Linear Wakefield Generation

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http://cas.web.cern.ch/schools/sesimbra-2019 https://indico.cern.ch/event/759579/overview

Wakefield generation equations

Starting from Maxwell's equations and Fluid equations:

$$\epsilon_0 \nabla \cdot \mathbf{E} = n_i q_i + n_e q_e \quad (1)$$

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} \quad (2)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (3)$$

$$\mu_0^{-1} \nabla \times \mathbf{B} = n_i q_i \mathbf{v_i} + n_e q_e \mathbf{v_e} + \epsilon_0 \dot{\mathbf{E}} \quad (4)$$

$$m_j n_j \left[\frac{\partial \mathbf{v_j}}{\partial t} + (\mathbf{v_j} \cdot \nabla)\mathbf{v_j} \right] = n_j q_j (\mathbf{E} + \mathbf{v_j} \times \mathbf{B}) - \nabla p_j + n_j \mathbf{F} \quad (5)$$

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \mathbf{v_j}) = 0 \quad (6)$$

$$p_j = C_j n_j^{\gamma} \quad (7)$$

$j \in \{i, e\}$ refers to the species; ions or electrons

Wakefield generation equations

Assume a *cold* (no temperature) hydrogen plasma ($q_i = +e$, $q_e = -e$), with no magnetic fields

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Wakefield generation equations

Assume a *cold* (no temperature) hydrogen plasma (q_i = +e, q_e = -e), with no magnetic fields

$$\epsilon_0 \nabla \cdot \mathbf{E} = e(n_i - n_e) \tag{1}$$

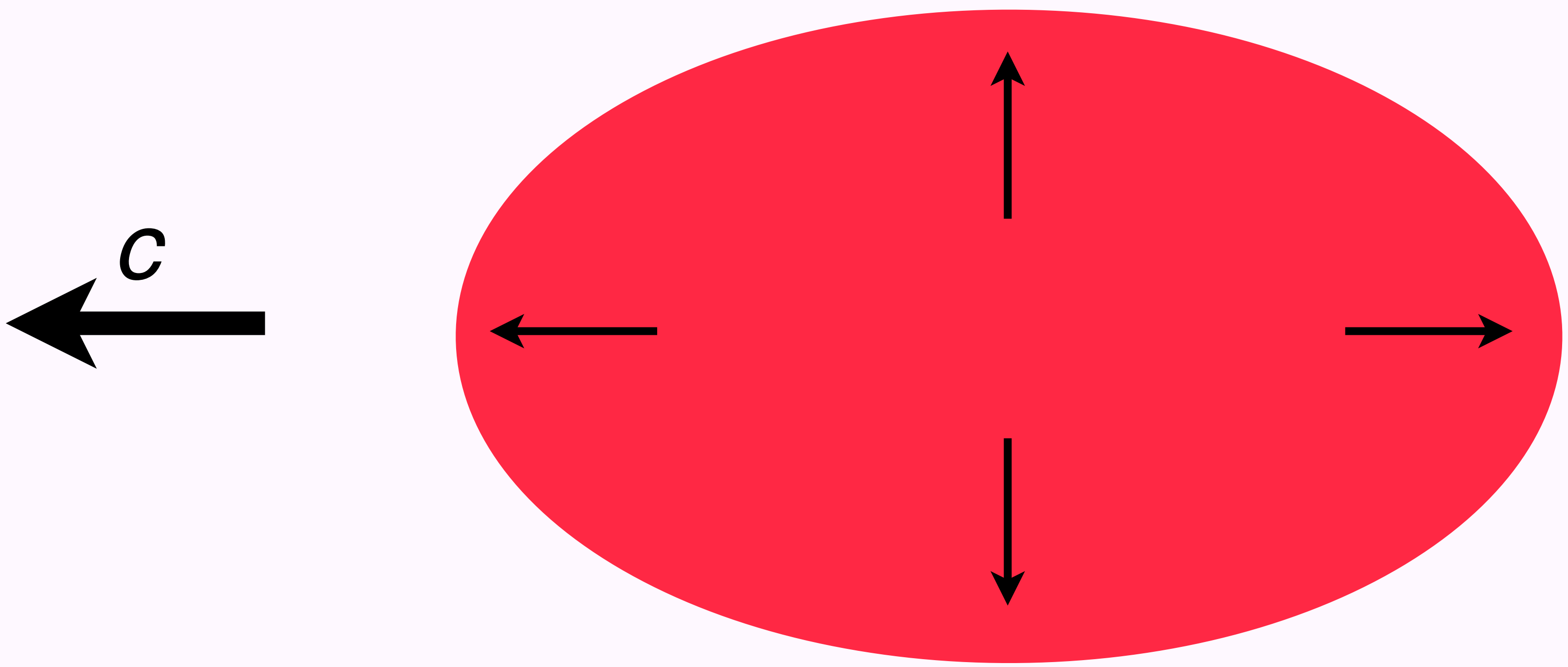
$$m_j \left[\frac{\partial \mathbf{v_j}}{\partial t} + (\mathbf{v_j} \cdot \nabla)\mathbf{v_j} \right] = q_j \mathbf{E} + \mathbf{F} \tag{5}$$

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \mathbf{v_j}) = 0 \tag{6}$$

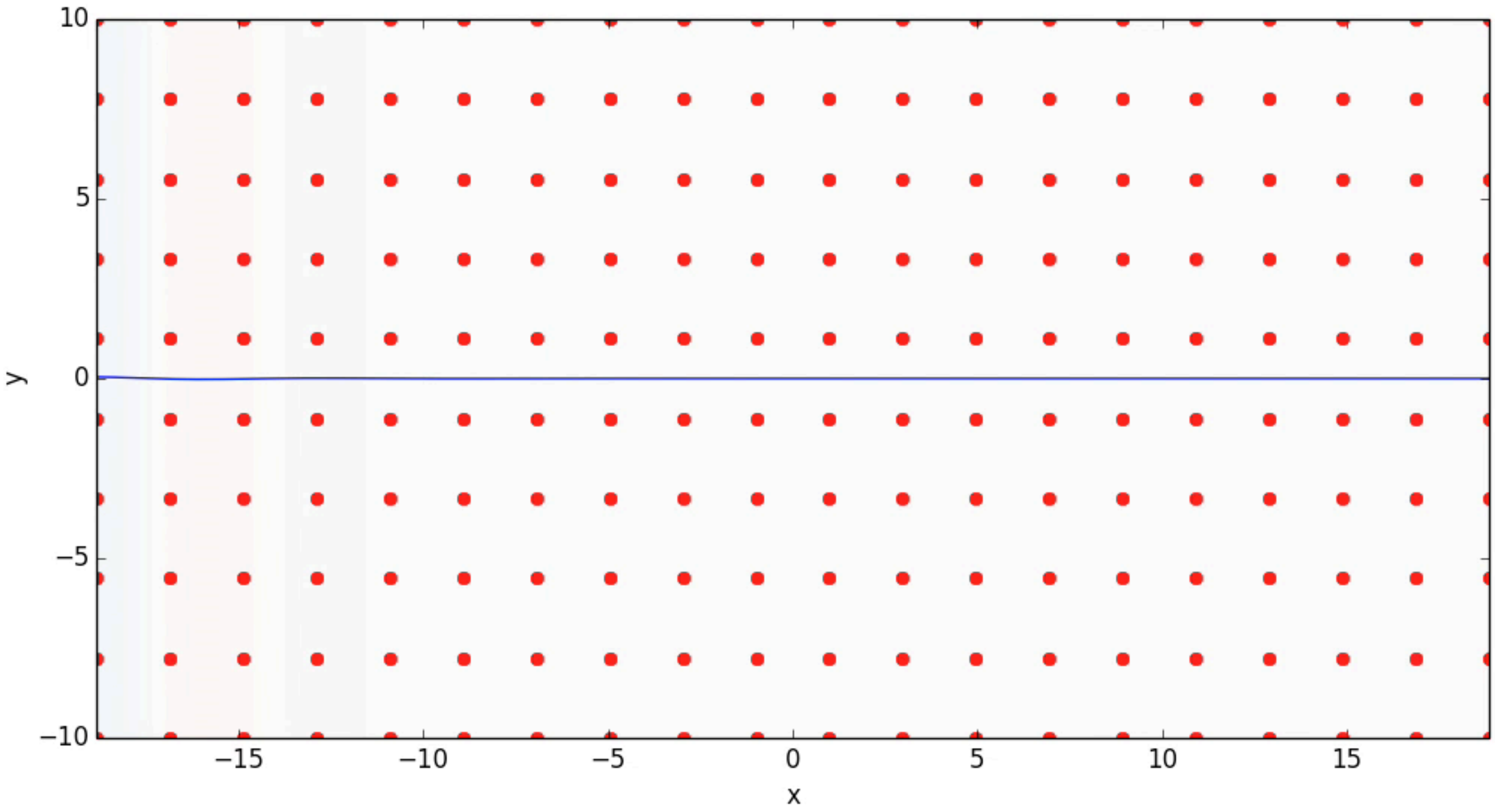
$j \in \{i, e\}$ refers to the species; ions or electrons

We are left with Gauss, Newton and Continuity Equations but what is **F**?

