# CAS: Acceleration of electrons in plasma I

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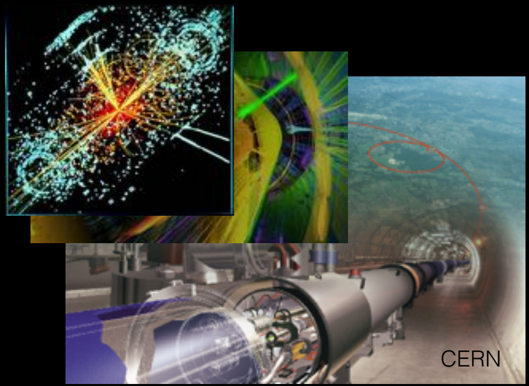
# Topics covered

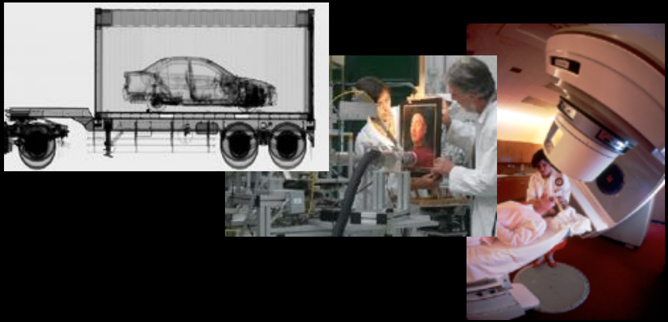
- Parameters for wakefield drivers
- Linear and nonlinear wakes
- Limits on Maximum energy gain of an electron beam
    - Dephasing
    - Depletion
- Wake Hamiltonian
- Wake phase space
- Trapped/untrapped orbits
- Phase space rotation
- Transformer ratio
- Beamloading and energy spread considerations
- Transverse emittance
- Betatron oscillations in wakefields

# Particle accelerators are drivers of science and technology




High energy colliders are at the forefront of fundamental physics discovery

Light sources enabled by particle accelerators are revolutionizing biotechnology, materials science and condensed matter physics research

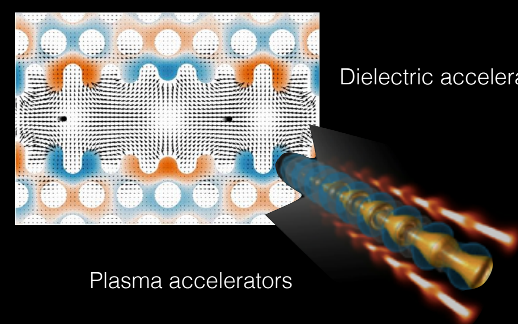
Particle accelerators are used throughout industry, homeland security and medicine, from materials engineering to cargo scanning to cancer treatment

# The limits of current technology and advanced accelerator concepts

- RF technology has been successful but accelerating gradient is limited to **100 MeV/m** because of breakdown limits
- Use of a circular machine ultimately limited by synchrotron radiation emission
- Advanced accelerator (high gradient concepts) are those that provide **>1 GeV/m** accelerating gradients
  - Candidate technologies: Dielectric accelerators, Plasma accelerators

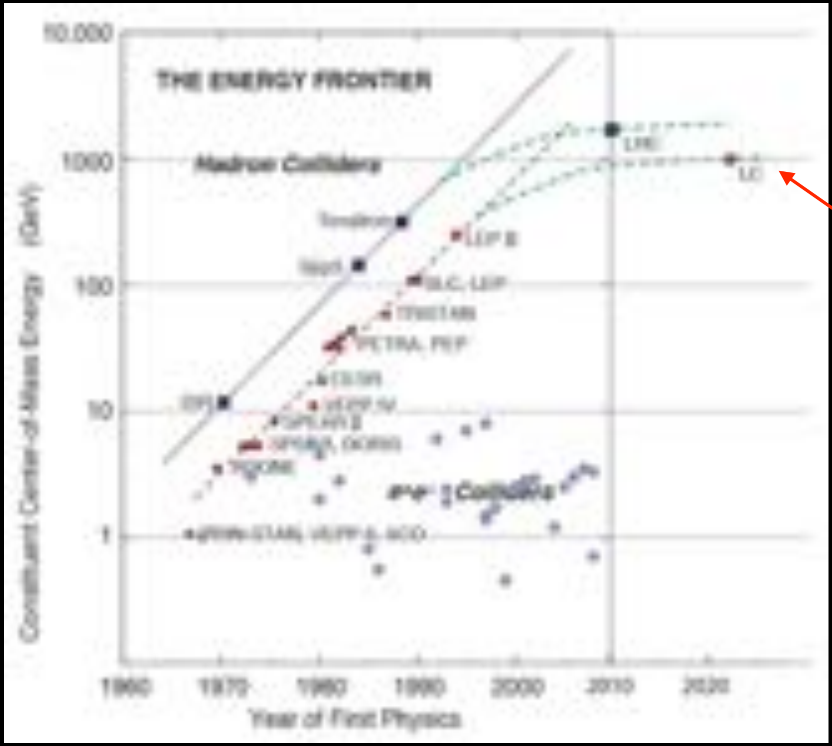

RF cavities


Diamond Synchrotron


Dielectric accelerators

Plasma accelerators

# Why do we need novel acceleration concepts?




31 miles long

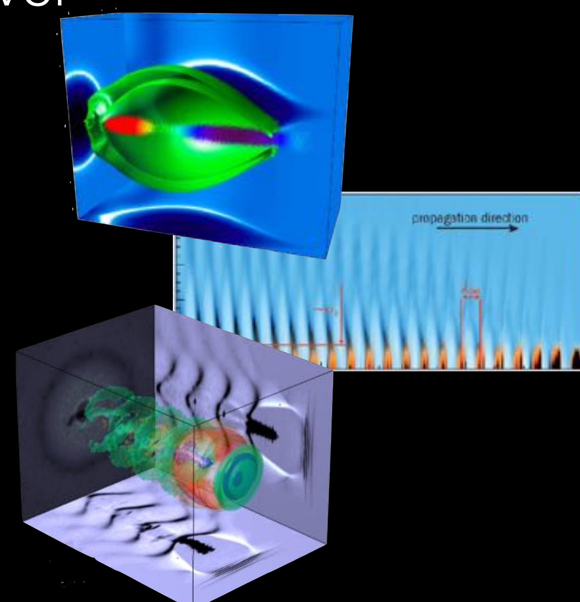
# Plasma based accelerators

- Plasma is ionized matter where collective dynamics dominate - in particular, **longitudinal electric waves**
- To create a **linear accelerating structure** using plasma, need a relativistic driver
  - Electron beam (FACET II)

