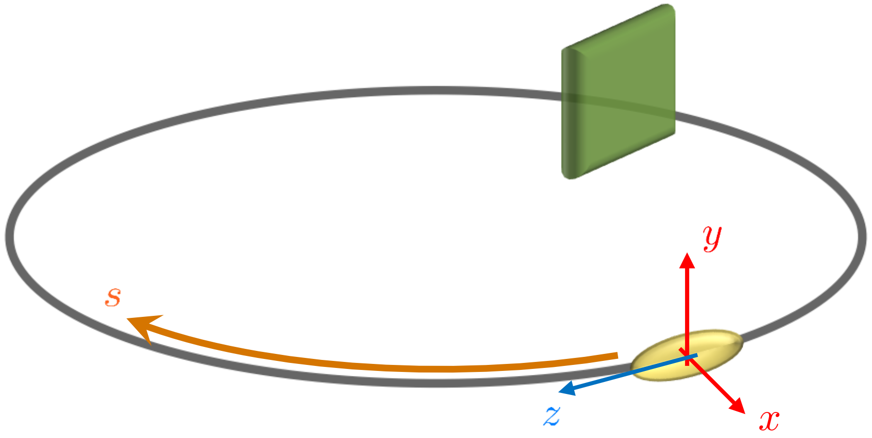
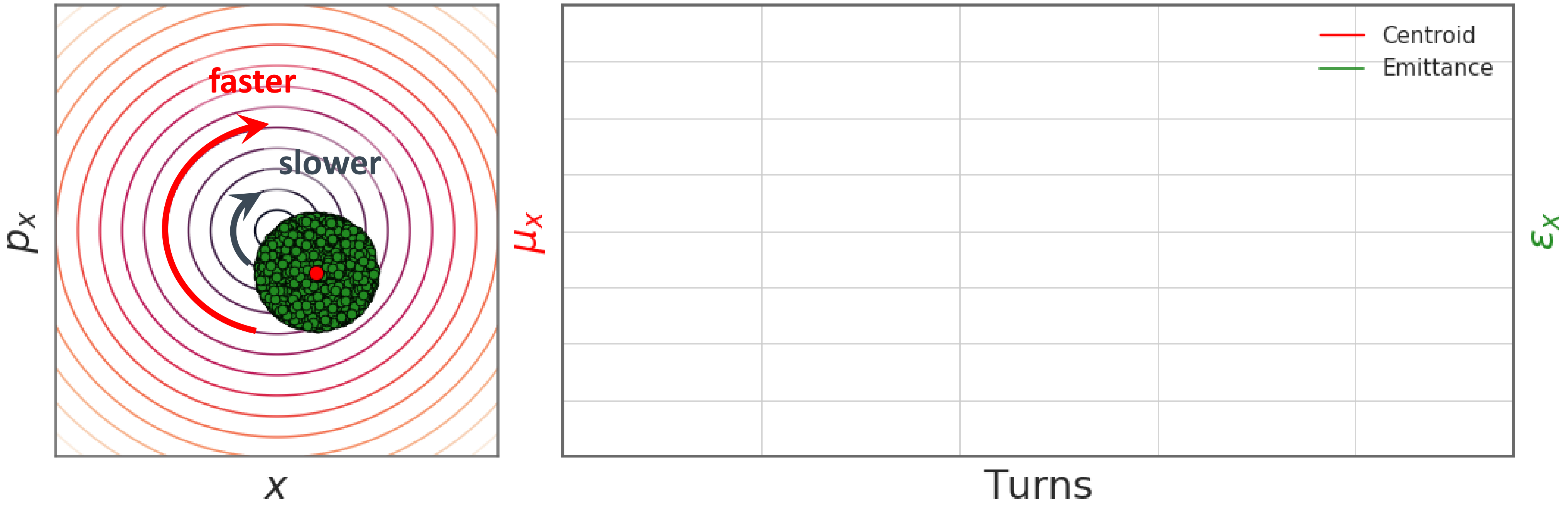




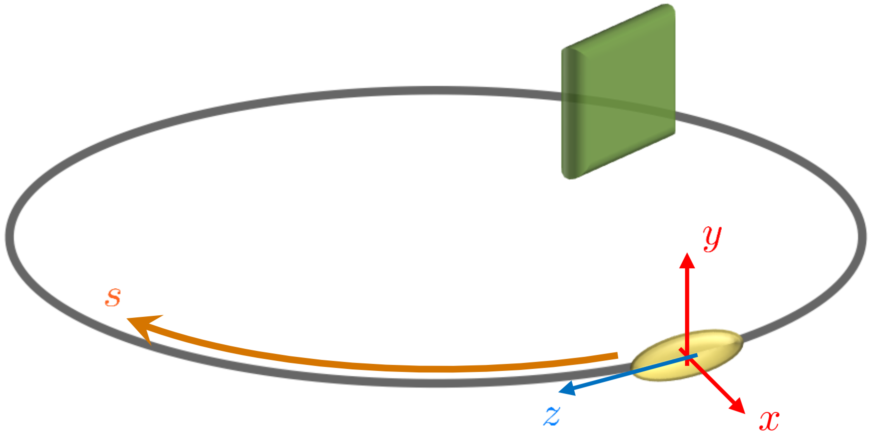
Incoherent vs. coherent motion – octupoles



- Let’s observe a system of five hundred particles:
  - When offset and in the presence of non-linearities (detuning with amplitude), all particles move **around the orbit at different tunes**. As a consequence, we can observe filamentation of the bunch in phase space. The **bunch decoheres** and the emittance increases.



Incoherent vs. coherent motion – octupoles



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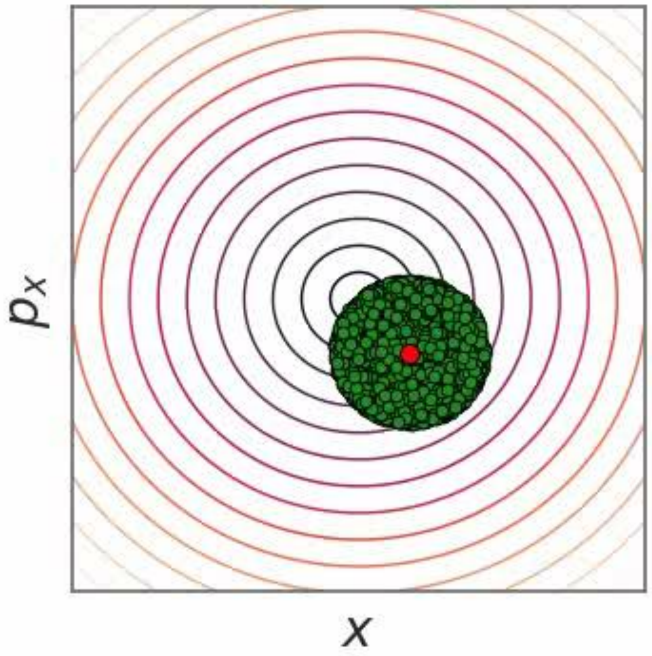
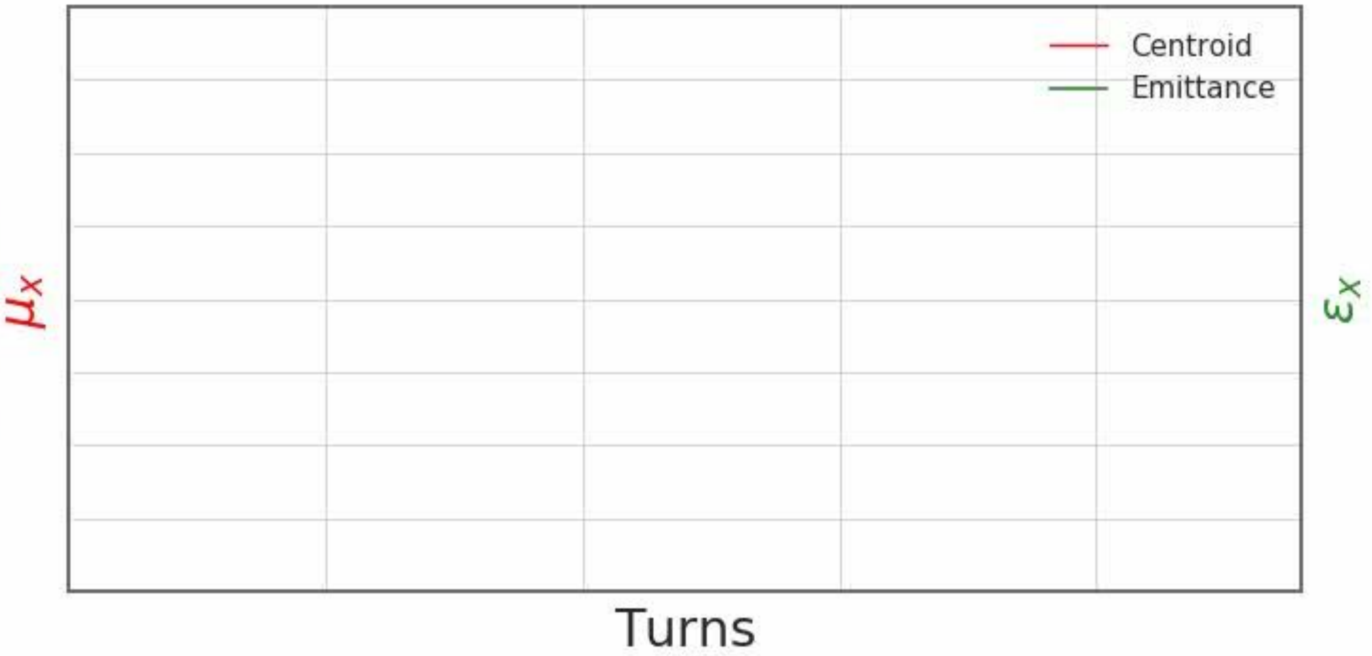
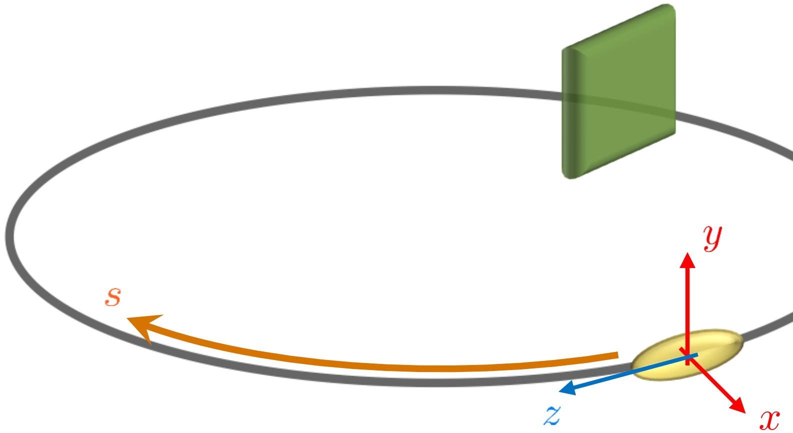




