LONGITUDINAL DYNAMICS

Frank Tecker
CERN, BE-OP

Basics of Accelerator Physics and Technology
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The goal of an accelerator is to provide a stable particle beam.

The particles nevertheless perform transverse betatron oscillations. We will see that they also perform oscillations in the longitudinal plane and in energy.

We will look at the stability of these oscillations, and their dynamics.

More related lectures:
- Linacs - Alessandra Lombardi
- RF Systems - myself

• Acceleration methods
• Accelerating structures
• Linac: Phase Stability + Energy-Phase oscillations
• Circular accelerators: Cyclotron / Synchrotron
• Stability in a Synchrotron
• Longitudinal Phase Space Motion
• Bunch and Bucket
• Injection Matching + Filamentation
• RF manipulations in the PS
• Linear accelerators scale in size and cost(!) \(~\text{linearly with the energy.}\)
• Circular accelerators can each turn reuse
  • the **accelerating system**
  • the vacuum chamber
  • the bending/focusing **magnets**
  • beam instrumentation, ...

-> **economic solution to reach higher particle energies**

**But each accelerator has a limited energy range.**
Particle types and acceleration

The accelerating system will depend upon the evolution of the particle velocity:
- electrons reach a constant velocity (~speed of light) at relatively low energy
- heavy particles reach a constant velocity only at very high energy
  -> we need different types of resonators, optimized for different velocities
  -> the revolution frequency will vary, so the RF frequency will be changing
  -> magnetic field needs to follow the momentum increase

Particle rest mass $m_0$:
- electron $0.511$ MeV
- proton $938$ MeV
- $^{239}$U $\sim 220000$ MeV

Total Energy: $E = \gamma m_0 c^2$

Relativistic gamma factor:
$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - \beta^2}}$$

Momentum:
$$p = m v = \frac{E}{c^2} \beta c = \beta \frac{E}{c} = \beta \gamma m_0 c$$
The revolution and RF frequency will be changing during acceleration. Much more important for lower energies (values are kinetic energy - protons).

**PS Booster:**
- Pre LS2: 50 MeV ($\beta=0.314$) $\rightarrow$ 1.4 GeV ($\beta=0.915$)
- Post LS2: 602 kHz $\rightarrow$ 1746 kHz $\Rightarrow$ **190% frequency increase**

**PS:**
- Pre LS2: 1.4 GeV ($\beta=0.915$) $\rightarrow$ 25.4 GeV ($\beta=0.9994$)
- Post LS2: 437 KHz $\rightarrow$ 477 KHz $\Rightarrow$ **9% increase**

**SPS:**
- 25.4 GeV $\rightarrow$ 450 GeV ($\beta=0.999998$)
  $\Rightarrow$ **0.06% frequency increase**

**LHC:**
- 450 GeV $\rightarrow$ 7 TeV ($\beta=0.999999991$)
  $\Rightarrow$ **only 2 $10^{-6}$ increase**

RF system needs more flexibility in lower energy accelerators.
Hence, it is necessary to have an electric field $E$ (preferably) along the direction of the initial momentum ($z$), which changes the momentum of the particle.

The 2nd term - larger at high velocities - is used for:
- **BENDING**: generated by a magnetic field perpendicular to the plane of the particle trajectory. The bending radius $\rho$ obeys to the relation:

$$ \frac{dp}{dt} = eE_z $$

- **FOCUSING**: the bending effect is used to bring the particles trajectory closer to the axis, hence to increase the beam density.
Energy Gain

The acceleration increases the momentum, providing kinetic energy to the charged particles.

In relativistic dynamics, total energy \( E \) and momentum \( p \) are linked by

\[
E^2 = E_0^2 + p^2 c^2
\]

\( (E = E_0 + W) \quad W \text{ kinetic energy} \)
\( E_0 \text{ rest energy} \)

Hence:

\[
dE = v dp
\]

\( (2E dE = 2c^2 p dp \iff dE = c^2 mv / E \: dp = v dp) \)

The rate of energy gain per unit length of acceleration (along \( z \)) is then:

\[
\frac{dE}{dz} = v \frac{dp}{dz} = \frac{dp}{dt} = eE_z
\]

and the kinetic energy gained from the field along the \( z \) path is:

\[
dW = dE = e E_z \: dz \quad \rightarrow \quad W = e \int E_z \: dz = eV
\]

where \( V \) is just a potential.
Unit of Energy

Today’s accelerators and future projects work/aim at the TeV energy range.

- **LHC**: 7 TeV -> 14 TeV
- **CLIC**: 3 TeV
- **HE/VHE-LHC**: 33/100 TeV

In fact, this energy unit comes from acceleration:

1 eV (electron Volt) is the energy that 1 elementary charge e (like one electron or proton) gains when it is accelerated in a potential (voltage) difference of 1 Volt.

**Basic Unit**: eV (electron Volt)

- keV = 1000 eV = $10^3$ eV
- MeV = $10^6$ eV
- GeV = $10^9$ eV
- TeV = $10^{12}$ eV

LHC = ~450 Million km of batteries!!!

3x distance Earth-Sun
Electrostatic Acceleration

Electrostatic Field:

Force: \( \vec{F} = \frac{d\vec{p}}{dt} = e \vec{E} \)

Energy gain: \( W = e \Delta V \)

used for first stage of acceleration:
particle sources, electron guns,
x-ray tubes

Limitation: insulation problems
maximum high voltage (~ 10 MV)

Van-de-Graaf generator at MIT
Radio-Frequency (RF) Acceleration

Electrostatic acceleration limited by insulation possibilities => use time-varying fields

1924: Ising suggests drift-tubes with time-varying fields

1928: Widerøe builds first demonstration linac

Prinzip einer Methode zur Herstellung von Kanalstrahlen hoher Voltzahl.