    Chen Phys. Rev. Lett. (1985)
  - Proton beam (AWAKE)

    Caldwell Nat. Phys. (2009)
  - Laser Pulse (Numerous)

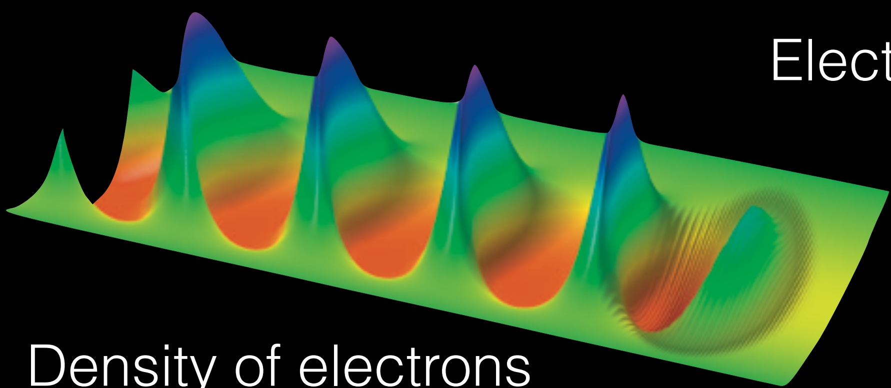
    Tajima Phys. Rev. Lett. (1977)



# Plasma Wakefield Acceleration

# Plasma waves generated by relativistic object

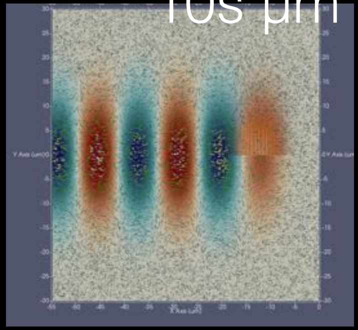
- When relativistic object* perturbs plasma we generate a plasma wave with relativistic phase velocity



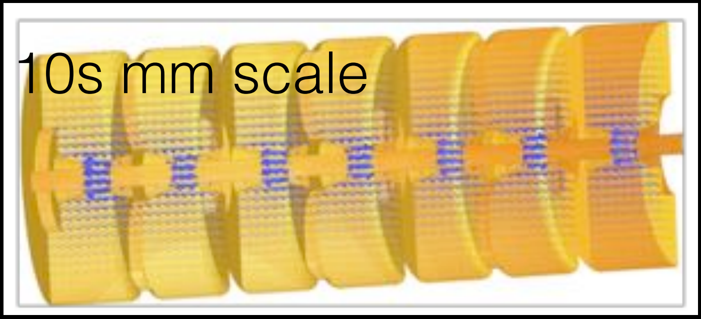
Density of electrons

Electric field

10s μm scale

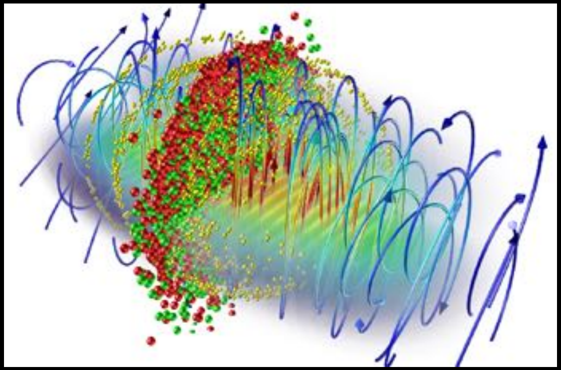


Accelerator cavity




*i.e. laser pulse or particle beam

# Why plasma?



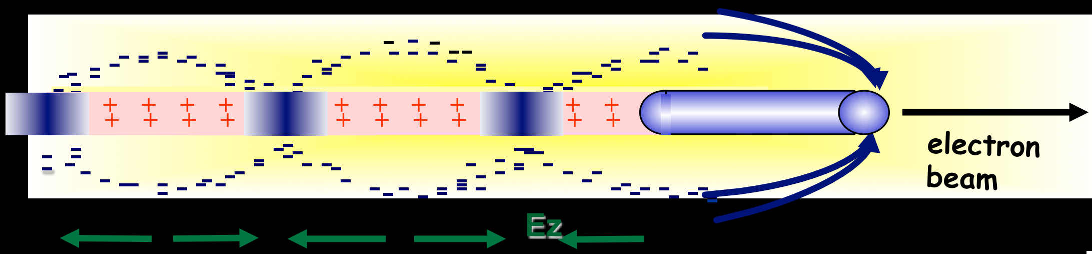
- Plasma is already ionized
  - **No classical breakdown limit**
  - If we can support much stronger fields, the length of the accelerator is reduced:
- Experiments now already routinely demonstrate GeV energies in cm-scale plasma accelerator
  - **i.e. 0.1 TeV/m accelerating gradient**
  - **…That's a pretty high gradient!**

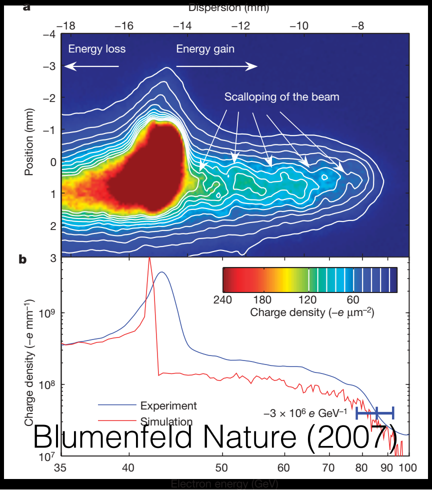
# A smorgasbord of acronyms

- LPA - Laser plasma accelerator
- LWFA - Laser wakefield accelerator
- PWFA - Plasma wakefield accelerator (but generally means beam driven)
- PWA - Plasma wakefield accelerator

# Beam driven accelerator: “PWFA”

- Beam driven - electric field of bunch displaces bunch



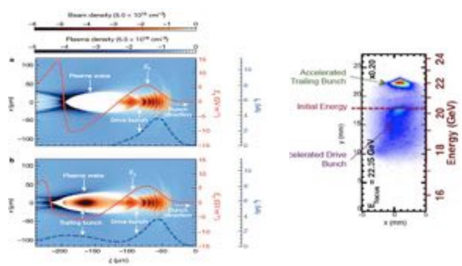




- Energy doubling of 40 GeV beam demonstrated in 1 m plasma cell 2007 / High efficiency 2014