Incoherent vs. coherent motion – octupoles



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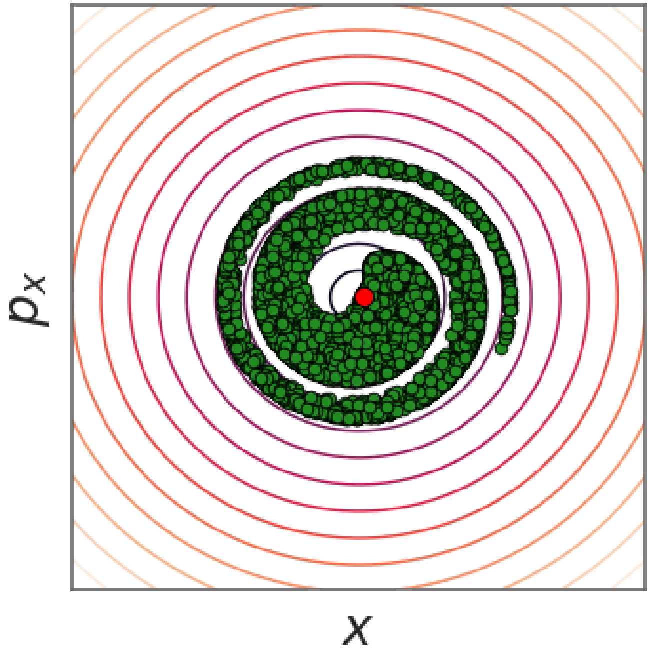