Von GUSTAF ISING.
Mit 2 Figuren im Texte.
Mitgeteilt am 12. März 1924 durch C. W. Oseen und M. Siegbahn.

To Pump R T F L

E C L R

Ground

To Pump R T F L

E C L R

Ground
Radio-Frequency (RF) Acceleration

Cylindrical electrodes (drift tubes) separated by gaps and fed by a RF generator, as shown above, lead to an alternating electric field polarity.

Synchronism condition \[ L = \frac{v}{2} T \]

\( v = \) particle velocity  \( T = \) RF period

Consequence: We can only accelerate bunched beam!

Similar for standing wave cavity as shown (with \( v \approx c \))

D. Schulte

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RF acceleration: Alvarez Structure

Synchronism condition \((g << L)\)

\[
L = v_s T_{RF} = \beta_s \lambda_{RF}
\]

\[
\omega_{RF} = 2\pi f_{RF} = 2\pi \frac{v_s}{L}
\]
Resonant RF Cavities

- Considering RF acceleration, it is obvious that when particles get high velocities the drift spaces get longer and one loses on the efficiency. => The solution consists of using a higher operating frequency.

- The power lost by radiation, due to circulating currents on the electrodes, is proportional to the RF frequency. => The solution consists of enclosing the system in a cavity which resonant frequency matches the RF generator frequency.

- The electromagnetic power is now constrained in the resonant volume

- Each such cavity can be independently powered from the RF generator

- Note however that joule losses will occur in the cavity walls (unless made of superconducting materials)
Some RF Cavity Examples

\[ L = \frac{vT}{2} \ (\pi \ mode) \]

Single Gap

\[ L = vT \ (2\pi \ mode) \]

Multi-Gap

More in RF Systems
1. For **circular accelerators**, the origin of time is taken at the **zero crossing** of the RF voltage with positive slope.

2. For **linear accelerators**, the origin of time is taken at the positive **crest** of the RF voltage.

Time $t = 0$ chosen such that:

\[ E_1(t) = E_0 \sin(\omega_{RF} t) \]

\[ E_2(t) = E_0 \cos(\omega_{RF} t) \]

3. I will stick to **convention 1** in the following to avoid confusion.
Let's consider a succession of accelerating gaps, operating in the $2\pi$ mode, for which the synchronism condition is fulfilled for a phase $\Phi_s$.

\[ eV_s = e\hat{V}\sin\Phi_s \]

is the energy gain in one gap for the particle to reach the next gap with the same RF phase: $P_1, P_2, \ldots$ are fixed points.

For a $2\pi$ mode, the electric field is the same in all gaps at any given time.

If an energy increase is transferred into a velocity increase =>

- $M_1$ & $N_1$ will move towards $P_1$ => stable
- $M_2$ & $N_2$ will go away from $P_2$ => unstable

(Highly relativistic particles have no significant velocity change)
The divergence of the field is zero according to Maxwell:
\[ \nabla \cdot \mathbf{E} = 0 \implies \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 0 \implies \frac{\partial E_x}{\partial x} = -\frac{\partial E_z}{\partial z} \]

Transverse fields
- focusing at the entrance and
- defocusing at the exit of the cavity.

Electrostatic case: Energy gain inside the cavity leads to focusing
RF case: Field increases during passage => transverse defocusing!

External focusing (solenoid, quadrupole) is then necessary
Energy-phase Oscillations (Small Amplitude) (1)

- Rate of energy gain for the synchronous particle:

\[
\frac{dE_s}{dz} = \frac{dp_s}{dt} = eE_0 \sin \phi_s
\]

- Use reduced variables with respect to synchronous particle

\[
w = W - W_s = E - E_s \quad \varphi = \phi - \phi_s
\]

Energy gain:

\[
\frac{dw}{dz} = eE_0 \left[ \sin(\phi_s + \varphi) - \sin \phi_s \right] \approx eE_0 \cos \phi_s \varphi \quad \text{(small } \varphi \text{)}
\]

- Rate of phase change with respect to the synchronous one:

\[
\frac{d\varphi}{dz} = \omega_{RF} \left( \frac{dt}{dz} - \left( \frac{dt}{dz} \right)_s \right) = \omega_{RF} \left( \frac{1}{v} - \frac{1}{v_s} \right) \approx - \omega_{RF}^2 \left( v - v_s \right)
\]

Leads finally to:

\[
\frac{d\varphi}{dz} = - \frac{\omega_{RF}}{m_0 v_s^3 \gamma_s^3} w
\]
Combining the two 1\textsuperscript{st} order equations into a 2\textsuperscript{nd} order equation gives the equation of a harmonic oscillator:

\[ \frac{d^2 \phi}{dz^2} + \Omega_s^2 \phi = 0 \]

with

\[ \Omega_s^2 = \frac{eE_0 \omega_{RF} \cos \phi_s}{m_0 v_s^3 \gamma_s^3} \]

Stable harmonic oscillations imply:

\[ \cos \phi_s > 0 \]

hence:

\[ \sin \phi_s > 0 \]

And since acceleration also means:

\[ 0 < \phi_s < \frac{\pi}{2} \]

You finally get the result for the stable phase range:

\[ 0 < \phi_s < \frac{\pi}{2} \]

Positive rising RF slope!

