A first taste of Non-Linear Beam Dynamics

Yannis PAPAPHILIPPOU
Accelerator and Beam Physics group
Beams Department
CERN

CERN Accelerator School
Introduction to Accelerator Physics 2019
Atrium Hotel, Vysoké Tatry, Slovakia
19-20 September 2019
Purpose of the lecture

- Introducing aspects of non-linear dynamics
  - Mathematical tools for modelling nonlinear dynamics
    - Power series (Taylor) maps and symplectic maps.
  - Effects of nonlinear perturbations
    - Resonances, tune shifts, dynamic aperture.
  - Analysis methods:
    - Normal forms, frequency map analysis.

- Employ two types of accelerator systems for illustrating the methods and tools
  - Bunch compressor (a single-pass system)
  - A storage ring (a multi-turn system).
Aim of the 2nd Lecture

- Describe some of the **phenomena** associated with nonlinearities in periodic beamlines (such as storage rings)
- Explain significance of **symplectic maps**, and describe some of the challenges in calculating and applying symplectic maps
- Outline some of the **analysis methods** that can be used to characterise nonlinear beam dynamics in periodic beamlines.
Example of a periodic system: a simple storage ring
As example, consider the transverse dynamics in a simple storage ring, assuming:

- The storage ring is constructed from some number of identical cells consisting of dipoles, quadrupoles and sextupoles.
- The phase advance per cell can be tuned from close to zero, up to about $0.5 \times 2\pi$.
- There is one sextupole per cell, which is located at a point where the horizontal beta function is 1 m, and the alpha function is zero.

Usually, storage rings will contain (at least) two sextupoles per cell, to correct horizontal and vertical chromaticity. To keep things simple, we will use only one sextupole per cell.
Sextupoles are needed in a storage ring to compensate for the fact that quadrupoles have lower focusing strength for particles of higher energy:

The change in focusing strength with particle energy has undesirable consequences, especially in storage rings: it can lead to particle motion becoming unstable because of resonances.
A **sextupole** can be regarded as a **quadrupole** with focusing strength that **increases** with horizontal offset from the axis.

If **sextupoles** are located where there is **non-zero dispersion**, they can be used to control the **chromaticity** in a storage ring.
The **chromaticity**, and hence the sextupole strength, will normally be a **function** of the **phase advance**.

However, just to investigate the nonlinear effects of the sextupoles, we shall keep the **sextupole strength** $k_2 L$ **fixed**, and **change** only the **phase advance**.

We can assume that the **map** from one sextupole to the next is **linear**, and corresponds to a **rotation** in phase space through an angle equal to the phase advance:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \mapsto \begin{pmatrix} \cos \mu_x & \sin \mu_x \\ -\sin \mu_x & \cos \mu_x \end{pmatrix} \begin{pmatrix} x \\ p_x \end{pmatrix}$$

Again to keep things simple, we shall consider only **horizontal motion**, and assume that the vertical coordinate $y = 0$.
Recall that the **vertical field component** in a sextupole magnet is:

\[
\frac{B_y}{B\rho} = \frac{1}{2} k_2 x^2
\]

with \(B\rho\) the beam rigidity and the normalized sextupole gradient is

\[
k_2 = \frac{1}{B\rho} \frac{\partial^2 B_y}{\partial x^2}
\]

In the “thin lens” approximation, the **deflection** of a particle passing through the sextupole of length \(L\) is

\[
\Delta p_x = - \int \frac{B_y}{B\rho} ds = - \frac{1}{2} k_2 L x^2
\]
The map for a particle moving through a short \textbf{sextupole} can be represented by a "kick" in the horizontal momentum:

\[ x \mapsto x , \]

\[ p_x \mapsto p_x - \frac{1}{2} k_2 L x^2 \]

Let us choose a fixed value \( k_2 L = -600 \text{ m}^{-2} \), and look at the effects of the maps for different phase advances.

For each case, we construct a \textbf{phase space portrait} by plotting the values of the dynamical variables after repeated application of the map (rotation + sextupole) for a range of initial conditions.

