
Advanced Accelerator Physics Course

Slangerup, Denmark.

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# Low Emittance Lattices

## Part 2: Equilibrium Emittance and Storage Ring Lattice Design

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# Lecture 1 summary

In Lecture 1, we:

- discussed the effects of synchrotron radiation on the (linear) motion of particles in storage rings;
- derived expressions for the damping times of the longitudinal, vertical and horizontal emittances;
- discussed the effects of quantum excitation;
- derived expressions for the equilibrium horizontal and longitudinal emittances in an electron storage ring in terms of the lattice functions and beam energy.

In this lecture, we shall:

- derive expressions for the natural (horizontal) emittance in four types of lattice:
  - FODO;
  - double-bend achromat (DBA);
  - multi-bend achromats (MBA);
  - theoretical minimum emittance (TME).

- discuss some aspects of the vertical emittance, including:
  - the fundamental lower limit;
  - generation of vertical emittance from betatron coupling and vertical dispersion;
  - calculation of the emittance in the presence of machine errors;
  - low-emittance tuning.

Our first goal is to calculate the natural emittance in a lattice with magnets of given strengths, lengths and positions.

In Lecture 1, we showed that the natural emittance in a storage ring is given by:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2}, \tag{1}$$

where $C_q$ is the quantum radiation constant, $\gamma$ is the relativistic factor, $j_x$ is the horizontal damping partition number, and $I_2$ and $I_5$ are the second and fifth synchrotron radiation integrals.

Note that $j_x$, $I_2$ and $I_5$ are all functions of the lattice, and are independent of the beam energy.

In most storage rings, if the bends have no quadrupole component, the horizontal damping partition number $j_x \approx 1$.

In this case, to find the natural emittance we just need to evaluate the two synchrotron radiation integrals:

$$I_2 = \oint \frac{1}{\rho^2}\, ds, \qquad I_5 = \oint \frac{\mathcal{H}_x}{|\rho|^3}\, ds. \tag{2}$$

If we know the strength and length of all the dipoles in the lattice, it is straightforward to calculate $I_2$.

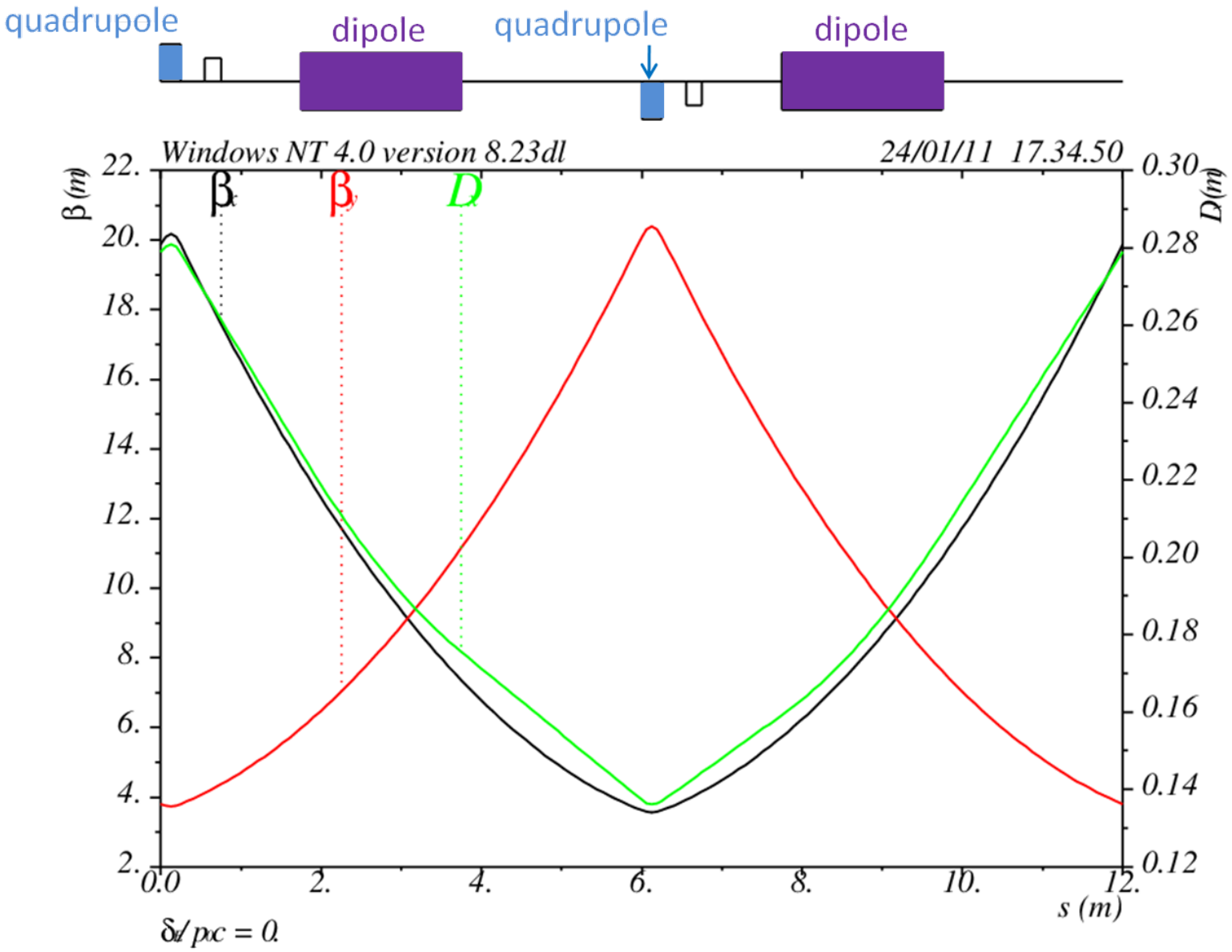
For example, if all the bends are identical, then in a complete ring (total bending angle $= 2\pi$):

$$I_2 = \oint \frac{1}{\rho^2} ds = \oint \frac{B}{(B\rho)} \frac{ds}{\rho} = 2\pi \frac{B}{(B\rho)} \approx 2\pi \frac{cB}{E/e}, \tag{3}$$

where $E$ is the beam energy.

$I_5$ is more complicated: it depends on the lattice functions...

# Case 1: natural emittance in a FODO lattice



Let us consider the case of a FODO lattice. To simplify the system, we use the following approximations:

- the quadrupoles are represented as thin lenses;
- the space between the quadrupoles is completely filled by the dipoles.

With these approximations, the lattice functions (Courant–Snyder parameters and dispersion) are completely determined by the following parameters:

- the focal length $f$ of a quadrupole;
- the bending radius $\rho$ of a dipole;
- the length $L$ of a dipole.

From the evolution of the lattice functions through a given FODO cell, we can find an (approximate) expression for $I_5/I_2$: the details are given in Appendix A.

For small $\theta$, and if $\rho \gg 2f$ (which is often the case) we find:

$$\frac{I_5}{I_2} \approx \left(1 - \frac{L^2}{16f^2}\right) \frac{8f^3}{\rho^3}. \tag{4}$$

This result can be further simplified if $4f \gg L$ (which is *not* always the case):

$$\frac{I_5}{I_2} \approx \frac{8f^3}{\rho^3}. \tag{5}$$

Making the approximation $j_x \approx 1$ (if there is no quadrupole component in the dipole), and writing $\rho = L/\theta$, we have:

$$\varepsilon_0 \approx C_q \gamma^2 \left(\frac{2f}{L}\right)^3 \theta^3. \tag{6}$$

Notice how the emittance scales with the beam and lattice parameters. The emittance:

- is proportional to the square of the energy;
- is proportional to the cube of the bending angle – increasing the number of cells in a storage ring reduces the dipole bending angle, and reduces the emittance;
- is proportional to the cube of the quadrupole focal length: stronger quads means lower emittance.
- is inversely proportional to the cube of the cell length.

The phase advance in a FODO cell is given by:

$$\cos(\mu_x) = 1 - \frac{L^2}{2f^2}. \tag{7}$$

This means that a stable lattice must have:

$$\frac{f}{L} \geq \frac{1}{2}. \tag{8}$$

In the limiting case, $\mu_x = 180^\circ$, and $f$ has the minimum value $f = L/2$. Using the approximation (6):

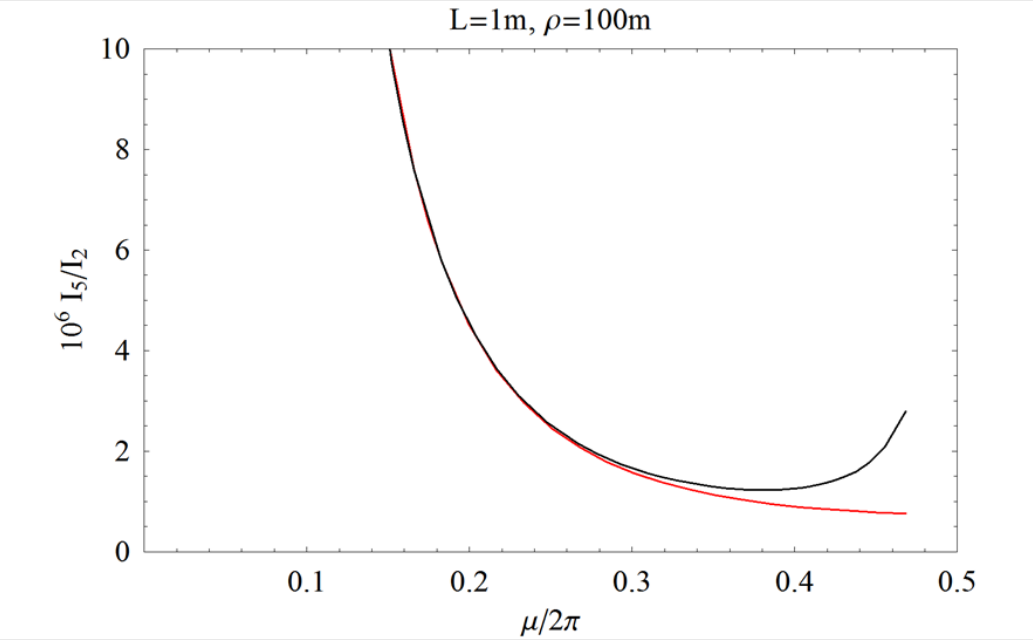
$$\varepsilon_0 \approx C_q \gamma^2 \left(\frac{2f}{L}\right)^3 \theta^3,$$

the *minimum emittance in a FODO lattice* is expected to be:

$$\varepsilon_0 \approx C_q \gamma^2 \theta^3. \tag{9}$$

However, as we increase the focusing strength, the approximations we used to obtain the simple expression for $\varepsilon_0$ start to break down...

Plotting the exact formula for $I_5/I_2$ as a function of the phase advance, we find that there is a minimum in the natural emittance, at $\mu_x \approx 137°$.




Black line:
exact formula.

Red line:
approximation,

$$\frac{I_5}{I_2} \approx \left(1 - \frac{L^2}{16f^2}\right)\frac{8f^3}{\rho^3}.$$

It turns out that the minimum value of the natural emittance in a FODO lattice is given by:

$$\varepsilon_{0,\mathrm{FODO,min}} \approx 1.2 C_q \gamma^2 \theta^3. \qquad (10)$$

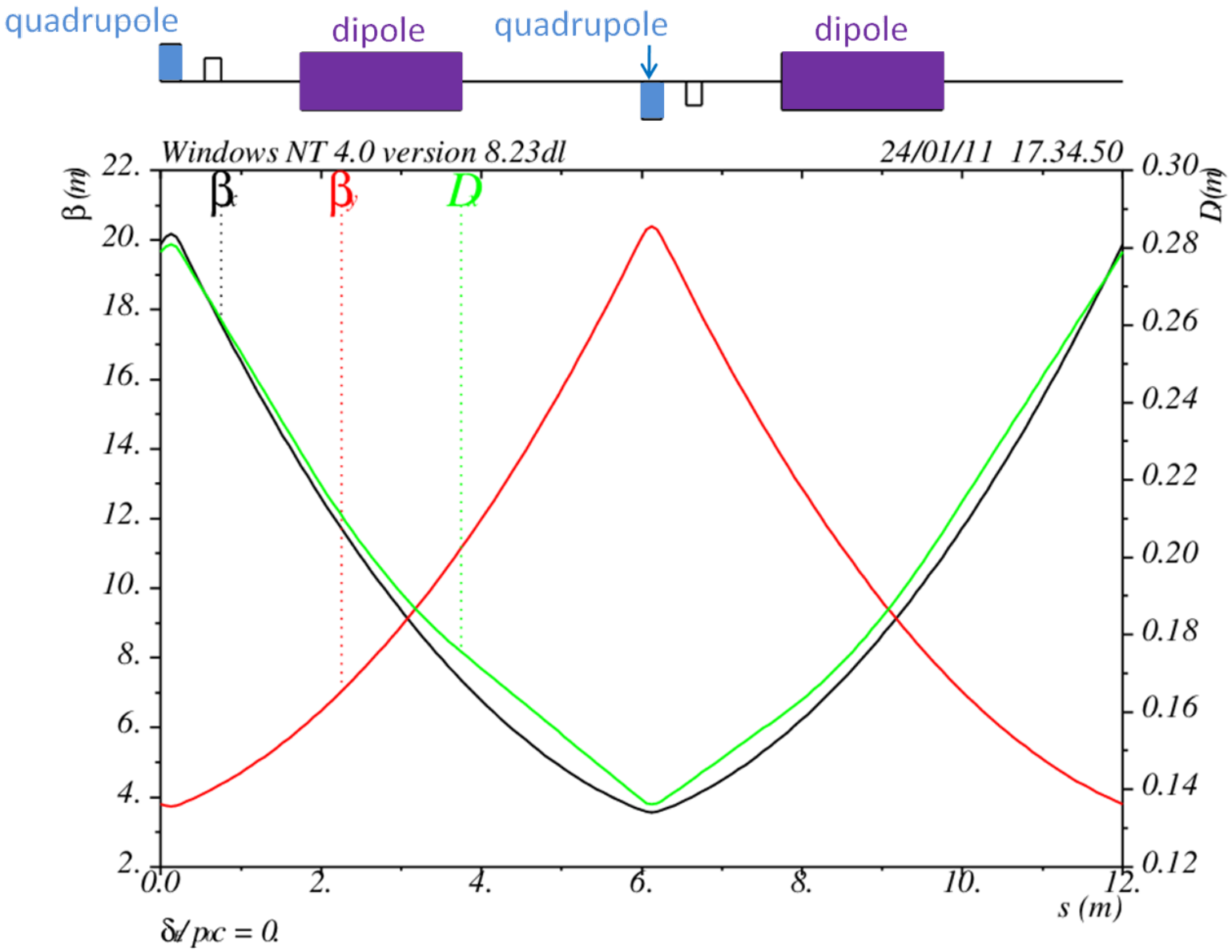
Using Eq. (6), we estimate that a storage ring constructed from 16 FODO cells (32 dipoles) with 90° phase advance per cell ($f = L/\sqrt{2}$), and storing beam at 2 GeV would have a natural emittance of around 125 nm.