Energy-phase Oscillations (Small Amplitude) (2)
• **Acceleration by electric fields**, static fields limited
  => time-varying fields

• **Synchronous condition** needs to be fulfilled for acceleration

• Particles perform oscillation around synchronous phase

• Stable acceleration on the rising slope in a linac.

• Electrons are quickly relativistic, speed does not change

• Protons and ions need changing structure geometry and certain RF frequency range
Circular accelerators

Betatron
Cyclotron
Synchrotron
Acceleration by Induction: The Betatron

A ramping magnetic field
- Guides particles on a circular trajectory and
- Creates a tangential electric field that accelerates the particles

Limited by saturation in iron (~300 MeV e-)

Used in industry and medicine, as they are compact accelerators for electrons

Donald Kerst with the first betatron, invented at the University of Illinois in 1940
Circular accelerators: Cyclotron

Courtesy: EdukiteLearning, https://youtu.be/cNnNM2ZqIsc
Circular accelerators: Cyclotron

Used for protons, ions

\[ B = \text{constant} \]
\[ \omega_{RF} = \text{constant} \]

Synchronism condition

\[ \omega_s = \omega_{RF} \]
\[ 2\pi \rho = v_s T_{RF} \]

Cyclotron frequency

\[ \omega = \frac{q B}{m_0 \gamma} \]

1. \( \gamma \) increases with the energy
   \( \Rightarrow \) no exact synchronism

2. if \( v << c \) \( \Rightarrow \gamma \approx 1 \)

Animation: http://www.sciences.univ-nantes.fr/sites/genevieve_tulloue/Meca/Charges/cyclotron.html

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Synchrocyclotron: Same as cyclotron, except a modulation of $\omega_{RF}$

- $B$ = constant
- $\gamma \omega_{RF}$ = constant
- $\omega_{RF}$ decreases with time

The condition:

$$\omega_s(t) = \omega_{RF}(t) = \frac{qB}{m_0 \gamma(t)}$$

Allows to go beyond the non-relativistic energies
Circular accelerators: The Synchrotron

1. **Constant orbit** during acceleration
2. To keep particles on the closed orbit, \( B \) should increase with time
3. \( \omega \) and \( \omega_{RF} \) increase with energy

RF frequency can be multiple of revolution frequency

\[
\omega_{RF} = h\omega
\]

\[
T_s = h T_{RF}
\]

\[
\frac{2\pi R}{v_s} = h T_{RF}
\]

\( h \) integer, harmonic number: number of RF cycles per revolution

\( h \) is the maximum number of bunches in the synchrotron. Normally less bunches due to gaps for kickers, collision constraints,...
Circular accelerators: The Synchrotron

Examples of different proton and electron synchrotrons at CERN

+ LHC (of course!)
The Synchrotron

The synchrotron is a synchronous accelerator since there is a synchronous RF phase for which the energy gain fits the increase of the magnetic field at each turn. That implies the following operating conditions:

\[ eV \sin \phi \] \rightarrow \text{Energy gain per turn}

\[ \phi = \phi_s = cte \] \rightarrow \text{Synchronous particle}

\[ \omega_{RF} = h\omega \] \rightarrow \text{RF synchronism (h - harmonic number)}

\[ \rho = cte \quad R = cte \] \rightarrow \text{Constant orbit}

\[ B \rho = \frac{P}{e} \Rightarrow B \] \rightarrow \text{Variable magnetic field}

If \( v \approx c \), \( \omega \) hence \( \omega_{RF} \) remain constant (ultra-relativistic e\(^{-}\))
The magnetic field (dipole current) is increased during the acceleration.

![Diagram of the Synchrotron - LHC Operation Cycle](image)

The magnetic field (dipole current) is increased during the acceleration.
The Synchrotron - Energy ramping

Energy ramping by increasing the B field (frequency has to follow \( v \)):

\[
p = eB\rho \quad \Rightarrow \quad \frac{dp}{dt} = e\rho \dot{B} \quad \Rightarrow \quad (\Delta p)_\text{turn} = e\rho \dot{B}T_r = \frac{2\pi e\rho R\dot{B}}{v}
\]

With

\[
E^2 = E_0^2 + p^2c^2 \quad \Rightarrow \quad \Delta E = v\Delta p
\]

\[
(\Delta E)_\text{turn} = (\Delta W)_s = 2\pi e\rho R\dot{B} = e\hat{V}\sin\phi_s
\]

Synchronous phase \( \phi_s \) changes during energy ramping

\[
\sin\phi_s = 2\pi \rho R \frac{\dot{B}}{\hat{V}_\text{RF}} \quad \Rightarrow \quad \phi_s = \arcsin\left(2\pi \rho R \frac{\dot{B}}{\hat{V}_\text{RF}}\right)
\]

- The synchronous phase depends on
  - the change of the magnetic field
  - and the RF voltage

\[
V_{\text{RF}}
\]

\[
\phi = \omega_{\text{RF}} t
\]

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The Synchrotron - Frequency change

During the energy ramping, the RF frequency increases to follow the increase of the revolution frequency:

\[ \omega = \frac{\omega_{RF}}{\hbar} = \omega (B, R_s) \]

Hence:
\[ \frac{f_{RF}(t)}{h} = \frac{v(t)}{2\pi R_s} = \frac{1}{2\pi} \frac{ec^2}{E_s(t) R_s} \rho B(t) \]

(using \( p(t) = eB(t)\rho, \quad E = mc^2 \))

Since \( E^2 = (m_0c^2)^2 + p^2c^2 \) the RF frequency must follow the variation of the B field with the law

\[ \frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_s} \left\{ \frac{B(t)^2}{(m_0c^2 / ec\rho)^2 + B(t)^2} \right\}^{1/2} \]

RF frequency program during acceleration determined by B-field!
During the energy ramping, the B-field and the revolution frequency increase.