First, let us look at the phase space portraits for a range of phase advances from \( 0.2 \times 2\pi \) to \( 0.5 \times 2\pi \).
Phase space: single sextupole

- **For small amplitudes**, particles trace out **closed loops** around the origin: this is what we expect for a purely **linear map**.

- As the **amplitude** is **increased**, there appear “**islands**” in phase space. The **phase advance** (for the linear map) is often close to $m/p$, where $m$ is an integer and $p$ is the number of islands.

- **Larger number** of islands appears at **larger amplitude**.
Usually, there is a **closed curve** that divides a region of **stable** from a region of **chaotic** motion.

Outside that curve, the **amplitude** of particles increases **without limit**

The area of the stable region **depends** strongly on the **phase advance**
- Usually, there is a **closed curve** that divides a region of **stable** from a region of **chaotic** motion.

- **Outside** that curve, the **amplitude** of particles increases **without limit**

- The area of the stable region **depends** strongly on the **phase advance**

- For **phase advance** close to $2\pi/3$, it appears that the **stable region** almost **vanishes** altogether
Usually, there is a **closed curve** that divides a region of **stable** from a region of **chaotic** motion.

**Outside** that curve, the **amplitude** of particles **increases without limit**.

The area of the stable region **depends** strongly on the **phase advance**.

When **phase advance** approaches $\pi$, the **stable area** becomes **large**, and distortions from the linear ellipse become less evident.
Usually, there is a **closed curve** that divides a region of **stable** from a region of **chaotic** motion.

**Outside** that curve, the **amplitude** of particles **increases without limit**.

The area of the stable region **depends** strongly on the **phase advance**.

When **phase advance** approaches \( \pi \), the **stable area** becomes **large**, and distortions from the linear ellipse become less evident.
Effect of phase advance on nonlinear dynamics
**Effect of phase advance**

- An important observation is that the **effect** of the sextupole in the periodic cell **depends strongly** on the **phase advance** across the cell.

- We can start to understand the significance of the phase advance by considering **two special cases**:
  - Phase advance equal to an **integer** times $2\pi$
  - Phase advance equal to a **half integer** times $2\pi$
Let us consider first a **phase advance** equal to an **integer**. In that case, the linear part of the map is just the identity

\[ x \mapsto x, \]

\[ p_x \mapsto p_x \]

So the **combined effect** of the **linear map** and the **sextupole kick** is:

\[ x \mapsto x, \]

\[ p_x \mapsto p_x - \frac{1}{2} k_2 L x^2 \]

Clearly, the **horizontal momentum** will **increase** without limit

There are **no stable regions** of phase space, apart from the line \( x = 0 \)
Now consider what happens if the phase advance of a cell is a **half integer** times $2\pi$, so the linear part of the map is just a rotation through $\pi$.

If a **particle** starts at the entrance of a sextupole with $x = x_0$ and $p_x = p_{x0}$, then at the **exit** of that sextupole:

$$x_1 = x_0,$$

$$p_{x1} = p_{x0} - \frac{1}{2}k_2Lx_0^2$$

Then, after passing to the **entrance** of the **next** sextupole, the co-ordinates will be:

$$x_2 = \cos(\pi)x_1 = -x_1 = -x_0,$$

$$p_{x2} = \cos(\pi)p_{x1} = -p_{x1} = -p_{x0} + \frac{1}{2}k_2Lx_0^2$$
Finally, on passing through the second sextupole:

\[ x_3 = x_2 = -x_0 , \]

\[ p_{x3} = p_{x2} - \frac{1}{2} k_2 L x_2^2 = -p_{x0} \]

In other words, the momentum kicks from the two sextupoles cancel each other exactly.

The resulting map is a purely linear phase space rotation by \( \pi \).

In this situation, we expect the motion to be stable (and periodic), no matter what the amplitude.
The effect of the phase advance on the sextupole “kicks” is similar to the effect on perturbations arising from dipole and quadrupole errors in a storage ring.

In the case of dipole errors, the kicks add up if the phase advance is an integer, and cancel if the phase advance is a half integer.
In the case of **quadrupole errors**, the kicks add up if the phase advance is a **half integer**.