- AWAKE project at CERN just demonstrated modulation of proton beam and 2 GeV electron acceleration

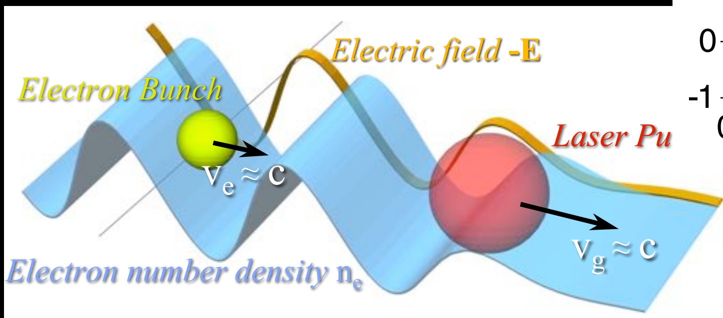
High-Efficiency acceleration of an electron beam in a plasma wakefield accelerator, M. Litos et al., doi, Nature, 6 Nov 2014, 10.1038/nature 13992

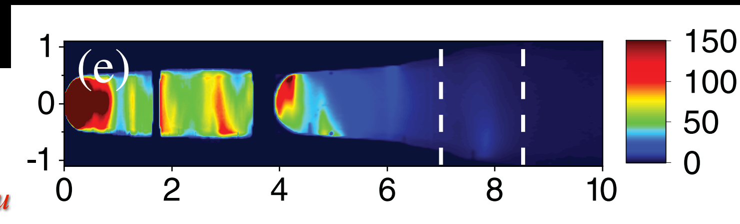
- 1.7 GeV energy gain in 30 cm of pre-ionized Li vapour plasma
- 6 GeV energy in 1.3 m of plasma
- Total efficiency is <29.1%> with a maximum of 50%.
- Final energy spread of 0.7 % (2% average)




- Electric field in plasma wake is loaded by presence of witness bunch
- Allows efficient energy extraction from the plasma wake

# Laser Wakefield Accelerator: "LWFA"

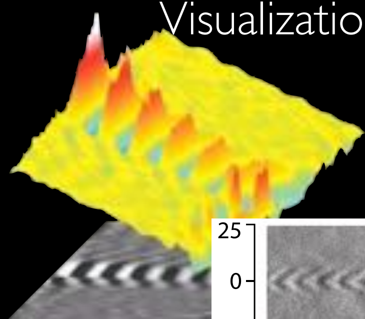


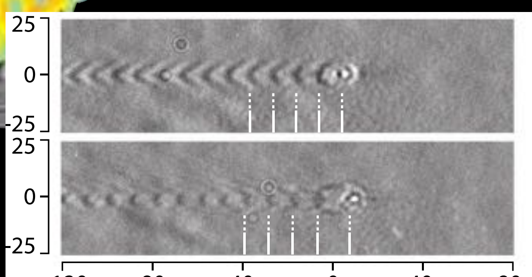

Momentum GeV/c

Electron energy spectrum
(Gonsalves, PRL 2019)

Visualization of wakefield



Matlis et al Nat. Phys. (2006)
Dong et al PRL (2010)

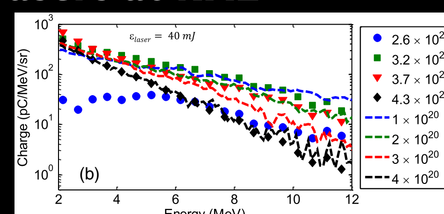



Savart et al PRL (2015)

Experiments now routinely demonstrate GeV energi
in a cm-scale plasma accelerator

**i.e. 100 GeV/m accelerating gradient**

**Also MeV energies with mJ class lasers at kHz**




Electron energy spectrum
(A. Goers, PRL 2015)

# Typical parameters of current experiments

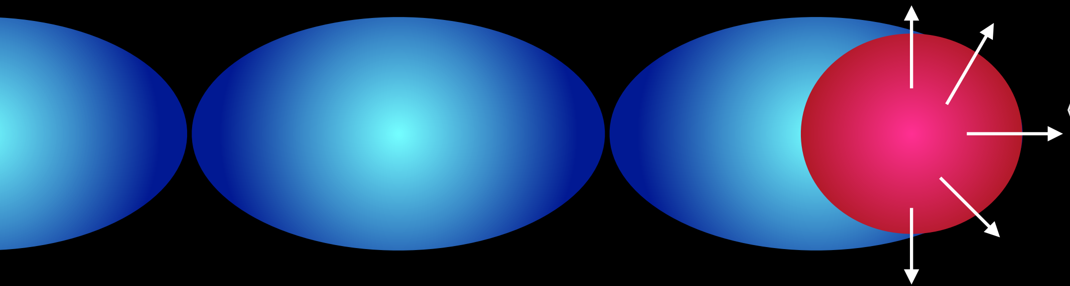
- PWFA
- Electron beam energy ~ 10s GeV (SLAC FACET)
- Proton beam energy ~100s GeV (CERN AWAKE - long bunches)
- ~$10^{16}$ particles per cc density in preionized Li plasma
- 10s um focus / 100s fs duration bunches with ~kA currents (nC charge)

- LWFA
- 1-10s Joule laser pulses
- ~30 fs / 10 TW - PW
- ~$10^{18}$-$10^{19}$ particles per cc density in He plasma
- 10s um focus

**10 GeV at $10^{-9}$ beam charge is 10 J beam energy total**

# Generation of plasma waves with relativistic phase velocity

- A laser generates a plasma wave from its ponderomotive force

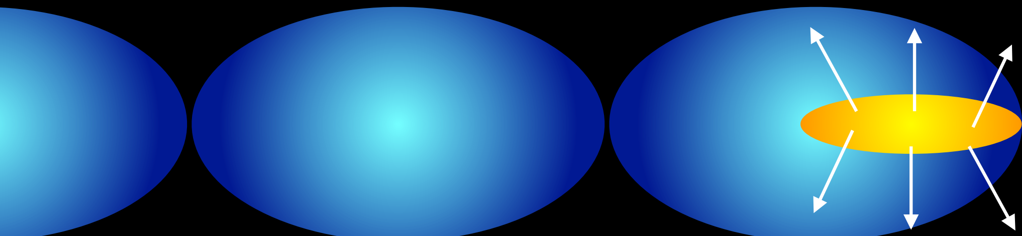


$$\langle F \rangle = -\frac{e^2}{2\langle\gamma\rangle m}\langle \nabla A^2 \rangle$$