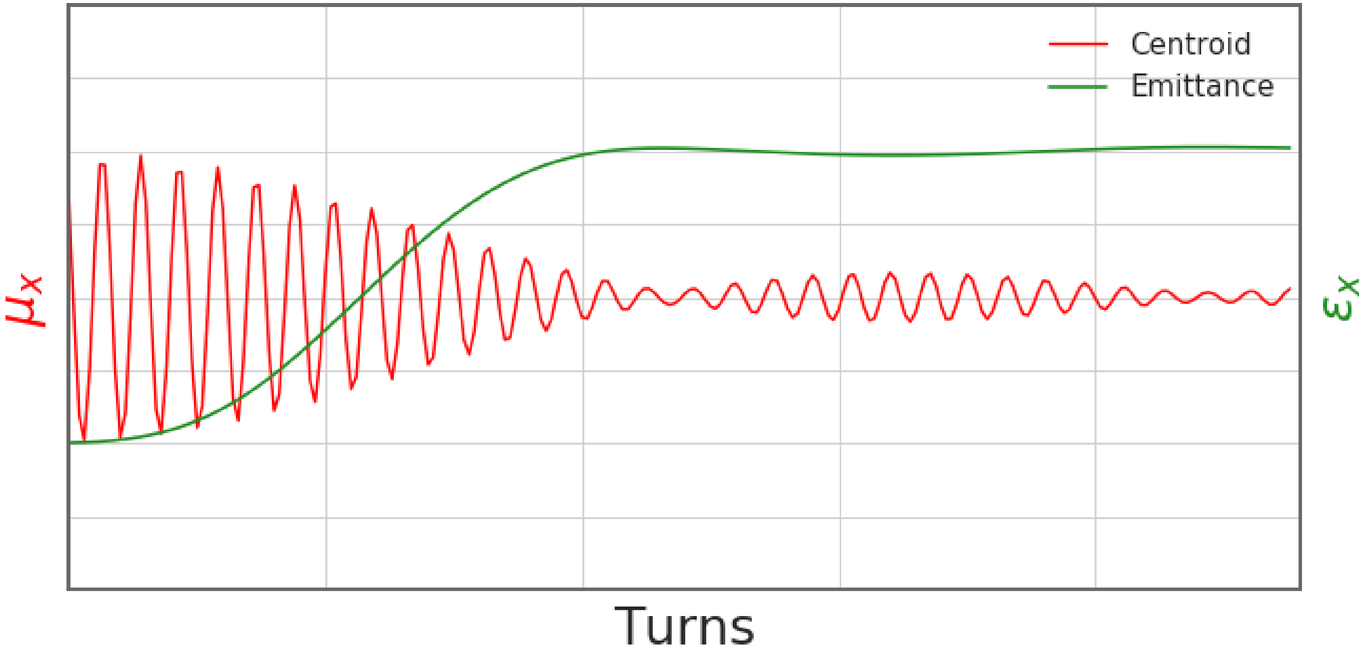

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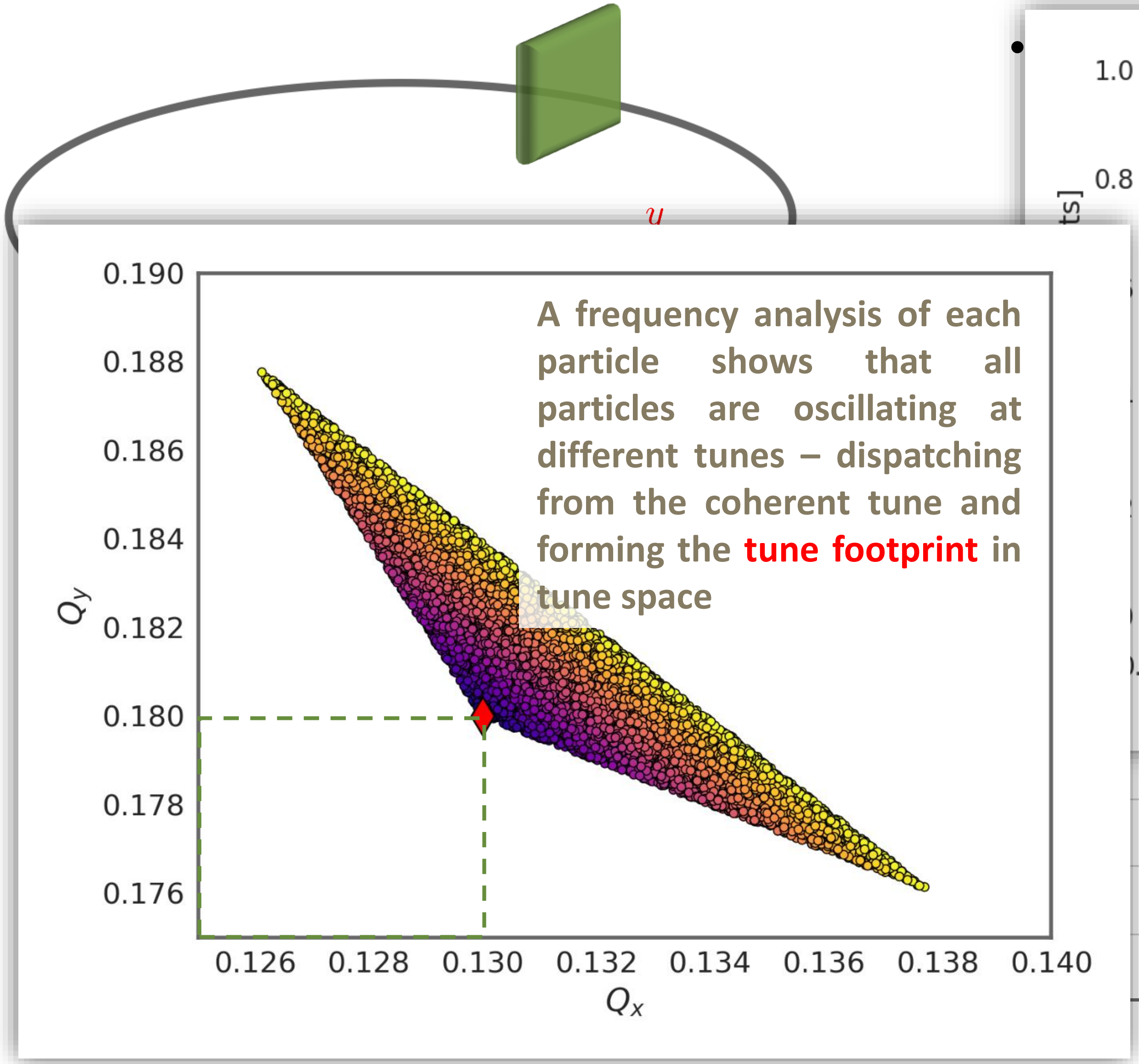
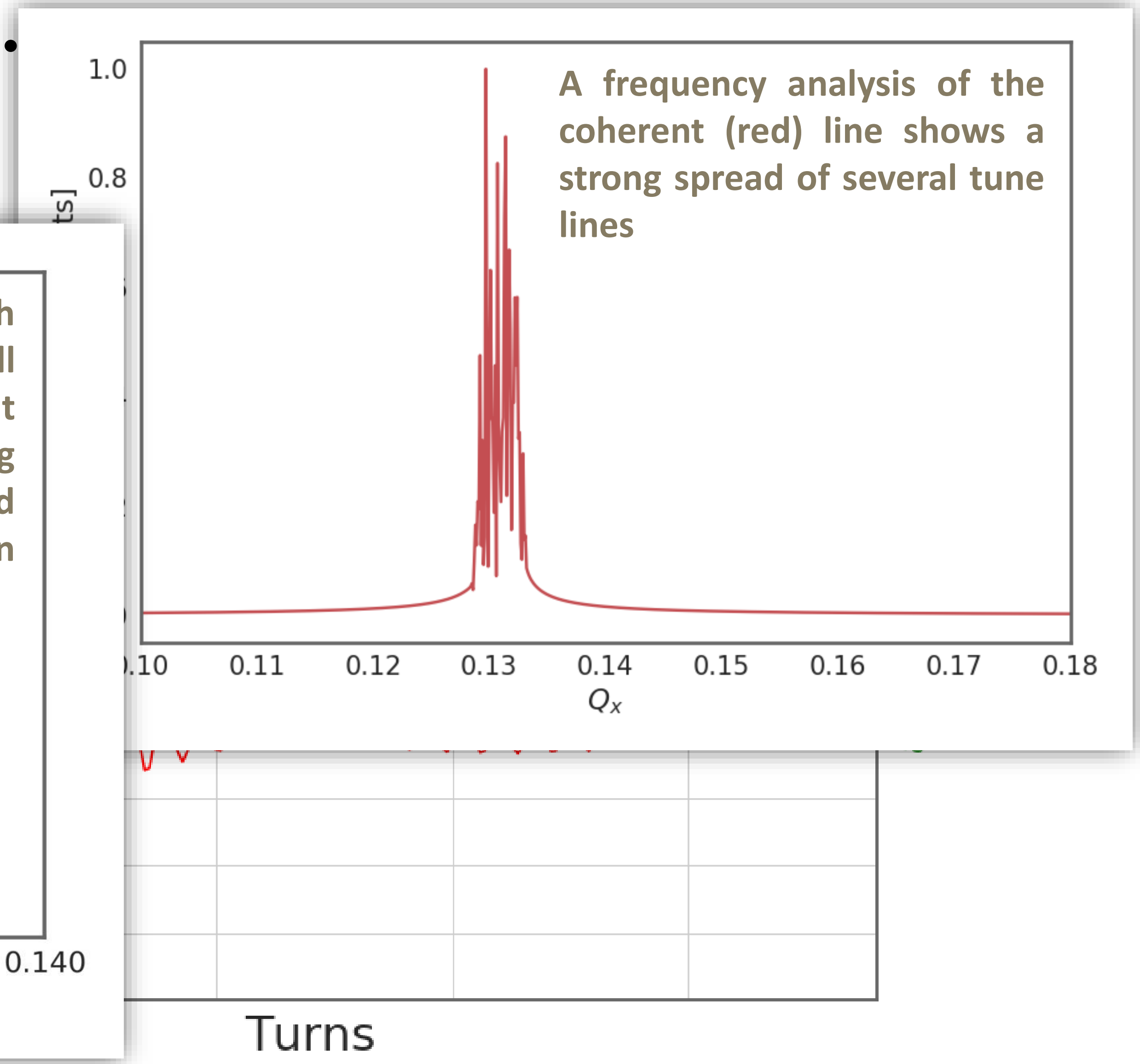
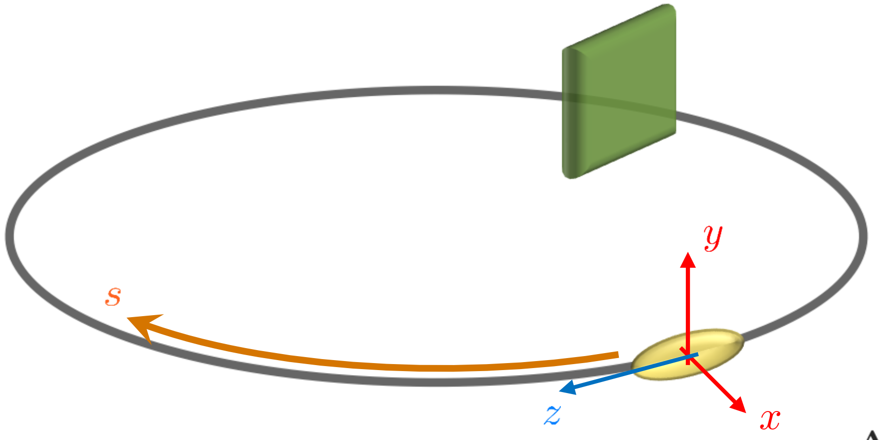




Incoherent vs. coherent motion – chromaticity



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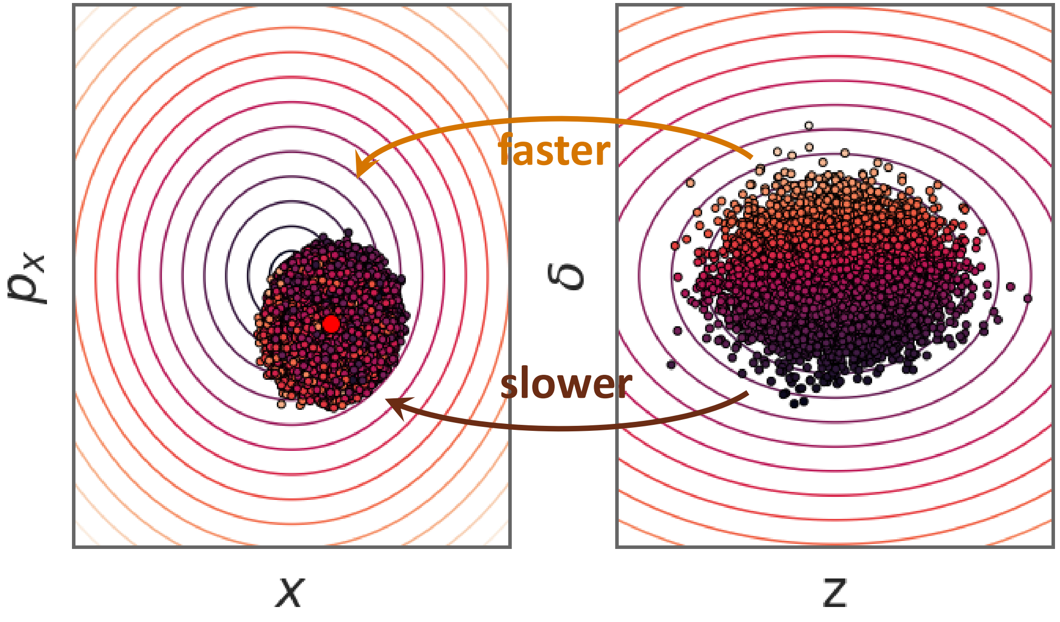