**Higher-order multipoles** drive **higher-order resonances** but the effects are less easily illustrated on a phase space diagram.
Resonances
If we include vertical as well as horizontal motion, then we find that resonances occur when the tunes satisfy

\[ n_x Q_x + n_y Q_y = r \]

where \( n_x, n_y \) and \( r \) are integers and resonance is of order \( |n_x| + |n_y| \)

Resonances up to order 2

- normal resonances (= even \( n_y \))
- skew resonances (= odd \( n_y \))
If we include vertical as well as horizontal motion, then we find that resonances occur when the tunes satisfy

\[ n_x Q_x + n_y Q_y = r \]

where \( n_x, n_y \) and \( r \) are integers and resonance is of order \(|n_x| + |n_y|\).

**Resonances up to order 3**

- normal resonances (= even \( n_y \))
- skew resonances (= odd \( n_y \))
If we include vertical as well as horizontal motion, then we find that resonances occur when the tunes satisfy

\[ n_x Q_x + n_y Q_y = r \]

where \( n_x \), \( n_y \) and \( r \) are integers and resonance is of order \(|n_x| + |n_y|\).

Resonances up to order 4

- normal resonances (= even \( n_y \))
- skew resonances (= odd \( n_y \))
If we include vertical as well as horizontal motion, then we find that resonances occur when the tunes satisfy

\[ n_x Q_x + n_y Q_y = r \]

where \( n_x \), \( n_y \) and \( r \) are integers and resonance is of order \( |n_x| + |n_y| \)

Resonances up to order 5

- normal resonances (= even \( n_y \))
- skew resonances (= odd \( n_y \))
Resonances in storage ring

- **Resonances** are associated with **chaotic motion** for particles in storage rings.

- However, the number of **resonance lines** in tune space is **infinite**: any point in tune space will be close to a resonance of some order.

- This observation raises two questions:
  - How do we know what the real effect of any given resonance line will be?
  - How can we design a storage ring to minimise the adverse effects of resonances?

- These are not easy questions to answer and usually necessitate numerical integration of the equations of motion.

- We shall discuss some of the issues in the remaining parts of this lecture.
Resonance cancellation by periodicity

- By imposing a **periodicity** $P$ in the lattice (i.e. building a machine from $P$ identical cells) the resonance condition becomes

  $$n_x Q_x + n_y Q_y = rP$$

- Resonances for which $r \times P$ is integer $\rightarrow$ systematic

- If $r \times P$ is NOT integer the resonance cancels $\rightarrow$ non-systematic

### Periodicity P=1

- Solid lines: normal resonances
- Dashed lines: skew resonances

### Periodicity P=2

### Periodicity P=3
Advanced Light Source design lattice periodicity: 12

Measurement of beam loss as function of tune

Synchrotron light beam spot

Uncorrected optics

Corrected optics

**Beta beating**

Before optics correction: ~30%

After optics correction: <1%

Real life example for periodicity: SPS

- SPS (hadron machine) has design lattice **periodicity** of 6

- Some indication for the **strength of individual resonance lines** can be inferred from the **beam loss rate** during dynamic tune scans, i.e. the derivative of the beam intensity at the moment of resonance crossing

- **Sextupole resonances** can be clearly identified although they should be suppressed by lattice periodicity ... but **SPS has no individual quadrupoles** to restore optics functions distortions

![Measured losses during tune scan](image1)

![Measured loss rate in 2D scan](image2)
Analytical methods for nonlinear dynamics
There are two approaches widely used in accelerator physics: perturbation theory and normal form analysis.

In both these techniques, the goal is to construct a quantity that is invariant under application of the single-turn transfer map. Unfortunately, in both cases the mathematics is complicated and fairly cumbersome.

In the case of a single sextupole in a storage ring, we find from normal form analysis the following expression for the betatron action as a function of the betatron phase (angle variable):

$$J_x \approx I_0 - \frac{k2L}{8} \left(2\beta_x I_0\right)^{3/2} \frac{\cos(3\mu_x/2 + 2\phi_x) + \cos(\mu_x/2)}{\sin(3\mu_x/2)} + O(I_0^2)$$

where $I_0$ is a constant (an invariant of the motion), $\phi_x$ is the angle variable, and $\mu_x$ is the phase advance per cell.