$\gamma \sim Ae/mc$

- A charged particle beam generates a plasma wave from its space-charge repulsion

$F_p \sim \nabla A$



$$F = e\nabla\left(\phi - v_b A_x\right) = e\nabla\psi$$

# Strength parameters

- **Beam** driver

$$\Lambda_0 = \frac{n_b}{n_0}\frac{k_p^2 r_0^2}{2}$$

  - normalized beam charge per unit length (current density)

- **Laser** driver

$$a_0 = \frac{eE_0\lambda_0}{2\pi mc^2}$$

  - normalized field strength

- In strongly nonlinear regime $a_0 \gg 1\,,\Lambda_0 \gg 1$

$$k_p r_b \approx 2\sqrt{\Lambda_0} \qquad k_p r_b \approx 2\sqrt{a_0}$$

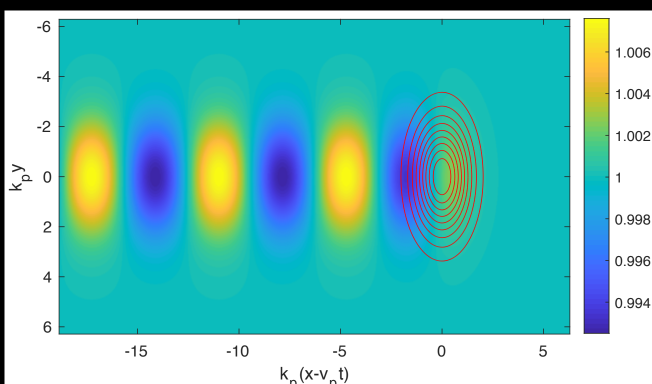
- In linear regime

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + k_p^2\right)\frac{\delta n}{n_0} = -\frac{2\Lambda}{r_0^2} - \nabla^2\frac{a^2}{2}$$

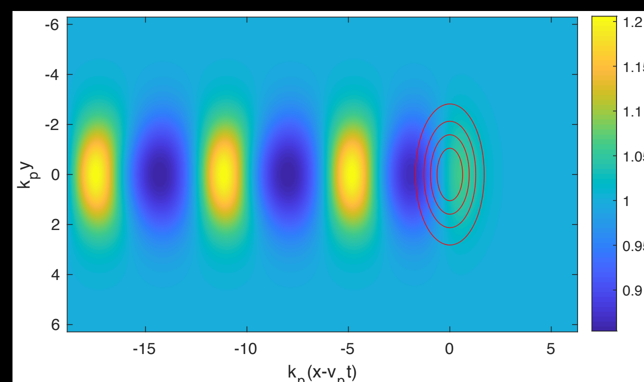
# Linear to quasilinear to nonlinear regimes

- As the driver beam intensity increases, the wakefield becomes more nonlinear

a= 0.1




a= 0.



# Linear to quasilinear to nonlinear regimes

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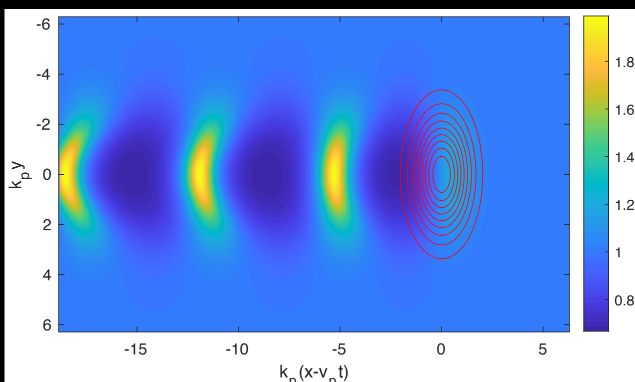
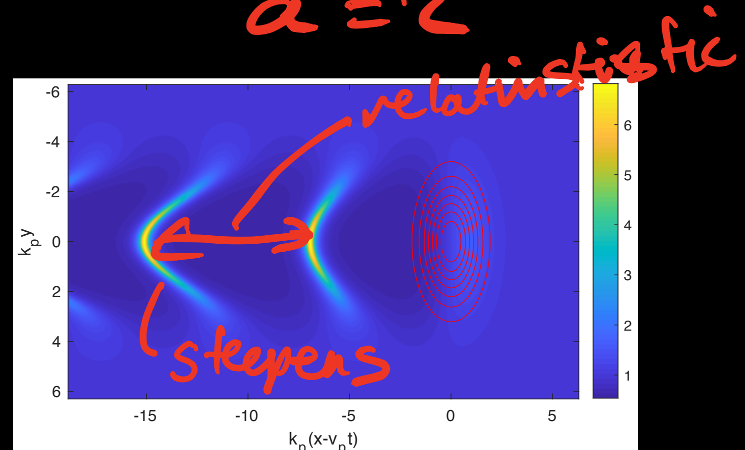
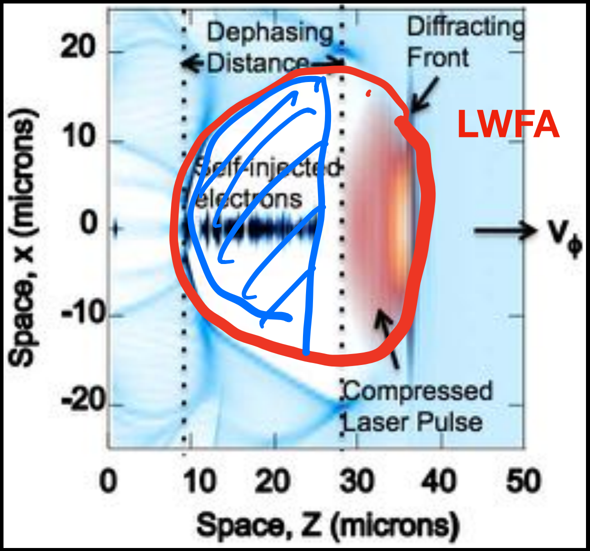