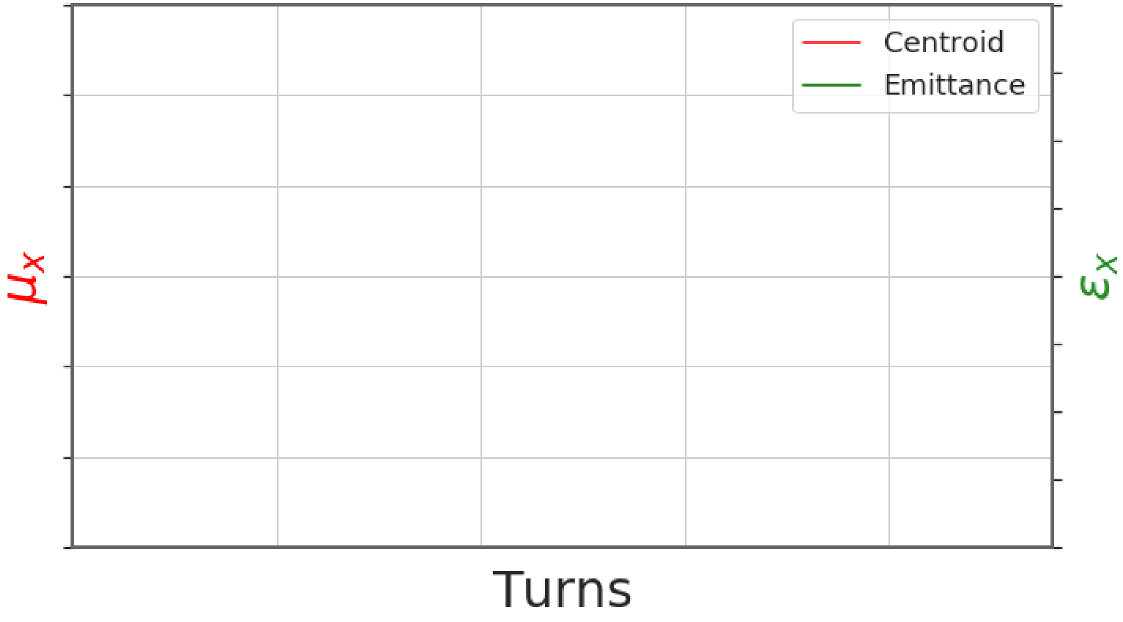

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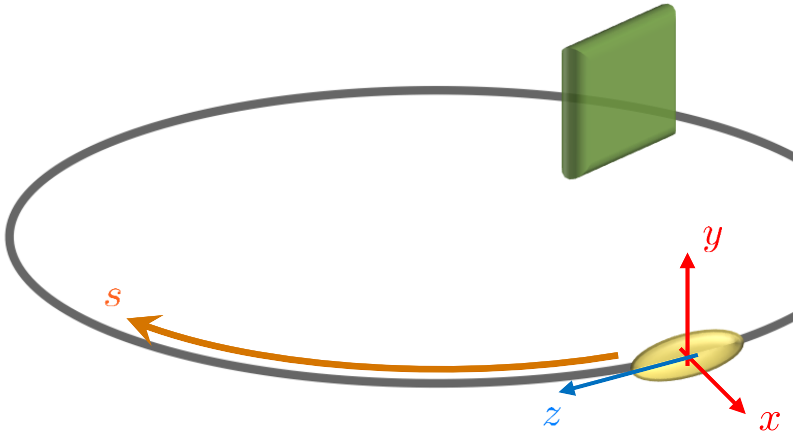

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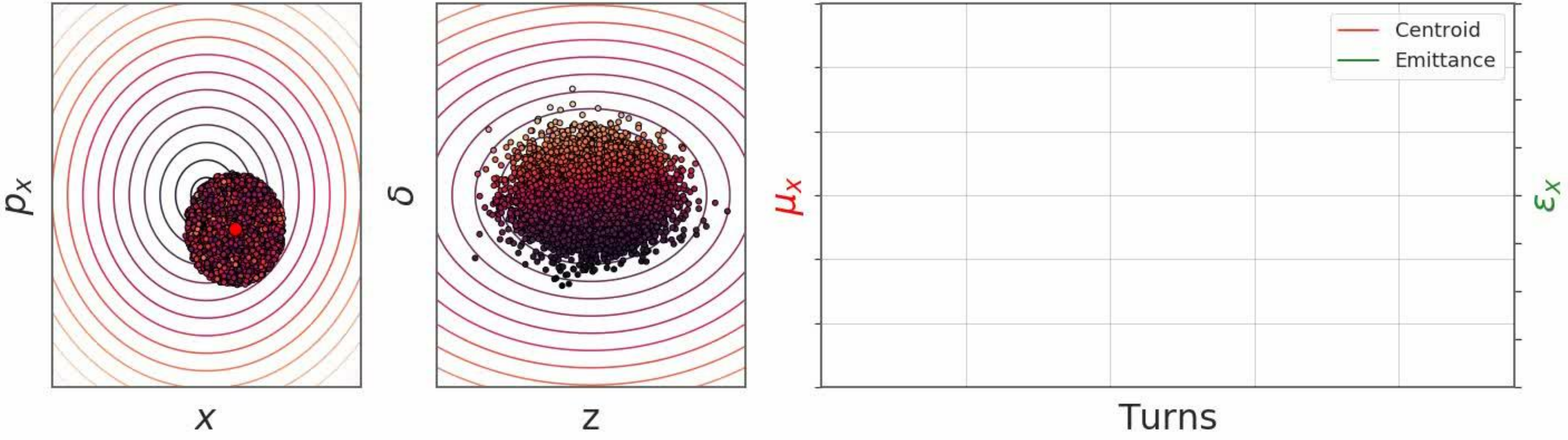

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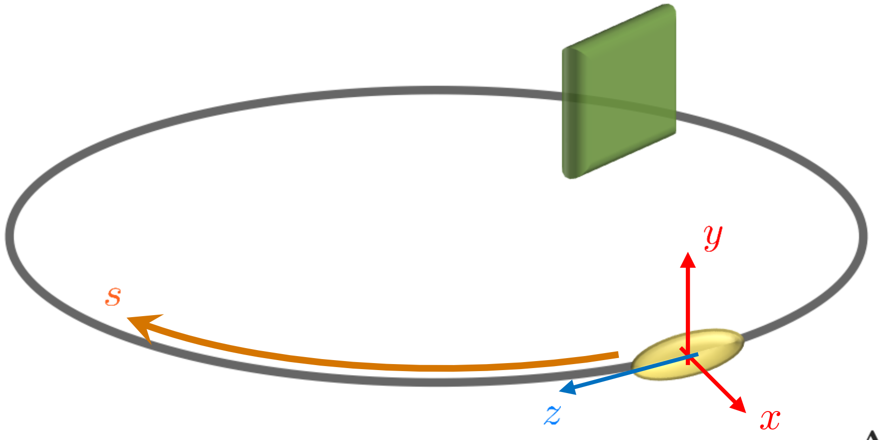

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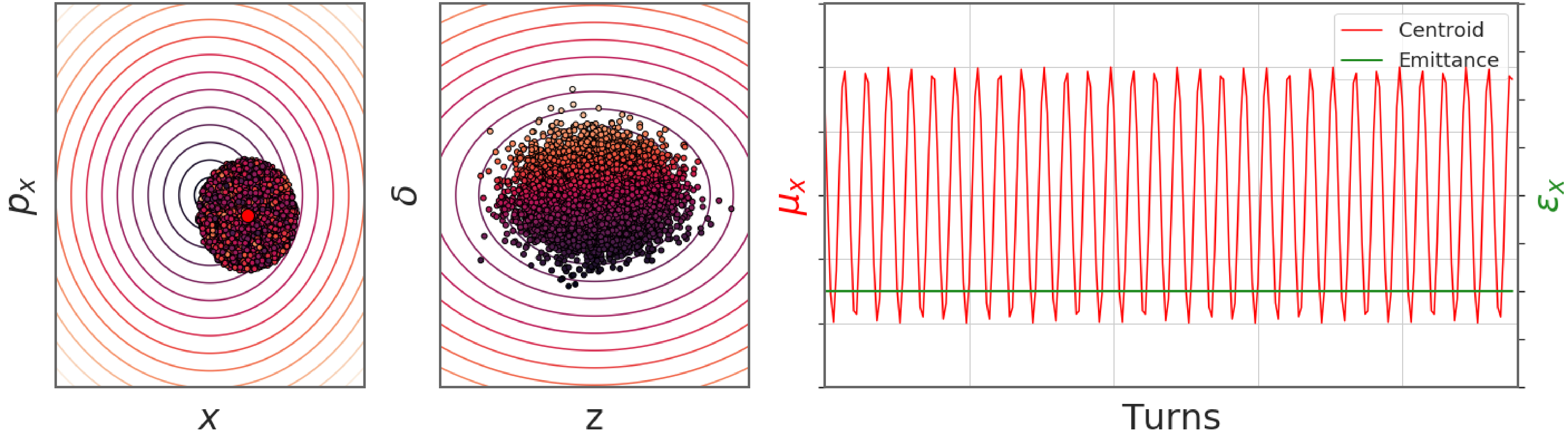

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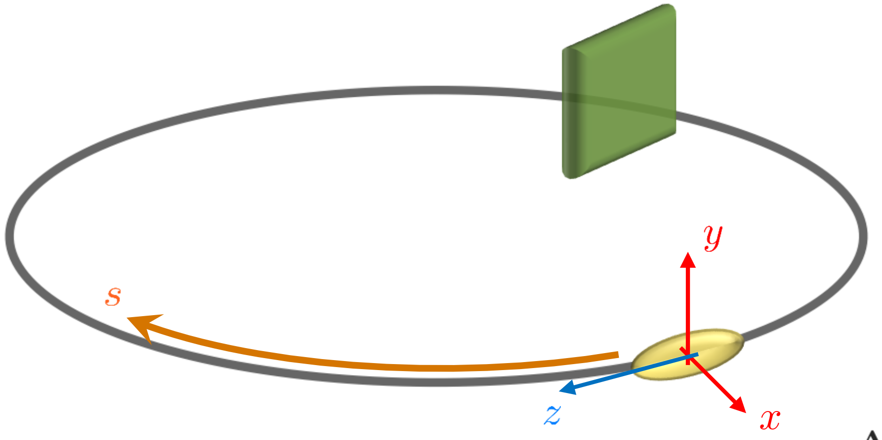