The second term becomes very large when $\mu_x$ is close to third integer.
Normal form for sextupole

phase advance $\mu_x = 0.28 \times 2\pi$
Normal form for sextupole

phase advance $\mu_x = 0.30 \times 2\pi$
phase advance $\mu_x = 0.315 \times 2\pi$
Close inspection of the plots on the previous slides reveals another effect, in addition to the obvious distortion of the phase space ellipses: the **phase advance** per turn (i.e. the tune) **varies** with increasing **betatron amplitude**.

Normal form analysis (and perturbation theory) can be used to obtain **estimates** for the **tune shift** with amplitude.

In the case of a sextupole, the **tune shift** is **higher-order** in the **sextupole strength**.

An octupole, however, does have a **first-order** in the octupole strength **tune shift** with amplitude, given by:

\[
\nu_x \approx \nu_{x0} + \frac{k3L\beta_x^2}{16\pi} J_x + O(J_x^2)
\]
Particle trapped in 4\textsuperscript{th} order resonance

- Simulation of simple storage ring with a \textbf{single octupole} close to 4\textsuperscript{th} order resonance
- **Detuning with amplitude** (linear in action)
- Particles in the stable islands have tune \textbf{locked} to resonance

![Graph showing particle behavior](image-url)
Simulation of simple storage ring with a **sextupole** and an **octupole** close to 3rd order resonance

The **amplitude detuning** induced by the octupole can create **stable islands** even for the 3rd order resonance

The tune of particles in islands is locked to the resonance
Non-linear map representation
Taylor maps

- For any dynamical variable $z_j$ the **Taylor map** up to 3rd order can be written as

$$z_j^{\text{new}} = \sum_{k=1}^{6} R_{jk} z_k + \sum_{k=1}^{6} \sum_{l=1}^{6} T_{jkl} z_k z_l + \sum_{k=1}^{6} \sum_{l=1}^{6} \sum_{m=1}^{6} U_{jklm} z_k z_l z_m$$

- **Taylor series** provide a convenient way of systematically representing **transfer maps** for **beamline components**, or sections of beamline.

- The **main drawback** of Taylor series is that in general, transfer maps can only be represented exactly by series with an **infinite number** of terms.

- In practice, we have to **truncate** a **Taylor map** at some order, and we then lose certain desirable properties of the map.

- In particular, a **truncated map** will be usually be **non-symplectic**.
Symplectic maps

- Consider two sets of canonical variables $\mathbf{z}, \mathbf{\bar{z}}$ which may be even considered as the evolution of the system between two points in phase space.

- A transformation from the one to the other set can be constructed through a map $\mathcal{M} : \mathbf{z} \mapsto \mathbf{\bar{z}}$.

- The Jacobian matrix of the map $M = M(\mathbf{z}, t)$ is composed by the elements $M_{ij} \equiv \frac{\partial \bar{z}_i}{\partial z_j}$.

- The map is symplectic if $M^T J M = J$, with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

- It can be shown that $\det(M) = 1$.

- Physically, a symplectic transfer map conserves phase space volumes when the map is applied.

- This is Liouville’s theorem, and is a property of charged particles moving in electromagnetic fields, in the absence of radiation.
The effect of **losing symplecticity** becomes apparent if we compare phase space portraits constructed using symplectic (below, left) and non-symplectic (below, right) transfer maps.
Consider a sextupole with equations of motion:
\[
\frac{dx}{ds} = px, \quad \frac{dp_x}{ds} = -\frac{1}{2}k_2x^2.
\]

Exact solutions using some elementary functions do not exist.