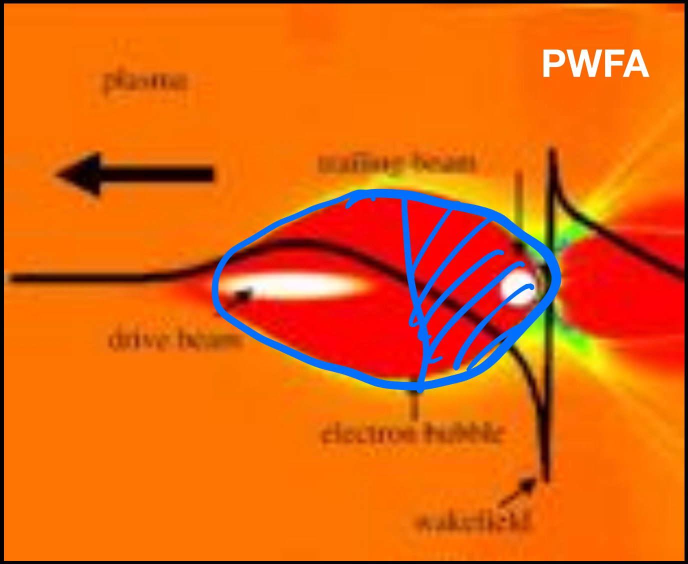

a=2



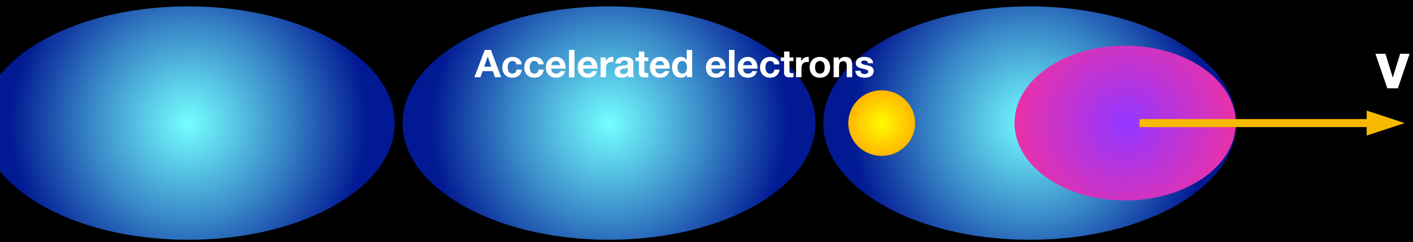
# Strongly nonlinear regime

- Dynamics becomes complicated and kinetic (multivalued, fluid approximation breaks down)



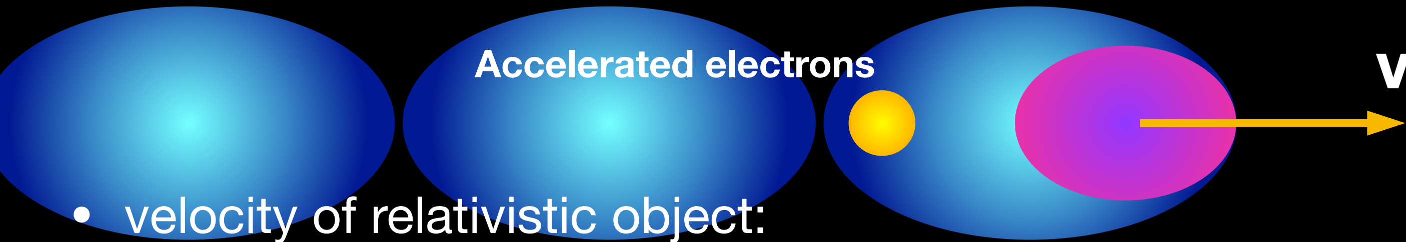
# Plasma wakefield accelerators




- All plasma wakefield accelerators involve the generation of a plasma wave with relativistic phase velocity by a perturbing object (laser pulse, charged particle beam) traveling at near light speed.

- These lectures will concentrate on the implications of a general object with an approximately **constant velocity and which don't change in amplitude** (complications such as refractive index/front erosion etc. are left for later lectures).

# Plasma wakefield accelerators




- velocity of relativistic object:
  - for **particle beam** of energy $\gamma_b mc^2$ velocity is $v_b = \sqrt{1 - \frac{1}{\gamma_b^2}}$
  - for a **laser pulse** traveling in plasma of density $n_0$ (**linear dispersion**), envelope velocity is

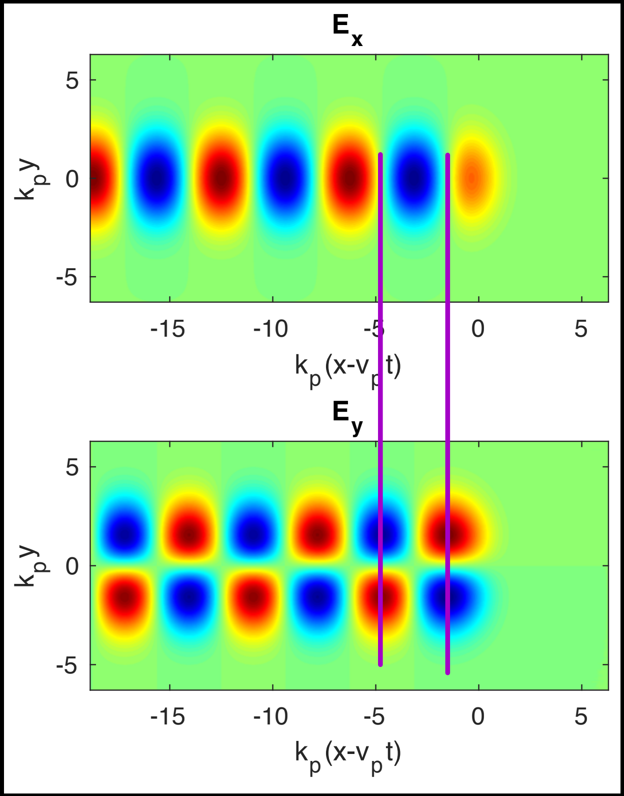
$$v_g = \sqrt{1 - \frac{n_0}{n_c}}$$

$\gamma_p$

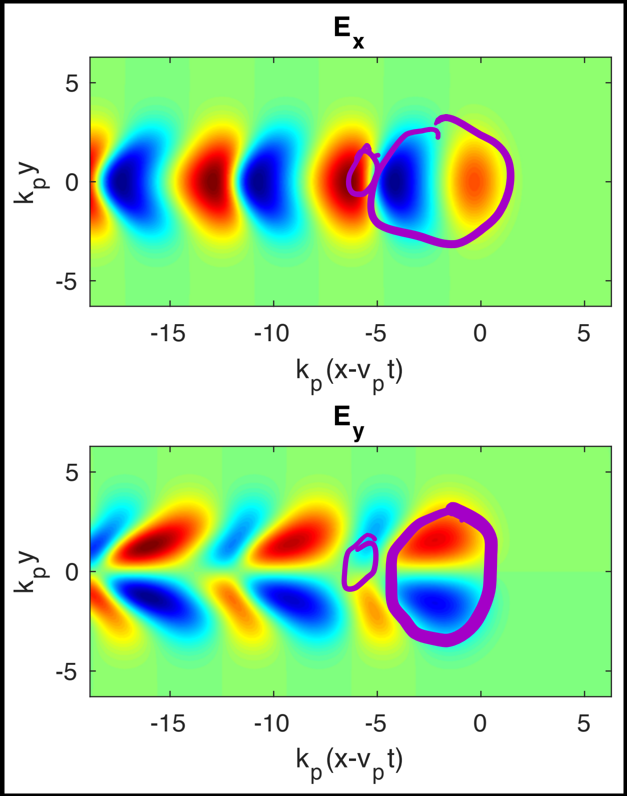
- So we can assign a Lorentz factor $\gamma_g = \sqrt{\frac{n_c}{n_0}} = \frac{\omega_0}{\omega_p}$