Example: PS - Field / Frequency change

- **B-field**
- **kinetic energy**
- **'B-dot'**
- **B-field change**
- **revolution frequency**

(time (ms))
Overtaking in a roundabout

Finally a real-life problem: what is the fastest way through a roundabout?

Most CERN people encounter this near the French entrance to CERN.
Empty Slide – Wait for the talk
Empty Slide - Wait for the talk
Empty Slide – Wait for the talk
Overtaking in a Synchrotron

A particle slightly shifted in momentum will have a

- dispersion orbit and a **different orbit length**
  (higher momentum => less bent in magnet)
- a **different velocity**.

As a result of both effects the revolution frequency changes with a “slip factor $\eta$”:

$$
\eta = \frac{df_r}{dp} \frac{f_r}{p} \Rightarrow \eta = \frac{p}{f_r} \frac{df_r}{dp}
$$

Note: you also find $\eta$ defined with a minus sign!

Effect from orbit defined by **Momentum compaction factor:**

$$
\alpha_c = \frac{dL/L}{dp/p}
$$

Property of the **transverse beam optics:**

(derivation in Appendix)
Dispersion Effects - Revolution Frequency

The two effects of the orbit length and the particle velocity change the revolution frequency as:

\[ f_r = \frac{\beta c}{2\pi R} \Rightarrow \frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R} = \frac{d\beta}{\beta} - \alpha_c \frac{dp}{p} \]

\[
\frac{df_r}{f_r} = \left( \frac{1}{\gamma^2 - \alpha_c} \right) \frac{dp}{p}
\]

\[
p = mv = \beta \gamma \frac{E_0}{c} \Rightarrow \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d(1 - \beta^2)^{-\frac{1}{2}}}{(1 - \beta^2)^{-\frac{1}{2}}}
\]

\[
\frac{d(1 - \beta^2)^{-\frac{1}{2}}}{(1 - \beta^2)^{-\frac{1}{2}}} = (1 - \beta^2)^{-1} \frac{d\beta}{\beta}
\]

Slip factor: \[ \eta = \frac{1}{\gamma^2 - \alpha_c} \]

or \[ \eta = \frac{1}{\gamma^2 - \gamma_t^2} \]

with \[ \gamma_t = \frac{1}{\sqrt{\alpha_c}} \]

At transition energy, \( \eta = 0 \), the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.
Phase Stability in a Synchrotron

From the definition of $\eta$ it is clear that an increase in momentum gives
- below transition ($\eta > 0$) a higher revolution frequency
  (increase in velocity dominates) while

- above transition ($\eta < 0$) a lower revolution frequency ($v \approx c$ and longer path)
  where the momentum compaction (generally $> 0$) dominates.

$$\eta = \frac{1}{\gamma^2 - \alpha_c}$$
Crossing Transition

At transition, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Crossing transition during acceleration makes the previous stable synchronous phase unstable. The RF system needs to make a rapid change of the RF phase, a ‘phase jump’.

\[ \alpha_c \sim \frac{1}{Q_x^2} \]

\[ \gamma_t = \frac{1}{\sqrt{\alpha_c}} \sim Q_x \]

In the PS: \( \gamma_t \) is at \( \sim 6 \) GeV
In the SPS: \( \gamma_t = 22.8 \), injection at \( \gamma = 27.7 \)
  => no transition crossing!
In the LHC: \( \gamma_t \) is at \( \sim 55 \) GeV, also far below injection energy

Transition crossing not needed in leptons machines, why?
**Dynamics: Synchrotron oscillations**

Simple case (no accel.): \( B = \text{const.}, \) below transition \( \gamma < \gamma_t \)

The phase of the synchronous particle must therefore be \( \phi_0 = 0 \).

\( \Phi_1 \) - The particle \( B \) is accelerated
  - Below transition, an energy increase means an increase in revolution frequency
  - The particle arrives earlier - tends toward \( \phi_0 \)

\( \phi_2 \) - The particle is decelerated
  - decrease in energy - decrease in revolution frequency
  - The particle arrives later - tends toward \( \phi_0 \)
Particle B performs **Synchrotron Oscillations** around synchronous particle A.

The amplitude depends on the initial phase and energy.

The oscillation frequency is much slower than in the transverse plane. It takes a large number of revolutions for one complete oscillation. The restoring electric force is smaller than the magnetic force.

- proton synchrotrons of the order of 1000 turns
- electron storage rings of the order of ~10 turns
The Potential Well

Cavity voltage

Potential well
Longitudinal phase space

The energy - phase oscillations can be drawn in phase space. Similar to transverse, but here it’s TIME and ENERGY!

The particle trajectory in the phase space ($\Delta p/p, \phi$) describes its longitudinal motion.