By splitting integration into three steps, it is possible to write an explicit and symplectic approximate solution:

\[
\begin{align*}
0 \leq s < \frac{L}{2} : & \quad x_1 = x_0 + px_0, \quad p_{x1} = px_0, \\
\frac{s}{2} = \frac{L}{2} : & \quad x_2 = x_1, \quad p_{x2} = p_{x1} - \frac{1}{2}k_2Lx_1^2, \\
\frac{L}{2} < s \leq L : & \quad x_3 = x_2 + px_2, \quad p_{x3} = p_{x2}.
\end{align*}
\]

This an example of a symplectic integrator known as a “drift–kick–drift” approximation.
Numerical methods: Dynamic aperture
The most direct way to evaluate the non-linear dynamics performance of a ring is the computation of **Dynamic Aperture** (short: DA)

Particle motion due to multi-pole errors is generally **non-bounded**, so chaotic particles can escape to infinity. This is not true for all non-linearities (e.g. the beam-beam force)

Need a **symplectic tracking code** to follow particle trajectories (a lot of initial conditions) for a number of turns (depending on the given problem) until the particles start getting lost. This boundary defines the **Dynamic aperture**

As multi-pole errors may not be completely known, one has to track through several machine models built by **random distribution** of these errors

One could start with **4D (only transverse)** tracking but certainly needs to simulate **5D (constant energy deviation)** and finally **6D (synchrotron motion included)**
Dynamic aperture plots show the **maximum initial values** of **stable trajectories** in x-y coordinate space at a particular point in the lattice, for a range of energy errors.

- The **beam size** can be shown on the same plot.

- Generally, the goal is to allow some **significant margin** in the design – the measured dynamic aperture is often smaller than the predicted dynamic aperture.
Numerical methods: Frequency map analysis
Building the frequency map

- Choose coordinates \((x_i, y_i)\) with \(p_x = p_y = 0\)
- Numerically integrate the phase trajectories through the lattice for sufficient number of turns
- Compute through advanced Fourier methods (\textbf{NAFF} algorithm) \(Q_x\) and \(Q_y\) after sufficient number of turns
- Plot them in the tune diagram

![Diagram showing the frequency map with coordinates and trajectories]
- All dynamics represented in two plots (Frequency Map / Diffusion Map)
- **Regular motion** represented by blue colors
- **Resonances** appear as distorted lines in frequency space (or curves in initial condition space)
- **Chaotic motion** is represented by red scattered particles and defines dynamic aperture of the machine
- FMA shows also nicely the detuning with amplitude
Frequency map analysis for HL-LHC in collision

Large tune footprint and DA reduction due to “long range beam-beam” forces (electromagnetic field of other beam in interaction region)

DA clearly improved when compensating long range beam-beam with a wire

S. Fartoukh et al., PRSTAB, 2015
Experimental frequency maps

- Frequency analysis of **turn-by-turn data** of beam oscillations produced by a **fast kicker magnet** and recorded on a **Beam Position Monitor**

- Reproduction of the **non-linear model** of the **Advanced Light Source** storage ring and working point optimization for increasing beam lifetime

Conclusions and Summary
Nonlinear dynamics appear in a wide variety of accelerator systems, including single-pass systems (such as bunch compressors) and multi-turn systems (such as storage rings).

It is possible to model nonlinear dynamics in a given component or section of beamline by representing the transfer map as a power series.

Conservation of phase space volumes is an important feature of the beam dynamics in many systems. To conserve phase space volumes, transfer maps must be symplectic.

In general, (truncated) power series maps are not symplectic.

To construct a symplectic transfer map, the equations of motion in a given accelerator component must be solved using a symplectic integrator (e.g. the “drift–kick–drift” approximation for a multipole magnet).
**Summary**

- **Common features** of nonlinear dynamics in accelerators include **phase space distortion**, **tune shifts with amplitude**, **resonances**, and **chaotic particle trajectories at large amplitudes** (**dynamic aperture** limits)

- Analytical methods such as **perturbation theory** and **normal form** analysis can be used to estimate the impact of nonlinear perturbations in terms of quantities such as **resonance strengths** and **tune shifts with amplitude**

- **Frequency map analysis** provides a useful numerical tool for characterising tune shifts and resonance strengths from tracking data.

- This can give some **insight** into **limitations** on the **dynamic aperture**