# Electric fields of wakefield

**a = 0.1**




**a = 1**

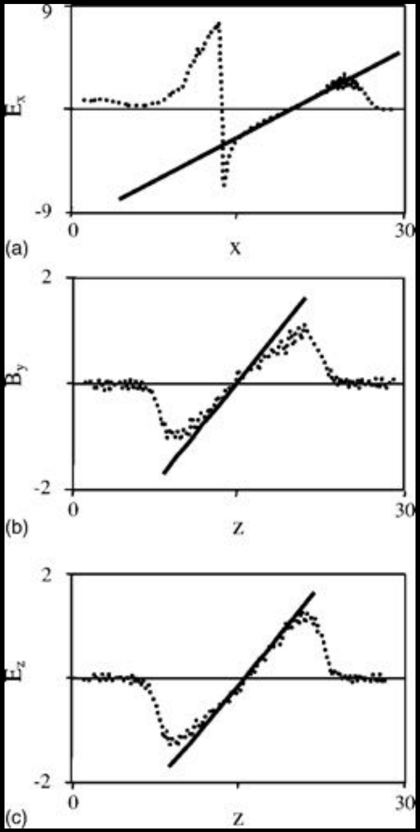


# Electric fields in strongly nonlinear regime

- In strongly nonlinear ('Bubble' or 'blowout') regime field structure is simply linear with a gradient

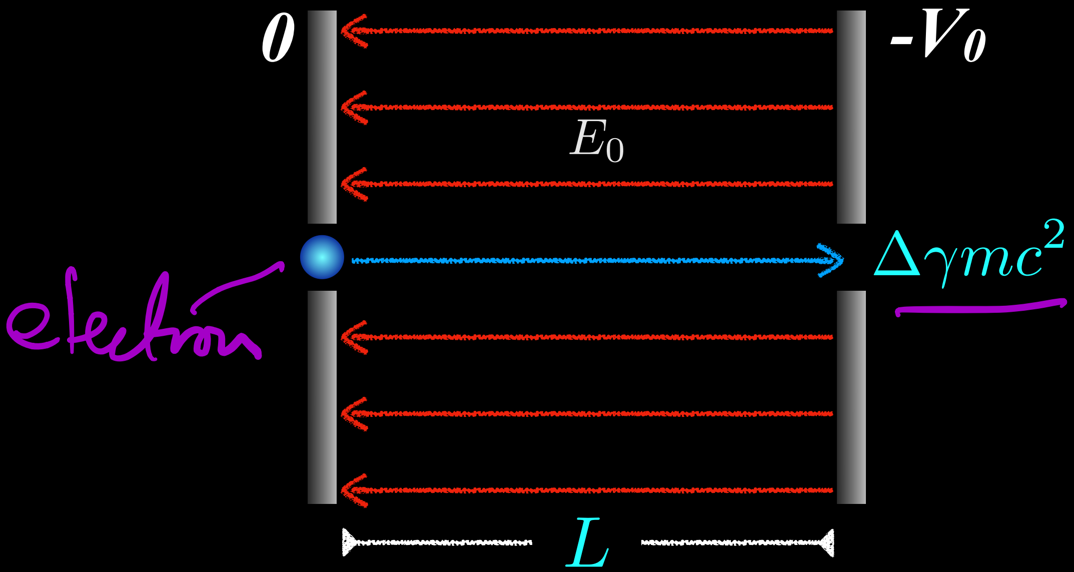
$$E_x = \frac{mc\omega_p}{e}\frac{\omega_p x}{c} = \frac{m\omega_p^2}{e}x$$

- For all components measured from the centre of the 'bubble'



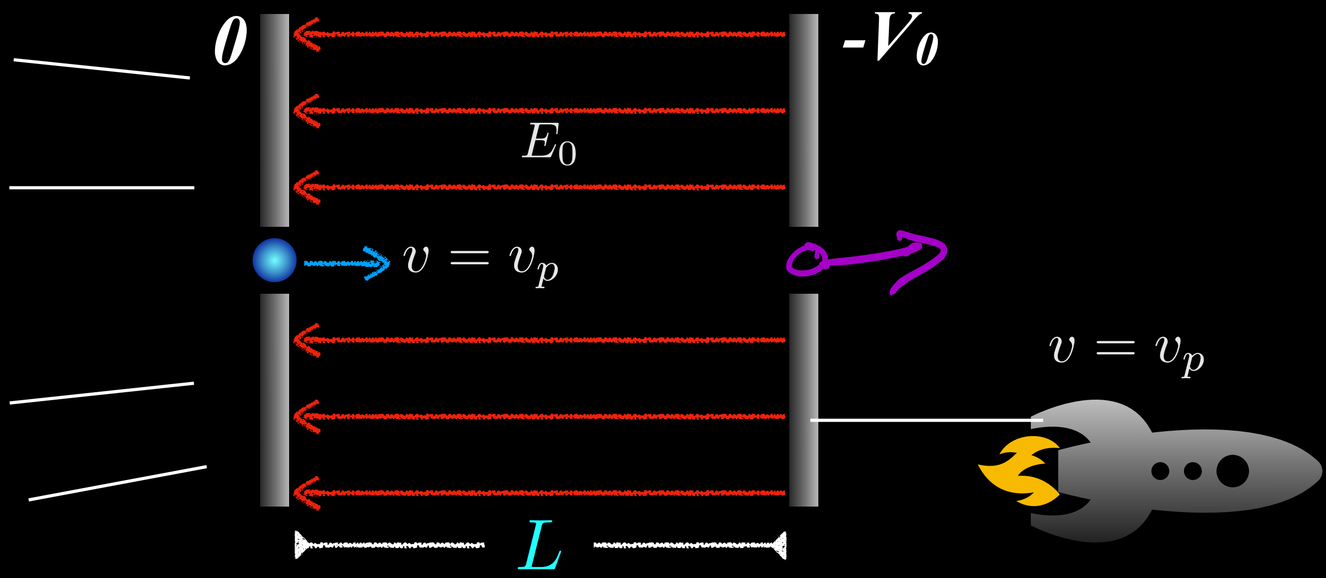

**Kostyukov Phys. Plasmas 2004**

# Acceleration of an electron between two parallel plates




- Energy gain is simply $\Delta\gamma mc^2 = eE_0L = eV_0$
- MV potential difference would be impressive, how do we get to 40 GeV?