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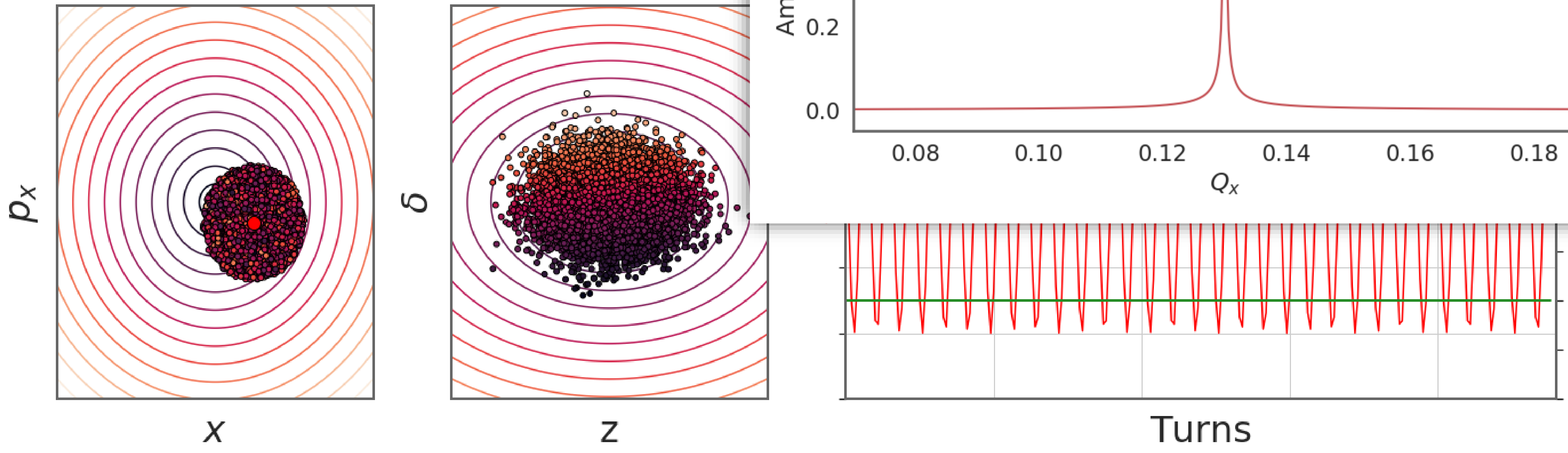
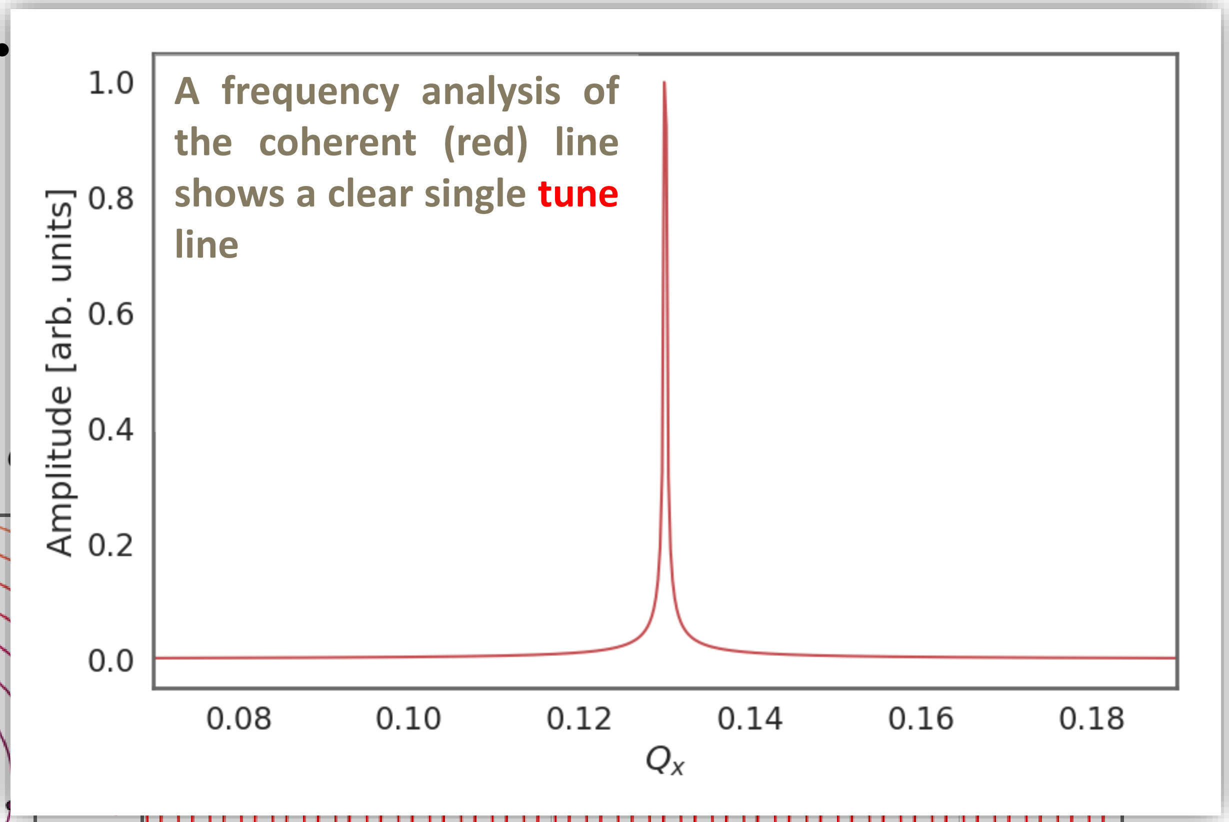
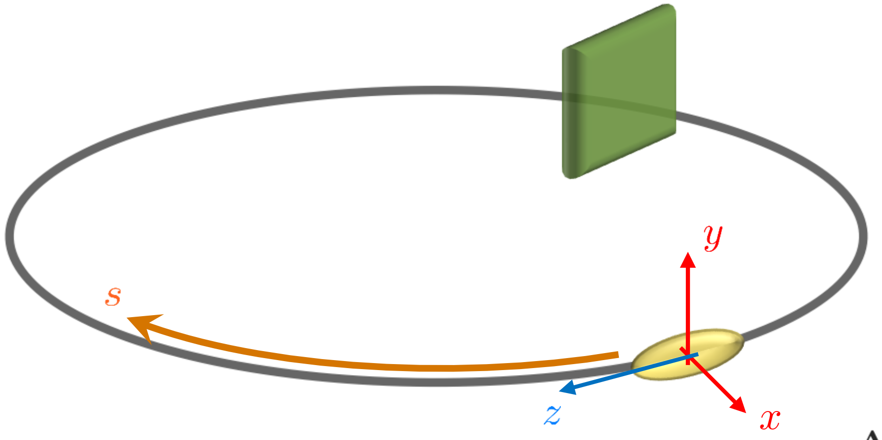




Incoherent vs. coherent motion – chromaticity



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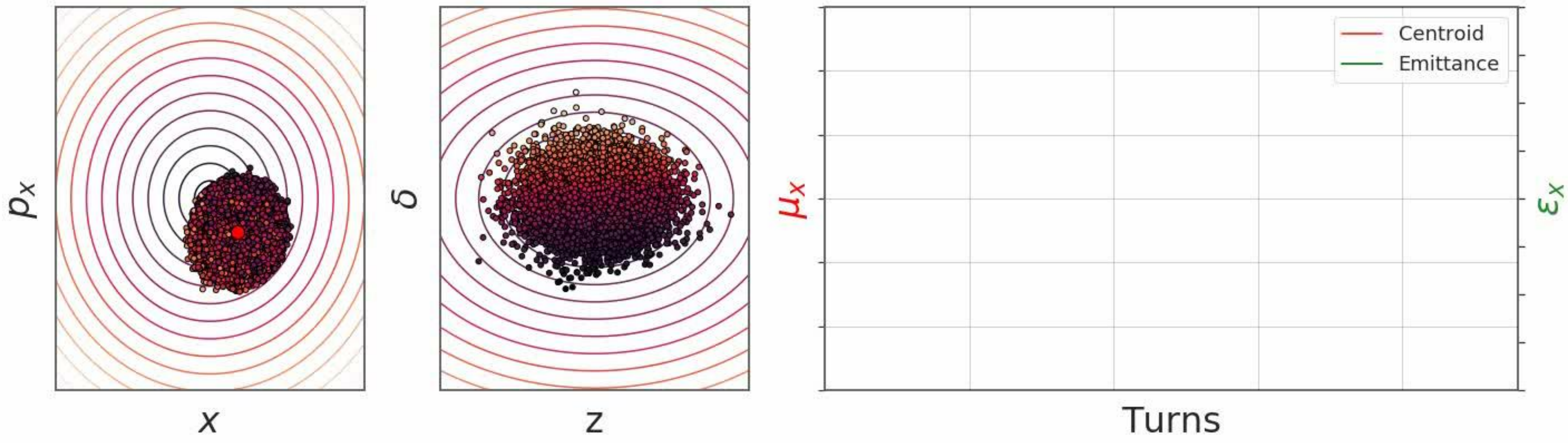

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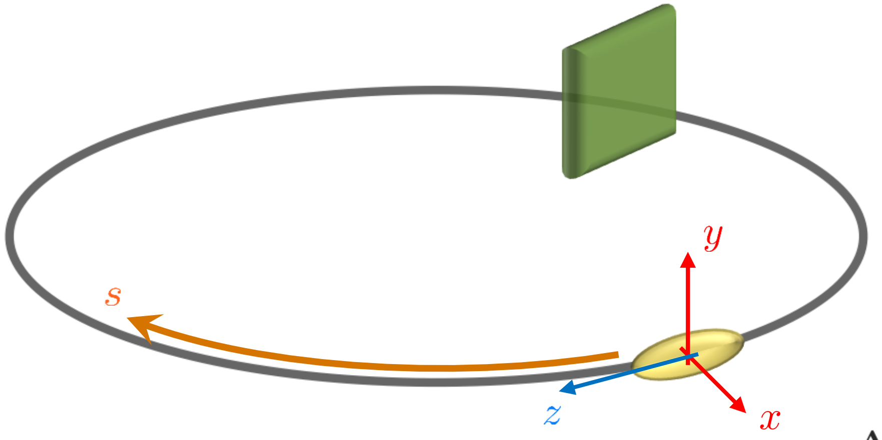

