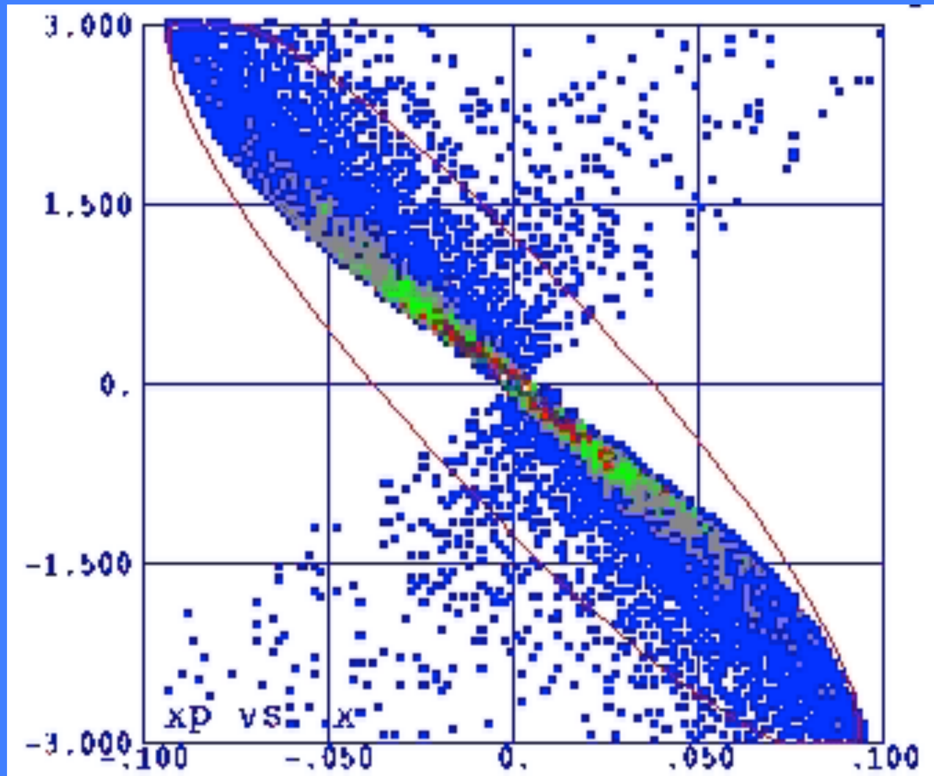
Statistical Description of Particle Beams

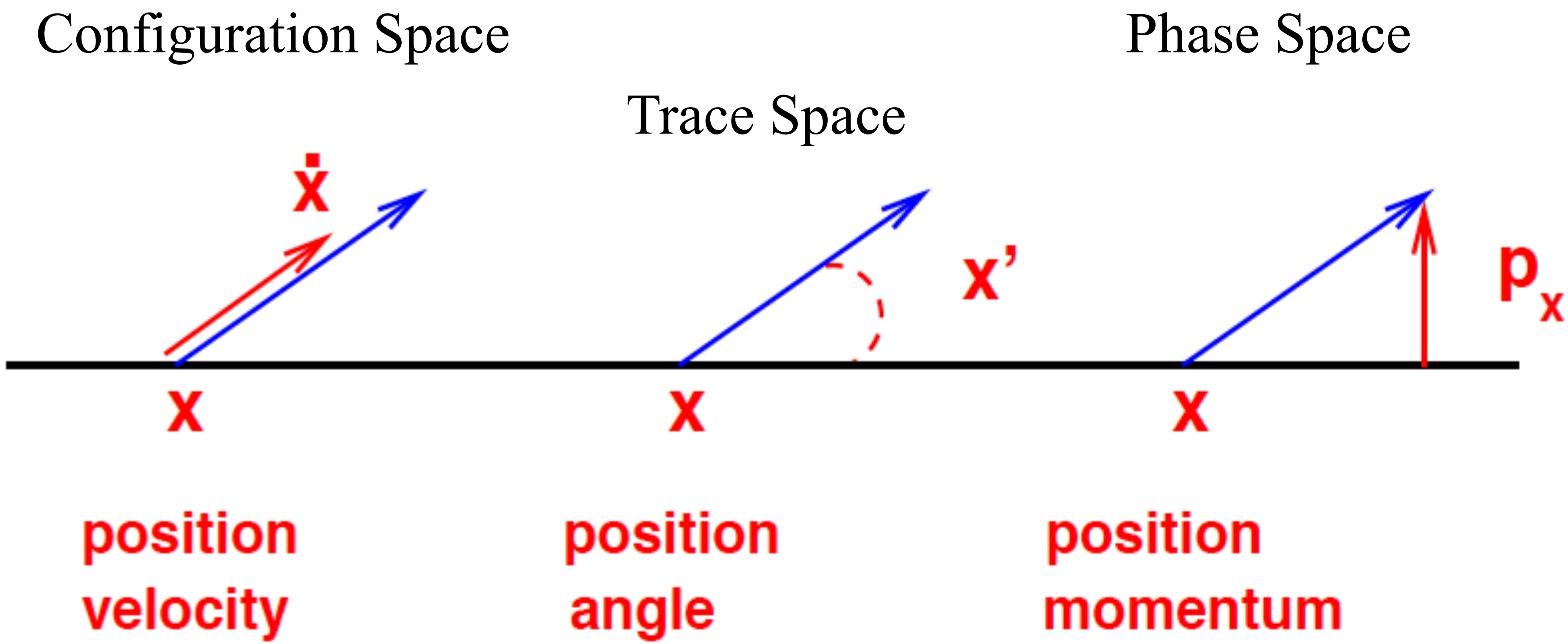
Massimo.Ferrario@LNF.INFN.IT



The CERN Accelerator School

Introduction to Accelerator Physics – Vysoke, Slovakia – 12 Sept. 2019

Typical coordinates to describe the particle motion
(6 per particle)



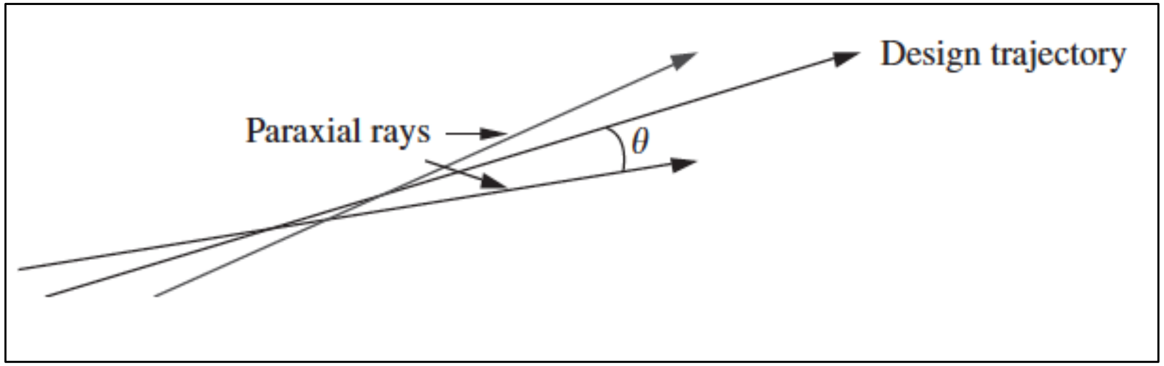
$$\begin{cases} x \\ x' = \dfrac{dx}{dz} = \dfrac{p_x}{p_z} \qquad p_x << p_z \end{cases}$$

$$\begin{aligned} p_z &= \gamma m_o v_z \\ &= \beta_z \gamma m_o c \end{aligned}$$

Paraxial Ray Approximation

paraxial rays => vector representations of the local trajectory which, by definition, have an angle with respect to a design trajectory that is much smaller than unity.

Trajectories of interest in beam physics are often paraxial: one must confine the beam inside of small, near-axis regions.



In a locally Cartesian coordinate system, we take the distance along the design trajectory to be z. The horizontal offset is designated by x and the horizontal angle is θ_x .