Driver is either laser or particle beam



Ponderomotive force



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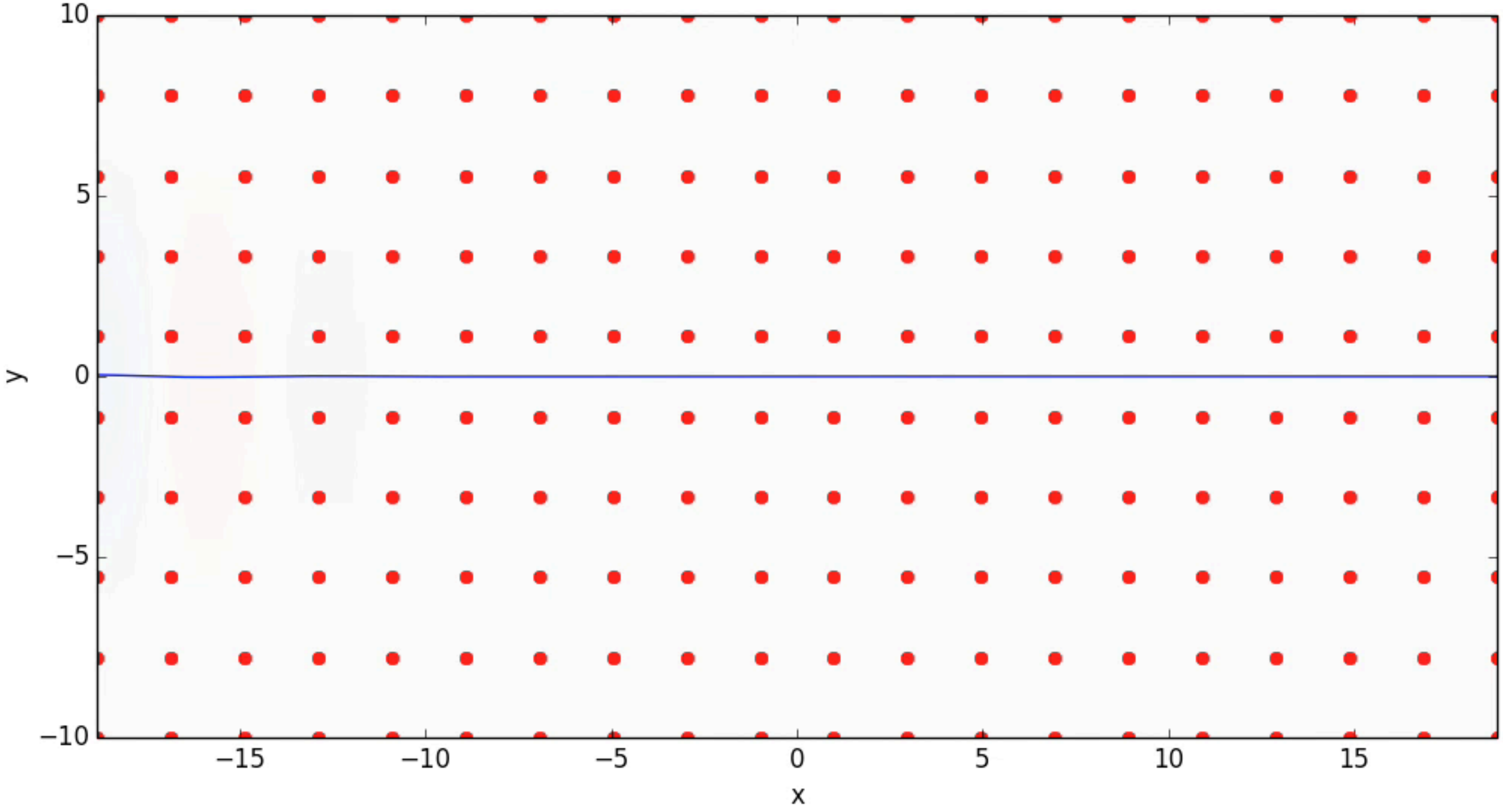

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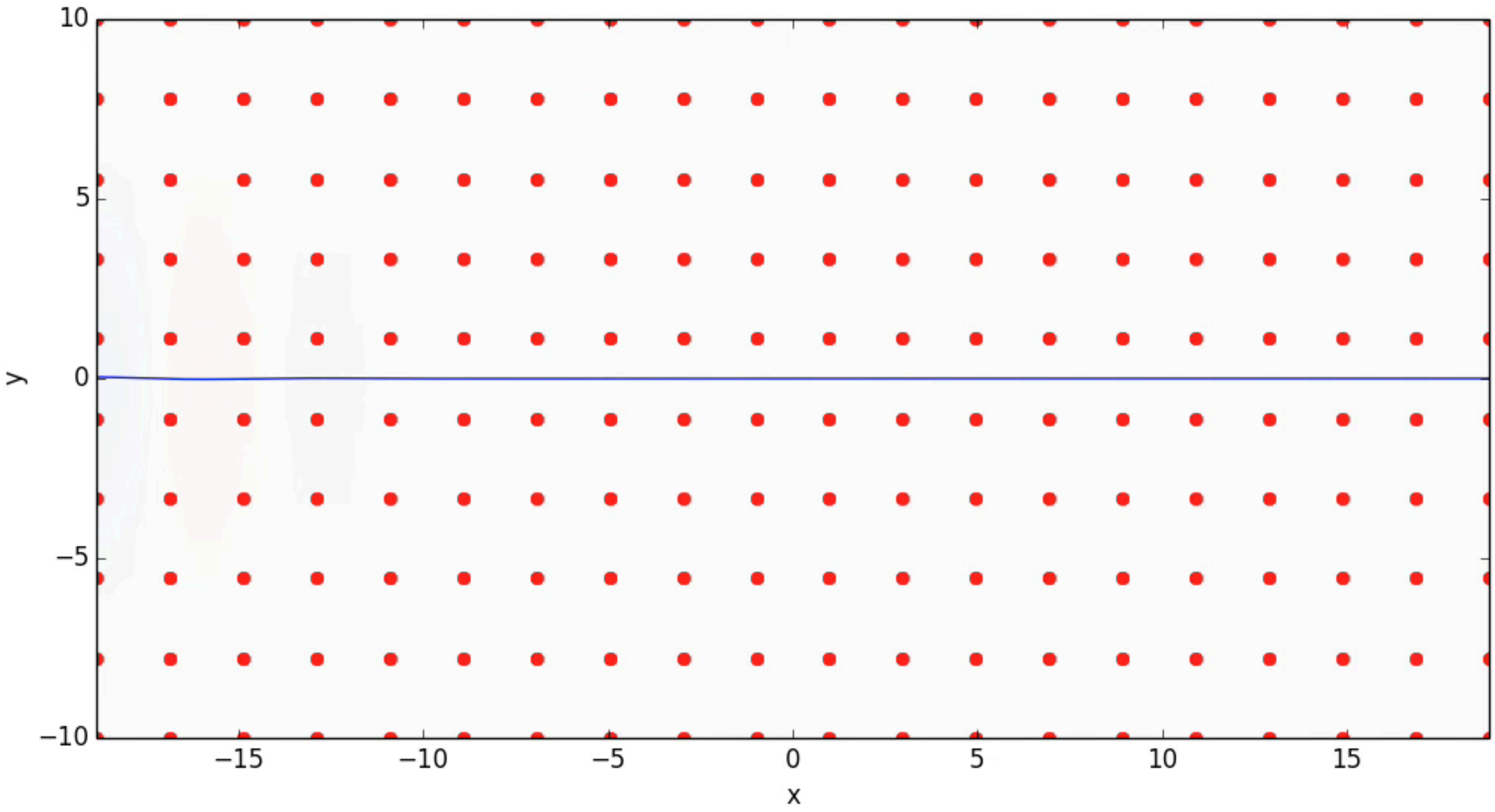


Consider the effect of a laser on plasma electrons, starting with:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{e}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

where the second term on each side is non-linear. To first order we can ignore these terms to give,

$$\frac{\partial \mathbf{v_1}}{\partial t} = -\frac{e}{m}\mathbf{E}$$

For a field with normalised vector potential,

$$\mathbf{a} = a_0 \sin(kz - \omega t)\hat{\mathbf{x}} = \frac{eE_0}{m\omega c} \sin(kz - \omega t)\hat{\mathbf{x}},$$

the velocity is $\mathbf{v_1} = c\,\mathbf{a}$.