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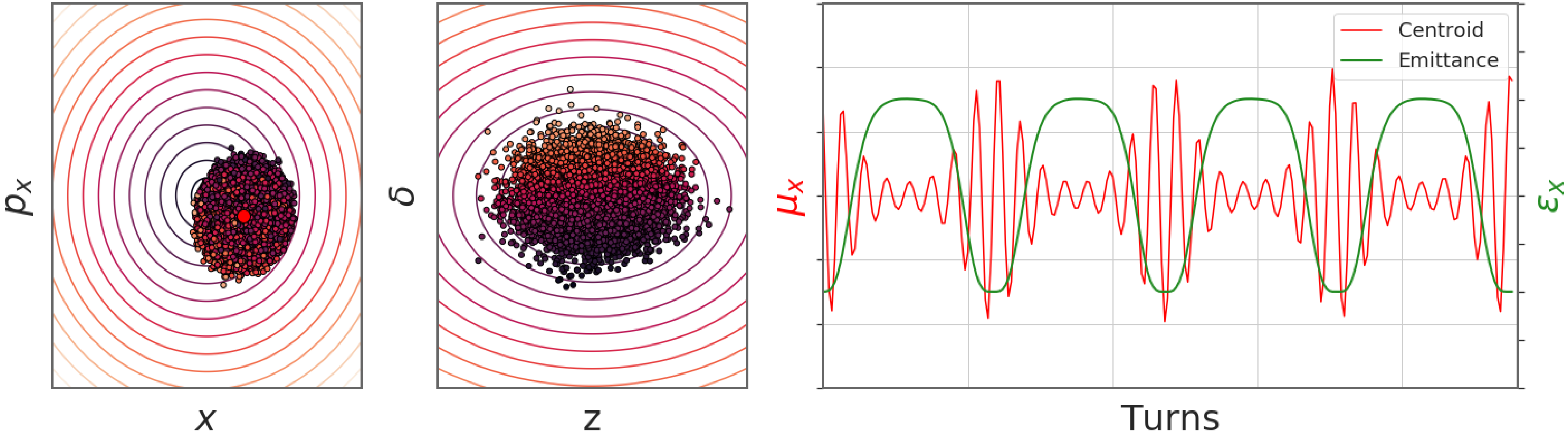

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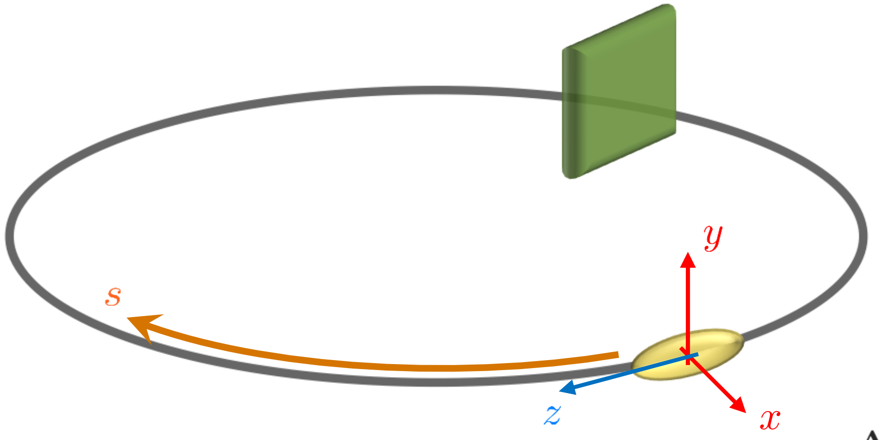

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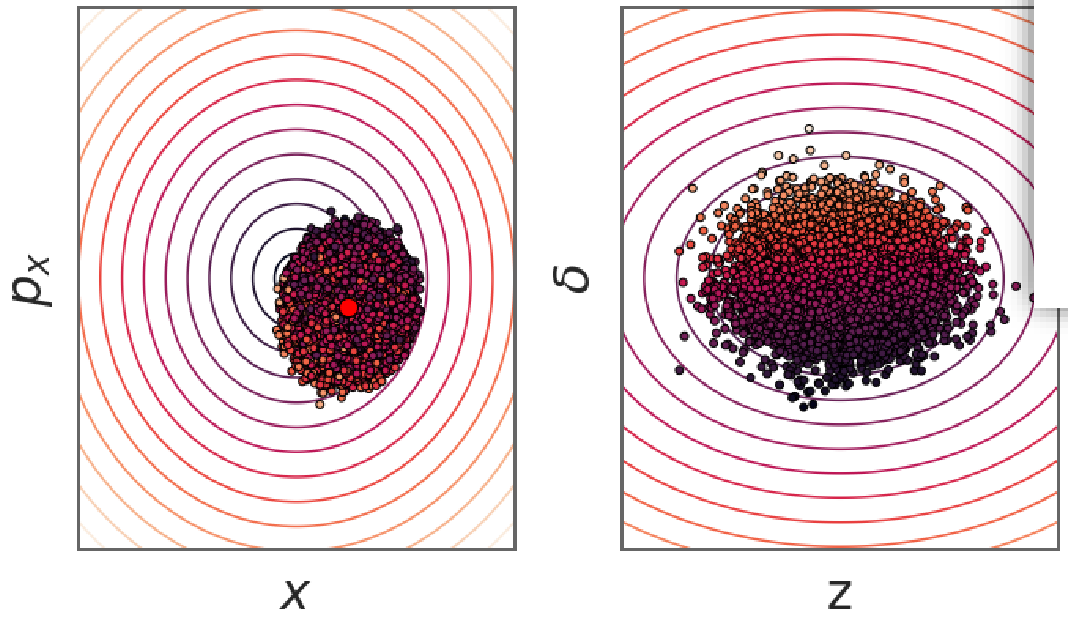
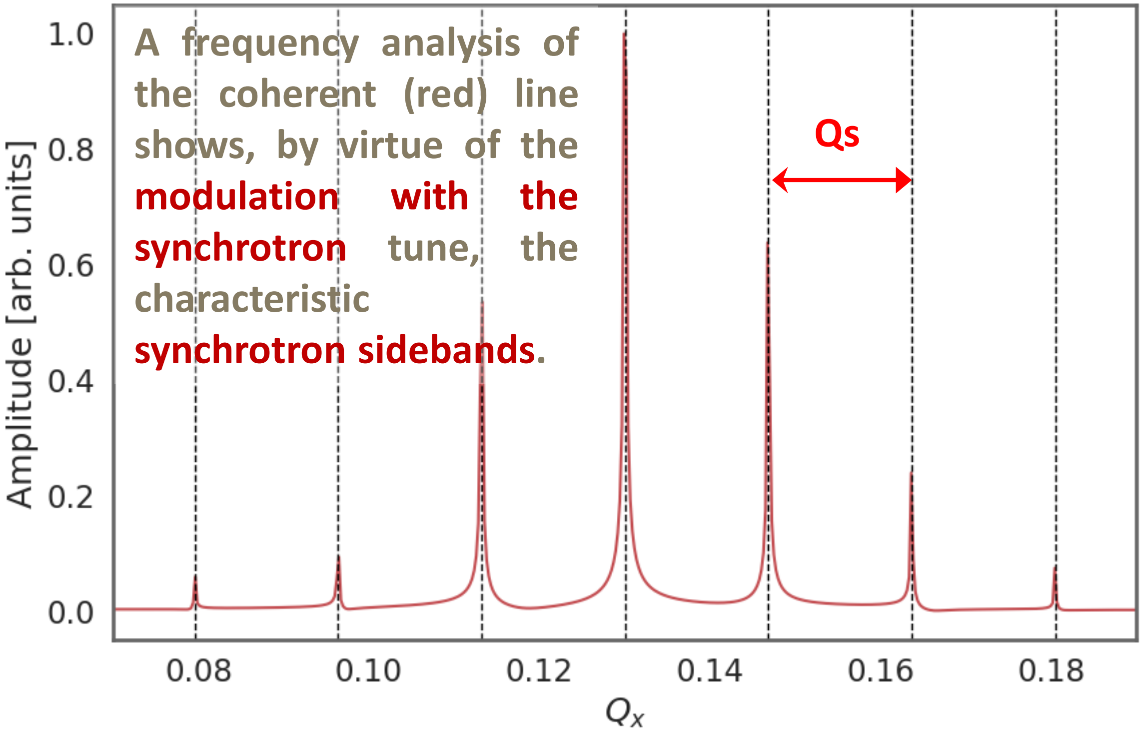
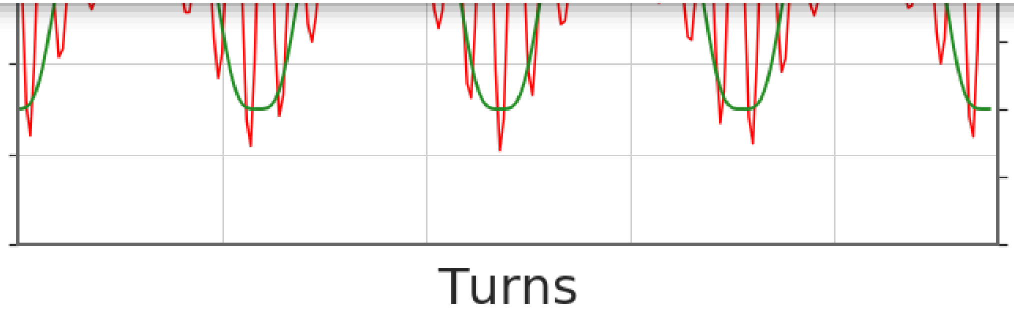




A frequency analysis of the coherent (red) line shows, by virtue of the **modulation with the synchrotron tune**, the characteristic **synchrotron sidebands**.