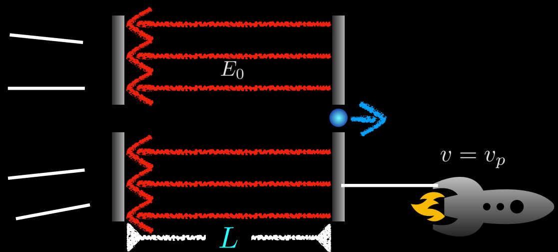
# Acceleration of an electron between two *moving* parallel plates




- Parallel plates moving at velocity $v_p$, **what is energy gain?** …how far does it travel before electron catches up with front plate? … does it reach the end before the (rocket) runs out of fuel?

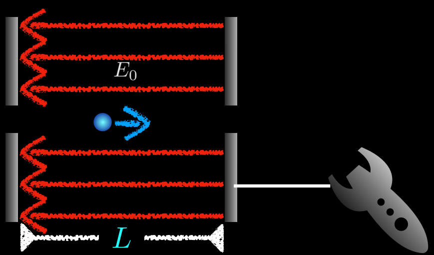
# Energy gain considerations

- In this example, the maximum electron energy is achieved either when…
  - it reaches the end of the parallel plate - "**dephasing**"




*Usually* limits LWFAs

  - the driver runs out of energy - "**depletion**"




*Usually* limits PWFAs

# Dephasing limited energy gain

# Energy gain limits of electron between *moving* parallel plates: dephasing

- EOM for electron assuming constant velocity for plates

$$\frac{dp}{dt} = eE_0 \qquad\qquad p = \gamma mc$$

- Can solve exactly to find dephasing length $x_d$ and maximum energy gain after some algebra:

$$x_d = \gamma_p^2 L \left(1 + \frac{v_p}{c}\sqrt{1 + \frac{2mc^2}{\gamma_p e V_0}}\right)$$

$$\Delta\gamma mc^2 = \gamma_p^2 e V_0 \left(1 + \frac{v_p}{c}\sqrt{1 + \frac{2mc^2}{\gamma_p e V_0}}\right)$$

# Energy gain limits of electron between *moving* parallel plates: dephasing

- Note that for a highly relativistic driver 

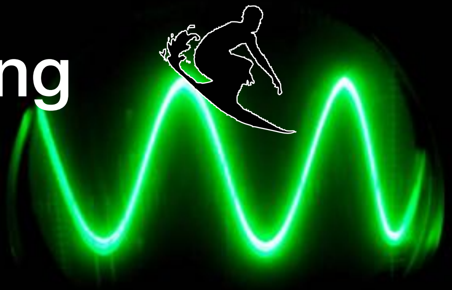
$$\gamma_p \to \infty$$

- acceleration length and energy gain therefore increased by

$$x_d \approx 2\gamma_p^2 L$$

$$\Delta\gamma mc^2 \approx 2\gamma_p^2 eV_0$$

# We need a better way of describing general accelerating gradients



- Note that for a **constant velocity, non-evolving** driver the fields, potentials etc., e.g.

$$E(x, t)$$

- Can be expressed in terms of a single coordinate only

$$\xi = x - v_p t \qquad \rightarrow E(\xi)$$

- This is the **wake phase**

# Change of variables

$$\xi = x - v_p t \qquad \tau = t$$

Chain rule

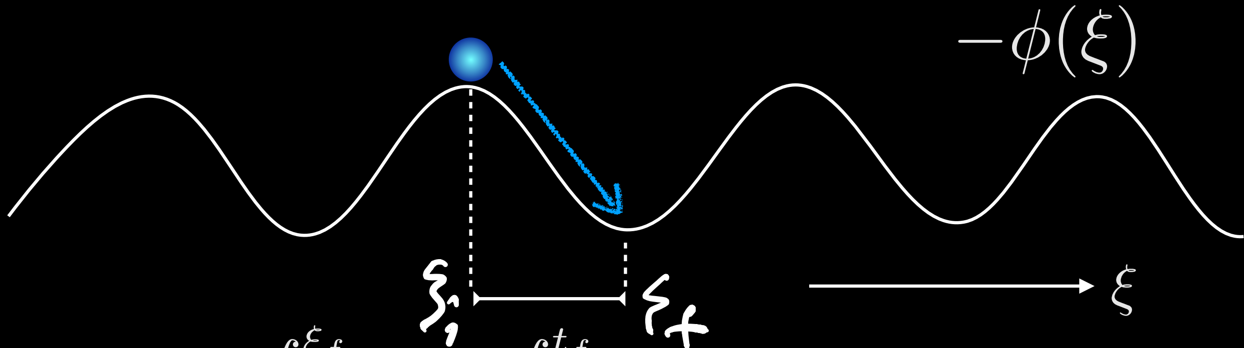
$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial f}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial t}$$

$$= \frac{\partial f}{\partial \tau} - v_p \frac{\partial}{\partial \xi}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \tau}\frac{\partial \tau}{\partial x} + \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial f}{\partial \xi}$$

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial \tau} - v_p \frac{\partial}{\partial \xi} \qquad \frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial \xi}$$

# Dephasing




$$\int_{\xi_i}^{\xi_f} d\xi = \int_{t_i}^{t_f} (v_x - v_p)\,dt = L_d - v_p t_d$$

$$t_d \approx L_d / c$$

$$\Delta\xi = L = L_d\left(1 - \frac{v_p}{c}\right)\frac{\left(1+\frac{v_p}{c}\right)}{\left(1+\frac{v_p}{c}\right)} = \frac{L_d}{\gamma_p^2\left(1+\frac{v_p}{c}\right)}$$

$$L_d \approx 2\gamma_p^2 L$$

# Hamiltonian for wakefield

- Start with Hamiltonian for a system that depends on the wake phase coordinate $\xi = x - v_p t$

$$H = \sqrt{(mc^2)^2 + p_x^2 c^2 + \left(p_\perp - eA_\perp(\xi)\right)^2 c^2} - e\phi(\xi) - p_x v_p$$