Writing $\mathbf{B} = \nabla \times \mathbf{A} = (mc/e)\nabla \times \mathbf{a}$ and using $\mathbf{v_1} \approx c\,\mathbf{a}$, the second term becomes:

$$\frac{e}{m}(\mathbf{v} \times \mathbf{B}) = \frac{e}{m}(c\mathbf{a} \times (mc/e)\nabla \times \mathbf{a}) = c^2(\mathbf{a} \times (\nabla \times \mathbf{a}))$$

$$= c^2 \left(\frac{1}{2}\nabla a^2 - (\mathbf{a} \cdot \nabla)\mathbf{a} \right)$$

So,

$$\frac{\partial \mathbf{v}'}{\partial t} = -c^2(\mathbf{a} \cdot \nabla)\mathbf{a} - c^2 \left(\frac{1}{2}\nabla a^2 - (\mathbf{a} \cdot \nabla)\mathbf{a} \right) = -\frac{1}{2}c^2 \nabla a^2$$

So that the *ponderomotive force* $F_p = m\frac{\partial \mathbf{v}'}{\partial t}$, is given by:

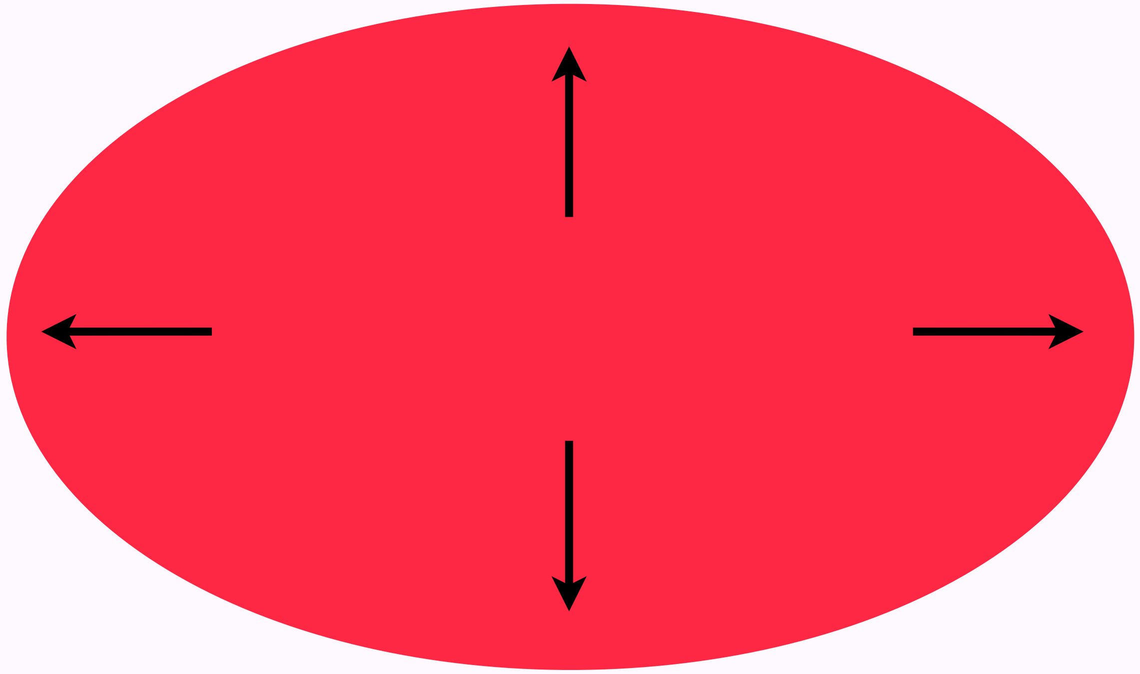
$$F_p = -\frac{1}{2}mc^2\nabla a^2.$$

Often we take the time-average of a^2, since we know that the fast motion tends to time-average to nothing, i.e. $\langle a^2 \rangle = \frac{1}{2}{a_0}^2$. So

$$F_p = -\frac{1}{4}mc^2\nabla {a_0}^2 = -\frac{e^2}{4m\omega^2}\nabla {E_0}^2 = -\frac{e^2}{4\epsilon_0 mc\omega^2}\nabla I_0$$

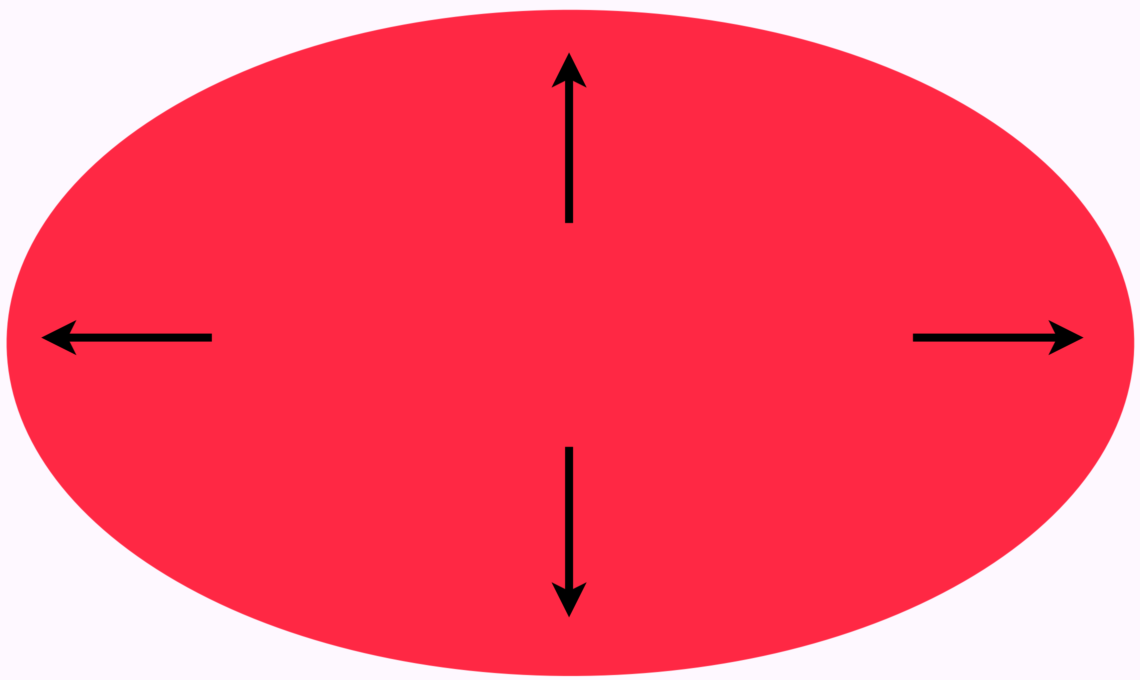
where I_0 is the peak intensity. The three expressions are identical and all say that the force acts away (due to the minus sign) from regions of high intensity.