Paraxial Ray Approximation

The angle is given in terms of the momenta as:

$$\tan\theta_x = \frac{p_x}{p_z} = \frac{v_x}{v_z} \qquad\qquad p_{x,y} \ll p_z$$

In order to use z as the independent variable, we must be able to write equations of motion in terms of z:

$$\frac{\mathrm{d}}{\mathrm{d}z} = \frac{1}{v_z}\frac{\mathrm{d}}{\mathrm{d}t}$$

The derivative of a horizontal offset with respect to z is given by:

$$x' = \frac{\mathrm{d}x}{\mathrm{d}z} = \tan(\theta_x) \qquad\qquad \boxed{x' = \tan(\theta_x) \cong \theta_x \cong \sin(\theta_x) \ll 1}$$

Example

Focusing systems often resemble simple harmonic oscillators, where the transverse force is **linear** in offset:

$$F_x = -Kx$$

And the equation of motion is:

$$\ddot{x} + \omega^2 x = 0 \qquad \omega^2 = K/\gamma m_0$$

in terms of a ray description:

$$x'' + k^2 x = 0 \qquad k \equiv \omega/v_z$$

The solutions are of the form:

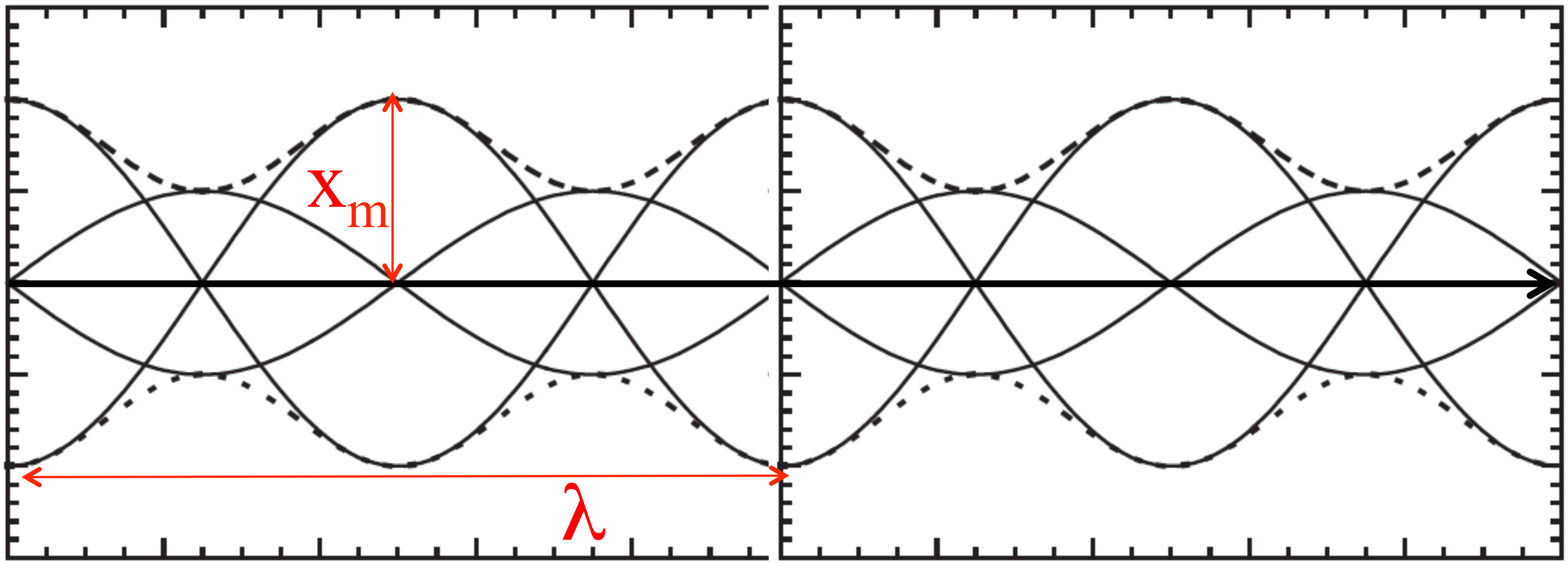
$$x = x_\mathrm{m} \cos(kz + \phi)$$

with an angle:

$$x' = \left| -k x_m \sin(kz + \phi) \right| << 1 \quad \Rightarrow \quad x_m << \frac{1}{k}$$

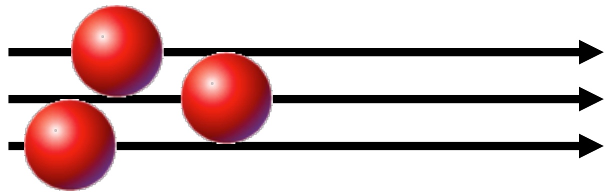
the paraxial approximation is valid for offsets smaller than the characteristic oscillation wavenumber.