Many modern applications (including synchrotron light sources) demand emittances smaller by one or two orders of magnitude.

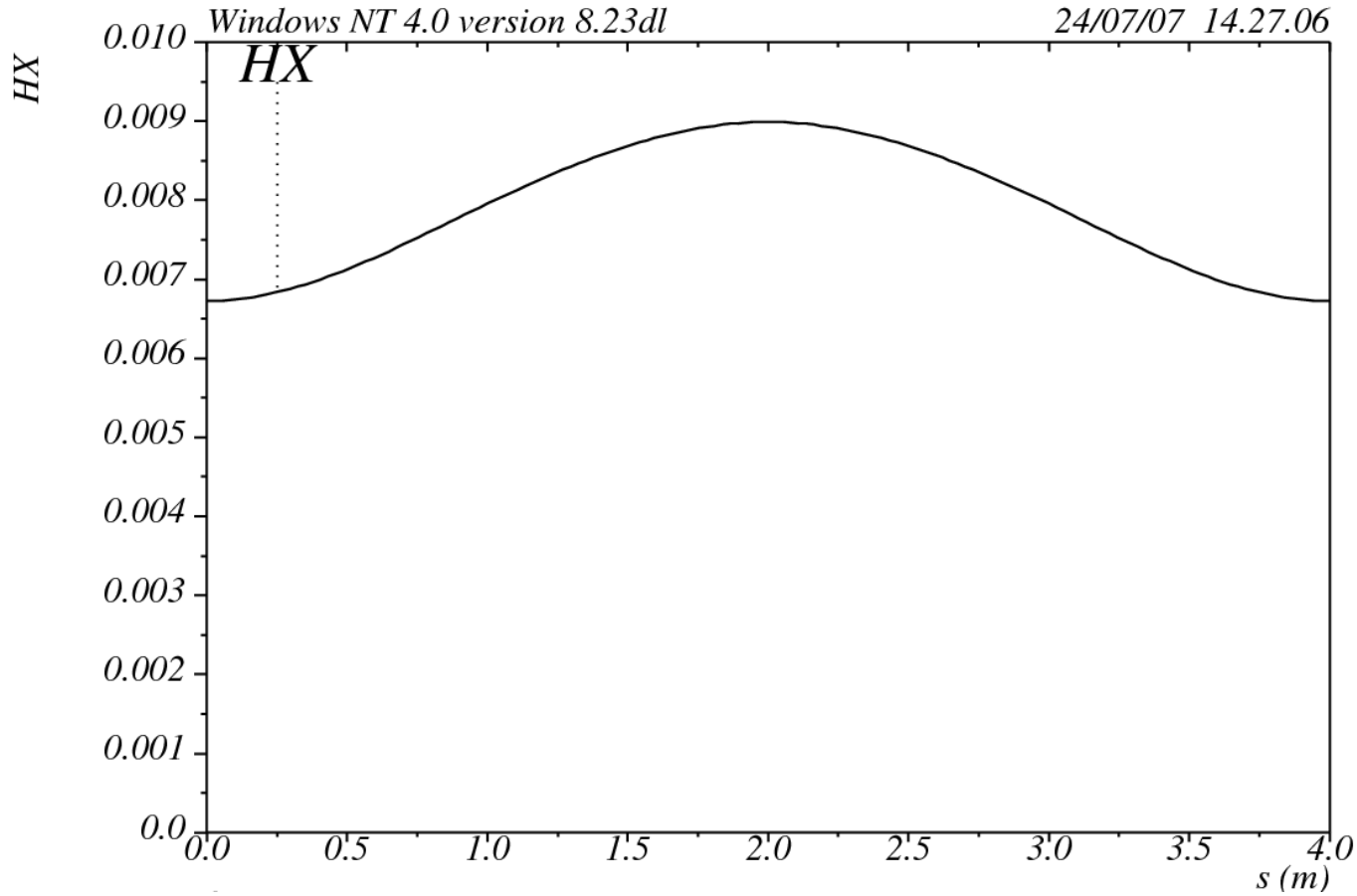
How can we design a lattice with a smaller natural emittance?

Looking at the function $\mathcal{H}_x$ ("curly-H") in a FODO cell provides a clue...

# Case 1: natural emittance in a FODO lattice

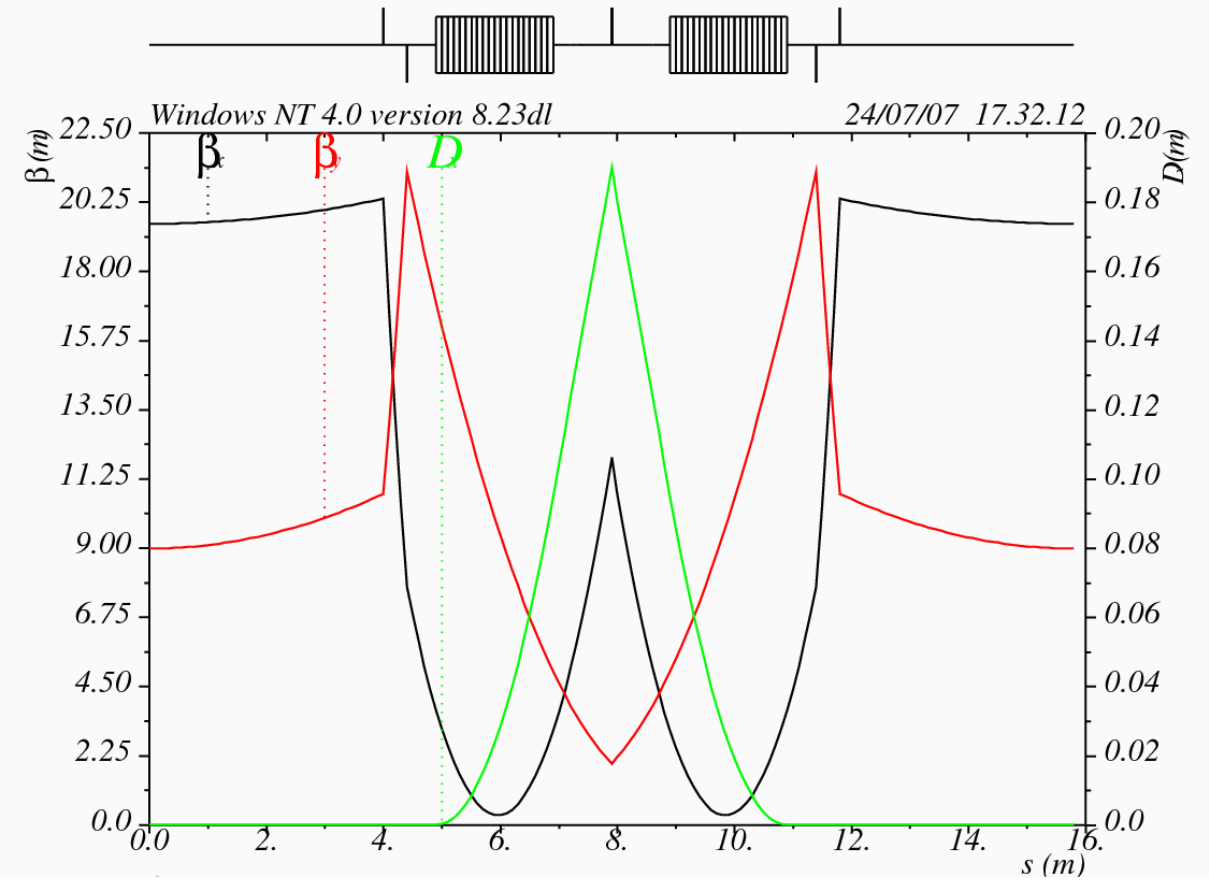


The function $\mathcal{H}_x$ remains at a relatively constant value throughout the lattice:



As a first attempt at reducing the natural emittance, we can try reducing the function $\mathcal{H}_x$ in the dipoles, by designing a lattice that has zero dispersion at either end of a dipole pair.

The result is a double bend achromat (DBA) cell:



To calculate the natural emittance in a DBA, let us begin by considering the conditions for zero dispersion at the start and the exit of the cell.

Assume that the dispersion is zero at the start of the cell.

Place a quadrupole midway between the dipoles, to reverse the gradient of the dispersion.

By symmetry, the dispersion at the exit of the cell will be zero.

In the thin lens approximation, the focal length of the quadrupole can be found from:

$$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} \eta_x \\ \eta_{px} \end{pmatrix} = \begin{pmatrix} \eta_x \\ \eta_{px} - \frac{\eta_x}{f} \end{pmatrix} = \begin{pmatrix} \eta_x \\ -\eta_{px} \end{pmatrix}, \qquad (11)$$

where $\eta_x$ and $\eta_{px}$ are the dispersion and gradient of the dispersion at the quadrupole.

Hence the central quadrupole must have focal length:

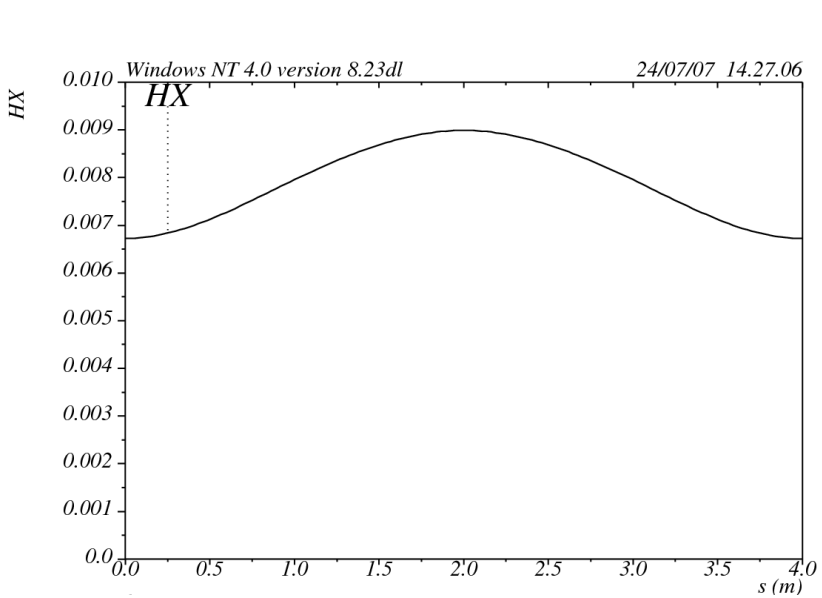
$$f = \frac{\eta_x}{2\eta_{px}}. \qquad (12)$$

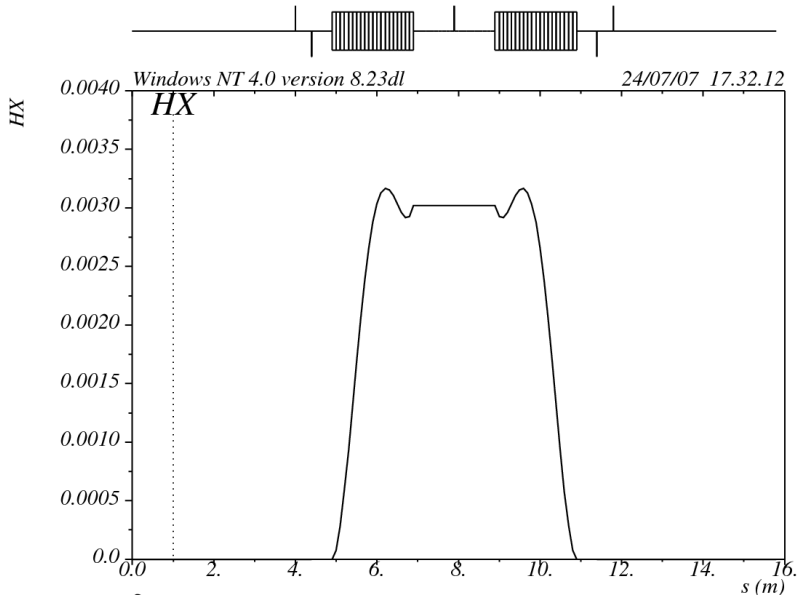
The dispersion is determined by the dipole bending angle $\theta$, the bending radius $\rho$, and the drift length $L$:

$$\eta_x = \rho(1 - \cos(\theta)) + L\sin(\theta), \qquad \eta_{px} = \sin(\theta). \qquad (13)$$

Is this style of lattice likely to have a lower natural emittance than a FODO lattice? We can get some idea by looking at the function $\mathcal{H}_x$...

# Case 2: natural emittance in a DBA lattice




The function $\mathcal{H}_x$ is much smaller in the DBA lattice (right) than in the FODO lattice (left).

Note that we use the same dipoles (bending angle and length) in both cases.

Let us calculate the minimum natural emittance of a DBA lattice, for given bending radius $\rho$ and bending angle $\theta$ in the dipoles.

To do this, we need to calculate the minimum value of:

$$I_5 = \int \frac{\mathcal{H}_x}{\rho^3}\, ds \tag{14}$$

in one dipole, subject to the constraints:

$$\eta_{x,0} = \eta_{px,0} = 0, \tag{15}$$

where $\eta_{x,0}$ and $\eta_{px,0}$ are the dispersion and gradient of the dispersion at the entrance of a dipole.

We know how the dispersion and the Courant–Snyder parameters evolve through the dipole, so we can calculate $I_5$ for one dipole, for given initial values of the Courant–Snyder parameters $\alpha_{x,0}$ and $\beta_{x,0}$.

Then, we simply have to minimise the value of $I_5$ with respect to $\alpha_{x,0}$ and $\beta_{x,0}$.

Again, the algebra is rather formidable, and the full expression for $I_5$ is not especially enlightening.

Therefore, we just quote the significant results...

We find that, for given $\rho$ and $\theta$ and with the constraints:

$$\eta_{x,0} = \eta_{px,0} = 0, \tag{16}$$

the minimum value of $I_5$ is given by:

$$I_{5,\mathrm{min}} = \frac{1}{4\sqrt{15}}\frac{\theta^4}{\rho} + O(\theta^6). \tag{17}$$

This minimum occurs for values of the Courant–Snyder parameters at the entrance to the dipole given by:

$$\beta_{x,0} = \sqrt{\frac{12}{5}}L + O(\theta^3), \qquad \alpha_{x,0} = \sqrt{15} + O(\theta^2), \tag{18}$$

where $L = \rho\theta$ is the length of a dipole.