For laser beam



$$\mathbf{F_p} = -\frac{1}{2} m c^2 \nabla a^2 \qquad \text{(Ponderomotive)}$$

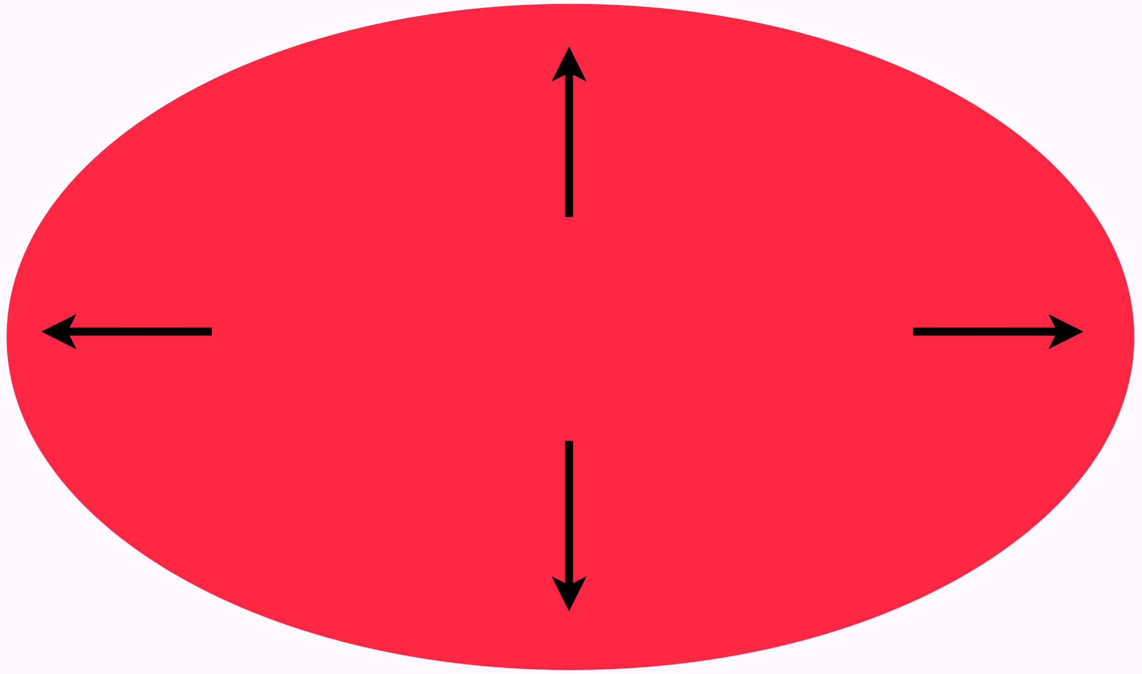
For particle beam



$$\mathbf{F_b} = q_j \mathbf{E_b} \qquad \text{(space charge force)}$$

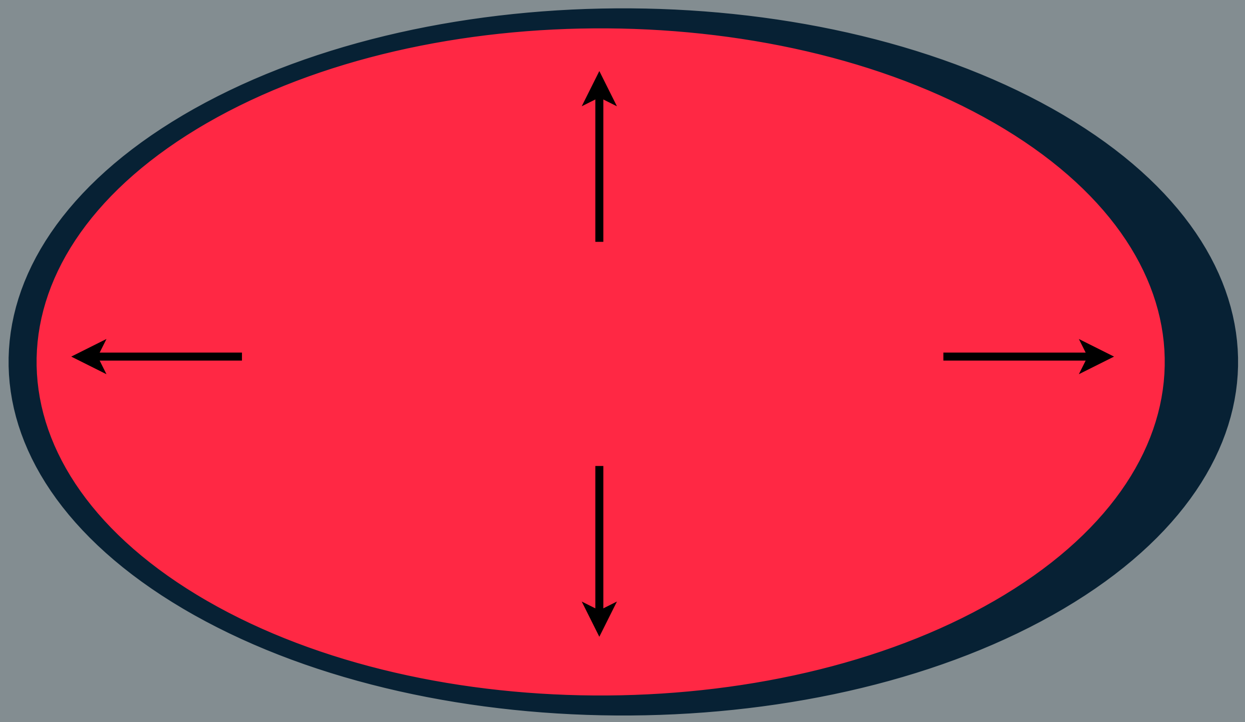
with $\epsilon_0 \nabla \cdot \mathbf{E_b} = \rho_b$

Ions assumed to be immobile



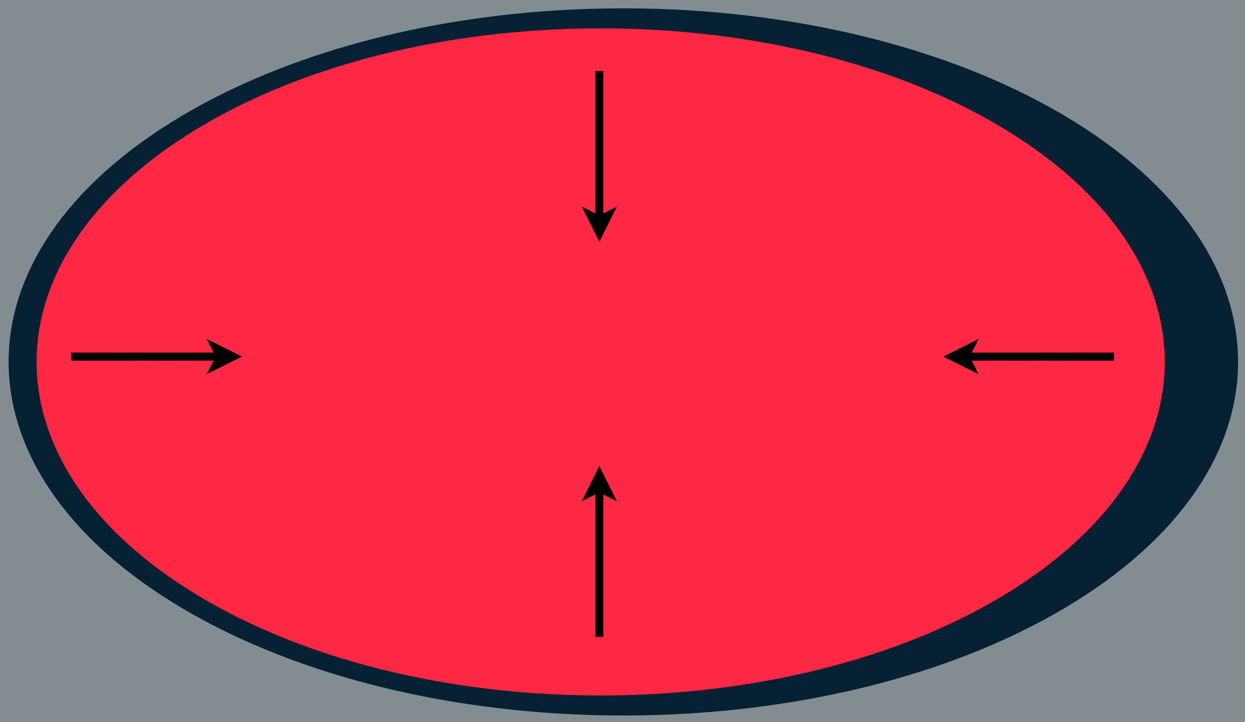
$$\frac{\partial \mathbf{p}}{\partial t} = \mathbf{F_p} + \mathbf{F_b} \qquad \text{(Motion)}$$

Ions assumed to be immobile



$$\frac{\partial \mathbf{p}}{\partial t} = \mathbf{F_p} + \mathbf{F_b} \qquad \text{(Motion)}$$

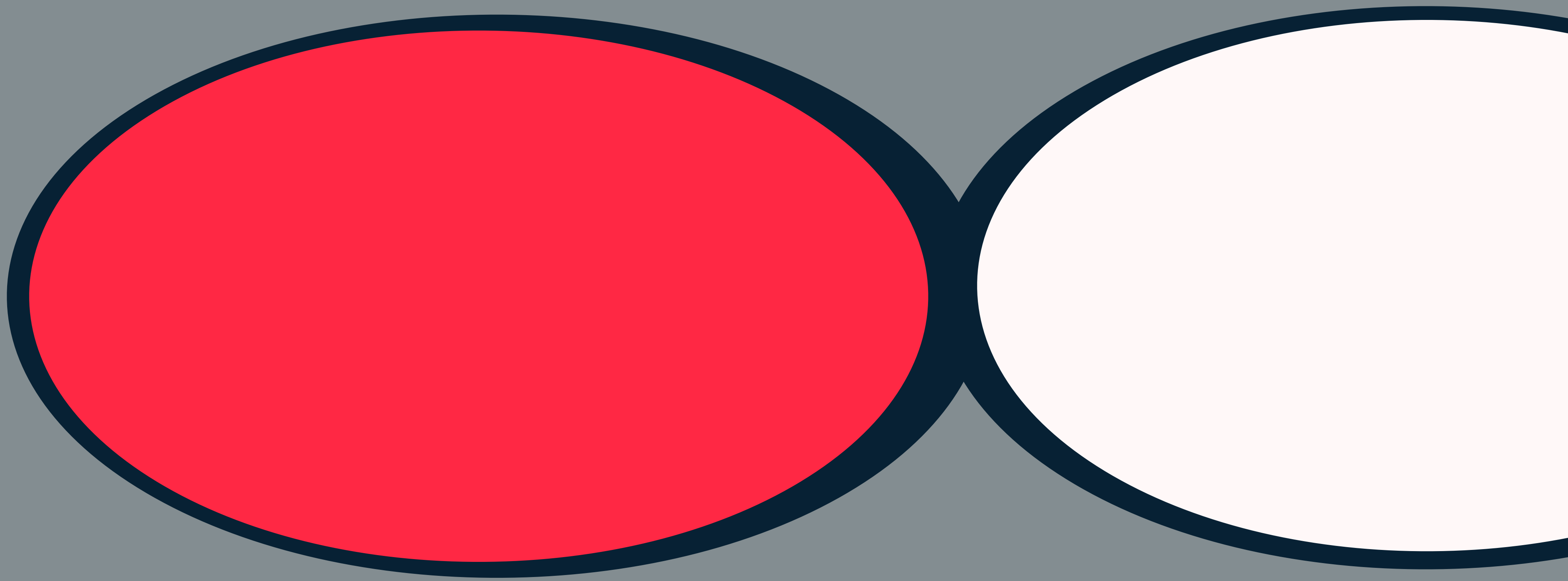
Space-charge forcing electrons back



$$\frac{\partial \mathbf{p}}{\partial t} = \mathbf{F_p} + \mathbf{F_b} \qquad \text{(Motion)}$$