Emittance: phase space area including all the particles

NB: if the emittance contour correspond to a possible orbit in phase space, its shape does not change with time (matched beam)
Particle B oscillates around particle A in a synchrotron oscillation. Plotting this motion in longitudinal phase space (time, energy) gives:

Longitudinal Phase Space Motion

- **Early arrival**
- **Late arrival**
- **Lower energy**
- **Higher energy**
Synchrotron oscillations - No acceleration

Phase space picture

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**Synchrotron motion in phase space**

\( \Delta E - \phi \) phase space of a stationary bucket
(when there is no acceleration)

**Dynamics of a particle**
Non-linear, conservative oscillator \( \rightarrow \) e.g. pendulum

---

Particle inside the separatrix:

Particle at the unstable fix-point

---

Bucket area: area enclosed by the separatrix

The area covered by particles is the longitudinal emittance

---

Particle outside the separatrix:
(Stationary) Bunch & Bucket

The bunches of the beam fill usually a part of the bucket area.

**Bucket area** = longitudinal Acceptance $[\text{eVs}]$

**Bunch area** = longitudinal beam emittance $= 4\pi \sigma_E \sigma_t \ [\text{eVs}]$

Attention: Different definitions are used!
The restoring force is non-linear.
\[ \Rightarrow \text{speed of motion depends on position in phase-space} \]

(here shown for a stationary bucket)
Synchrotron oscillations (with acceleration)

Case with acceleration $B$ increasing

$\gamma < \gamma_t$

$\phi = \omega_{RF} t$

$\phi_s < \phi < \pi - \phi_s$

Phase space picture

The symmetry of the case $B = \text{const.}$ is lost
The areas of stable motion (closed trajectories) are called “BUCKET”. The number of circulating buckets is equal to “h”.

The phase extension of the bucket is maximum for $\phi_s = 180^\circ$ (or $0^\circ$) which means no acceleration. During acceleration, the buckets get smaller, both in length and energy acceptance.

=> Injection preferably without acceleration.
Now we will look more quantitatively at the “synchrotron motion”.

The RF acceleration process clearly emphasizes two coupled variables, the energy gained by the particle and the RF phase experienced by the same particle.

Since there is a well defined synchronous particle which has always the same phase $\phi_s$, and the nominal energy $E_s$, it is sufficient to follow other particles with respect to that particle.

So let’s introduce the following reduced variables:

- revolution frequency: $\Delta f_r = f_r - f_{rs}$
- particle RF phase: $\Delta \phi = \phi - \phi_s$
- particle momentum: $\Delta p = p - p_s$
- particle energy: $\Delta E = E - E_s$
- azimuth angle: $\Delta \theta = \theta - \theta_s$

Look at difference from synchronous particle
Equations of Longitudinal Motion

In these reduced variables, the equations of motion are (see Appendix):

\[
\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h \eta \omega_{rs}} \frac{d(\Delta \phi)}{dt} = -\frac{p_s R_s}{h \eta \omega_{rs}} \dot{\phi}
\]

\[
2\pi \frac{d}{dt} \left( \frac{\Delta E}{\omega_{rs}} \right) \dot{e} \hat{V} (\sin \phi - \sin \phi_s)
\]

This second order equation is non linear. Moreover the parameters within the bracket are in general slowly varying with time.

We will simplify in the following...
Small Amplitude Oscillations

Let's assume constant parameters $R_s$, $p_s$, $\omega_s$ and $\eta$:

\[
\ddot{\phi} + \frac{\Omega_s^2}{\cos \phi_s} (\sin \phi - \sin \phi_s) = 0 \quad \text{with} \quad \Omega_s^2 = \frac{h \eta \omega_{rs} e \hat{V} \cos \phi_s}{2 \pi R_s p_s}
\]

Consider now small phase deviations from the reference particle:

\[
\sin \phi - \sin \phi_s = \sin (\phi_s + \Delta \phi) - \sin \phi_s \approx \cos \phi_s \Delta \phi \quad \text{(for small $\Delta \phi$)}
\]

and the corresponding linearized motion reduces to a harmonic oscillation:

\[
\ddot{\phi} + \Omega_s^2 \Delta \phi = 0 \quad \text{where $\Omega_s$ is the synchrotron angular frequency.}
\]

The synchrotron tune $\nu_s$ is the number of synchrotron oscillations per revolution:

\[
\nu_s = \frac{\Omega_s}{\omega_s}
\]

Typical values are $<<1$, as it takes several 10 - 1000 turns per oscillation.

- proton synchrotrons of the order $10^{-3}$
- electron storage rings of the order $10^{-1}$

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Stability condition for $\phi_s$

$$\Omega_s^2 = \frac{e \hat{V}_{RF} \eta h \omega_s}{2\pi R_s p_s} \cos \phi_s \quad \iff \quad \Omega_s^2 = \omega_s^2 \frac{e \hat{V}_{RF} \eta h}{2\pi \beta^2 E} \cos \phi_s \quad \text{with} \quad R_p = \frac{\beta^2 E}{\omega}$$

Stability is obtained when $\Omega_s$ is real and so $\Omega_s^2$ positive:

$$\Omega_s^2 > 0 \quad \updownarrow \quad \eta \cos \phi_s > 0$$

Stable in the region if $\eta > 0$ and $\gamma > \gamma_{tr}$.
Energy Acceptance

From the equation of the separatrix, we can calculate (see appendix) the acceptance in energy:

\[
\left( \frac{\Delta E}{E_s} \right)_{\text{max}} = \pm \beta \sqrt{\frac{e\hat{V}}{\pi \hbar \eta E_s}} G(\phi_s)
\]

\[
G(\phi_s) = \left[ 2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s \right]
\]

This “RF acceptance” depends strongly on \( \phi_s \) and plays an important role for the capture at injection, and the stored beam lifetime.