Since we know that $I_2$ in a single dipole is given by:

$$I_2 = \int \frac{1}{\rho^2}\, ds = \frac{\theta}{\rho}, \tag{19}$$

we can now write down an expression for the minimum emittance in a DBA lattice:

$$\varepsilon_{0,\mathrm{DBA,min}} = C_q \gamma^2 \frac{I_{5,\mathrm{min}}}{j_x I_2} \approx \frac{1}{4\sqrt{15}} C_q \gamma^2 \theta^3. \tag{20}$$

The approximation is valid for small $\theta$. Note that we have again assumed that there is no quadrupole component in the dipole, so $j_x \approx 1$.

Compare the above expression with that for the minimum emittance in a FODO lattice (10):

$$\varepsilon_{0,\mathrm{FODO,min}} \approx 1.2 C_q \gamma^2 \theta^3. \tag{21}$$

We see that in both cases (FODO and DBA), the emittance scales with the square of the beam energy, and with the cube of the bending angle.

However, the emittance in a DBA lattice is smaller than that in a FODO lattice (for given energy and dipole bending angle) by a factor of roughly $4\sqrt{15} \approx 15.5$.

This is a significant improvement... *but can we do even better?*

For a DBA lattice, we imposed the constraints:

$$\eta_{x,0} = \eta_{px,0} = 0. \tag{22}$$

To get a lower emittance, we can consider relaxing these constraints.

To derive the conditions for a "theoretical minimum emittance" (TME) lattice, we write down an expression for:

$$I_5 = \int \frac{\mathcal{H}_x}{\rho}\, ds, \tag{23}$$

with *arbitrary* dispersion $\eta_{x,0}$, $\eta_{px,0}$ and Courant–Snyder parameters $\alpha_{x,0}$ and $\beta_{x,0}$ in a dipole with given bending radius $\rho$ and angle $\theta$.

Then, we minimise $I_5$ with respect to $\eta_{x,0}$, $\eta_{px,0}$, $\alpha_{x,0}$ and $\beta_{x,0}$...

The result is:

$$\varepsilon_{0,\mathrm{TME,min}} \approx \frac{1}{12\sqrt{15}} C_q \gamma^2 \theta^3. \tag{24}$$

The minimum emittance is obtained with dispersion at the entrance to the dipole given by:

$$\eta_{x,0} = \frac{1}{6} L\theta + O(\theta^3), \qquad \eta_{px,0} = -\frac{\theta}{2} + O(\theta^3), \tag{25}$$

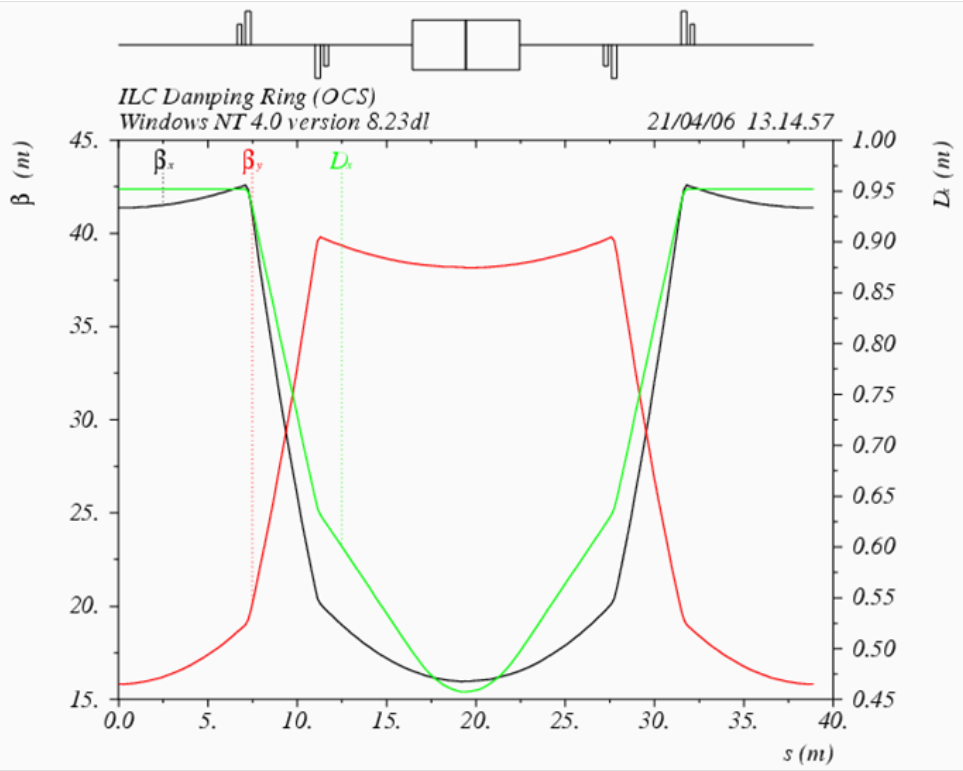
and with Courant–Snyder functions at the entrance:

$$\beta_{x,0} = \frac{8}{\sqrt{15}} L + O(\theta^2), \qquad \alpha_{x,0} = \sqrt{15} + O(\theta^2). \tag{26}$$

The dispersion and beta function reach minimum values in the centre of the dipole:

$$\eta_{x,\mathrm{min}} = \rho\left(1 - \frac{2}{\theta}\sin\left(\frac{\theta}{2}\right)\right) = \frac{L\theta}{24} + O(\theta^3), \quad \beta_{x,\mathrm{min}} = \frac{L}{2\sqrt{15}} + O(\theta^2). \tag{27}$$

By symmetry, we can consider a single TME cell to contain a single dipole, rather than a pair of dipoles as was necessary for the FODO and DBA cells.




Outside the dipole, the dispersion is relatively large.

This is not ideal for a light source, since insertion devices at locations with large dispersion will blow up the emittance.

Note that the cell shown here does not achieve the exact conditions for a TME lattice: a more complicated design would be needed for this.

## Summary: natural emittance in FODO, DBA and TME lattices

| Lattice style | Minimum emittance | Conditions/comments |
|---|---|---|
| 90° FODO | $\varepsilon_0 \approx 2\sqrt{2}C_q\gamma^2\theta^3$ | $\frac{f}{L} = \frac{1}{\sqrt{2}}$ |
| 137° FODO | $\varepsilon_0 \approx 1.2C_q\gamma^2\theta^3$ | minimum emittance FODO |
| DBA | $\varepsilon_0 \approx \frac{1}{4\sqrt{15}}C_q\gamma^2\theta^3$ | $\eta_{x,0} = \eta_{px,0} = 0$<br>$\beta_{x,0} \approx \sqrt{12/5}L \quad \alpha_{x,0} \approx \sqrt{15}$ |
| TME | $\varepsilon_0 \approx \frac{1}{12\sqrt{15}}C_q\gamma^2\theta^3$ | $\eta_{x,\mathrm{min}} \approx \frac{L\theta}{24} \quad \beta_{x,\mathrm{min}} \approx \frac{L}{2\sqrt{15}}$ |

The results we have derived have been for "ideal" lattices that perfectly achieve the stated conditions in each case.

Practical lattice designs rarely achieve the ideal conditions. In particular, the beta function in an achromat is usually not optimal for low emittance; and it is difficult to tune the dispersion for the ideal TME conditions.

The main reasons for this are:

- Beam dynamics phenomena (nonlinearities, collective effects...) generally impose a number of strong constraints on the design.
- Optimizing the lattice functions while meeting all the various constraints can require complex configurations of quadrupoles.

A particularly challenging constraint on design of a low-emittance lattice is the dynamic aperture.

Storage rings require a large dynamic aperture in order to achieve good injection efficiency and good beam lifetime.

However, low emittance lattices generally need low dispersion and beta functions, and hence require strong quadrupoles. As a result, the chromaticity can be large, and must be corrected by strong sextupoles.

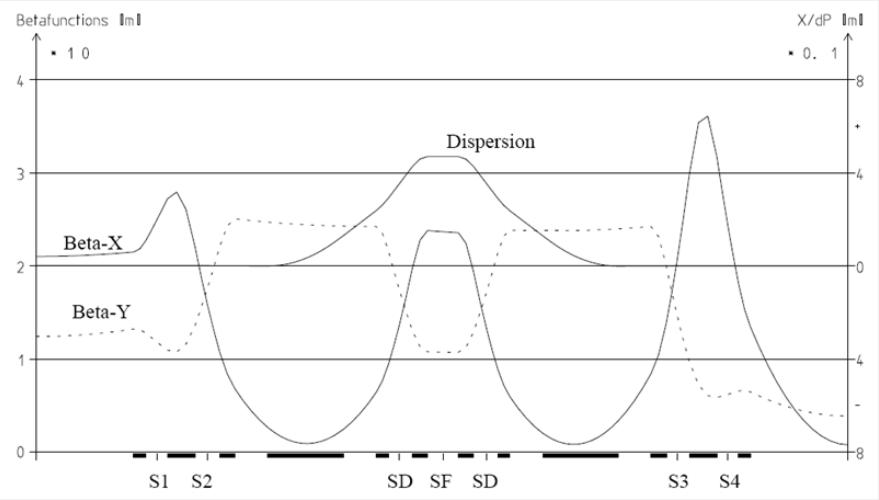
Strong sextupoles lead to strongly nonlinear motion, and limit the dynamic aperture (the trajectories of particles at large betatron amplitudes or large energy deviations become unstable).

There are many other options besides FODO, DBA and TME for the style of the lattice.

Here, we will discuss (briefly):

- the use of the DBA lattice in third-generation synchrotron light sources;
- "detuning" a DBA lattice to reduce the emittance;
- the use of multi-bend achromats.

Lattices composed of DBA cells have been a popular choice for third generation synchrotron light sources, e.g. the ESRF.




The DBA structure provides a lower natural emittance than a FODO lattice with the same number of dipoles.

The long, dispersion-free straight sections provide ideal locations for insertion devices such as undulators and wigglers.

## "Detuning" a DBA

If an insertion device (undulator or wiggler) is placed in a storage ring at a location with large dispersion, then the dipole fields in the device can make a significant contribution to the quantum excitation ($I_5$).

The insertion device can then increase the natural emittance of the storage ring.

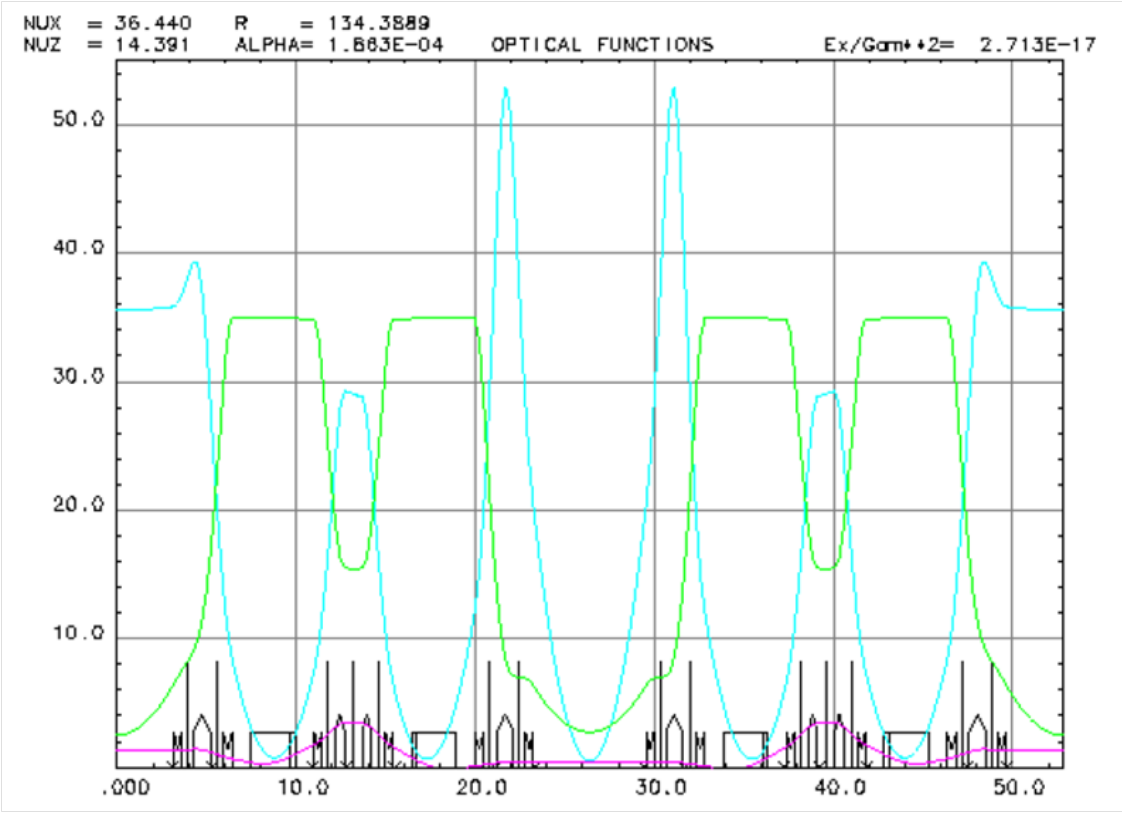
A DBA lattice provides dispersion-free straights in which undulators and wigglers can be placed without blowing up the natural emittance.

However, there is some tolerance. It is often possible to "detune" a lattice from the strict DBA conditions, allowing some reduction in natural emittance at the cost of some dispersion in the straights.

The insertion devices will then contribute to the quantum excitation; but depending on the lattice and the insertion devices, there can still be a net benefit.

Some light sources that were originally designed with zero-dispersion straights take advantage of tuning flexibility to operate with non-zero dispersion in the straights.

This provides a lower natural emittance, and better output for users. For example, the ESRF:



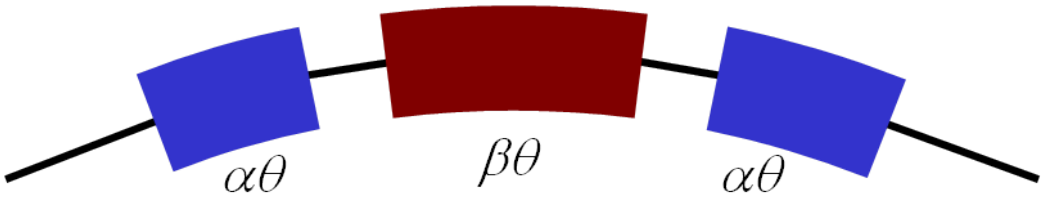
In principle, it is possible to combine the DBA and TME lattices by having an arc cell consisting of more than two dipoles.