$$\nabla \cdot \mathbf{E} = -e(n_e - n_i)/\epsilon_0 \qquad \text{(Gauss)}$$

Electrons stream back in behind pulse creating wakefield



$$\frac{\partial \mathbf{p}}{\partial t} = -e\mathbf{E} + \mathbf{F_p} + \mathbf{F_b} \qquad \text{(motion)}$$

$$\nabla \cdot \mathbf{E} = -e(n_e - n_i)/\epsilon_0 \qquad \text{(Gauss)}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e v) = 0 \qquad \text{(continuity)}$$

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In one (x-) dimension;

$$F_p = -\frac{1}{2} m c^2 \frac{\partial a^2}{\partial x}$$

and

$$F_b = -e \int \rho(x)/\epsilon_0 \, \mathrm{d}x.$$

Ions immobile, and electrons move only in *x*

one dimensional : $E = E_x;\ p = mv_x$

also take $n_i = n_0$

$$m\frac{\partial v}{\partial t} = -eE - \frac{1}{2}mc^2\frac{\partial a^2}{\partial x} - e\int \rho(x)/\epsilon_0 \, \mathrm{d}x \qquad \text{(motion)}$$

$$\frac{\partial E}{\partial x} = e(n_0 - n_e)/\epsilon_0 \qquad \text{(Gauss)}$$

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e v) = 0 \qquad \text{(continuity)}$$

Linearise; assume small perturbations:

$$n_e = n_0 + n_1; \qquad v = v_1; \qquad E = E_1$$

$$m\frac{\partial v_1}{\partial t} = -eE_1 - \frac{1}{2}mc^2\frac{\partial a^2}{\partial x} - e\int \rho(x)/\epsilon_0 \, \mathrm{d}x \qquad \text{(motion)}$$

$$\frac{\partial E_1}{\partial x} = -en_1/\epsilon_0 \qquad \text{(Gauss)}$$

$$\frac{\partial n_1}{\partial t} + \frac{\partial}{\partial x}((n_0 + n_1)v_1) = 0 \qquad \text{(continuity)}$$

Assume products of perturbations are negligible:

$$m\frac{\partial v_1}{\partial t} = -eE_1 - \frac{1}{2}mc^2\frac{\partial a^2}{\partial x} - e\int \rho(x)/\epsilon_0 \, \mathrm{d}x \qquad \text{(motion)}$$

$$\frac{\partial E_1}{\partial x} = -en_1/\epsilon_0 \qquad \text{(Gauss)}$$

$$\frac{\partial n_1}{\partial t} + n_0\frac{\partial v_1}{\partial x} = 0 \qquad \text{(continuity)}$$

Take spatial derivative of (motion) and time derivative of (continuity)

$$m\frac{\partial^2 v_1}{\partial x \partial t} = -e\frac{\partial E_1}{\partial x} - \frac{1}{2}mc^2\frac{\partial^2 a^2}{\partial x^2} - e\rho_b/\epsilon_0 \qquad \text{(motion)}$$

$$\frac{\partial E_1}{\partial x} = -en_1/\epsilon_0 \qquad \text{(Gauss)}$$

$$\frac{\partial^2 n_1}{\partial t^2} + n_0\frac{\partial^2 v_1}{\partial x \partial t} = 0 \qquad \text{(continuity)}$$

Eliminate E_1 and v_1

$$m\frac{\partial^2 v_1}{\partial x \partial t} = -e\frac{\partial E_1}{\partial x} - \frac{1}{2}mc^2\frac{\partial^2 a^2}{\partial x^2} - e\rho_b/\epsilon_0 \qquad \text{(motion)}$$

$$\frac{\partial E_1}{\partial x} = -en_1/\epsilon_0 \qquad \text{(Gauss)}$$

$$\frac{\partial^2 n_1}{\partial t^2} + n_0\frac{\partial^2 v_1}{\partial x \partial t} = 0 \qquad \text{(continuity)}$$

$$-m\frac{\partial^2 n_1}{\partial t^2} = e^2 n_0 n_1/\epsilon_0 - \frac{1}{2}n_0 mc^2\frac{\partial^2 a^2}{\partial x^2} - en_0\rho_b/\epsilon_0$$

Simplify:

$$-m\frac{\partial^2 n_1}{\partial t^2} = e^2 n_0 n_1/\epsilon_0 - \frac{1}{2} n_0 m c^2 \frac{\partial^2 a^2}{\partial x^2} - e n_0 \rho_b/\epsilon_0$$

$$\frac{\partial^2 n_1}{\partial t^2} + \frac{n_0 e^2}{\epsilon_0 m} n_1 = \frac{1}{2} n_0 c^2 \frac{\partial^2 a^2}{\partial x^2} + \frac{n_0 e}{\epsilon_0 m}\rho_b$$

Writing ${\omega_p}^2 = \dfrac{n_0 e^2}{\epsilon_0 m}$ and $\rho_b = -e n_b$,

$$\frac{\partial^2 n_1}{\partial t^2} + {\omega_p}^2 n_1 = \frac{1}{2} n_0 c^2 \frac{\partial^2 a^2}{\partial x^2} - {\omega_p}^2 n_b$$

Quasistatic approximation : $\xi = x - ct$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} = -c\frac{\partial}{\partial \xi}; \quad \frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi}$$

$$\frac{\partial^2}{\partial t^2} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi}\left(-c\frac{\partial}{\partial \xi}\right) = c^2\frac{\partial^2}{\partial \xi^2}; \quad \frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi}\frac{\partial}{\partial \xi} = \frac{\partial^2}{\partial \xi^2}$$

Linear wakefield generation equation in quasi static frame:

$$c^2 \frac{\partial^2 n_1}{\partial \xi^2} + {\omega_p}^2 n_1 = \frac{1}{2} n_0 c^2 \frac{\partial^2 a^2}{\partial \xi^2} - {\omega_p}^2 n_b$$

$$\frac{\partial^2 n_1}{\partial \xi^2} + {k_p}^2 n_1 = \frac{1}{2} n_0 \frac{\partial^2 a^2}{\partial \xi^2} - {k_p}^2 n_b$$

$$\text{where} \quad k_p = \omega_p / c = 2\pi / \lambda_p$$

Can be rewritten in terms of *E* and ϕ:

$$\text{Using } E = -\int \frac{e}{\epsilon_0} n_1 \, \mathrm{d}x \text{ and } \phi = -\int E \, \mathrm{d}x$$

$$\frac{\partial^2 n_1}{\partial \xi^2} + {k_p}^2 n_1 = \frac{1}{2} n_0 \frac{\partial^2 a^2}{\partial \xi^2} - {k_p}^2 n_b$$

$$\frac{\partial^2 E_1}{\partial \xi^2} + {k_p}^2 E_1 = -\frac{1}{2} \frac{e n_0}{\epsilon_0} \frac{\partial a^2}{\partial \xi} - {k_p}^2 E_b$$

$$\frac{\partial^2 \phi_1}{\partial \xi^2} + {k_p}^2 \phi_1 = \frac{1}{2} \frac{e n_0}{\epsilon_0} a^2 - {k_p}^2 \phi_b$$

Solving for *E* with a laser driver:

$$\frac{\partial^2 E_1}{\partial \xi^2} + {k_p}^2 E_1 = -\frac{1}{2}\frac{en_0}{\epsilon_0}\frac{\partial a^2}{\partial \xi}$$

Can solve directly using Green's functions:

$$E = -\frac{1}{2}\int_0^{\xi} \sin\left[k_p\left(\xi - \xi'\right)\right]\,\frac{\partial\left(a^2\left(\xi'\right)\right)}{\partial \xi'}\,\mathrm{d}\xi'$$

Solving for *E* with a laser driver:

$$\frac{\partial^2 E_1}{\partial \xi^2} + {k_p}^2 E_1 = -\frac{1}{2}\frac{en_0}{\epsilon_0}\frac{\partial a^2}{\partial \xi}$$