It’s largest for \( \phi_s=0 \) and \( \phi_s=\pi \) (no acceleration, depending on \( \eta \)).

It becomes smaller during acceleration, when \( \phi_s \) is changing.

Need a higher RF voltage for higher acceptance => need more $€ 😞
The areas of stable motion (closed trajectories) are called “BUCKET”. The number of circulating buckets is equal to “h”.

The phase extension of the bucket is maximum for $\phi_s = 180^\circ$ (or $0^\circ$) which means no acceleration.

During acceleration, the buckets get smaller, both in length and energy acceptance.

$\Rightarrow$ Injection preferably without acceleration.
Injection: Bunch-to-bucket transfer

- Bunch from sending accelerator into the bucket of receiving

Advantages:
- Particles always subject to longitudinal focusing
- No need for RF capture of de-bunched beam in receiving accelerator
- No particles at unstable fixed point
- Time structure of beam preserved during transfer
Effect of a Mismatch

Injected bunch: short length and large energy spread after 1/4 synchrotron period: longer bunch with a smaller energy spread.

For larger amplitudes, the angular phase space motion is slower (1/8 period shown below) => can lead to filamentation and emittance growth.

restoring force is non-linear

stationary bucket

accelerating bucket

W.Pirkl
Effect of a Mismatch (2)

- Long. emittance is only preserved for correct RF voltage

- Matched case
  - Bunch is fine, longitudinal emittance remains constant

- Longitudinal mismatch
  - Dilution of bunch results in increase of long. emittance
Effect of a Mismatch (3)

Evolution of an injected beam for the first 100 turns.
For a mismatched transfer, the emittance increases (right).
Bunch Rotation

Phase space motion can be used to make short bunches.
Start with a long bunch and extract or recapture when it's short.

initial beam
Capture of a Debunched Beam with Fast Turn-On
Capture of a Debunched Beam with Adiabatic Turn-On
Generating a 25ns Bunch Train in the PS

- **Longitudinal bunch splitting (basic principle)**
  - Reduce voltage on principal RF harmonic and simultaneously rise voltage on multiple harmonics (adiabatically with correct phase, etc.)

Use double splitting at 25 GeV to generate 50ns bunch trains instead
1. Inject four bunches  \( \sim 180 \text{ ns}, 1.3 \text{ eVs} \)

Wait 1.2 s for second injection

2. Inject two bunches

3. Triple split after second injection  \( \sim 0.7 \text{ eVs} \)

4. Accelerate from 1.4 GeV (\( E_{\text{kin}} \)) to 26 GeV
Production of the LHC 25 ns beam

5. During acceleration: longitudinal emittance blow-up: 0.7 – 1.3 eVs

6. Double split ($h_{21} \rightarrow h_{42}$)

7. Double split ($h_{42} \rightarrow h_{84}$) \~ 0.35 eVs, 4 ns

10. Fine synchronization, bunch rotation $\rightarrow$ Extraction!
The LHC25 (ns) cycle in the PS

Inject 4+2 bunches  
$h = 7$

Split in four at flat-top energy

Eject 72 bunches

$h = 21$

$h = 84$

Each bunch from the Booster divided by $12 \rightarrow 6 \times 3 \times 2 \times 2 = 72$
Triple splitting in the PS
Two times double splitting and bunch rotation:

- Bunch is divided twice using RF systems at $h = 21/42$ (10/20 MHz) and $h = 42/84$ (20/40 MHz)
- Rotation: first part $h84$ only + $h168$ (80 MHz) for final part
Summary

- **Cyclotrons/Synchrocyclotrons** for low energy
- **Synchrotrons** for high energies, constant orbit, rising field and frequency synchronously
- Particles with higher energy have a longer orbit (normally) but a higher velocity
  - at low energies (below transition) velocity increase dominates
  - at high energies (above transition) velocity almost constant
- Particles perform *oscillations around synchronous phase*
  - synchronous phase depending on acceleration
  - below or above transition
- **Bucket** is the stable region in phase space inside the *separatrix*
- **Bunch** is the area filled with beam
- **Matching** the shape of the bunch to the bucket is essential
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And CERN Accelerator Schools (CAS) Proceedings  
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Appendix

- Summary Relativity and Energy Gain
- Velocity, Energy, and Momentum
- Momentum compaction factor
- Synchrotron energy-phase oscillations
- Stability condition
- Separatrix stationary bucket
- Large amplitude oscillations
- Bunch matching into stationary bucket
Appendix: Relativity + Energy Gain

Newton-Lorentz Force: \[ \vec{F} = \frac{d\vec{p}}{dt} = e \left( \vec{E} + \vec{v} \times \vec{B} \right) \]

2nd term always perpendicular to motion \(\Rightarrow\) no acceleration

Relativistic Dynamics

\[ \beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \quad \gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - \beta^2}} \]

\[ p = mv = \frac{E}{c^2} \beta c = \beta \frac{E}{c} = \beta \gamma m_0 c \]

\[ E^2 = E_0^2 + p^2 c^2 \quad \rightarrow \quad dE = v dp \]

\[ \frac{dE}{dz} = v \frac{dp}{dz} = \frac{dp}{dt} = eE_z \]

\[ dE = dW = e E_z \, dz \quad \rightarrow \quad W = e \int E_z \, dz \]

RF Acceleration

\[ E_z = \hat{E}_z \sin \omega_{RF} t = \hat{E}_z \sin \phi(t) \]

\[ \int \hat{E}_z \, dz = \hat{V} \]

\[ W = e\hat{V} \sin \phi \]