$$x = x_m \cos(kz + \phi)$$

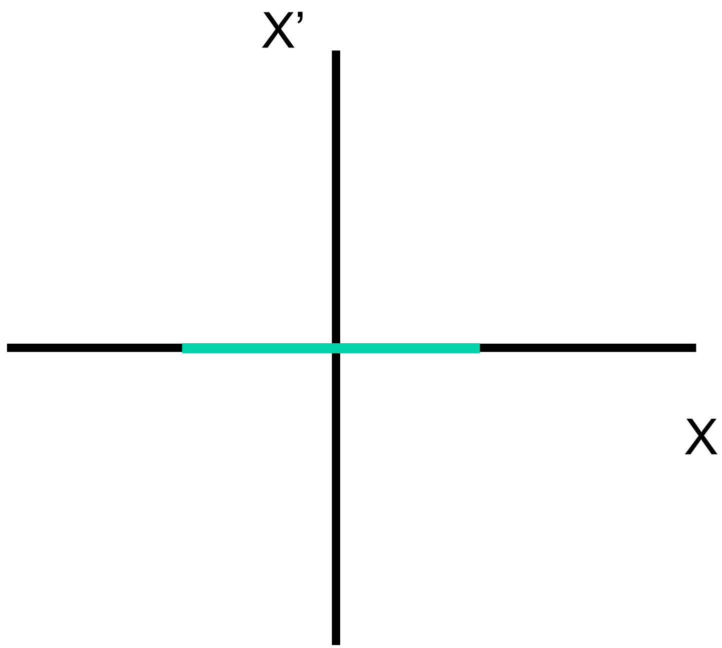


$$x_m << \frac{1}{k} = \frac{\lambda}{2\pi}$$

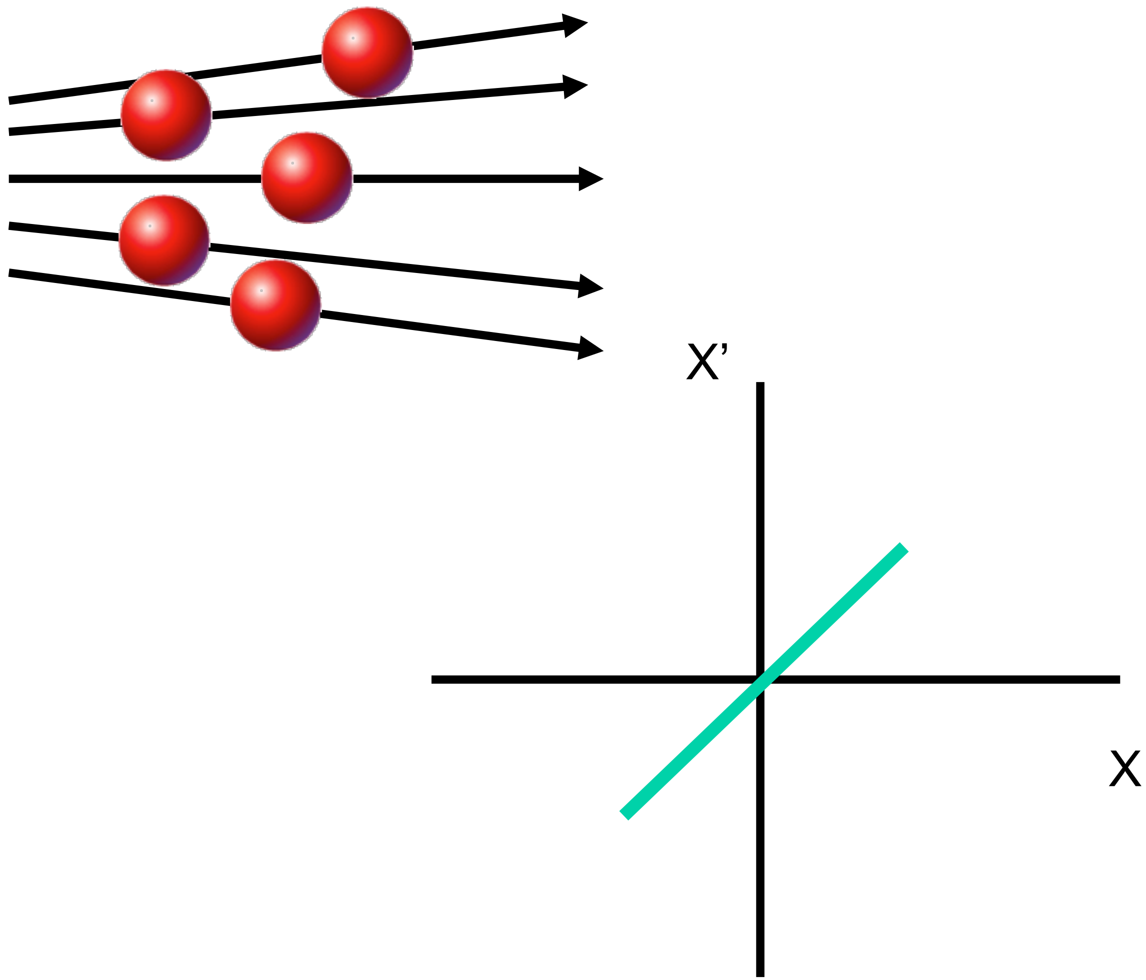
Trace space of an ideal beam



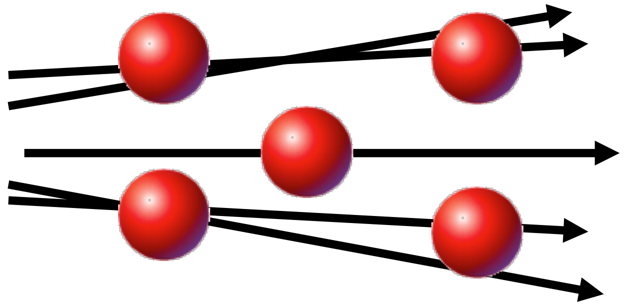
$$\begin{cases} x \\ x' = \dfrac{dx}{dz} = \dfrac{p_x}{p_z} \end{cases}$$

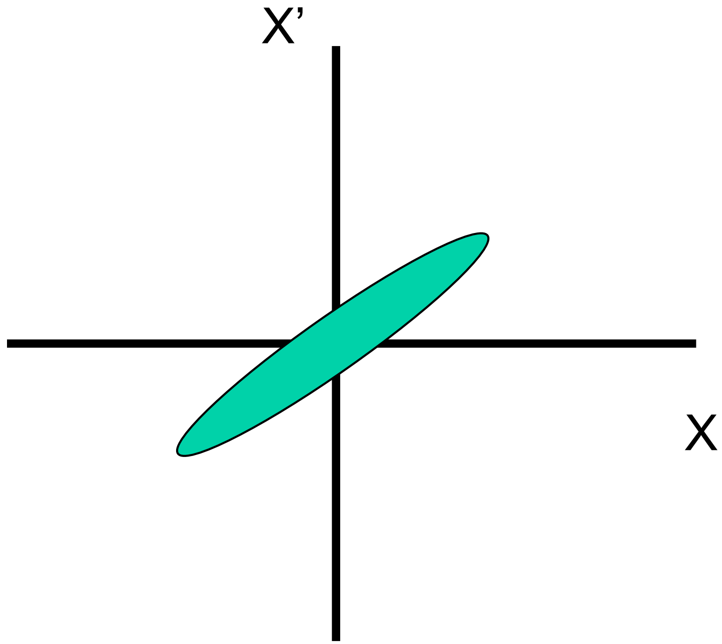


Trace space of a laminar beam



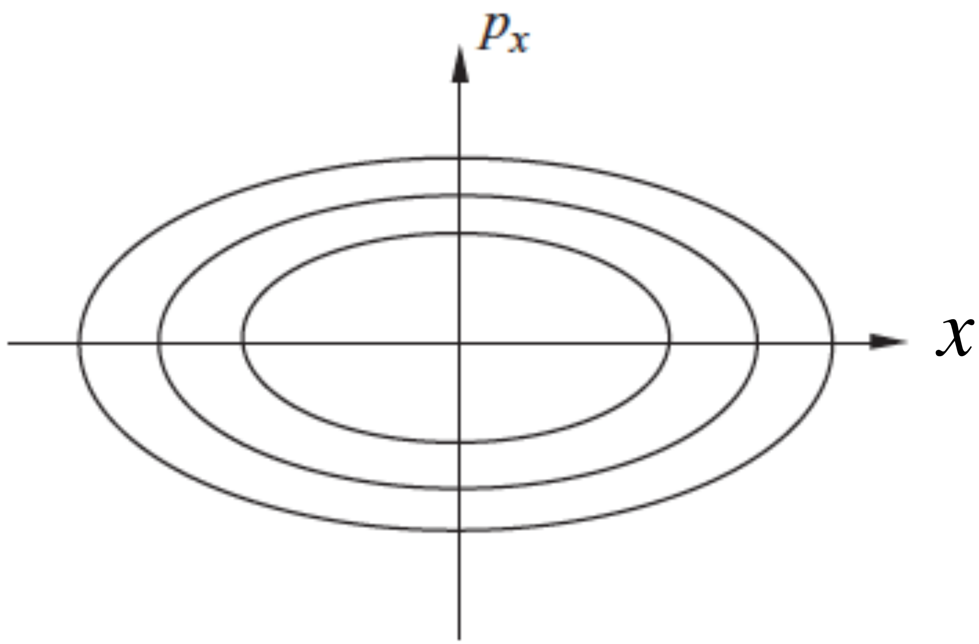
Trace space of non laminar beam





In a system where all the forces acting on the particles are linear (i.e., proportional to the particle's displacement x from the beam axis), it is useful to assume an elliptical shape for the area occupied by the beam in x-x' trace space or x-p_x phase space.

$$\ddot{x} + k^2 x = 0$$



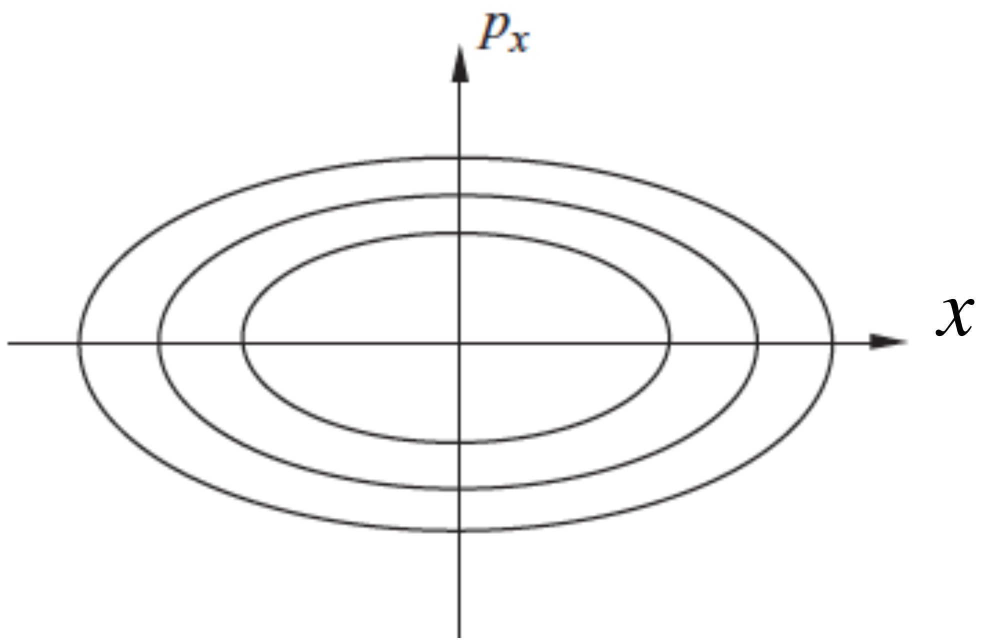
$$H = \frac{1}{2m}[p_x^2 + m^2\omega^2 x^2]$$

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i},$$

The **action** is the area enclosed by the phase space trajectory and it is related to the energy of the particle.

$$J = \frac{1}{2\pi} \oint p_x \, \mathrm{d}x.$$

The action is also generally known to be an *adiabatic invariant*, in that when the parameters of an oscillatory system are changed slowly, the action remains a constant.