- We can show that this is the correct Hamiltonian for the coordinates

$$\xi, y, z, p_x, p_y, p_z$$

- using Hamilton's equations…

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

**Show Hamiltonian is correct**

$$\frac{d\xi}{dt} = \frac{d}{dt}(x - v_p t) = v_x - v_p$$

$$\frac{d\xi}{dt} = \frac{\partial H}{\partial p_x} = v - v_p \checkmark$$

$$\frac{dp_x}{dt} = -\frac{\partial H}{\partial \xi} = -\frac{e^2 A c^2}{\gamma m c^2}\frac{\partial A}{\partial \xi} - \frac{e\partial \varphi}{\partial \xi}$$

$$= \frac{e^2}{2\gamma m}\frac{\partial A^2}{\partial \xi} - \frac{e\partial \varphi}{\partial \xi} \checkmark$$

# Hamiltonian for wakefield

- The ***Hamiltonian is conserved***, hence

$$H = \gamma mc^2 - e\phi(\xi) - p_x v_p = \text{const}$$

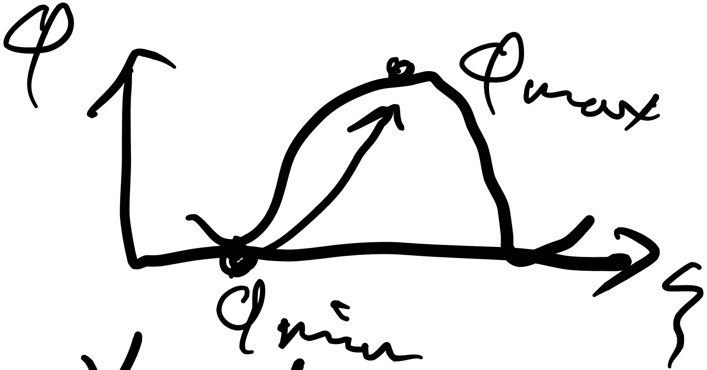
- is a useful constant of motion, or

$$\gamma(1 - \beta_x \beta_p) - \frac{e\phi}{mc^2} = H_0$$

## Show Hamiltonian predicts electron energy

$$\gamma m c^2 - e\varphi - p_x v_p = \gamma_p m c^2 - \gamma_p v_p^2 m - e\varphi_{min}$$

$$\gamma_p m c^2 - \gamma_p v_p^2 m = \frac{mc^2}{\gamma_p}$$



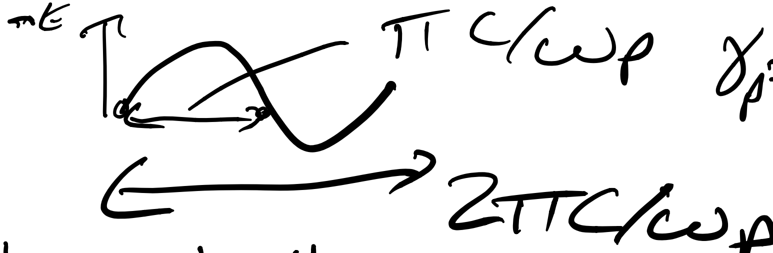

$v_e = v_p$

$$\gamma(1 - \beta_x \beta_p) \approx \frac{e\Delta\varphi}{mc^2}$$

if $\beta_x \to 1$

$$\Delta\gamma \approx 2\gamma_p^2 \frac{e\Delta\varphi}{mc^2}$$

**Show Hamiltonian predicts electron energy**




$\gamma_p = \frac{\omega_0}{\omega_p}$

Dephasing length

$$L_d \approx 2\gamma_p^2 \Delta\xi = \frac{2\pi c}{\omega_p}\frac{\omega_0^2}{\omega_p^2}$$

Energy gain Gauss's law

$$\frac{e\phi}{mc^2} = \frac{\delta n}{n_0}\cos k_p \xi$$

$$\frac{\delta n}{n_0} = \frac{a^2}{2}$$

$$\Delta\gamma = \cancel{2}\,\frac{\omega_0^2\, a^2}{\omega_p^2\, \cancel{2}}$$

# Summary of scalings (LWFA)

Lu PRZ 2007

| | $a_0$ | $k_p w_0$ | $\epsilon_{LW}$ | $k_p L_d$ |
|---|---|---|---|---|
| Linear: | $<1$ | $2\pi$ | $a_0^2$ | $\frac{\omega_0^2}{\omega_p^2}$ |
| 1D Nonlinear: | $>1$ | $2\pi$ | $a_0$ | $4a_0^2\frac{\omega_0^2}{\omega_p^2}$ |
| 3D Nonlinear: | $>2$ | $2\sqrt{a_0}$ | $\frac{1}{2}\sqrt{a_0}$ | $\frac{4}{3}\frac{\omega_0^2}{\omega_p^2}\sqrt{a_0}$ |
| Ref. [24]: | $>20$ | $\sqrt{a_0}$ | $\sqrt{a_0}$ | |

$\gamma_p(\text{laser}) \propto \frac{\omega_0}{\omega_p}$

plasma wavelength

| $k_p L_{pd}$ | $\lambda_W$ | $\gamma_\phi$ | $\Delta W/(mc^2)$ |
|---|---|---|---|
| $\frac{\omega_0^2}{\omega_p^2}\frac{\omega_p\tau}{a_0^2}$ | $\frac{2\pi}{k_p}$ | $\frac{\omega_0}{\omega_p}$ | $a_0^2\frac{\omega_0^2}{\omega_p^2}$ |
| $\frac{1}{3}\frac{\omega_0^2}{\omega_p^2}\omega_p\tau$ | $\frac{4a_0}{k_p}$ | $\sqrt{a_0}\frac{\omega_0}{\omega_p}$ | $4a_0^2\frac{\omega_0^2}{\omega_p^2}$ |
| $\frac{\omega_0^2}{\omega_p^2}\omega_p\tau$ | $\sqrt{a_0}\frac{2\pi}{k_p}$ | $\frac{1}{\sqrt{3}}\frac{\omega_0}{\omega_p}$ | $\frac{2}{3}\frac{\omega_0^2}{\omega_p^2}a_0$ |
| $a_0\frac{\omega_0^2}{\omega_p^2}\omega_p\tau$ | | | $\frac{\omega_0^2}{\omega_p^2}a_0^{3/2}\omega_p\tau$ |

$L \propto \frac{1}{k_p}$

3D nonlinear from scaling simulations

# LWFA scaling

Lu PRZ 2007