(neglecting transit time factor)

The field will change during the passage of the particle through the cavity
\(\Rightarrow\) effective energy gain is lower
Appendix: Velocity, Energy and Momentum

normalized velocity \[ \beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \]

=> electrons almost reach the speed of light very quickly (few MeV range)

total energy \[ E = \gamma m_0 c^2 \]

rest energy

\[ \gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \beta^2}} \]

Momentum \[ p = mv = \frac{E}{c^2} \beta c = \beta \frac{E}{c} = \beta \gamma m_0 c \]

=> Magnetic field needs to follow the momentum increase

CAS@ESI, 7-11 October 2019
Appendix: Momentum Compaction Factor

\[ \alpha_c = \frac{p \, dL}{L \, dp} \]

\[ ds_0 = \rho d\theta \]

\[ ds = (\rho + x) \, d\theta \]

The elementary path difference from the two orbits is:

\[ \frac{dl}{ds_0} = \frac{ds - ds_0}{ds_0} = \frac{x}{\rho} = \frac{D_x}{\rho} \frac{dp}{p} \]

leading to the total change in the circumference:

\[ dL = \int_c dl = \int_c \frac{x}{\rho} \, ds_0 = \int_c \frac{D_x}{\rho} \frac{dp}{p} \, ds_0 \]

\[ \alpha_c = \frac{1}{L} \int_c \frac{D_x(s)}{\rho(s)} \, ds_0 \]

With \( \rho = \infty \) in straight sections we get:

\[ \alpha_c = \frac{\langle D_x \rangle_m}{R} \]

< >_m means that the average is considered over the bending magnet only.
Appendix: First Energy-Phase Equation

\[ f_{RF} = h f_r \quad \Rightarrow \quad \Delta \phi = -h \Delta \theta \quad \text{with} \quad \theta = \int \omega \ dt \]

particle ahead arrives earlier
=> smaller RF phase

For a given particle with respect to the reference one:

\[ \Delta \omega = \frac{d}{dt} (\Delta \theta) = - \frac{1}{h} \frac{d}{dt} (\Delta \phi) = - \frac{1}{h} \frac{d\phi}{dt} \]

Since:

\[ \eta = \frac{p_s}{\omega_{rs}} \left( \frac{d\omega}{dp} \right)_s \]

and

\[ E^2 = E_0^2 + p^2 c^2 \]

\[ \Delta E = v_s \Delta p = \omega_{rs} R_s \Delta p \]

one gets:

\[ \frac{\Delta E}{\omega_{rs}} = \frac{p_s R_s}{h \eta \omega_{rs}} \frac{d(\Delta \phi)}{dt} = \frac{p_s R_s}{h \eta \omega_{rs}} \dot{\phi} \]
Appendix: Second Energy-Phase Equation

The rate of energy gained by a particle is:

\[ \frac{dE}{dt} = e \hat{V} \sin \phi \frac{\omega_r}{2\pi} \]

The rate of relative energy gain with respect to the reference particle is then:

\[ 2\pi \Delta \left( \frac{\dot{E}}{\omega_r} \right) = e \hat{V} (\sin \phi - \sin \phi_s) \]

Expanding the left-hand side to first order:

\[ \Delta \left( \dot{E} T_r \right) \approx \dot{E} \Delta T_r + T_{rs} \Delta \dot{E} = \Delta E \dot{T}_r + T_{rs} \Delta \dot{E} = \frac{d}{dt} \left( T_{rs} \Delta E \right) \]

leads to the second energy-phase equation:

\[ 2\pi \frac{d}{dt} \left( \frac{\Delta E}{\omega_{rs}} \right) = e \hat{V} \left( \sin \phi - \sin \phi_s \right) \]
Appendix: Stability condition for $\phi_s$

Stability is obtained when $\Omega_s$ is real and so $\Omega_s^2$ positive:

$$\Omega_s^2 = \frac{e \hat{V}_{RF} \eta h \omega_s}{2\pi R_s p_s} \cos \phi_s \Rightarrow \Omega_s^2 > 0 \iff \eta \cos \phi_s > 0$$

Stable in the region if:

- $\gamma < \gamma_{tr}$ and $\eta > 0$
- $\gamma > \gamma_{tr}$ and $\eta < 0$
- $\gamma > \gamma_{tr}$ and $\eta < 0$
- $\gamma < \gamma_{tr}$ and $\eta > 0$

acceleration  \hspace{1cm} deceleration
Appendix: Stationary Bucket - Separatrix

This is the case $\sin \phi_s = 0$ (no acceleration) which means $\phi_s = 0$ or $\pi$. The equation of the separatrix for $\phi_s = \pi$ (above transition) becomes:

\[
\frac{\dot{\phi}^2}{2} + \Omega_s^2 \cos \phi = \Omega_s^2
\]

Replacing the phase derivative by the (canonical) variable $W$:

\[
W = \frac{\Delta E}{\omega_{rf}} = - \frac{p_s R_s}{h \eta \omega_{rf}} \dot{\phi}
\]

and introducing the expression for $\Omega_s$ leads to the following equation for the separatrix:

\[
W = \pm \frac{C}{\pi hc} \sqrt{-e \hat{V} E_s} \sin \frac{\phi}{2} = \pm W_{bk} \sin \frac{\phi}{2}
\]

with $C = 2\pi R_s$.
Stationary Bucket (2)