Example: slowly varying oscillator:

$$\ddot{x} + K(t)x = 0$$

Substituting: $x(t) = C\sqrt{a(t)}\cos(\psi(t))$

$$\begin{cases} \dot{\psi} = \dfrac{1}{a} \\ \dfrac{\ddot{a}}{a} - \dfrac{\dot{a}^2}{2a^2} - \dfrac{4}{2a^2} + 2K = 0 \end{cases}$$

$$\psi = \int \frac{\mathrm{d}t}{a(t)}$$

Assuming the first and second time derivatives of a are small.

Condition of adiabaticity :

$$\begin{cases} \ddot{a}/a \ll K \\ (\dot{a}/a)^2 \ll K \end{cases}$$

$$-\frac{2}{a^2} + 2K = 0$$

$$a(t) \cong (K(t))^{-1/2}$$

$$\begin{cases} x(t) = C(K(t))^{-1/4}\cos(\psi(t)) \\ p_x(t) = \dot{x} \cong C(K(t))^{1/4}\sin(\psi(t)) \end{cases}$$

$$J = \tfrac{1}{2} p_{x,\max} x_{\max} = \tfrac{1}{2} C^2$$

The area of the phase space ellipse, whose semimajor and semiminor axes are the maximum excursion in x and p_x respectively, is the value of the action at that time.

Analytical Geometry: Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ Canonical Ellipse equation $$Area = \pi ab$$

$$Ax^2 + Bxy + Cy^2 = 1$$ Rotated Ellipse $$Area = \frac{2\pi}{\sqrt{4AC - B^2}}$$

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon$$ Emittance Ellipse

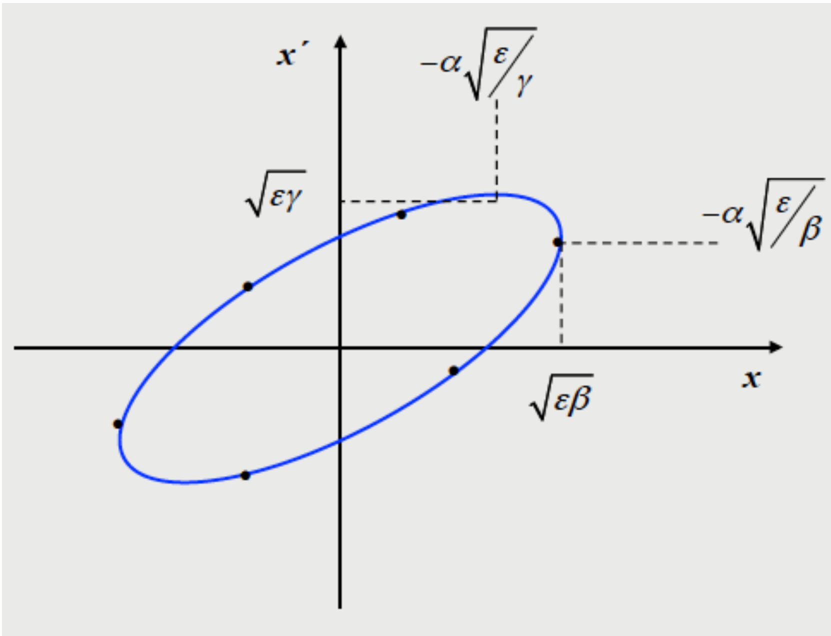
$$Area = \frac{\pi\varepsilon}{\sqrt{\gamma\beta - \alpha^2}} = \pi\varepsilon \Leftrightarrow \gamma\beta - \alpha^2 = 1$$

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon$$

$$\gamma\beta - \alpha^2 = 1$$

Solving for x=f(x') and computing df(x')/dx'=0 =>

$$\begin{cases} x'_{x_{\max}} = -\alpha\sqrt{\dfrac{\varepsilon}{\beta}} \\ x_{\max} = \sqrt{\varepsilon\beta} \end{cases}$$



From: $$\frac{dx_{\max}}{dz} = x'_{x_{\max}} \qquad \frac{\beta'}{2}\sqrt{\frac{\varepsilon}{\beta}} = -\alpha\sqrt{\frac{\varepsilon}{\beta}}$$