The dispersion is zero at the entrance and exit of each cell, thus satisfying the "achromat" condition.

In the dipoles within the body of the cell, the Courant–Snyder parameters and dispersion satisfy the TME conditions.

This kind of lattice is known as a "multi-bend achromat" (MBA).

Suppose that the dipoles all have the same bending radius (i.e. the same field strength), but can have different lengths.




In this case, we find (see Appendix B) that the minimum natural emittance in an $M$-bend achromat is given by:

$$\varepsilon_0 \approx C_q \gamma^2 \frac{1}{12\sqrt{15}} \left(\frac{M+1}{M-1}\right) \theta^3, \qquad 2 < M < \infty, \qquad (28)$$

where $\theta$ is the *average* bending angle per dipole.

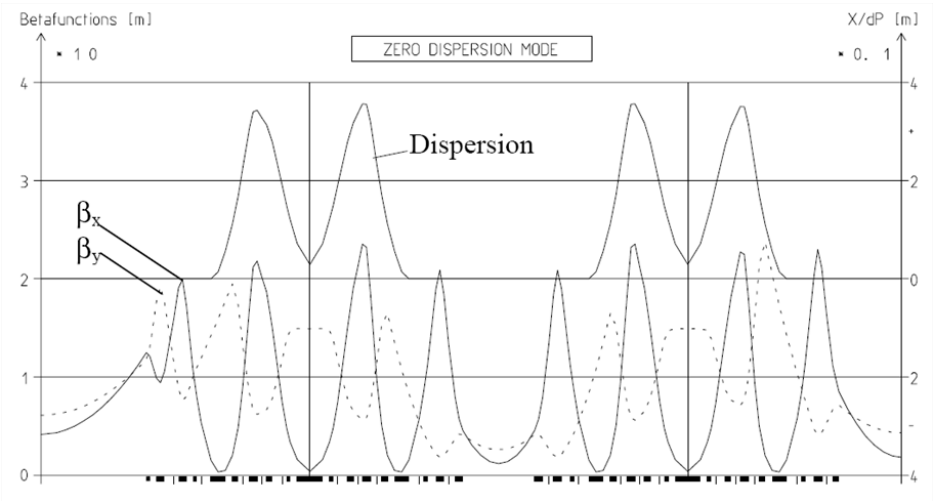
The minimum emittance is achieved when the central dipoles are longer than the outer dipoles by a factor $\sqrt[3]{3}$.

# Example of a triple-bend achromat: the Swiss Light Source

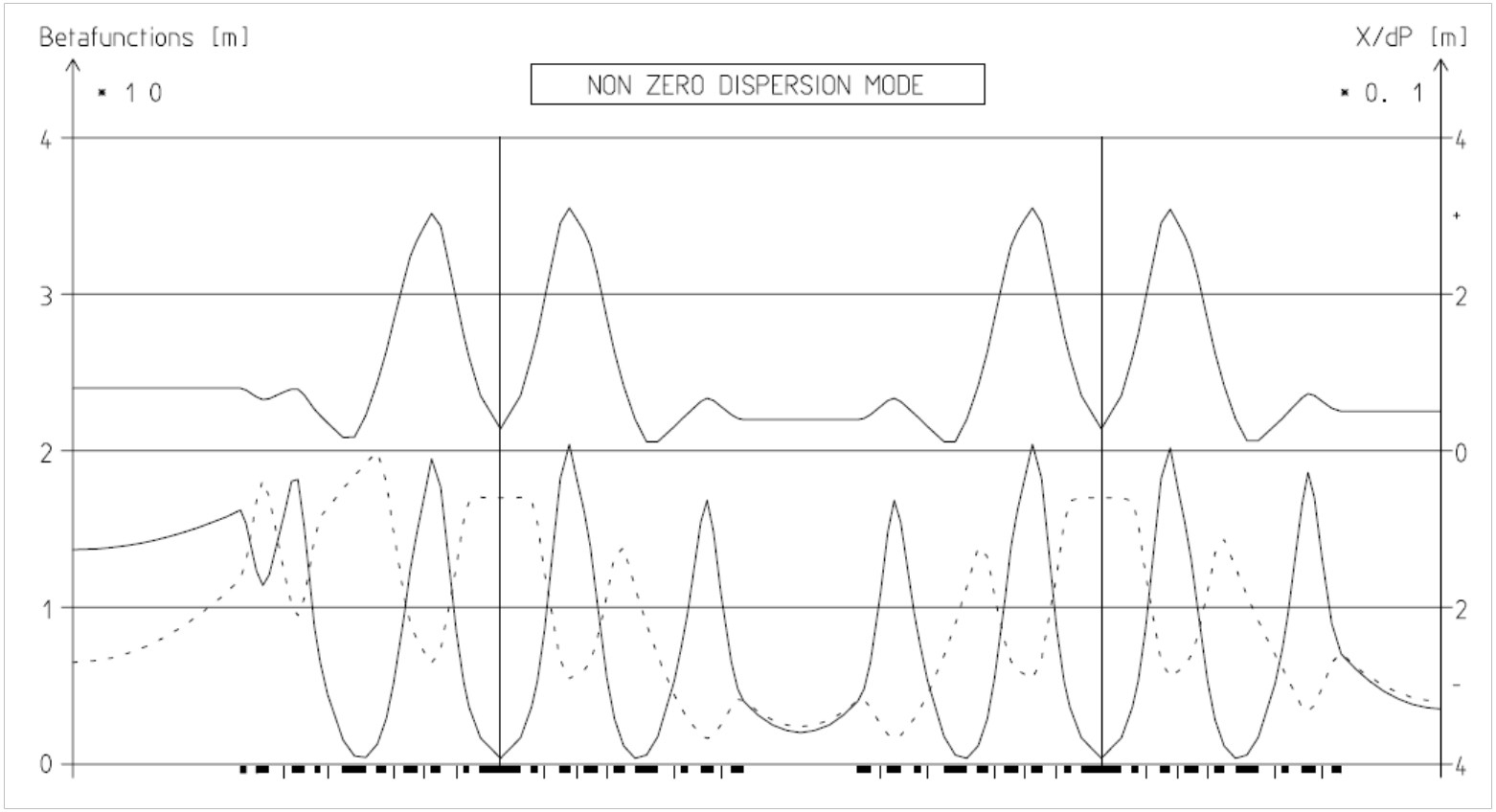
The Swiss Light Source storage ring consists of 12 triple-bend achromat (TBA) cells.

The circumference is 288 m, and the beam energy is 2.4 GeV.

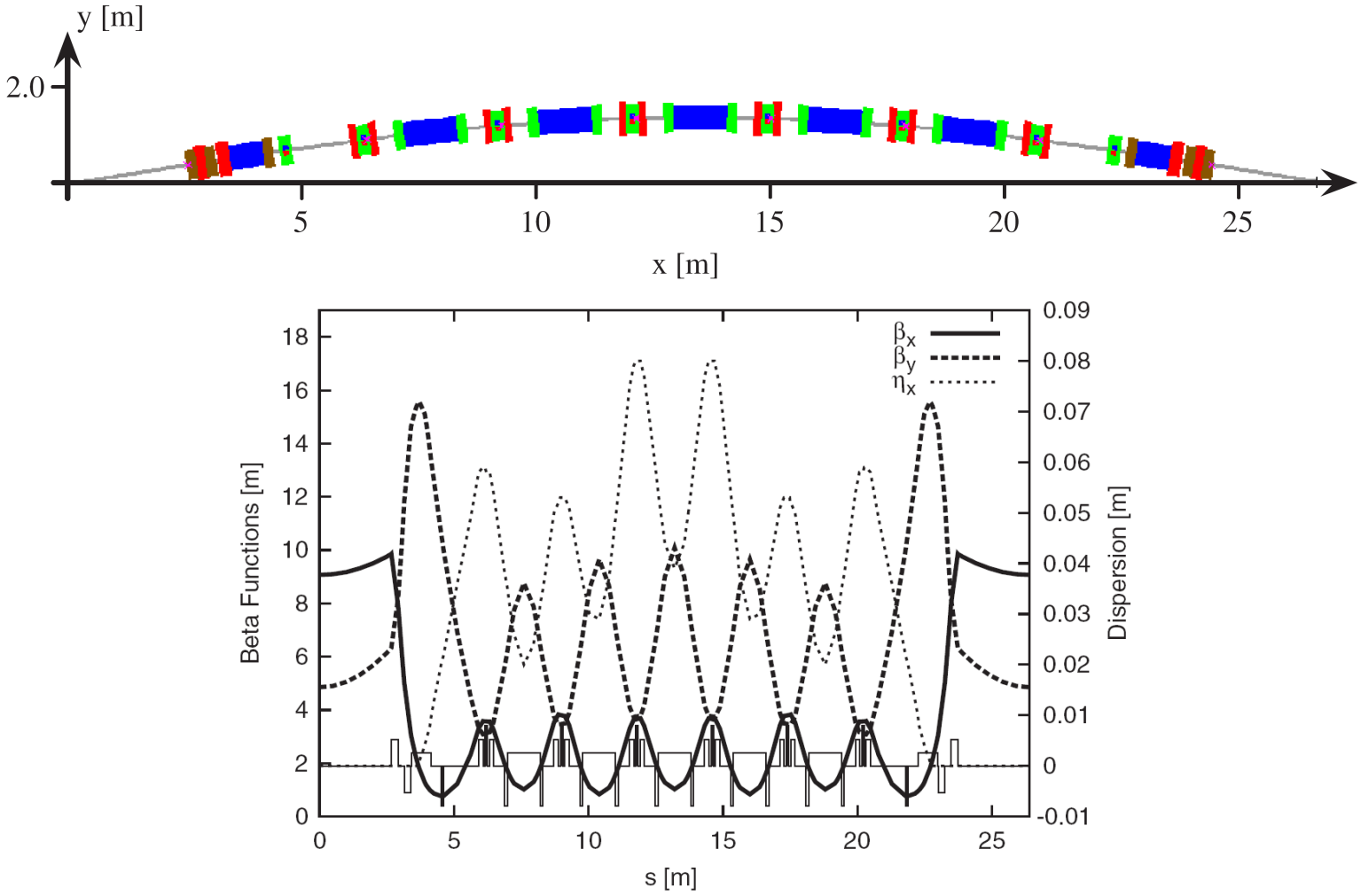
In the "zero-dispersion" mode, the natural emittance is 4.8 nm-rad.




Detuning the achromat to allow dispersion in the straights reduces the natural emittance by about 20% (to 3.9 nm-rad).



# A 7-bend achromat: MAX IV




Note: vertical focusing is provided by a gradient in the bending magnets.

S.C. Leeman et al, "Beam dynamics and expected performance of Sweden's new storage-ring light source: MAX IV," PRST-AB 12, 120701 (2009).

# A 7-bend achromat: MAX IV

| | |
|---|---|
| Beam energy | 3 GeV |
| Circumference | 528 m |
| Number of cells | 20 |
| Horizontal emittance (no IDs) | 0.326 nm |
| Horizontal emittance (with IDs) | 0.263 nm |

# Variational (longitudinal gradient) bends

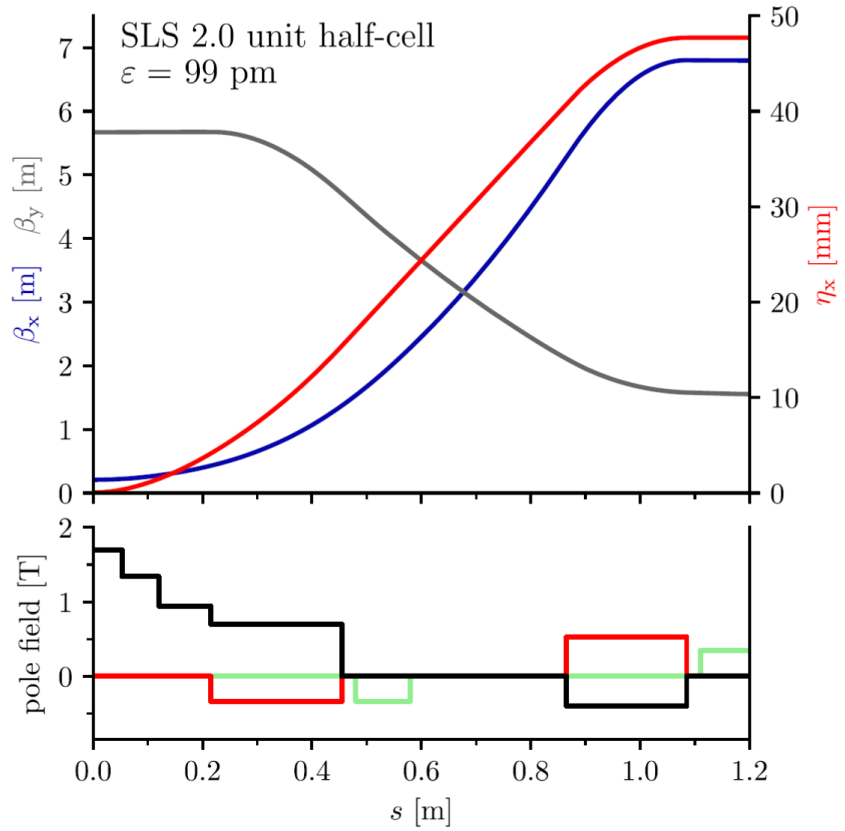



FIG. 17. The unit cell of the SLS upgrade lattice "SLS 2.0" represents a fully optimized, real LGB/RB cell (see Fig. 4 for legend).

B. Riemann and A. Streun, PRAB 22, 021601 (2019).

Allowing the magnetic field in a dipole to vary along the length of the magnet provides another degree of freedom in reducing the emittance.

We expect an optimised design to have the strongest field at the centre of the dipole, where the dispersion can be minimised.

Recall that the natural (horizontal) emittance in a storage ring is given by:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2}. \tag{29}$$

If the vertical motion is independent of the horizontal motion (i.e. if there is no betatron coupling) then we can apply the same analysis to the vertical motion as we did to the horizontal.