But its instructive to take a trial laser shape:

$$a = a_0 \sin(b\xi) \qquad 0 < \xi < \pi/b$$

$$a^2 = a_0^2 \sin^2\left(b^2\xi\right)$$

$$\begin{aligned}\frac{\partial a^2}{\partial \xi} =& 2a_0^2 b \sin(b\xi)\cos(b\xi) \\ &= ba_0^2 \sin(2b\xi)\end{aligned}$$

Here L = π/b is the pulse length

Substituting for the ponderomotive force:

$$\frac{\partial^2 E_1}{\partial \xi^2} + {k_p}^2 E_1 = -\frac{1}{2}\frac{e n_0}{\epsilon_0} b a_0^2 \sin(2b\xi)$$

Clearly only resonant if $2b = k_p$

$$\frac{\partial^2 E_1}{\partial \xi^2} + {k_p}^2 E_1 = -\frac{1}{4}\frac{e n_0}{\epsilon_0} k_p a_0^2 \sin(k_p\xi)$$

A trial solution is:

$$E_1 = A \sin k_p \xi + B\xi \cos k_p \xi$$

Solving:

$$
\begin{aligned}
E_1 &= \frac{1}{8}\left(\frac{en_0}{\varepsilon_0}\frac{a_0^2}{k_p}\right)(k_p\xi\cos k_p\xi - \sin k_p\xi) \\
&= \frac{a_0^2}{8}\left(\frac{mc\omega_p}{e}\right)(k_p\xi\cos k_p\xi - \sin k_p\xi) \\
&= \frac{a_0^2}{8}E_0\,(k_p\xi\cos k_p\xi - \sin k_p\xi)
\end{aligned}
$$

reaches maximum value when $\xi = \pi/b = 2\pi/\xi$:

$$
E_{max} = \frac{\pi}{4}{a_0}^2 E_0
$$

Wakefield generation

Solving (in 1D):

$$\frac{\partial E}{\partial \zeta} = -n_1 \qquad \text{(Gauss' Law)}$$

$$\frac{\partial n_1}{\partial \zeta} = \frac{\partial (n_e \beta)}{\partial \zeta} \qquad \text{(Continuity)}$$

$$(1-\beta)\frac{\partial \beta}{\partial \zeta} = eE - \frac{1}{\gamma}\frac{\partial (a^2)}{\partial \zeta} \qquad \text{(Motion)}$$

where $\beta = v/c$, $n_1 = \delta n / n_0$, and $E = E_{wf}/E_0$

(or alternatively m_e, c, ϵ_0, c all normalised to 1).

Wakefield generation

Solving (in 1D):

$$\frac{\partial E}{\partial \zeta} = -n_1 \qquad \text{(Gauss' Law)}$$

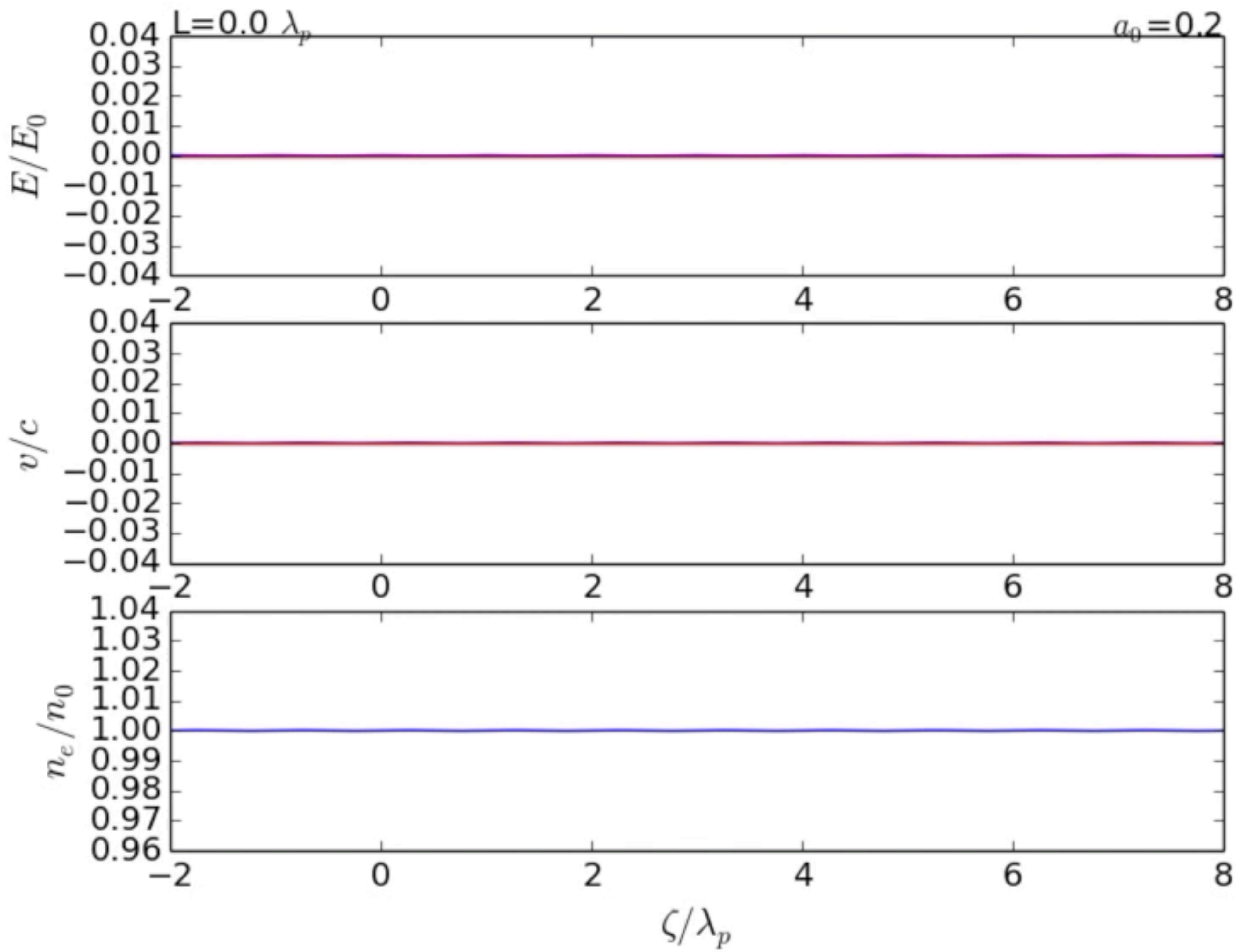
$$n_1 = n_0 \beta \qquad \text{(Continuity)}$$

$$\frac{\partial \beta}{\partial \zeta} = eE - \frac{\partial (a^2)}{\partial \zeta} \qquad \text{(Motion)}$$

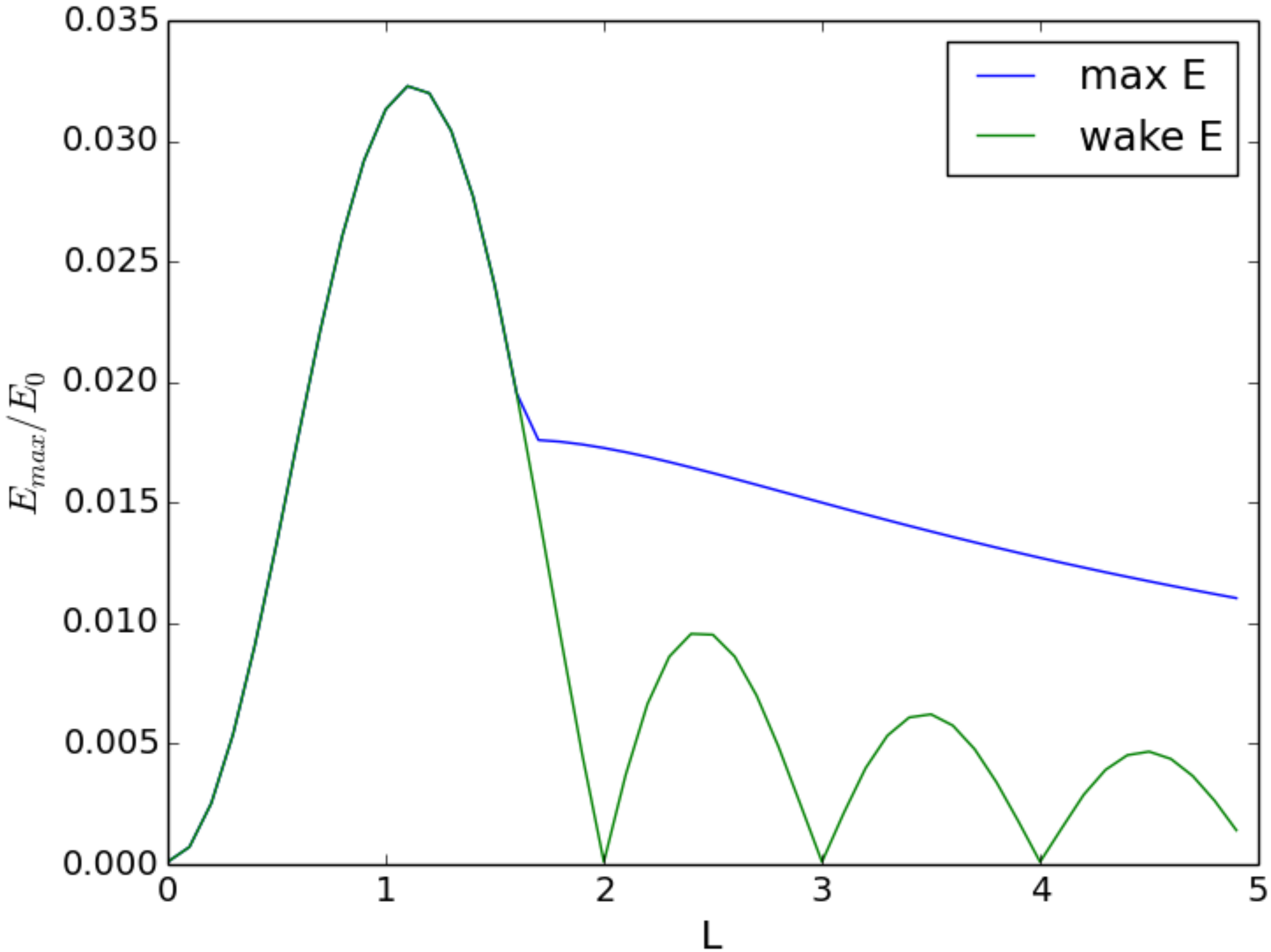
Assuming $\beta \ll 1$, $n_1 \ll n_0$ $\qquad n_e = n_0(1+\beta)$