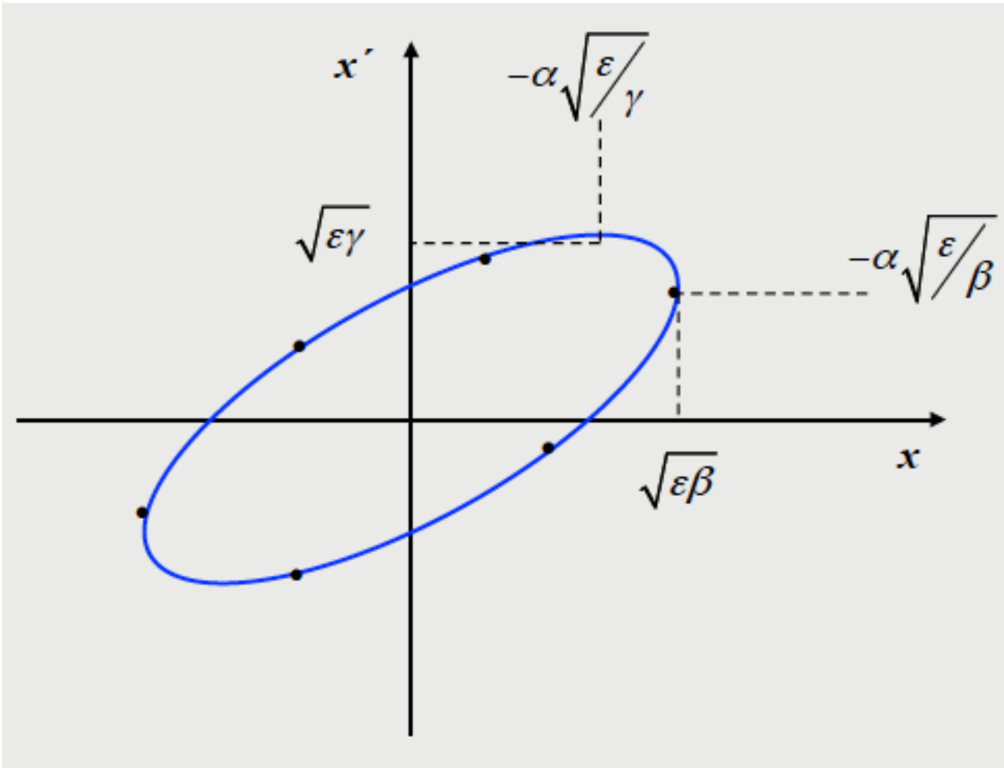
$$\beta' = -2\alpha$$

Geometric emittance: ε_g

Ellipse equation: $\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_g$

Twiss parameters: $\beta\gamma - \alpha^2 = 1$ $\beta' = -2\alpha$

Ellipse area: $A = \pi\varepsilon_g$



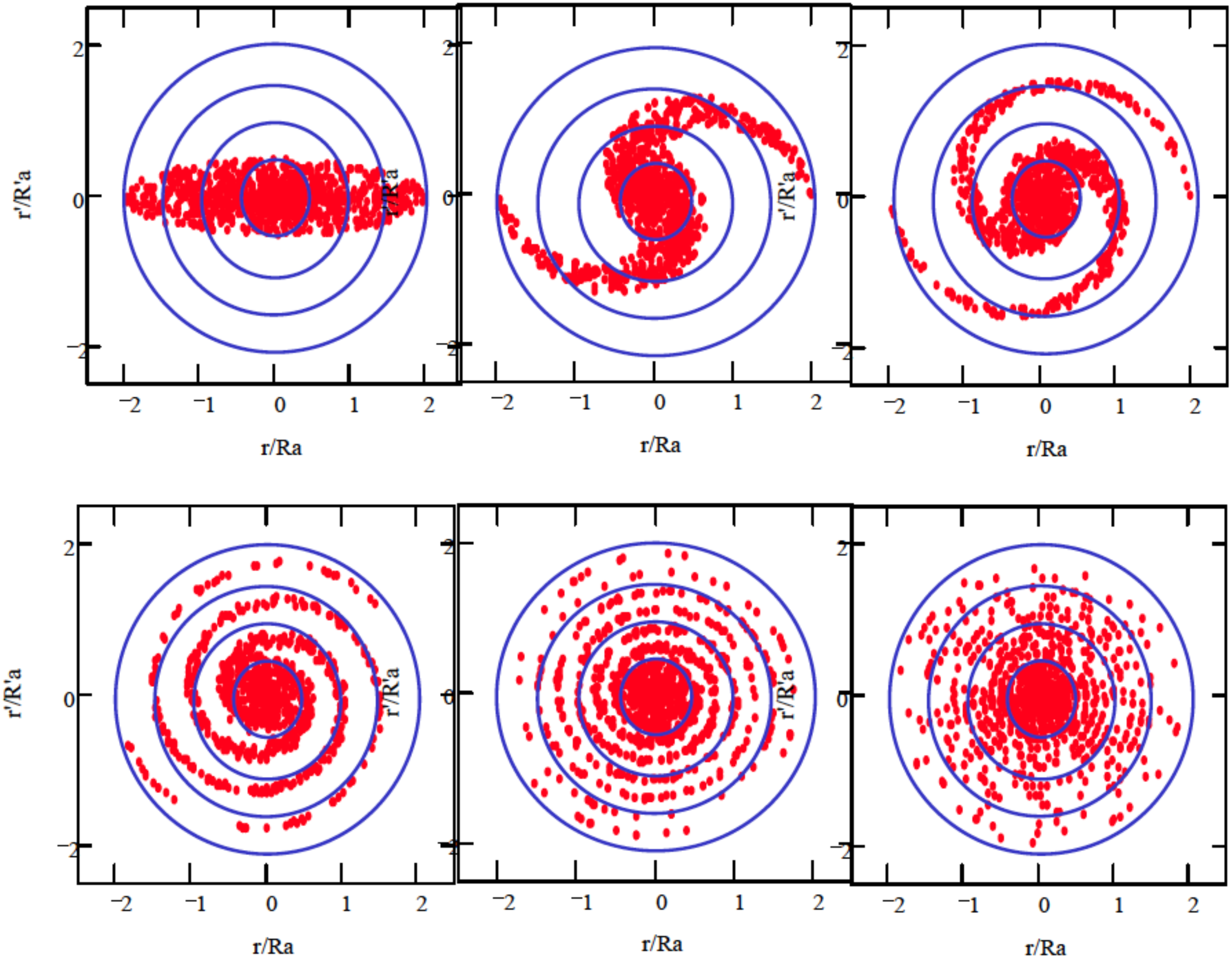
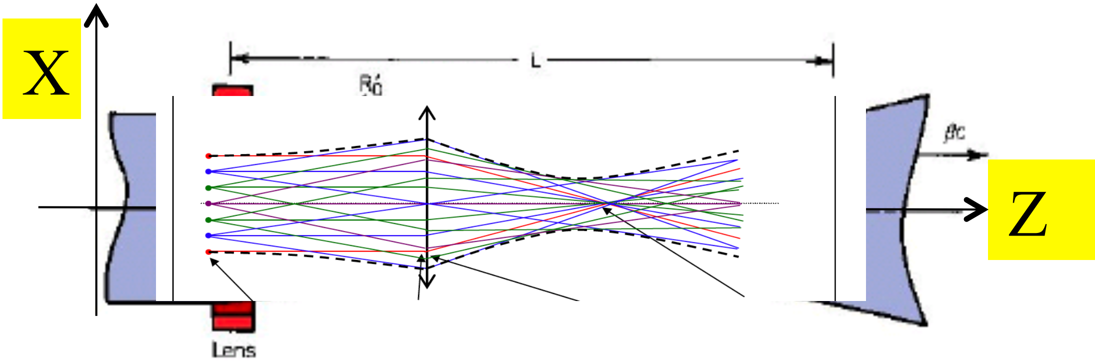
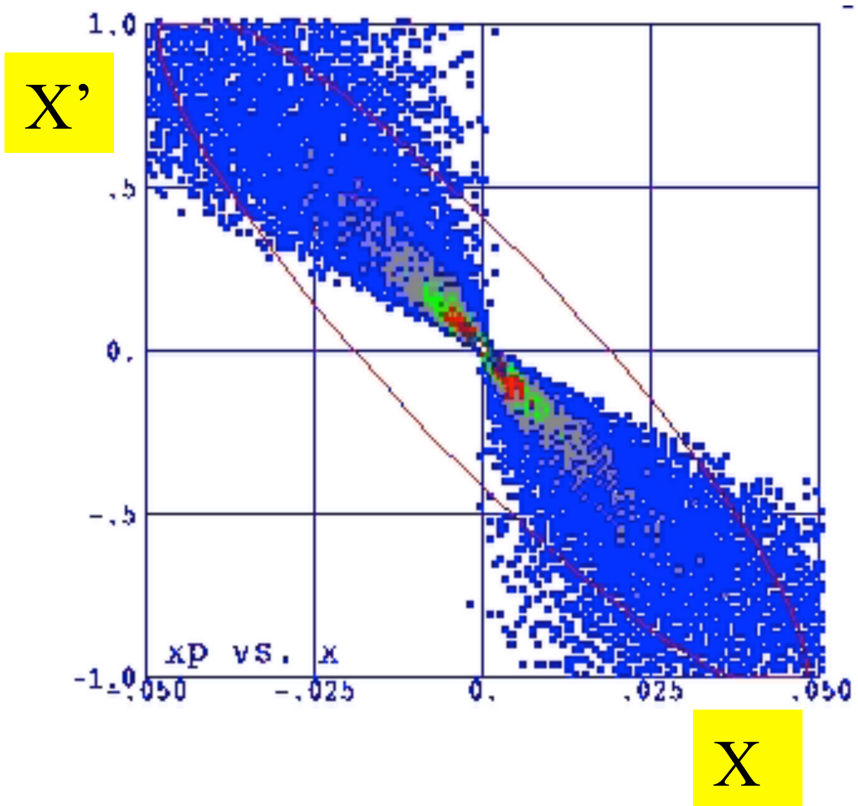