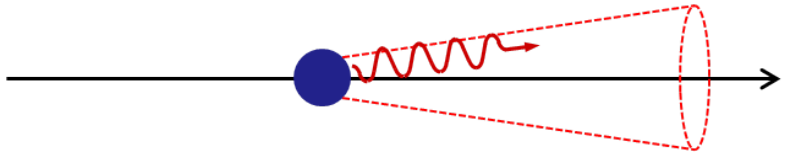
Then, if we build a ring that is completely flat (i.e. no vertical bending), then there is no vertical dispersion:

$$\eta_y = \eta_{py} = 0 \qquad \therefore \quad \mathcal{H}_y = 0 \qquad \therefore \quad I_{5y} = 0. \tag{30}$$

This implies that the vertical emittance will damp to zero.

However, in deriving equation (29) for the natural emittance, we assumed that all photons were emitted directly along the instantaneous direction of motion of the electron.

In fact, emitted photons have a distribution with angular width $1/\gamma$ about the direction of motion of the electron.



This leads to some vertical "recoil" that excites vertical betatron motion, resulting in a non-zero vertical emittance.

Detailed analysis* leads to the following formula for the fundamental lower limit on the vertical emittance:

$$\varepsilon_{y,\mathrm{min}} = \frac{13}{55}\frac{C_q}{j_y I_2}\oint \frac{\beta_y}{|\rho|^3}\,ds. \tag{31}$$

Making some approximations, this can be written:

$$\varepsilon_{y,\mathrm{min}} \approx \frac{C_q\langle\beta_y\rangle}{4 j_y I_2}\oint \frac{1}{|\rho|^3}\,ds = \frac{\langle\beta_y\rangle}{4}\frac{j_z}{j_y}\frac{\sigma_\delta^2}{\gamma^2}. \tag{32}$$

Using some typical values ($\langle\beta_y\rangle = 20\,\mathrm{m}$, $j_z = 2$, $j_y = 1$, $\sigma_\delta = 10^{-3}$, $\gamma = 6000$), we find:

$$\varepsilon_{y,\mathrm{min}} \approx 0.3\,\mathrm{pm}. \tag{33}$$

The lowest vertical emittance achieved so far in a storage ring is around a picometer, several times larger than the fundamental lower limit.

*T. Raubenheimer, SLAC Report 387 (1992)

In practice, vertical emittance in a (nominally flat) storage ring is dominated by two effects:

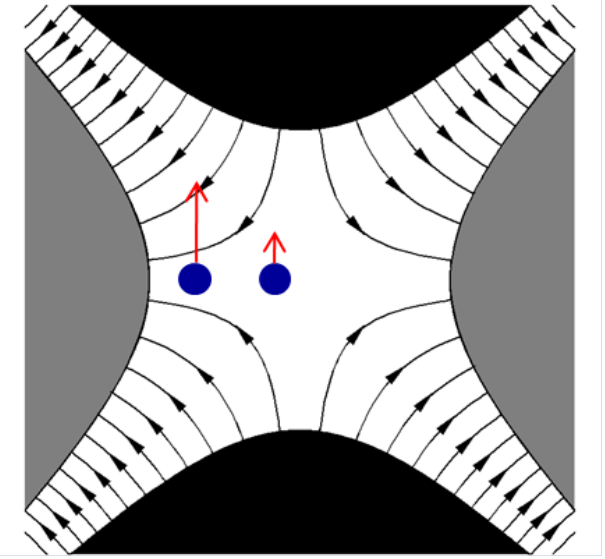
- residual vertical dispersion, which couples longitudinal and vertical motion;
- betatron coupling, which couples horizontal and vertical motion.

The dominant causes of residual vertical dispersion and betatron coupling are magnet alignment errors, in particular:

- tilts of the dipoles around the beam axis;
- vertical alignment errors on the quadrupoles;
- tilts of the quadrupoles around the beam axis;
- vertical alignment errors of the sextupoles.

Betatron coupling describes the dependence of the vertical motion of a particle on its horizontal motion (and vice-versa).

In a storage ring, betatron coupling often comes from skew quadrupole fields generated by rotation (tilt) errors on quadrupoles and vertical alignment errors on sextupoles.



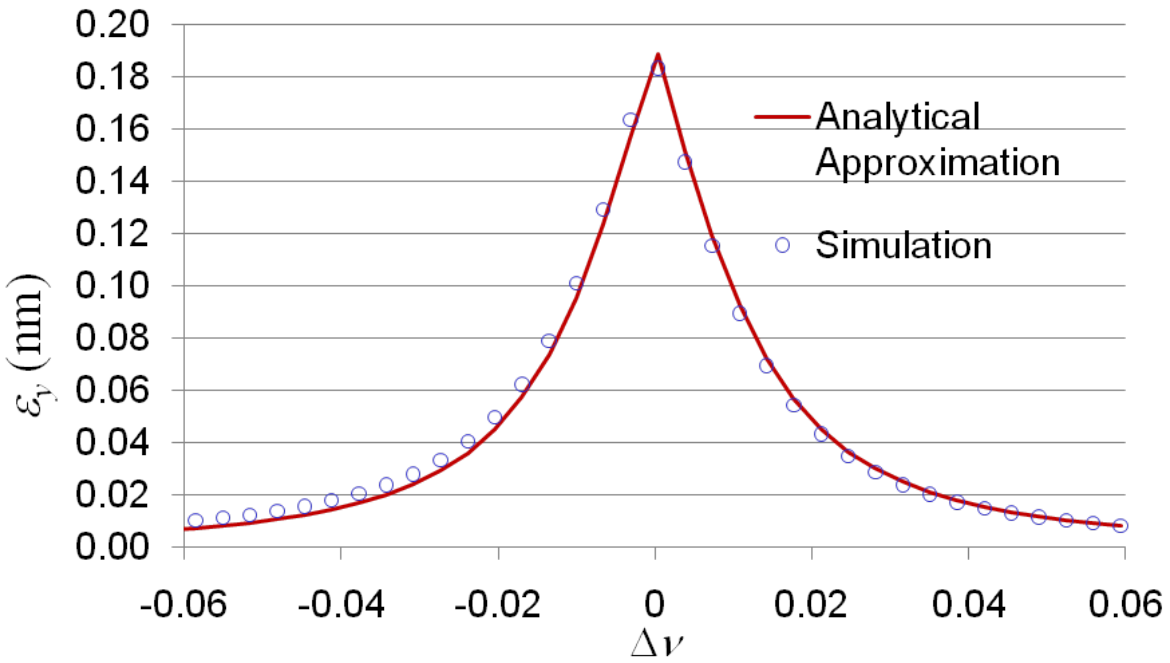
A rigorous treatment of coupling can be complex; but it is possible to use simplified models to derive approximate expressions for the equilibrium emittances in the presence of coupling.

# Betatron coupling: example

One approach to analysis of betatron coupling is described in Appendix C.

As an illustration, we can plot the vertical emittance in a model of the ILC damping rings, as a function of the "tune split" $\Delta\nu$, with a single skew quadrupole (located at a point of zero dispersion...)

The tunes are controlled by adjusting the regular (normal) quadrupoles:




Note: the "simulation" results are based on emittance calculation using Chao's method, which we shall discuss later.

Vertical emittance can also be generated by vertical dispersion, in the same way that horizontal emittance can be generated by horizontal dispersion.

If we know the vertical dispersion all around the ring, then to calculate the vertical emittance we can simply modify the formula for the natural emittance (see Lecture 1).

Further details are given in Appendix D, where we show that, if the vertical dispersion comes from random errors in the lattice, the contribution to the vertical emittance is given by:

$$\varepsilon_y \approx 2\frac{j_z}{j_y}\left\langle\frac{\eta_y^2}{\beta_y}\right\rangle\sigma_\delta^2. \tag{34}$$

Since the energy spread, beta function and damping partition numbers are usually well known, this gives a convenient expression for the relationship between the rms vertical dispersion and the vertical emittance.

The formulae we have derived so far are useful for making rough estimates of the sensitivity to particular types of error.

For detailed studies, including modelling and simulations, we need more accurate formulae for computing the vertical emittance in a storage ring with a given set of alignment errors.

Methods for computing the equilibrium emittances in complex lattices (including lattices with errors), include:

- radiation integrals generalised to the normal modes;
- Chao's method: A. Chao, "Evaluation of beam distribution parameters in an electron storage ring," J. Appl. Phys. 50, 595-598 (1979);
- the 'envelope' method (Appendix E): K. Ohmi, K. Hirata, K. Oide, "From the beam-envelope matrix to the synchrotron radiation integrals," Phys. Rev. E 49, 751-765 (1994).

In practice, tuning a storage ring to achieve a vertical emittance of no more than a few picometres is a considerable challenge.

This cannot be done just by survey alignment of the magnets: beam-based methods are also required. However, precise alignment of the magnets is always the first step.

A variety of beam-based methods for tuning storage rings have been developed over the years.

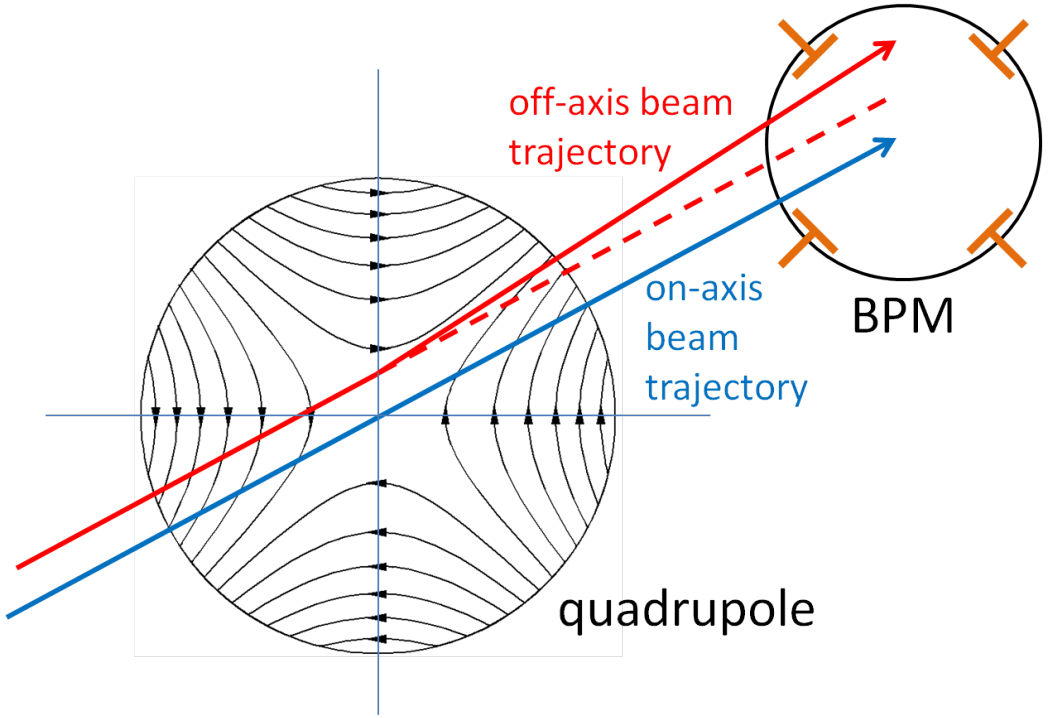
A typical procedure might look as follows...

Step 1: Align the magnets by a survey of the ring.

Typically, quadrupoles need to be aligned to better than a few tens of microns, and sextupoles to better than a couple of hundred microns.

Step 2: Determine the positions of the BPMs relative to the quadrupoles.

This is known as "beam-based alignment" (BBA): the beam is steered to a position in each quadrupole such that changing the quadrupole strength has no effect on the orbit.



Step 3: Correct the orbit (using steering magnets) so that it is as close as possible to the centres of the quadrupoles.

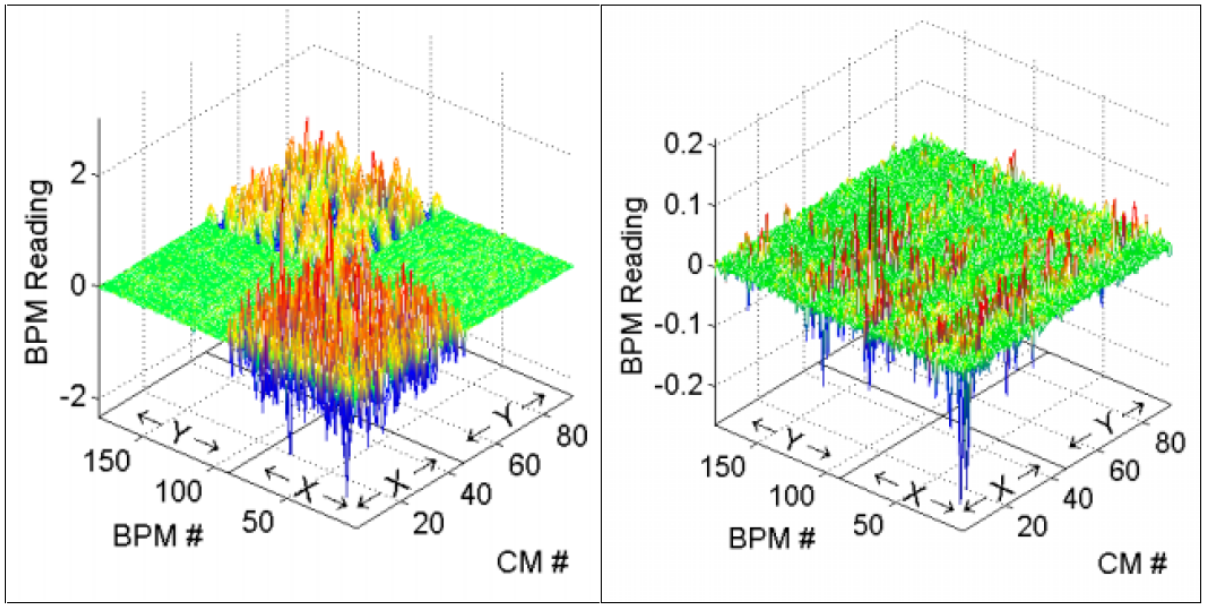
Step 4: Correct the vertical dispersion (using steering magnets and/or skew quadrupoles, and measuring at the BPMs) as close to zero as possible.

Step 5: Correct the coupling, by adjusting skew quadrupoles so that an orbit "kick" in one plane (from any orbit corrector) has no effect on the orbit in the other plane.

Usually, these last three steps need to be iterated several (or even many) times.

Results from the tuning procedure described above can be limited by errors on the BPMs, which can affect dispersion and coupling measurements.

A useful technique for overcoming such limitations is to apply Orbit Response Matrix (ORM) analysis. This can be used to determine a wide range of magnet and diagnostics parameters, including coupling errors and BPM tilts.