Have coupled equations in E and β to solve

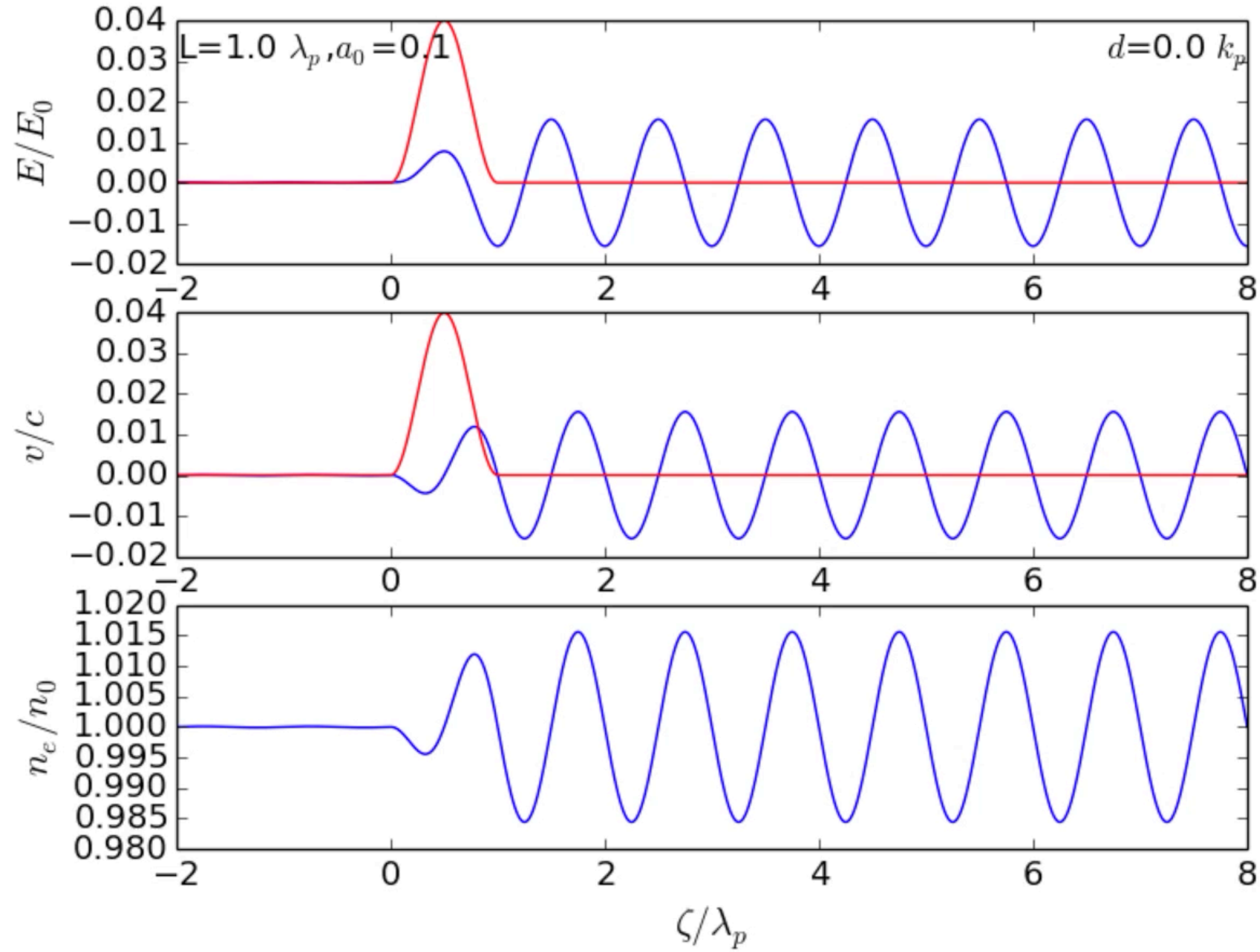
Wakefield generation



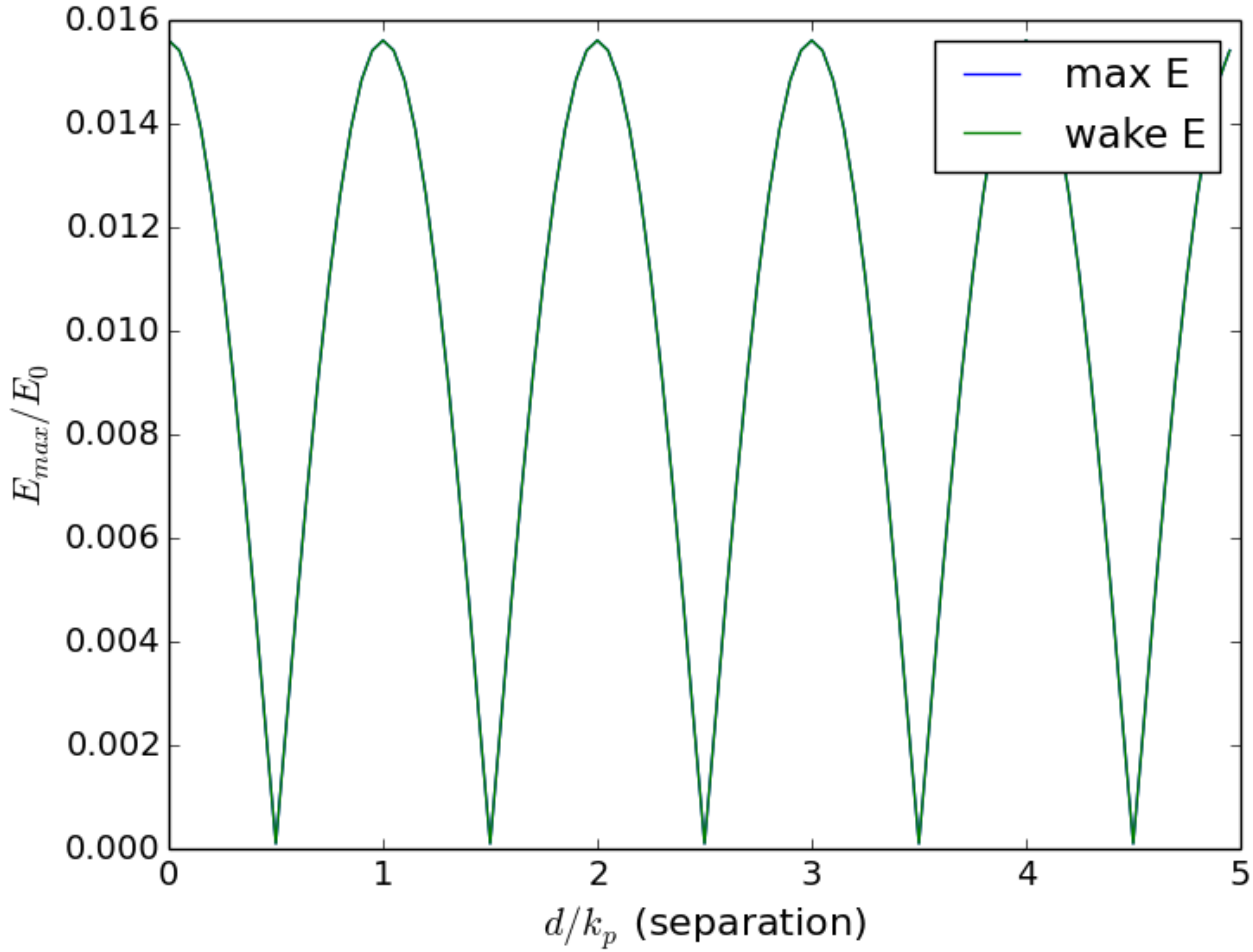
Wakefield generation



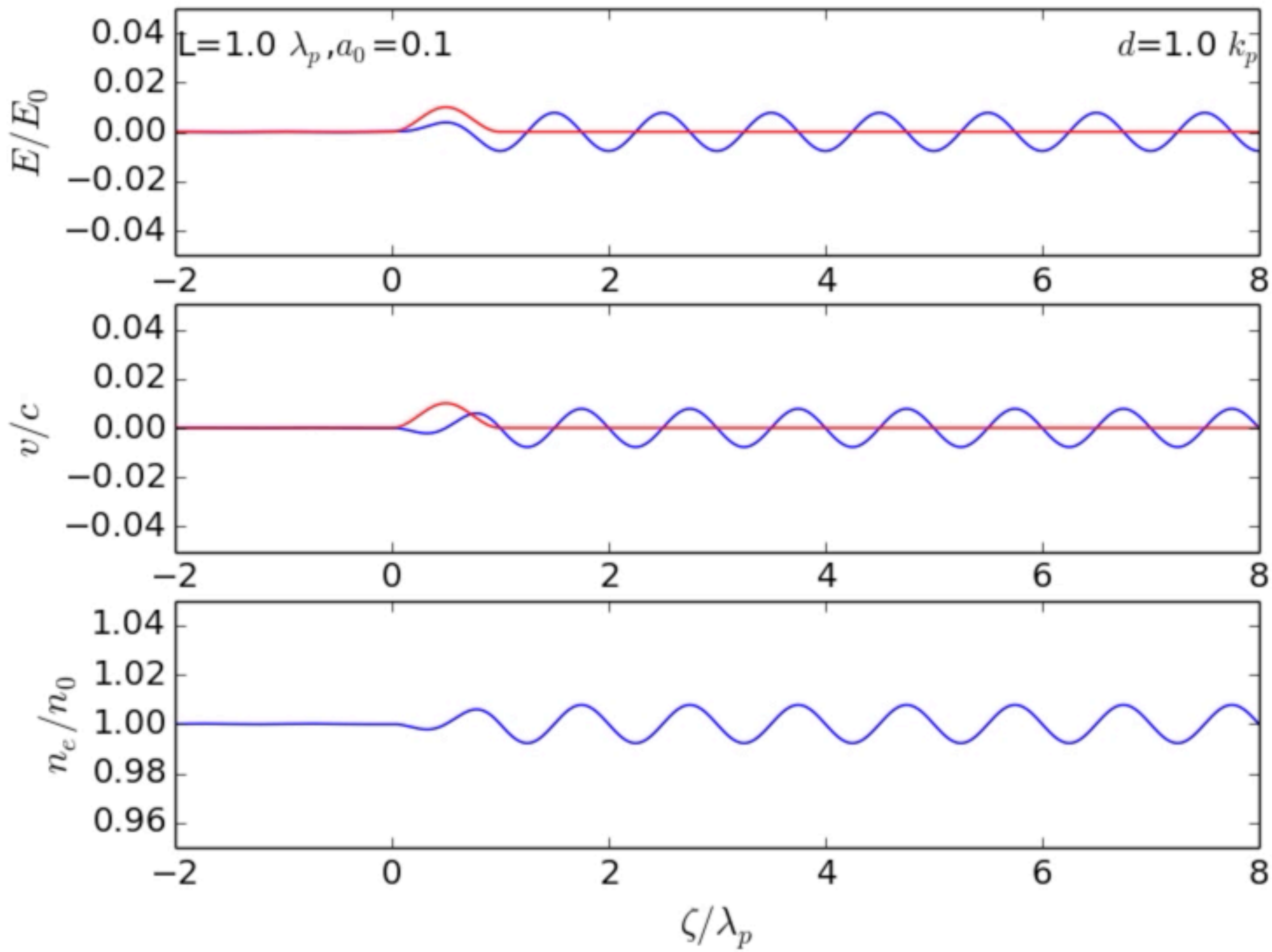
Double pulse excitation



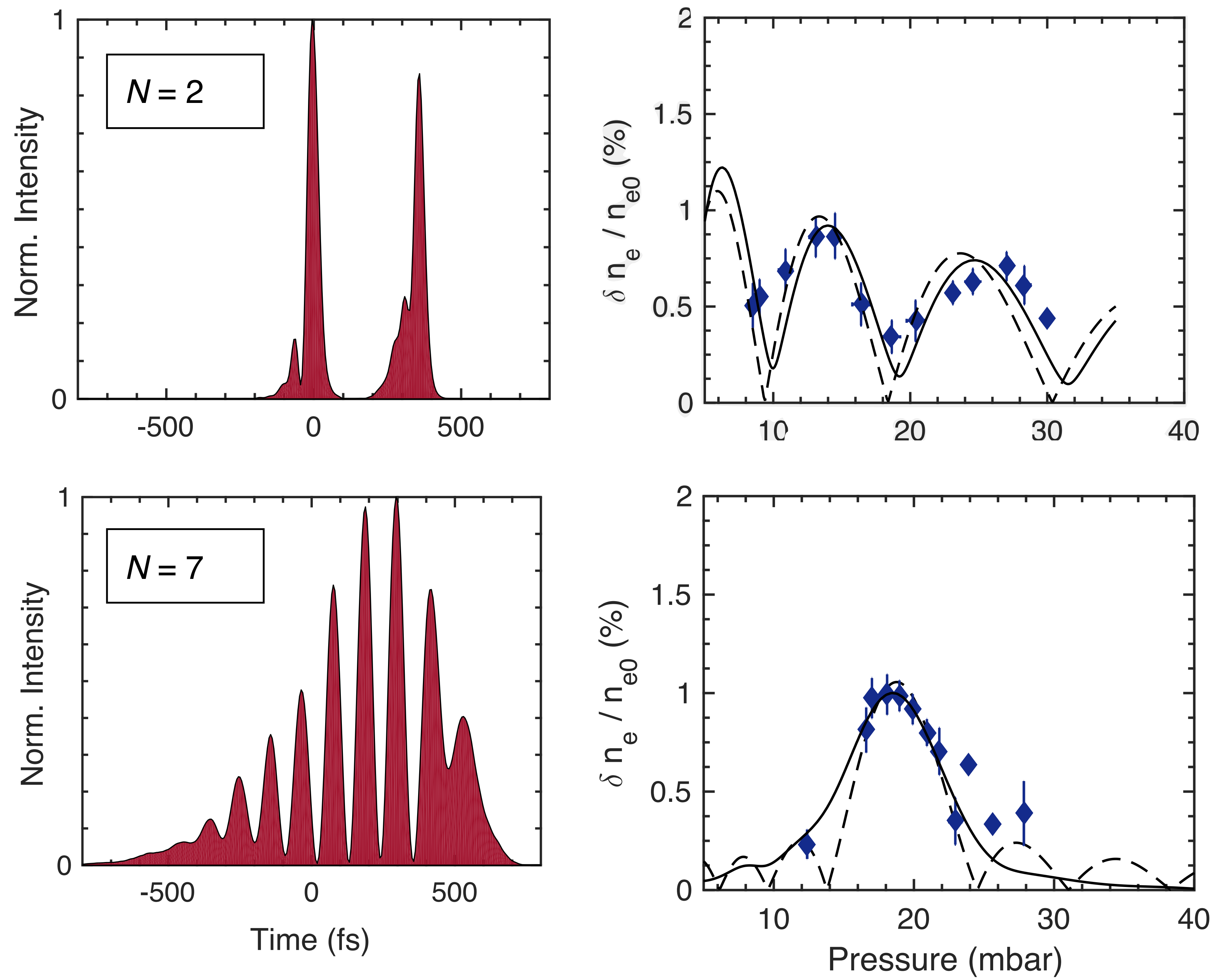
Double pulse excitation



Multi pulse excitation



Multi pulse excitation



J. Cowley, et al, Excitation and Control of Plasma Wakefields by Multiple Laser Pulses, Phys. Rev. Lett. 119, 044802 (2017).

Wakefield Generation Summary

Wake can be driven by ponderomotive force of laser or space-charge force of particle beam

Wake amplitude maximised for $L \sim \lambda_p$ ($L_{fwhm} \sim \lambda_p/2$)

In linear regime, at resonance wake amplitude $E/E_0 \sim a_0^2$

Secondary pulses can be used to enhance wakefield or to eliminate it.