Fig. 17: Filamentation of mismatched beam in non-linear force

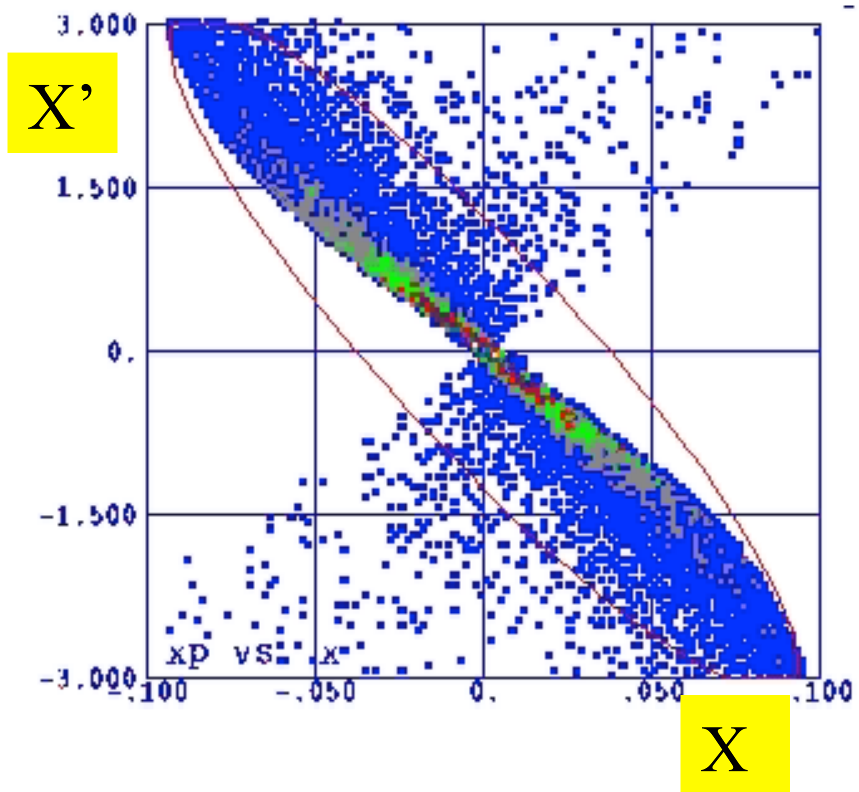
Phase space evolution



No space charge => cross over

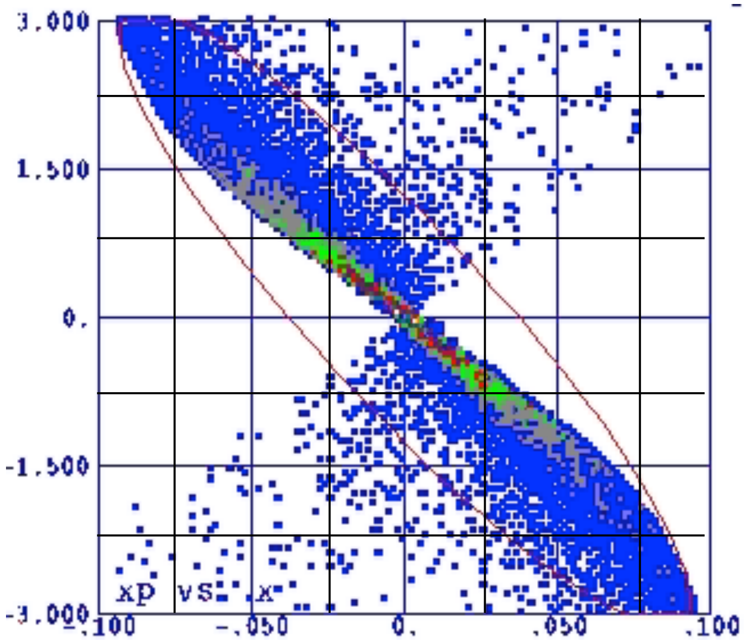


With space charge => no cross over



In charged particle beam dynamics, we are commonly not interested in the phase-space location of individual particles, a statistical mechanics approach is appropriate using function f (x , p , t)

The distribution function f is viewed as a smooth probability function in a 2M – dimensional space f (x , p , t)



The number of particles found in a differential volume dV = d^3x d^3p in the neighborhood of a location x , p at a time t is simply given by f (x , p , t) dV .

The total time derivative of the phase space distribution function is

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \sum_i \left(\frac{\mathrm{d}x_i}{\mathrm{d}t} \frac{\partial f}{\partial x_i} + \frac{\mathrm{d}p_i}{\mathrm{d}t} \frac{\partial f}{\partial p_i} \right)$$

If the forces are derivable from a Hamiltonian, then:

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i},$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial H}{\partial p_i} \frac{\partial f}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial f}{\partial p_i} \right)$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial H}{\partial p_i} \frac{\partial H}{\partial x_i} \frac{\mathrm{d}f}{\mathrm{d}H} - \frac{\partial H}{\partial x_i} \frac{\partial H}{\partial p_i} \frac{\mathrm{d}f}{\mathrm{d}H} \right) = \frac{\partial f}{\partial t}.$$

If neither creation nor destruction of the particles are allowed, we also have:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0.$$

This result is termed Liouville' s theorem, and it implies that the phase space density encountered as one travels with a particle in a Hamiltonian system is conserved.

The full distribution function contains all of the information needed to describe the state of a non-interacting ensemble of beam particles.

One may not need all of this information.

Moments of the distribution are used to arrive at a simpler description of the distribution's evolution.

These moments are formally written in the Trace Space as:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n (x')^m f_x(x,x')\,\mathrm{d}x\,\mathrm{d}x'.$$

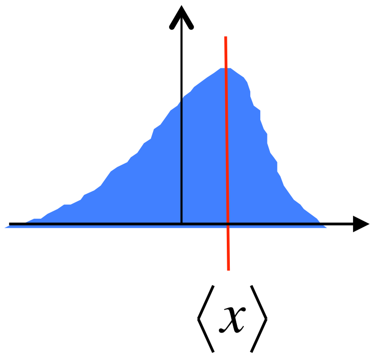
where m and n are equal to zero or a positive integer, and the quantity m + n is referred to as the order of the moment.

The zeroth-order moment is simply the normalization condition on the distribution:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_x(x, x')\,\mathrm{d}x\,\mathrm{d}x' = 1.$$

The first-order moments are the centroids of the distribution:

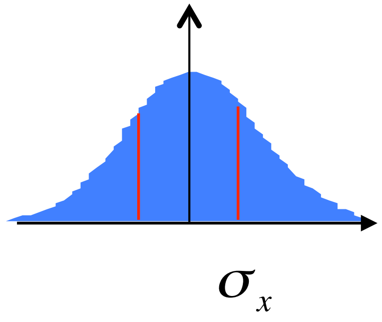
$$\langle x\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_x(x, x')\,\mathrm{d}x\,\mathrm{d}x',$$



$$\langle x'\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x' f_x(x, x')\,\mathrm{d}x\,\mathrm{d}x',$$

which vanish when a beam is simmetric to its design axis.

The second moments are written in standard notation as the distribution variances (spread and correlations):



$$\sigma_x^2 = \langle x^2 \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_x(x, x')\,\mathrm{d}x\,\mathrm{d}x',$$

$$\sigma_x'^2 = \langle x'^2 \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x'^2 f_x(x, x')\,\mathrm{d}x\,\mathrm{d}x',$$

$$\sigma_{xx'} = \langle xx' \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xx' f(x, x')\,\mathrm{d}x\,\mathrm{d}x',$$

Moments of a Distribution

- First Moment:
  - ***mean*** **- measure of location**

$$\mu_1 = \frac{1}{n}\sum_{i=1}^{n} x_i$$

- Second Moment:
  - ***standard deviation*** **- measure of spread**

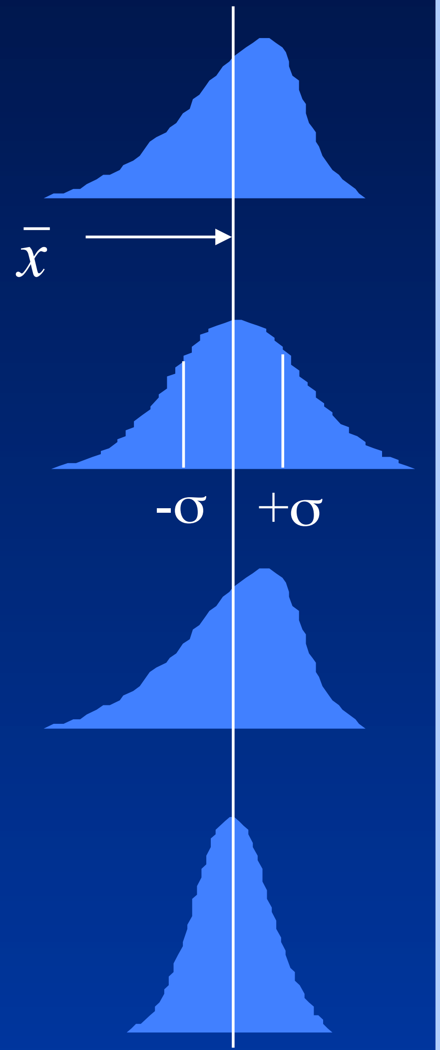
$$\mu_2 = \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_1)^2$$

- Third Moment:
  - ***skewness*** **- measure of symmetry**

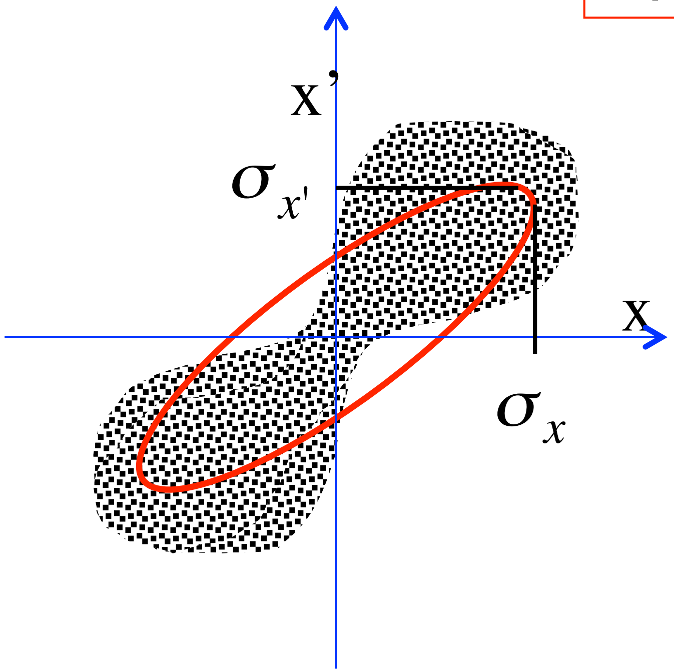
$$\mu_3 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_1)^3$$