ORM analysis in KEK ATF.

Left: measured orbit response matrix.

Right: residuals between measured ORM and machine model.

The natural emittance in a storage ring is determined by the balance between the radiation damping (given by $I_2$) and the quantum excitation (given by $I_5$).

The quantum excitation depends on the lattice functions. Different "styles" of lattice can be used, depending on the emittance specification for the storage ring.

In general, for small bending angle $\theta$ the natural emittance can be written as:

$$\varepsilon_0 \approx F C_q \gamma^2 \theta^3, \tag{35}$$

where $\theta$ is the bending angle of a single dipole, and the numerical factor $F$ is determined by the lattice style...

$$\varepsilon_0 \approx F C_q \gamma^2 \theta^3$$

| Lattice style | $F$ |
|---|---|
| 90° FODO | $2\sqrt{2}$ |
| 137° FODO | 1.2 |
| Double-bend achromat (DBA) | $\frac{1}{4\sqrt{15}}$ |
| Multi-bend achromat | $\frac{1}{12\sqrt{15}}\left(\frac{M+1}{M-1}\right)$ |
| TME | $\frac{1}{12\sqrt{15}}$ |

# Summary (3)

Achromats have been popular choices for storage ring lattices in third-generation synchrotron light sources for two reasons:

- they provide lower natural emittance than FODO lattices;
- they provide zero-dispersion locations appropriate for insertion devices (wigglers and undulators).

Light sources have been built using double-bend achromats (e.g. ESRF, APS, SPring-8, DIAMOND, SOLEIL...) and triple-bend achromats (e.g. ALS, SLS).

Increasing the number of bends in an achromat cell ("multiple-bend achromats") and "detuning" an achromat (to allow some dispersion in the straights) can help to achieve a lower emittance.

The opening angle of the synchrotron radiation places a lower limit (typically, a fraction of a picometre) on the vertical emittance.

In practice, the vertical emittance is dominated by alignment and tuning errors (betatron coupling and vertical dispersion).

Natural emittances of a few nanometres are typical in storage rings for third generation light sources.

Storage rings for light sources often operate with vertical emittances of order 1% (or less) of the horizontal (natural) emittance: this requires careful tuning and correction of alignment errors.

# Appendices

# Appendix A: Evaluating $I_5$ in a FODO lattice

In this Appendix, we shall derive an expression for the fifth synchrotron radiation integral, $I_5$, in a FODO cell, in terms of the dipole length $L$ and bending radius $\rho$, and the quadrupole focal length $f$.

It is assumed that the dipoles completely fill the space between the (thin) quadrupoles.

In terms of $f$, $\rho$ and $L$, the horizontal beta function at the horizontally-focusing quadrupole is given by:

$$\beta_x = \frac{4f\rho\sin(\theta)(2f\cos(\theta)+\rho\sin(\theta))}{\sqrt{16f^4-[\rho^2-(4f^2+\rho^2)\cos 2\theta]^2}}, \tag{36}$$

where $\theta = L/\rho$ is the bending angle of a single dipole.

The dispersion at a horizontally-focusing quadrupole is given by:

$$\eta_x = \frac{2f\rho(2f+\rho\tan\frac{\theta}{2})}{4f^2+\rho^2}. \tag{37}$$

By symmetry, at the centre of a quadrupole, $\alpha_x = \eta_{px} = 0$.

# Appendix A: Evaluating $I_5$ in a FODO lattice

We can evolve the lattice functions through a lattice using the transfer matrices, $M$.

For the Courant–Snyder parameters:

$$A(s_1) = MA(s_0)M^{\mathsf{T}}, \tag{38}$$

where $M = M(s_1; s_0)$ is the transfer matrix from $s_0$ to $s_1$, and:

$$A = \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix}. \tag{39}$$

The dispersion can be evolved (over a distance $\Delta s$, with constant bending radius $\rho$) using:

$$\begin{pmatrix} \eta_x \\ \eta_{px} \end{pmatrix}_{s_1} = M \begin{pmatrix} \eta_x \\ \eta_{px} \end{pmatrix}_{s_0} + \begin{pmatrix} \rho\left(1 - \cos\left(\frac{\Delta s}{\rho}\right)\right) \\ \sin\left(\frac{\Delta s}{\rho}\right) \end{pmatrix}. \tag{40}$$

For a thin quadrupole, the transfer matrix is: $M = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$.

For a dipole, the transfer matrix is: $M = \begin{pmatrix} \cos\left(\frac{s}{\rho}\right) & \rho\sin\left(\frac{s}{\rho}\right) \\ -\frac{1}{\rho}\sin\left(\frac{s}{\rho}\right) & \cos\left(\frac{s}{\rho}\right) \end{pmatrix}$.

# Appendix A: Evaluating $I_5$ in a FODO lattice

We now have all the information we need to find an expression for $I_5$ in the FODO cell.

However, the algebra is rather formidable. The result is most easily expressed as a power series in the dipole bending angle, $\theta$:

$$\frac{I_5}{I_2} = \left(4 + \frac{\rho^2}{f^2}\right)^{-\frac{3}{2}} \left[8 - \frac{\rho^2}{2f^2}\theta^2 + O(\theta^4)\right]. \tag{41}$$

For small $\theta$, the expression for $I_5/I_2$ can be written:

$$\frac{I_5}{I_2} \approx \left(1 - \frac{\rho^2}{16f^2}\theta^2\right)\left(1 + \frac{\rho^2}{4f^2}\right)^{-\frac{3}{2}} = \left(1 - \frac{L^2}{16f^2}\right)\left(1 + \frac{\rho^2}{4f^2}\right)^{-\frac{3}{2}}. \tag{42}$$

This can be further simplified if $\rho \gg 2f$ (often the case):

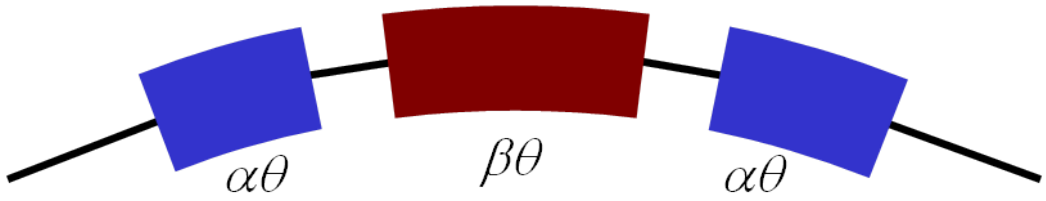
$$\frac{I_5}{I_2} \approx \left(1 - \frac{L^2}{16f^2}\right)\frac{8f^3}{\rho^3}, \tag{43}$$

and still further simplified if $4f \gg L$ (less often the case):

$$\frac{I_5}{I_2} \approx \frac{8f^3}{\rho^3}. \tag{44}$$

In this appendix, we shall derive a formula for the natural emittance in a multi-bend achromat (MBA) lattice.

For simplicity, we consider the case where the dipoles all have the same bending radius (i.e. they all have the same field strength), but they vary in length.




Assuming that each arc cell has a fixed number, $M$, of dipoles, and $\theta = 2\pi/MN_{\text{cells}}$, the bending angles satisfy:

$$2\alpha + (M-2)\beta = M. \tag{45}$$

Since the synchrotron radiation integrals are additive, for an $M$-bend achromat, we can write:

$$I_{5,\text{cell}} \approx \frac{2}{4\sqrt{15}}\frac{(\alpha\theta)^4}{\rho} + \frac{(M-2)}{12\sqrt{15}}\frac{(\beta\theta)^4}{\rho} = \frac{6\alpha^4 + (M-2)\beta^4}{12\sqrt{15}}\frac{\theta^4}{\rho}, \tag{46}$$

$$I_{2,\text{cell}} \approx 2\frac{\alpha\theta}{\rho} + (M-2)\frac{\beta\theta}{\rho} = [2\alpha + (M-2)\beta]\,\frac{\theta}{\rho}. \tag{47}$$

Hence, in an $M$-bend achromat:

$$\frac{I_{5,\text{cell}}}{I_{2,\text{cell}}} \approx \frac{1}{12\sqrt{15}} \left[ \frac{6\alpha^4 + (M-2)\beta^4}{2\alpha + (M-2)\beta} \right] \theta^3. \tag{48}$$

Minimising the ratio $I_5/I_2$ with respect to $\alpha$ gives:

$$\frac{\alpha}{\beta} = \frac{1}{\sqrt[3]{3}}, \qquad \frac{6\alpha^4 + (M-2)\beta^4}{2\alpha + (M-2)\beta} \approx \frac{M+1}{M-1}. \tag{49}$$

The central bending magnets should be longer than the outer bending magnets by a factor $\sqrt[3]{3}$.

Then, the minimum natural emittance in an $M$-bend achromat is given by:

$$\varepsilon_0 \approx C_q \gamma^2 \frac{1}{12\sqrt{15}} \left( \frac{M+1}{M-1} \right) \theta^3, \qquad 2 < M < \infty. \tag{50}$$

Note that $\theta$ is the *average* bending angle per dipole.

# Appendix C: Equations of motion in a coupled storage ring

Our goal is to find the equations of motion for a particle in a coupled storage ring, and by solving these equations, to show how the horizontal and vertical emittances depend on the natural emittance, the betatron tunes, and the strengths of the skew quadrupole fields distributed around the lattice.

We will use Hamiltonian mechanics. In this formalism, the equations of motion for the action-angle variables (with path length $s$ as the independent variable) are derived from the Hamiltonian:

$$H = H(\phi_x, J_x, \phi_y, J_y; s), \tag{51}$$

using Hamilton's equations:

$$\frac{dJ_x}{ds} = -\frac{\partial H}{\partial \phi_x}, \qquad \frac{dJ_y}{ds} = -\frac{\partial H}{\partial \phi_y}, \tag{52}$$

$$\frac{d\phi_x}{ds} = \frac{\partial H}{\partial J_x}, \qquad \frac{d\phi_y}{ds} = \frac{\partial H}{\partial J_y}. \tag{53}$$

For a particle moving along a linear, uncoupled beamline, the Hamiltonian is:

$$H = \frac{J_x}{\beta_x} + \frac{J_y}{\beta_y}. \tag{54}$$

# Appendix C: Equations of motion in a coupled storage ring

The first step is to derive an appropriate form for the Hamiltonian in a storage ring with skew quadrupole perturbations.

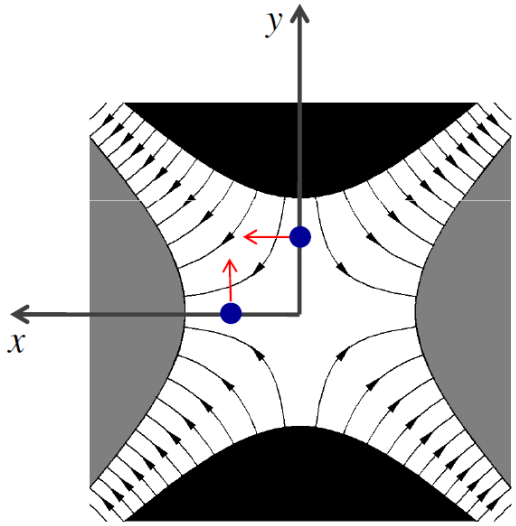
In Cartesian variables, the equations of motion in a skew quadrupole can be written:

$$\frac{dp_x}{ds} = k_s y, \qquad \frac{dp_y}{ds} = k_s x, \tag{55}$$

$$\frac{dx}{ds} = p_x, \qquad \frac{dy}{ds} = p_y, \tag{56}$$

where:

$$k_s = \frac{1}{B\rho}\frac{\partial B_x}{\partial x}. \tag{57}$$




These equations can be derived from the Hamiltonian:

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 - k_s xy. \tag{58}$$

# Appendix C: Equations of motion in a coupled storage ring

We are interested in the case where there are skew quadrupoles distributed around a storage ring.

The "focusing" effect of a skew quadrupole is represented by a term in the Hamiltonian:

$$k_s xy = 2k_s\sqrt{\beta_x\beta_y}\sqrt{J_x J_y}\cos(\phi_x)\cos(\phi_y). \tag{59}$$

This implies that the Hamiltonian for a beam line with distributed skew quadrupoles can be written:

$$H = \frac{J_x}{\beta_x} + \frac{J_y}{\beta_y} - 2k_s(s)\sqrt{\beta_x\beta_y}\sqrt{J_x J_y}\cos(\phi_x)\cos(\phi_y). \tag{60}$$

The beta functions and the skew quadrupole strength are functions of the position $s$. This makes it difficult to solve the equations of motion exactly.

Therefore, we simplify the problem by "averaging" the Hamiltonian:

$$H = \omega_x J_x + \omega_y J_y - 2\bar{\kappa}\sqrt{J_x J_y}\cos(\phi_x)\cos(\phi_y). \tag{61}$$

Here, $\omega_x$, $\omega_y$ and $\bar{\kappa}$ are constants.

$\omega_x$ and $\omega_y$ are the betatron frequencies, given by:

$$\omega_{x,y} = \frac{1}{C}\int_0^C \frac{ds}{\beta_{x,y}}. \tag{62}$$

For reasons that will become clear shortly, we re-write the coupling term, to put the Hamiltonian in the form:

$$H = \omega_x J_x + \omega_y J_y - \bar{\kappa}_- \sqrt{J_x J_y}\cos(\phi_x - \phi_y) - \bar{\kappa}_+ \sqrt{J_x J_y}\cos(\phi_x + \phi_y). \tag{63}$$

The constants $\bar{\kappa}_\pm$ represent the skew quadrupole strength averaged around the ring. However, we need to take into account that the kick from a skew quadrupole depends on the betatron phase. Thus, we write:

$$\bar{\kappa}_\pm e^{i\chi} = \frac{1}{C}\int_0^C e^{i(\mu_x \pm \mu_y)} k_s \sqrt{\beta_x \beta_y}\, ds, \tag{64}$$

where $\mu_x$ and $\mu_y$ are the betatron phase advances from the start of the ring.