Setting $\phi = \pi$ in the previous equation gives the height of the bucket:

$$W_{bk} = \frac{C}{\pi hc} \sqrt{\frac{-e \hat{V} E_s}{2\pi h\eta}}$$

This results in the maximum energy acceptance:

$$\Delta E_{\text{max}} = \omega_{rf} W_{bk} = \beta_s \sqrt{2 \frac{-e \hat{V}_{RF} E_s}{\pi \eta h}}$$

The area of the bucket is:

$$A_{bk} = 2 \int_0^{2\pi} W d\phi$$

Since:

$$\int_0^{2\pi} \sin \frac{\phi}{2} d\phi = 4$$

one gets:

$$A_{bk} = 8W_{bk} = 8 \frac{C}{\pi hc} \sqrt{\frac{-e \hat{V} E_s}{2\pi h\eta}} \quad \rightarrow \quad W_{bk} = \frac{A_{bk}}{8}$$
Appendix: Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

\[ \ddot{\phi} + \frac{\Omega_s^2}{\cos \phi_s} (\sin \phi - \sin \phi_s) = 0 \]  

(\(\Omega_s\) as previously defined)

Multiplying by \(\dot{\phi}\) and integrating gives an invariant of the motion:

\[ \frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos \phi_s} (\cos \phi + \phi \sin \phi_s) = I \]

which for small amplitudes reduces to:

\[ \frac{\dot{\phi}^2}{2} + \Omega_s^2 \frac{(\Delta \phi)^2}{2} = I' \]  

(the variable is \(\Delta \phi\), and \(\phi_s\) is constant)

Similar equations exist for the second variable: \(\Delta E \propto d\phi/dt\)
When $\phi$ reaches $\pi-\phi_s$ the force goes to zero and beyond it becomes non restoring.
Hence $\pi-\phi_s$ is an extreme amplitude for a stable motion which in the phase space $(\frac{\dot{\phi}}{\Omega_s}, \Delta \phi)$ is shown as closed trajectories.

Equation of the separatrix:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos \phi_s} (\cos \phi + \phi \sin \phi_s) = - \frac{\Omega_s^2}{\cos \phi_s} (\cos(\pi - \phi_s) + (\pi - \phi_s) \sin \phi_s)$$

Second value $\phi_m$ where the separatrix crosses the horizontal axis:

$$\cos \phi_m + \phi_m \sin \phi_s = \cos(\pi - \phi_s) + (\pi - \phi_s) \sin \phi_s$$
Energy Acceptance

From the equation of motion it is seen that $\dot{\phi}$ reaches an extreme when $\phi = 0$, hence corresponding to $\phi = \phi_s$.
Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}_{\text{max}}^2 = 2\Omega_s^2 \left\{ 2 + (2\phi_s - \pi)\tan\phi_s \right\}$$

That translates into an acceptance in energy:

$$\left( \frac{\Delta E}{E_s} \right)_{\text{max}} = \pm \beta \sqrt{\frac{e\dot{V}}{\pi \hbar \eta E_s}} G(\phi_s)$$

$$G(\phi_s) = \left[ 2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s \right]$$

This "RF acceptance" depends strongly on $\phi_s$ and plays an important role for the capture at injection, and the stored beam lifetime.

It's largest for $\phi_s=0$ and $\phi_s=\pi$ (no acceleration, depending on $\eta$).

Need a higher RF voltage for higher acceptance.
Bunch Matching into a Stationary Bucket

A particle trajectory inside the separatrix is described by the equation:

\[
\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos \phi_s} (\cos \phi + \phi \sin \phi_s) = I
\]

The points where the trajectory crosses the axis are symmetric with respect to \( \phi_s = \pi \)

\[
\frac{\dot{\phi}^2}{2} + \Omega_s^2 \cos \phi = I
\]

\[
\dot{\phi} = \pm \Omega_s \sqrt{2 (\cos \phi_m - \cos \phi)}
\]

\[
W = \pm W_{bk} \sqrt{\frac{\cos^2 2 \phi_m}{2} - \cos^2 \frac{\phi}{2}}
\]

\[
\cos(\phi) = 2 \cos^2 \frac{\phi}{2} - 1
\]
Setting $\phi = \pi$ in the previous formula allows to calculate the bunch height:

$$W_b = W_{bk} \cos \frac{\phi_m}{2} = W_{bk} \sin \frac{\phi}{2}$$

or:

$$W_b = \frac{A_{bk}}{8} \cos \frac{\phi_m}{2}$$

$$\left( \frac{\Delta E}{E_s} \right)_b = \left( \frac{\Delta E}{E_s} \right)_{RF} \cos \frac{\phi_m}{2} = \left( \frac{\Delta E}{E_s} \right)_{RF} \sin \frac{\phi}{2}$$

This formula shows that for a given bunch energy spread the proper matching of a shorter bunch ($\phi_m$ close to $\pi$, $\hat{\phi}$ small) will require a bigger RF acceptance, hence a higher voltage.

For small oscillation amplitudes the equation of the ellipse reduces to:

$$W = \frac{A_{bk}}{16} \sqrt{\hat{\phi}^2 - (\Delta \phi)^2} \quad \rightarrow \quad \left( \frac{16 W}{A_{bk} \hat{\phi}} \right)^2 + \left( \frac{\Delta \phi}{\hat{\phi}} \right)^2 = 1$$

Ellipse area is called longitudinal emittance

$$A_b = \frac{\pi}{16} A_{bk} \hat{\phi}^2$$