- Fourth Moment:
  - ***kurtosis*** **- measure of peakedness**

$$\mu_4 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_1)^4$$



rms emittance ε_{rms}



$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,x')\,dx\,dx' = 1$$

$$f'(x,x') = 0$$

rms beam envelope:

$$\sigma_x^2 = \langle x^2 \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f(x,x')\,dx\,dx'$$

Define rms emittance:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_{rms}$$

such that:

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$

$$\sigma_{x'} = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

Since: $\alpha = -\frac{\beta'}{2}$

it follows: $\alpha = -\frac{1}{2\varepsilon_{rms}}\frac{d}{dz}\langle x^2 \rangle = -\frac{\langle x x' \rangle}{\varepsilon_{rms}} = -\frac{\sigma_{xx'}}{\varepsilon_{rms}}$

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$

$$\sigma_{x'} = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

$$\sigma_{xx'} = \langle x x' \rangle = -\alpha \varepsilon_{rms}$$

It holds also the relation: $\gamma\beta - \alpha^2 = 1$

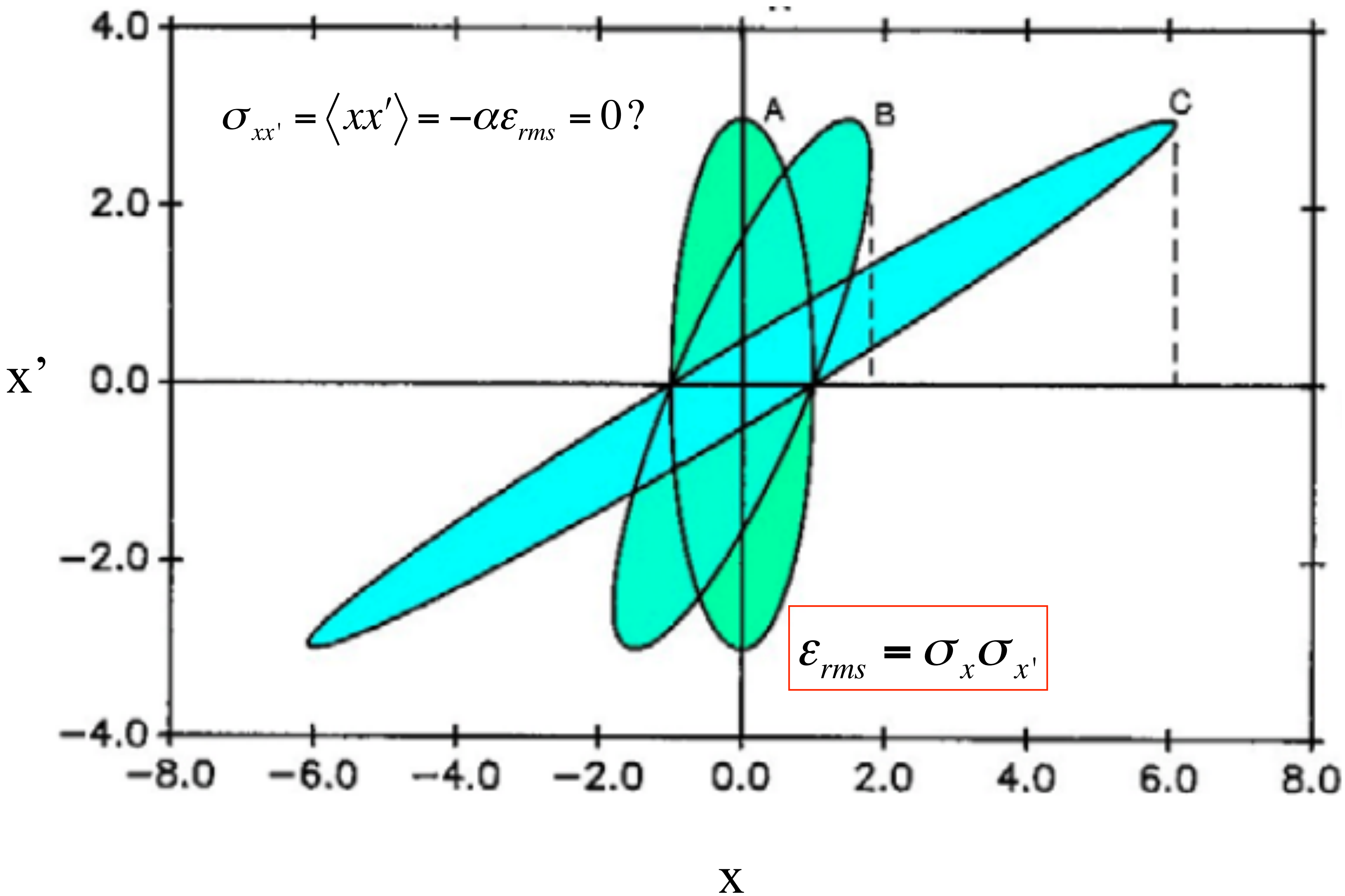
Substituting α, β, γ we get $\frac{\sigma_{x'}^2}{\varepsilon_{rms}} \frac{\sigma_x^2}{\varepsilon_{rms}} - \left(\frac{\sigma_{xx'}}{\varepsilon_{rms}}\right)^2 = 1$

We end up with the definition of rms emittance in terms of the second moments of the distribution:

$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2\right)}$$

$$x' = \frac{p_x}{p_z}$$

Which distribution has no correlations?

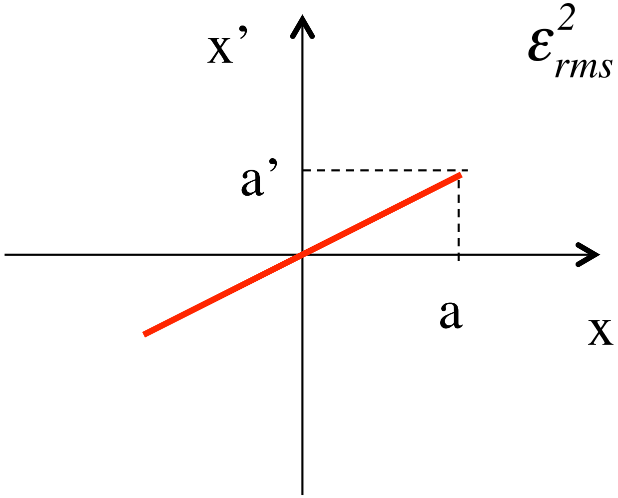


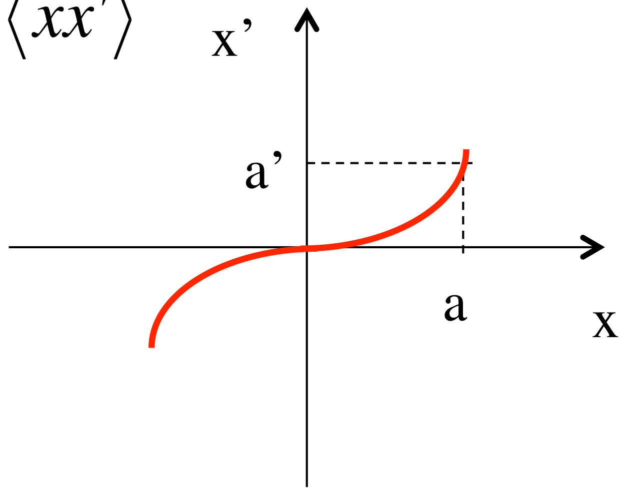
$$\sigma_{xx'} = \langle xx' \rangle = -\alpha\varepsilon_{rms} = 0\,?$$

$$\varepsilon_{rms} = \sigma_x \sigma_{x'}$$

What does rms emittance tell us about phase space distributions under linear or non-linear forces acting on the beam?