Now suppose that $\bar{\kappa}_- \gg \bar{\kappa}_+$. (This can occur, for example, if $\omega_x \approx \omega_y$, in which case all the skew quadrupole perturbations will add together in phase.) Then, we can simplify things further by dropping the term in $\bar{\kappa}_+$ from the Hamiltonian:

$$H = \omega_x J_x + \omega_y J_y - \bar{\kappa}_- \sqrt{J_x J_y} \cos(\phi_x - \phi_y). \tag{65}$$

We can now write down the equations of motion:

$$\frac{dJ_x}{ds} = -\frac{\partial H}{\partial \phi_x} = \bar{\kappa}_- \sqrt{J_x J_y} \sin(\phi_x - \phi_y), \tag{66}$$

$$\frac{dJ_y}{ds} = -\frac{\partial H}{\partial \phi_y} = -\bar{\kappa}_- \sqrt{J_x J_y} \sin(\phi_x - \phi_y), \tag{67}$$

$$\frac{d\phi_x}{ds} = \frac{\partial H}{\partial J_x} = \omega_x + \frac{\bar{\kappa}_-}{2} \sqrt{\frac{J_x}{J_y}} \cos(\phi_x - \phi_y), \tag{68}$$

$$\frac{d\phi_y}{ds} = \frac{\partial H}{\partial J_y} = \omega_y + \frac{\bar{\kappa}_-}{2} \sqrt{\frac{J_y}{J_x}} \cos(\phi_x - \phi_y). \tag{69}$$

Even after all the simplifications we have made, the equations of motion are still rather difficult to solve. Fortunately, however, we do not require the general solution. In fact, we are only interested in the properties of some special cases.

First of all, we note that the sum of the actions is constant:

$$\frac{dJ_x}{ds} + \frac{dJ_y}{ds} = 0 \qquad \therefore \quad J_x + J_y = \text{constant}. \tag{70}$$

This is true in all cases.

Going further, we notice that if $\phi_x = \phi_y$, then the rate of change of each action falls to zero, i.e.:

$$\text{if} \quad \phi_x = \phi_y \quad \text{then} \quad \frac{dJ_x}{ds} = \frac{dJ_y}{ds} = 0. \tag{71}$$

This implies that if we can find a solution to the equations of motion with $\phi_x = \phi_y$ for all $s$, then the actions will remain constant.

From the equations of motion, we find that if:

$$\phi_x = \phi_y \quad \text{and} \quad \frac{d\phi_x}{ds} = \frac{d\phi_y}{ds}, \tag{72}$$

then:

$$\frac{J_y}{J_x} = \frac{\sqrt{1 + \bar{\kappa}_-^2/\Delta\omega^2} - 1}{\sqrt{1 + \bar{\kappa}_-^2/\Delta\omega^2} + 1}, \tag{73}$$

where $\Delta\omega = \omega_x - \omega_y$.

If we further use $J_x + J_y = J_0$, where $J_0$ is a constant, then we have the *fixed point solution*:

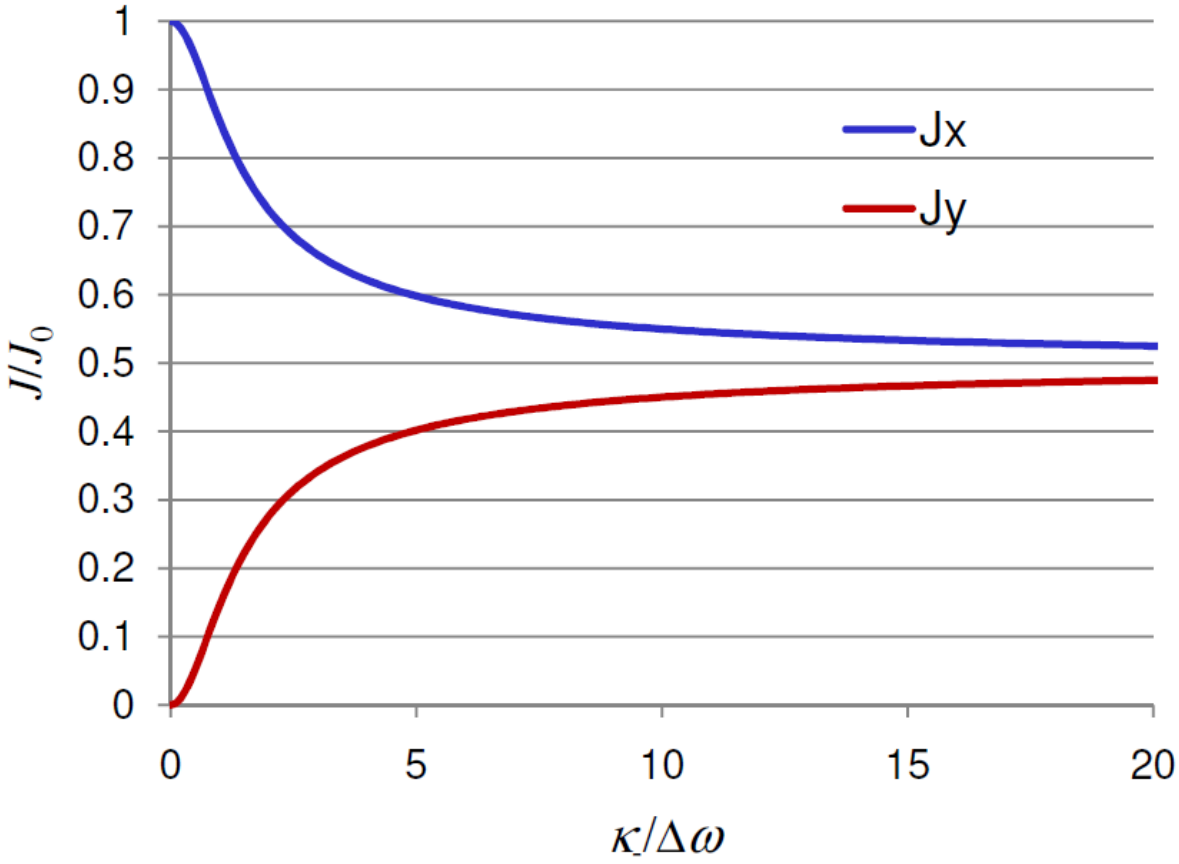
$$J_x = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1 + \bar{\kappa}_-^2/\Delta\omega^2}}\right) J_0, \tag{74}$$

$$J_y = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1 + \bar{\kappa}_-^2/\Delta\omega^2}}\right) J_0. \tag{75}$$

Note the behaviour of the fixed-point actions as we vary the "coupling strength" $\bar{\kappa}_-$ and the betatron tunes (betatron frequencies).

The fixed-point actions are well-separated for $\bar{\kappa}_- \ll \Delta\omega$, but approach each other for $\bar{\kappa}_- \gg \Delta\omega$.

The condition at which the tunes are equal (or differ by an exact integer) is known as the *difference coupling resonance*.



# Appendix C: Equations of motion in a coupled storage ring

Recall that the emittance may be defined as the betatron action averaged over all particles in the beam:

$$\varepsilon_x = \langle J_x \rangle, \qquad \text{and} \qquad \varepsilon_y = \langle J_y \rangle. \tag{76}$$

Now, synchrotron radiation will damp the beam towards an equilibrium distribution. In this equilibrium, we expect the betatron actions of the particles to change only slowly, i.e. on the timescale of the radiation damping, whis is much longer than the timescale of the betatron motion.

In that case, the actions of most particles must be in the correct ratio for a fixed-point solution to the equations of motion. Then, if we assume that $\varepsilon_x + \varepsilon_y = \varepsilon_0$, where $\varepsilon_0$ is the natural emittance of the storage ring, we must have for the equilibrium emittances:

$$\varepsilon_x = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1 + \bar{\kappa}_-^2/\Delta\omega^2}}\right)\varepsilon_0, \tag{77}$$

$$\varepsilon_y = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1 + \bar{\kappa}_-^2/\Delta\omega^2}}\right)\varepsilon_0. \tag{78}$$

These equations give the horizontal and vertical emittances in terms of the natural emittance $\epsilon_0$, the betatron tunes (represented by $\Delta\omega$) and the skew quadrupole strengths (characterised by $\bar{\kappa}_-$).

To estimate the effect of a skew quadrupole perturbation on the betatron tunes, we use the Hamiltonian (65). If we consider a particle close to the fixed point solution, we can assume that $\phi_x = \phi_y$, so that the Hamiltonian becomes:

$$H = \omega_x J_x + \omega_y J_y - \bar{\kappa}_- \sqrt{J_x J_y}. \tag{79}$$

The normal modes describe motion that is periodic with a single well-defined frequency. In the absence of coupling, the transverse normal modes correspond to motion in just the horizontal or vertical plane. When coupling is present, the normal modes involve combination of horizontal and vertical motion.

Let us write the Hamiltonian (79) in the form:

$$H = \begin{pmatrix} \sqrt{J_x} & \sqrt{J_y} \end{pmatrix} A \begin{pmatrix} \sqrt{J_x} \\ \sqrt{J_y} \end{pmatrix}, \text{ where } A = \begin{pmatrix} \omega_x & -\frac{1}{2}\bar{\kappa}_- \\ -\frac{1}{2}\bar{\kappa}_- & \omega_y \end{pmatrix}. \tag{80}$$

The normal modes can be constructed from the eigenvectors of the matrix $A$, and the frequency of each mode is given by the corresponding eigenvalue.

From the eigenvalues of $A$, we find that the normal mode frequencies are:

$$\omega_\pm = \frac{1}{2}\left(\omega_x + \omega_y \pm \sqrt{\bar{\kappa}_-^2 + \Delta\omega^2}\right). \tag{81}$$

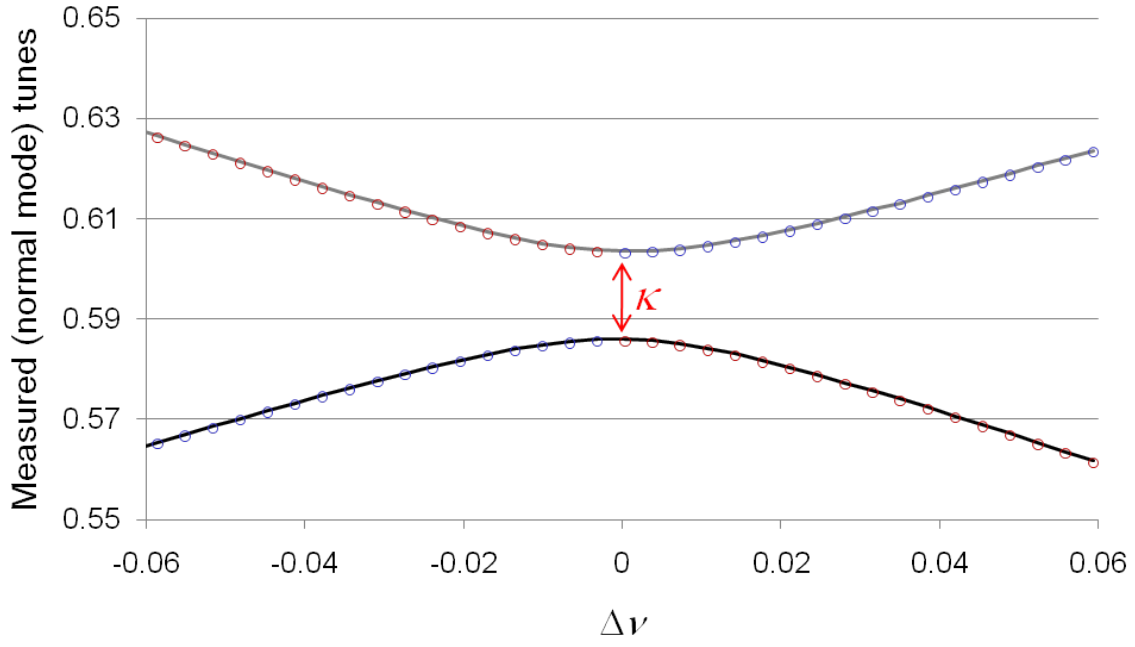
If transverse beam oscillations are driven by a kicker (or similar component), the beam will resonate at the normal mode frequencies $\omega_{\pm}$.

Hence, measurement of the betatron tunes (by resonant excitation of transverse beam oscillations) will give values:

$$\nu_{\pm} = \frac{1}{2}\left(\nu_x + \nu_y \pm \sqrt{\bar{\kappa}_-^2 + \Delta\nu^2}\right), \tag{82}$$

where $\nu_x$ and $\nu_y$ are the betatron tunes in the absence of errors.

Thus, measurement of the tunes provides a way to characterise the coupling strength in a real lattice: the coupling strength $\bar{\kappa}_-$ is the minimum tune separation observed if the tunes are scanned across a coupling resonance.



# Appendix D: Vertical emittance from vertical dispersion

Vertical emittance can be generated by vertical dispersion, in the same way that horizontal emittance can be generated by horizontal dispersion.

Vertical dispersion can come from vertical steering (for example, from vertical alignment errors on the quadrupoles), or from coupling of the horizontal dispersion into the vertical plane (for example, by quadrupole tilt errors, or vertical alignment errors on the sextupoles).

If we know the vertical dispersion all around the ring, then to calculate the vertical emittance we can simply modify the formula for the natural emittance (see Lecture 1):

$$\varepsilon_y = C_q \gamma^2 \frac{I_{5y}}{j_y I_2}, \tag{83}$$

where $j_y$ is the vertical damping partition number (usually, $j_y = 1$), and the synchrotron radiation integrals are given by:

$$I_2 = \oint \frac{1}{\rho^2}\, ds, \tag{84}$$

and:

$$I_{5y} = \oint \frac{\mathcal{H}_y}{|\rho|^3}\, ds, \qquad \text{where} \qquad \mathcal{H}_y = \gamma_y \eta_y^2 + 2\alpha_y \eta_y \eta_{py} + \beta_y \eta_{py}^2. \tag{85}$$

# Appendix D: Vertical emittance from vertical dispersion

If the vertical dispersion is generated randomly (for example, by alignment errors on the quadrupole and sextupole magnets), then we can assume that it will *not* be correlated with the curvature $1/\rho$ of the reference trajectory[†].

Then, we can write:

$$I_{5y} \approx \langle \mathcal{H}_y \rangle \oint \frac{1}{|\rho|^3}\, ds = \langle \mathcal{H}_y \rangle I_3. \tag{86}$$

Hence, we can write for the vertical emittance:

$$\varepsilon_y \approx C_q \gamma^2 \langle \mathcal{H}_y \rangle \frac{I_3}{j_y I_2}. \tag{87}$$

It is convenient to use:

$$\sigma_\delta^2 = C_q \gamma^2 \frac{I_3}{j_z I_2}, \tag{88}$$

which gives:

$$\varepsilon_y \approx \frac{j_z}{j_y} \langle \mathcal{H}_y \rangle \sigma_\delta^2. \tag{89}$$

[†]This is not the case for the horizontal dispersion!