Sources and impact of transverse nonlinearities

- We have learned or we may know from operational experience that there are a set of **crucial machine parameters to influence beam stability** – among them **chromaticity and amplitude detuning**

- Chromaticity
  - Controlled with sextupoles – provides **chromatic shift** of bunch spectrum wrt. impedance
  - Changes interaction of beam with impedance
  - Damping or excitation of **headtail modes**

- Amplitude detuning
  - Controlled with octupoles – provides (incoherent) **tune spread**
  - Leads to absorption of coherent power into the incoherent spectrum → **instability mitigation**

We have seen the difference between incoherent and coherent motion. For single particles, these are identical. However, for multi-particle systems and their dynamics **there are important differences**.

In this context, we have seen effects such as **decoherence, filamentation and emittance blow-up**. We have learned about the concept of the **tune footprint**.

So far, we have never taken into account any interaction among the particles. We have therefore not yet seen any collective effects. We will now look at **a first collective effects** which is intuitively very natural to grasp: the direct space charge effect.

- Part I: Multi-particle effects – direct space charge
  - Multi-particle systems and their representation
  - Incoherent and coherent motion
  - Direct space charge

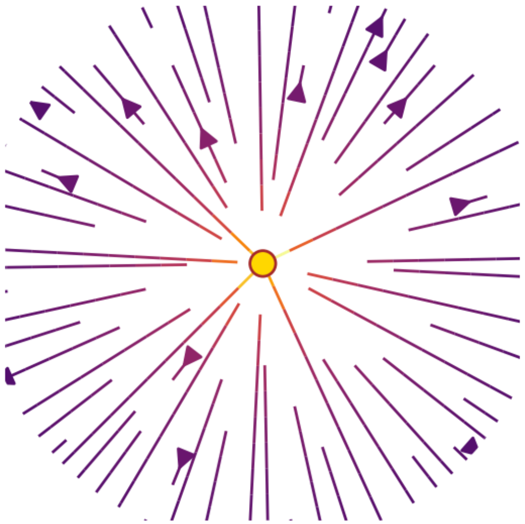
Single particle dynamics – reminder coordinates

- A beam of charged particles **induces electromagnetic fields** when circulating inside the vacuum chamber.

Space charge effects:

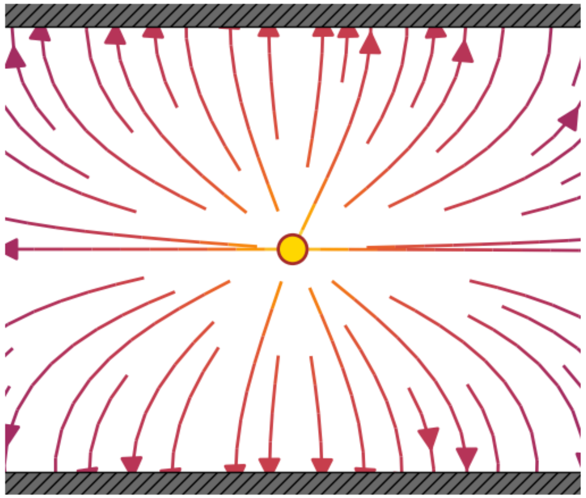
- cause **tune shifts** (transverse and longitudinal both incoherent (direct) and coherent (direct and indirect))
- can result in **longitudinal instability** (negative mass instability)

- When we talk about space charge we think about



Direct space charge:

Interaction of charged particles in free space

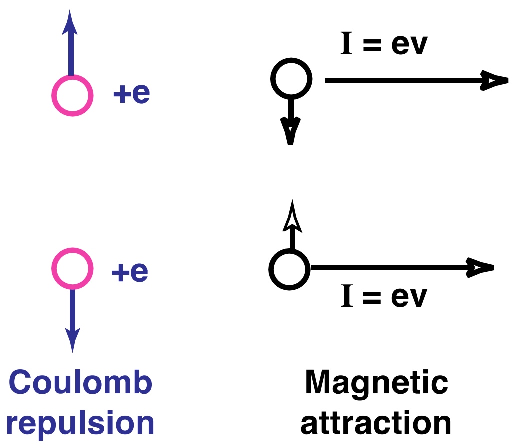


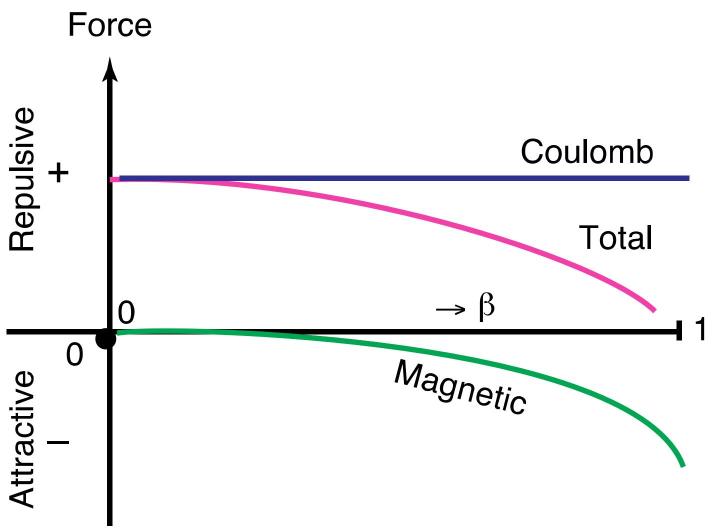
Indirect space charge:

Interaction with image charges and currents induced in perfect conducting walls and ferromagnetic materials close to the beam pipe

Two point charges with same velocity

- Consider two point charges with the same charge q and with same velocity $v_1=v_2=v$ on parallel trajectories
  - In the rest frame, we know already the electric and magnetic fields generated by a "source" particle.
  - The force on the "test" particle is given by **the Lorentz force**.
- The attractive magnetic force tends to compensate the repulsive electric force
  - At rest the two particles experience only the repulsive Coulomb force
  - When travelling with velocity v the particles represent two parallel currents which attract each other by the induced magnetic field
  - The forces **become equal at the speed of light and thus cancel**





Using the correct Lorentz transforms one can compute the fields in the lab frame generated by a source particle with velocity v

$$E_r = \frac{e}{4\pi\epsilon_0}\frac{\gamma}{r^2} \qquad B_\phi = \frac{\beta E_r}{c}$$

→ Lorentz force acting on the test particle

$$F_r = e\left(E_r - v\,B_\phi\right) = e\left(E_r - \beta^2\,E_r\right)$$
$$= \frac{eE_r}{\gamma^2} = \frac{e^2}{4\pi\epsilon_0\,\gamma r^2}$$

Example: coasting beam with uniform charge density

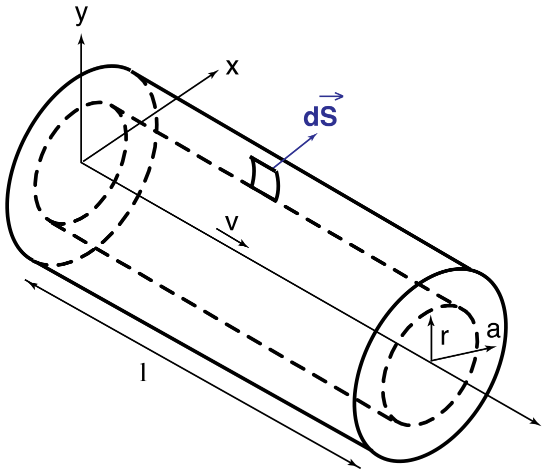
- Assume a coasting beam of circular cross section with radius a and uniform charge density $\rho = \lambda/(\pi\, a^2)$ [C/m^3] moving at constant velocity $v = \beta c$

Maxwell's equation: $\vec{\nabla} \cdot \vec{E} = \dfrac{\rho}{\epsilon_0}$

Gauss' law

$$\iiint \vec{\nabla} \cdot \vec{E}\, dV = \iint \vec{E}\, d\vec{S}$$

$$\Rightarrow \ \pi r^2 l \frac{\rho}{\epsilon_0} = 2\pi r l E_r$$



Electric field –

With $\lambda = \rho\, \pi a^2$ it follows:

$$E_r = \frac{\lambda}{2\pi\epsilon_0} \frac{r}{a^2}\,, \qquad r < a$$

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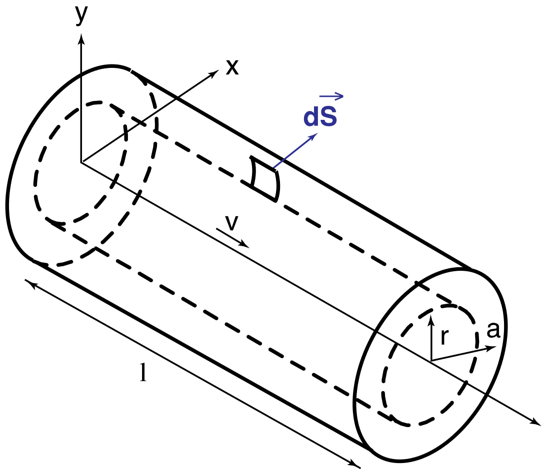