Now, note the similarity between the action:

$$2J_y = \gamma_y y^2 + 2\alpha_y y p_y + \beta_y p_y^2, \tag{90}$$

and the function $\mathcal{H}_x$:

$$\mathcal{H}_y = \gamma_y \eta_y^2 + 2\alpha_y \eta_y \eta_{py} + \beta_y \eta_{py}^2. \tag{91}$$

This implies that we can write:

$$\eta_y = \sqrt{\beta_y \mathcal{H}_y}\cos\phi_{\eta y}, \qquad \therefore \quad \left\langle \frac{\eta_y^2}{\beta_y} \right\rangle = \frac{1}{2}\langle \mathcal{H}_y \rangle. \tag{92}$$

Combining equations (89) and (92) gives a useful (approximate) relationship, between the vertical dispersion and the vertical emittance (34):

$$\varepsilon_y \approx 2\frac{j_z}{j_y} \left\langle \frac{\eta_y^2}{\beta_y} \right\rangle \sigma_\delta^2.$$

Since the energy spread, beta function and damping partition numbers are usually fixed by the lattice design, equation (34) gives a convenient formula for the rms vertical dispersion that can be tolerated, to achieve a vertical emittance below a given upper limit.

# Appendix E: The envelope method for computing emittances

It is often useful to be able to make an accurate calculation of the equilibrium beam emittances in a storage ring including the effects of magnet errors (for example, to estimate the size of errors that can be tolerated, to achieve specified emittance goals).

One way of computing the emittances, the "envelope method", is based on finding the equilibrium distribution described by the covariance matrix (the matrix of second order moments of the phase space variables):

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x p_x \rangle & \langle xy \rangle & \langle x p_y \rangle & \langle xz \rangle & \langle x\delta \rangle \\ \langle p_x x \rangle & \langle p_x^2 \rangle & \langle p_x y \rangle & \langle p_x p_y \rangle & \langle p_x z \rangle & \langle p_x \delta \rangle \\ \langle yx \rangle & \langle y p_x \rangle & \langle y^2 \rangle & \langle y p_y \rangle & \langle yz \rangle & \langle y\delta \rangle \\ \langle p_y x \rangle & \langle p_y p_x \rangle & \langle p_y y \rangle & \langle p_y^2 \rangle & \langle p_y z \rangle & \langle p_y \delta \rangle \\ \langle zx \rangle & \langle z p_x \rangle & \langle zy \rangle & \langle z p_y \rangle & \langle z^2 \rangle & \langle z\delta \rangle \\ \langle \delta x \rangle & \langle \delta p_x \rangle & \langle \delta y \rangle & \langle \delta p_y \rangle & \langle \delta z \rangle & \langle \delta^2 \rangle \end{pmatrix}. \tag{93}$$

This can be conveniently written as:

$$\Sigma_{ij} = \langle x_i x_j \rangle, \tag{94}$$

where $\Sigma_{ij}$ is the $(i,j)$ component of the Sigma matrix, and the set $x_i$ (for $i = 1 \ldots 6$) are the phase space variables. The brackets $\langle \cdot \rangle$ indicate an average over all particles in the bunch.

In the absence of coupling, the covariance matrix will be block diagonal. We are interested in the more general case, where coupling is present.

# Appendix E: The envelope method for computing emittances

The emittances and the lattice functions can be calculated from the covariance matrix, and vice-versa.

Consider the (simpler) case of motion in one degree of freedom. The covariance matrix in this case is:

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x p_x \rangle \\ \langle p_x x \rangle & \langle p_x^2 \rangle \end{pmatrix} = \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} \varepsilon_x. \tag{95}$$

Given a covariance matrix, we can compute the emittance as follows. First, define the matrix $S$:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{96}$$

Then, the eigenvalues of $\Sigma S$ are $\pm i\varepsilon_x$. (The proof of this is left as an exercise.)

Now, we can show that (under certain assumptions) the emittance is conserved as a bunch is transported along a beam line.

The linear transformation in phase space coordinates of a particle in the bunch between two points in the beam line can be represented by a matrix $M$:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \mapsto M \begin{pmatrix} x \\ p_x \end{pmatrix}, \tag{97}$$

where the symbol $\mapsto$ means "is mapped to".

If (for the moment) we neglect radiation and certain other effects, and consider only the Lorentz force on particles from the external electromagnetic fields, then the transport is *symplectic*.

Physically, this means that the phase-space volume of the bunch is conserved as the bunch moves along the beam line.

Mathematically, it means that $M$ is a *symplectic matrix*, i.e. $M$ satisfies:

$$M^{\mathsf{T}} S M = S. \tag{98}$$

Now consider how the covariance matrix transforms. Since it is written as the product of the phase-space coordinates averaged over the bunch, we have:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \mapsto M \begin{pmatrix} x \\ p_x \end{pmatrix}, \qquad \therefore \quad \Sigma \mapsto M \Sigma M^{\mathsf{T}}. \tag{99}$$

Since $S$ is a constant matrix, it immediately follows that:

$$\Sigma S \mapsto M \Sigma M^{\mathsf{T}} S. \tag{100}$$

Then, using the fact that $M$ is symplectic, we have:

$$\Sigma S \mapsto M \Sigma S M^{-1}. \tag{101}$$

This is a similarity transformation of $\Sigma S$: the eigenvalues of any matrix are conserved under a similarity transformation. Therefore, since the eigenvalues of $\Sigma S$ give the emittance of the bunch, it follows that the emittances are conserved under linear, symplectic transport.

The above discussion immediately generalises to three degrees of freedom.

We define the matrix $S$ in three degrees of freedom by:

$$\Sigma = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{102}$$

The six eigenvalues of $\Sigma S$ are then:

$$\pm\, i\varepsilon_x, \quad \pm i\varepsilon_y, \quad \pm i\varepsilon_z. \tag{103}$$

These quantities are all conserved under linear, symplectic transport.

Even if, as is generally the case, the covariance matrix is not block-diagonal (i.e. if there is coupling present), then we can still find three invariant emittances using this method, without any modification.

# Appendix E: The envelope method for computing emittances

If $M$ is a matrix that represents the linear single-turn transformation at some point in a storage ring, then an invariant or "matched" distribution is one that satisfies:

$$\Sigma \mapsto M \Sigma M^{\mathsf{T}} = \Sigma. \tag{104}$$

(In general, all the particles in the bunch change position in phase space after one turn around the ring: but for a matched distribution, the second order moments remain the same.)

This is not sufficient to determine the beam emittances – though this condition will determine the lattice functions (which can be found from the *eigenvectors* of $\Sigma S$).

In other words, the matched distribution condition determines the *shape* of the bunch, but not the *size* of the bunch. This makes sense: after all, in a proton storage ring, we can have a matched bunch of any emittance.

However, in an electron storage ring, we know that radiation effects will damp the emittances to some equilibrium values.

We shall now show how to apply the concept of a matched distribution, when radiation effects are included, to find the equilibrium emittances in an electron storage ring.

# Appendix E: The envelope method for computing emittances

In an electron storage ring, we must make two modifications to the single-turn transformation to account for radiation effects:

1. The matrix $M$ will no longer be symplectic: this accounts for radiation damping.

2. As well as first-order terms in the transformation (represented by the matrix $M$), there will be zeroth-order terms: these will turn out to correspond to the quantum excitation.

The condition for a matched distribution should then be written:

$$\Sigma = M\Sigma M^{\mathsf{T}} + D, \tag{105}$$

where $M$ and $D$ are constant, non-symplectic matrices that represent the first-order and zeroth-order terms in the single-turn transformation, respectively.

This equation is sufficient to determine the covariance matrix uniquely - in other words, using just this equation (with known $M$ and $D$) we can find the bunch emittances and the matched lattice functions.

# Appendix E: The envelope method for computing emittances

The envelope method for finding the equilibrium emittances in a storage ring consists of three steps:

1. Find the first-order terms $M$ and zeroth-order terms $D$ in the single-turn transformation:

$$\Sigma_1 = M\Sigma_0 M^{\mathsf{T}} + D, \tag{106}$$

   where $\Sigma_0$ is the initial covariance matrix, and $\Sigma_1$ is the covariance matrix after one turn.

2. Use the condition:

$$\Sigma_1 = \Sigma_0, \tag{107}$$

   to determine the *matched* covariance matrix (i.e. the covariance matrix that remains the same after one turn around the ring).

3. Find the equilibrium emittances from the eigenvalues of $\Sigma S$.

Note: strictly speaking, since $M$ is not symplectic, the emittances are not invariant as the bunch moves around the ring. Therefore, we may expect to find a different emittance at each point around the ring. However, if radiation effects are fairly small, then the variations in the emittances will also be small.

# Appendix E: The envelope method for computing emittances

As an illustration of the transformation matrices $M$ and $D$, we shall consider a thin "slice" of a dipole.

The thin slice of dipole is an important case:

- in most storage rings, radiation effects are only significant in dipoles;
- "complete" dipoles can be constructed by concatenating the maps for a number of slices.

Once we have a map for a thin slice of dipole, we simply need to concatenate the maps for all the elements in the ring, to construct the map for a complete turn starting at any point.

Recall that the transformations for the phase space variables in the emission of radiation carrying momentum $dp$ are:

$$\begin{array}{ccc} x \mapsto x & y \mapsto y & z \mapsto z \\ p_x \mapsto \left(1 - \frac{dp}{P_0}\right) p_x & p_y \mapsto \left(1 - \frac{dp}{P_0}\right) p_y & \delta \mapsto \delta - \frac{dp}{P_0} \end{array} \tag{108}$$

where $P_0$ is the reference momentum. In general, $dp$ is a function of the coordinates.

To find the transformation matrices $M$ and $D$, we first find an explicit expression for $dp/P_0$, and then write down the above transformations to first order in the phase space variables.

# Appendix E: The envelope method for computing emittances

For an ultra-relativistic particle, the momentum lost through radiation can be expressed in terms of the synchrotron radiation power, $P_\gamma$ (energy loss per unit time):

$$\frac{dp}{P_0} \approx \frac{P_\gamma}{E_0}\,dt \approx \frac{P_\gamma}{E_0}\left(1+\frac{x}{\rho}\right)\frac{ds}{c}, \tag{109}$$

where $\rho$ is the radius of curvature of the trajectory of a particle with the reference energy $E_0$.

Recall (from Lecture 1) that the radiation power from a particle of charge $e$ and energy $E$ in a magnetic field $B$ is:

$$P_\gamma = \frac{C_\gamma}{2\pi} e^2 c^3 E^2 B^2, \tag{110}$$

where the radiation constant $C_\gamma$ is (for electrons):

$$C_\gamma = \frac{e^2}{3\epsilon_0 (mc^2)^4} \approx 8.846 \times 10^{-5}\,\mathrm{m/GeV^3}. \tag{111}$$

The dipole may have a quadrupole gradient:

$$B = B_0 + B_1 x, \tag{112}$$

and the particle may have some energy deviation $\delta$:

$$E = (1+\delta)E_0. \tag{113}$$

Using (112) and (113) in (110), we find:

$$P_\gamma = c\frac{C_\gamma}{2\pi}\left(\frac{1}{\rho^2} + 2k_1\frac{x}{\rho}\right)(1+\delta)^2 E_0^4, \tag{114}$$

where $k_1$ is the normalised quadrupole gradient:

$$k_1 = \frac{e}{P_0}B_1. \tag{115}$$

Hence, the normalised momentum loss can be written:

$$\frac{dp}{P_0} \approx \frac{C_\gamma}{2\pi}\left(\frac{1}{\rho^2} + 2k_1\frac{x}{\rho}\right)(1+\delta)^2 E_0^3\, ds. \tag{116}$$

Expanding to first order in the phase space variables, we find:

$$\frac{dp}{P_0} \approx \frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}\, ds + \frac{C_\gamma}{2\pi}\left(\frac{1}{\rho^2} + 2k_1\right)\frac{E_0^3}{\rho}x\, ds + 2\frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}\delta\, ds. \tag{117}$$

Given the expression (117) for the momentum loss, the transformations (108) for the canonical momenta become:

$$p_x \;\mapsto\; \left(1 - \frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}\,ds\right) p_x, \tag{118}$$

$$p_y \;\mapsto\; \left(1 - \frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}\,ds\right) p_y, \tag{119}$$

$$\delta \;\mapsto\; \left(1 - 2\frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}\,ds\right)\delta - \frac{C_\gamma}{2\pi}\left(\frac{1}{\rho^2} + 2k_1\right)\frac{E_0^3}{\rho}x\,ds - \frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}\,ds. \tag{120}$$

The first-order terms in these transformations give us elements of the transfer matrix $M$.

The zeroth-order term (in the transformation of $\delta$) gives an element of $D$ that is of order $ds^2$: this vanishes in the limit $ds \to 0$.

However, we need to include the effects of quantum excitation: this will lead to an element in $D$ that is non-zero, even in the limit $ds \to 0$...

The only non-zero zeroth-order term in the transformation (through a thin slice of dipole) of the covariance matrix is:

$$D_{66} = \left\langle \left(\frac{dp}{P_0}\right)^2 \right\rangle \approx \frac{\langle u^2 \rangle}{E_0^2}, \tag{121}$$

where $\langle u^2 \rangle$ represents the mean square energy of the photons emitted in a slice of dipole of length $ds$.

We now use the result (see Lecture 1 Appendix D):

$$\langle \dot{N}(u) u^2 \rangle = 2 C_q \gamma^2 \frac{E_0}{\rho} P_\gamma, \tag{122}$$

where

$$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \approx 3.832 \times 10^{-13}\,\mathrm{m}, \tag{123}$$

is the quantum radiation constant, and $\dot{N}(u)\,du$ is the rate of emission of photons in the energy range $u$ to $u + du$.

Using (122) in (121) we find that, to zeroth-order in the phase space variables:

$$D_{66} \approx 2 C_q \gamma^2 \, \frac{C_\gamma}{2\pi} \frac{E_0^3}{\rho^3} \, ds. \tag{124}$$

Hence, in a thin slice of dipole of length $ds$, the radiation effects can be represented by the matrices:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 - \frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}ds & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 - \frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}ds & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{C_\gamma}{2\pi}\left(\frac{1}{\rho^2} + 2k_1\right)\frac{E_0^3}{\rho}ds & 0 & 0 & 0 & 0 & 1 - 2\frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}ds \end{pmatrix}, \tag{125}$$

and:

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2C_q\gamma^2\frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^3}\,ds \end{pmatrix}. \tag{126}$$

These results (together with the transfer matrices for standard storage ring components, neglecting radiation effects) provide all the ingredients we need to apply the envelope method, to find the equilibrium emittances in a computational model of a storage ring, including effects of alignment errors.