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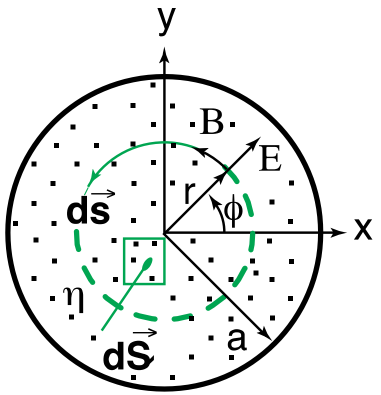
Maxwell's equation: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$



Stokes' law

$$\iint \vec{\nabla} \times \vec{B}\, d\vec{S} = \oint \vec{B}\, d\vec{s}$$

$$\Rightarrow\ \pi r^2 \mu_0 J = 2\pi r B_\phi$$

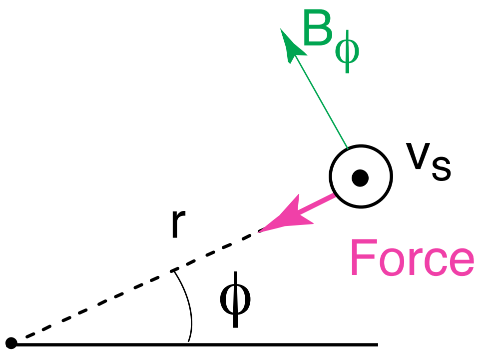
Magnetic field –

With $J = \beta c \rho = \beta c \dfrac{\lambda}{\pi a^2}$ and $\mu_0 = \dfrac{1}{\epsilon_0 c^2}$ it follows:

$$B_\phi = \frac{\lambda \beta}{2\pi\epsilon_0 c} \frac{r}{a^2}, \qquad r < a$$

Example: coasting beam with uniform charge density

- Assume a coasting beam of circular cross section with radius a and uniform charge density $\rho = \lambda/(\pi\, a^2)$ $[\mathrm{C/m^3}]$ moving at constant velocity $v = \beta c$
- Calculate the resulting force on a test particle with charge e



Electric and magnetic components have opposite signs and scale between each other with $\beta^2 \rightarrow$ there is perfect cancellation when $\beta = 1$

$$E_r = \frac{\lambda}{2\pi\epsilon_0}\frac{r}{a^2}$$

$$B_\phi = \frac{\lambda\beta}{2\pi\epsilon_0 c}\frac{r}{a^2}$$

Lorenz force for the geometry studied

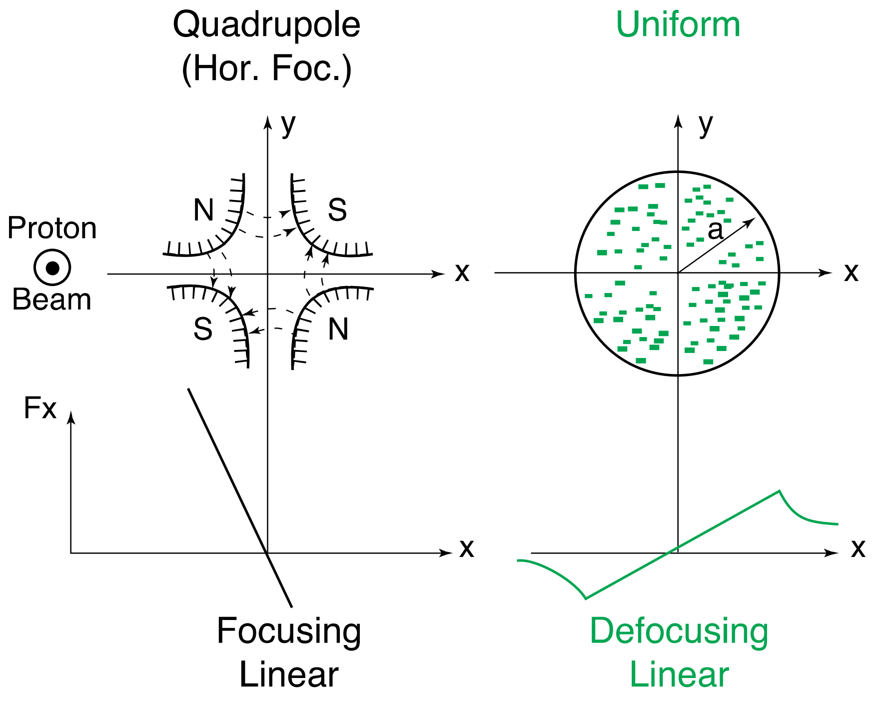
$$\begin{aligned} F_r &= e\,(E_r - v_s\,B_\phi) \\ &= \frac{e\lambda}{2\pi\epsilon_0}\left(1-\beta^2\right)\frac{r}{a^2} \\ &= \frac{e\lambda}{2\pi\epsilon_0}\frac{1}{\gamma^2}\frac{r}{a^2} \end{aligned}$$

The direct space charge force is linear in x and in y:

$$F_x = \frac{e\lambda}{2\pi\epsilon_0\gamma^2 a^2}\,x \qquad F_y = \frac{e\lambda}{2\pi\epsilon_0\gamma^2 a^2}\,y$$

Example: coasting beam with uniform charge density

- Assume a coasting beam of circular cross section with radius a and uniform charge density $\rho = \lambda/(\pi\, a^2)$ [C/m^3] moving at constant velocity $v = \beta c$
- In this case the direct space charge force is linear in x and y



$F_x = \dfrac{e\lambda}{2\pi\epsilon_0\gamma^2 a^2}\, x$	$F_y = \dfrac{e\lambda}{2\pi\epsilon_0\gamma^2 a^2}\, y$

Direct space charge is **like a defocusing quadrupole**…

… however, direct space charge is always **defocusing in both planes**, while quadrupole is focusing in one and defocusing in the other plane

Direct space charge tune shift

- Since the uniformly charged coasting beam acts **like an additional quadrupole**, it will contribute to the normal transverse focusing with an **additional quadrupole focusing term**:

    - Hill's equation

$$y'' + \left(K_y(s) + K_y^{SC}(s)\right)\, y = 0$$

$$F_y = \frac{e\lambda}{2\pi\epsilon_0\gamma^2 a^2}\, y\,, \qquad r_0 = \frac{e^2}{4\pi\epsilon_0 mc^2}$$

Linear force — Classical particle radius

    - Extra focusing term:

$$K_y^{SC}(s) = -\frac{1}{m\gamma\beta^2c^2}\frac{F_y^{\mathrm{SC}}}{y} = -\frac{2r_0\lambda}{e\beta^2\gamma^3a^2(s)}$$

    - Tune shift:

$$\Delta Q_y = \frac{1}{4\pi}\oint K_y^{SC}(s)\beta_y(s)\,ds = -\frac{1}{4\pi}\oint \frac{2r_0\lambda\beta_y(s)}{e\beta^2\gamma^3\,a^2(s)}\,ds = -\frac{r_0 R\lambda}{e\beta^2\gamma^3}\left\langle \frac{\beta_y(s)}{a^2(s)} \right\rangle$$

Tune shift of a particle subject to the direct space charge fields of a uniform charge distribution

Direct space charge tune shift

After some reshuffling we notice some of the **fundamental properties of the direct space charge tune shift**:

$$\left.\begin{aligned} \Delta Q_{x,y} &= -\frac{r_0 R \lambda}{e\beta^2\gamma^3}\left\langle \frac{\beta_{x,y}(s)}{a^2(s)} \right\rangle \\ a(s) &= \sqrt{\frac{\beta_{x,y}(s)\hat{\varepsilon}^n_{x,y}}{\beta\gamma}} \end{aligned}\right\} \Longrightarrow \boxed{\Delta Q_{x,y} = -\frac{r_0 R \lambda}{e\beta\gamma^2 \hat{\varepsilon}^n_{x,y}}}$$

$$r_0 = \frac{e^2}{4\pi\epsilon_0 mc^2} = \begin{cases} 1.54 \cdot 10^{-18} \text{ m (proton)} \\ 2.82 \cdot 10^{-15} \text{ m (electron)} \end{cases}$$

- is negative, because space charge transversely always defocuses
- is proportional to the line density and thus to the number of particles in the beam
- decreases with energy like $1/(\beta\gamma^2)$ (when expressed in terms of normalized emittance) and therefore vanishes in the ultra-relativistic limit
- does not depend on the local beta functions or beam sizes but is inversely proportional to the normalized emittance (here the emittance includes all particles!)

We have seen space charge as a first real collective effect. We learned that there **is the direct and the indirect space charge effect**. We looked at the case of a **circular uniformly charged coasting beam** to study an example of the direct space charge effect. We learned that the induced **direct space charge fields induce tune shifts** on witness particles.

We are ready to summarize our findings from this first lecture…

- Part I: Multi-particle effects – direct space charge
  - Multi-particle systems and their representation
  - Incoherent and coherent motion
  - Space charge

In this lecture, we have learned about the **dynamics and the representation of multiparticle systems**. We have seen how we differentiate between **incoherent and coherent motion**. Linked to this, we looked at the phenomenon of **filamentation with decoherence and emittance blow-up**.

We also discussed a first collective effect – **direct space charge**. We saw that direct space charge, for the case of a uniform coasting beam, leads to a shift of the tune of all witness particles.

Next, we will see the **space charge induced tune footprint** and look at some **mitigation methods** that can be put in place against direct space charge. Finally, we will discuss some of the effects of **indirect space charge**.

- Part I: Multi-particle effects – direct space charge
    - Multi-particle systems and their representation
    - Incoherent and coherent motion
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End part 1

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