$$\varepsilon_{rms}^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2$$





Assuming a generic x, x' correlation of the type: $x' = Cx^n$

$$\varepsilon_{rms}^2 = C^2 \left(\langle x^2 \rangle \langle x^{2n} \rangle - \langle x^{n+1} \rangle^2 \right)$$

When $n = 1$ ==> $\varepsilon_{rms} = 0$

When $n \neq 1$ ==> $\varepsilon_{rms} \neq 0$

Constant under linear transformation only

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right] = 2\langle xx'\rangle\langle x'^2\rangle + 2\langle x^2\rangle\langle x'\rangle\langle x''\rangle - 2\langle xx''\rangle\langle xx'\rangle = 0$$

For linear transformations, $x'' = -k_x^2 x$, and the right-hand side of the equation is

$$2k_x^2\langle x^2\rangle\langle xx'\rangle - 2\langle x^2\rangle\langle xx'\rangle k_x^2 = 0,$$

so

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right] = 0$$

And without acceleration:

$$x' = \frac{p_x}{p_z}$$

Normalized rms emittance: $\varepsilon_{n,rms}$

Canonical transverse momentum: $p_x = p_z x' = m_o c\beta\gamma x'$ $p_z \approx p$

$$\varepsilon_{n,rms} = \frac{1}{m_o c}\sqrt{\sigma_x^2\sigma_{p_x}^2 - \sigma_{xp_x}^2} = \frac{1}{m_o c}\sqrt{\left(\langle x^2\rangle\langle p_x^2\rangle - \langle xp_x\rangle^2\right)}$$

Liouville theorem: the density of particles *n*, or the volume V occupied by a given number of particles in phase space (x,p_x,y,p_y,z,p_z) remains invariant under conservative forces.

$$\frac{dn}{dt} = 0$$

It hold also in the projected phase spaces (x,p_x),(y,p_y)(,z,p_z) provided that there are no couplings.

But rms emittance is not Liouvillian!

Limit of single particle emittance

Limits are set by Quantum Mechanics on the knowledge of the two conjugate variables (x,p_x). According to Heisenberg:

$$\sigma_x \sigma_{p_x} \geq \frac{\hbar}{2}$$

This limitation can be expressed by saying that the state of a particle is not exactly represented by a point, but by a small uncertainty volume of the order of $\hbar^3$ in the 6D phase space.

In particular for a single electron in 2D phase space it holds:

$$\varepsilon_{n,rms} = \frac{1}{m_o c}\sqrt{\sigma_x^2 \sigma_{p_x}^2 - \sigma_{xp_x}^2} \Rightarrow \begin{cases} = 0 & \text{classical limit} \\ \geq \dfrac{1}{2}\dfrac{\hbar}{m_o c} = \dfrac{\lambda\!\!\!^{-}_c}{2} = 1.9 \times 10^{-13}\, m & \text{quantum limit} \end{cases}$$

Where $\lambda\!\!\!^{-}_c$ is the reduced Compton wavelength.

Normalized and un-normalized emittances

$$p_x = p_z x' = m_o c \beta\gamma x'$$

$$\varepsilon_{n,rms} = \frac{1}{m_o c}\sqrt{\left(\langle x^2\rangle\langle p_x^2\rangle - \langle x p_x\rangle^2\right)} = \sqrt{\left(\langle x^2\rangle\langle(\beta\gamma x')^2\rangle - \langle x\beta\gamma x'\rangle^2\right)} = \langle\beta\gamma\rangle\varepsilon_{rms}$$

Assuming **small energy** spread within the beam, the normalized and un-normalized emittances can be related by the above approximated relation.

This approximation that is often used in conventional accelerators may be strongly misleading when adopted to describe beams with significant energy spread, as the one at present produced by plasma accelerators.

When the correlations between the energy and transverse positions are negligible (as in a drift without collective effects) we can write:

$$\varepsilon_{n,rms}^2 = \langle \beta^2\gamma^2 \rangle \langle x^2 \rangle \langle x'^2 \rangle - \langle \beta\gamma \rangle^2 \langle xx' \rangle^2$$

Considering now the definition of relative energy spread:

$$\sigma_\gamma^2 = \frac{\langle \beta^2\gamma^2 \rangle - \langle \beta\gamma \rangle^2}{\langle \beta\gamma \rangle^2}$$

which can be inserted in the emittance definition to give:

$$\varepsilon_{n,rms}^2 = \langle \beta^2\gamma^2 \rangle \sigma_\gamma^2 \langle x^2 \rangle \langle x'^2 \rangle + \langle \beta\gamma \rangle^2 \left(\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 \right)$$

Assuming relativistic particles (β=1) we get:

$$\varepsilon_{n,rms}^2 = \langle \gamma^2 \rangle \left(\sigma_\gamma^2 \sigma_x^2 \sigma_{x'}^2 + \varepsilon_{rms}^2 \right)$$

$$\varepsilon_{n,rms}^2 = \langle \gamma^2 \rangle \left(\sigma_\gamma^2 \sigma_x^2 \sigma_{x'}^2 + \varepsilon_{rms}^2 \right)$$

Geometric emittance

At the plasma-vacuum interface is of the same order of magnitude as for conventional accelerators at low energies; however, due to the rapid increase of the bunch size, it becomes predominant compared to the second term.

Migliorati et al., Phys. Rev. STAB 16, 011302 (2013)

Envelope Equation without Acceleration

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$

$$\sigma'_x = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

$$\sigma_{xx'} = \langle x x' \rangle = -\alpha \varepsilon_{rms}$$

It holds also the relation: $\gamma\beta - \alpha^2 = 1$

Substituting α, β, γ we get $\dfrac{\sigma_{x'}^2}{\varepsilon_{rms}} \dfrac{\sigma_x^2}{\varepsilon_{rms}} - \left(\dfrac{\sigma_{xx'}}{\varepsilon_{rms}} \right)^2 = 1$

We end up with the definition of rms emittance in terms of the second moments of the distribution:

$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2 \right)}$$

$$x' = \frac{p_x}{p_z}$$

Envelope Equation without Acceleration

Now take the derivatives:

$$\frac{d\sigma_x}{dz} = \frac{d}{dz}\sqrt{\langle x^2\rangle} = \frac{1}{2\sigma_x}\frac{d}{dz}\langle x^2\rangle = \frac{1}{2\sigma_x}2\langle xx'\rangle = \frac{\sigma_{xx'}}{\sigma_x}$$

$$\frac{d^2\sigma_x}{dz^2} = \frac{d}{dz}\frac{\sigma_{xx'}}{\sigma_x} = \frac{1}{\sigma_x}\frac{d\sigma_{xx'}}{dz} - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{1}{\sigma_x}\left(\langle x'^2\rangle + \langle xx'\rangle\right) - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{\sigma_{x'}^2 + \langle xx''\rangle}{\sigma_x} - \frac{\sigma_{xx'}^2}{\sigma_x^3}$$

And simplify:

$$\sigma_x'' = \frac{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2}{\sigma_x^3} + \frac{\langle xx''\rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3} + \frac{\langle xx''\rangle}{\sigma_x}$$

We obtain the rms envelope equation in which the rms emittance enters as defocusing pressure like term.

$$\sigma_x'' - \frac{\langle xx''\rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}$$

$$\frac{\varepsilon_{rms}^2}{\sigma_x^3} \approx \frac{T}{V} \approx P$$

Beam Thermodynamics

Kinetic theory of gases defines temperatures in each directions and global as:

$$k_B T_x = m\left\langle v_x^2 \right\rangle \qquad T = \frac{1}{3}\left(T_x + T_y + T_z\right) \qquad E_k = \frac{1}{2}m\left\langle v^2 \right\rangle = \frac{3}{2}k_B T$$

Definition of beam temperature in analogy:

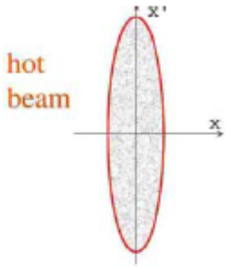
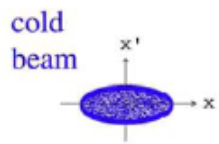
$$k_B T_{beam,x} = \gamma m_o \left\langle v_x^2 \right\rangle \qquad \left\langle v_x^2 \right\rangle = \beta^2 c^2 \left\langle x'^2 \right\rangle = \beta^2 c^2 \sigma_{x'}^2 = \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_x^2} = \beta^2 c^2 \frac{\varepsilon_{rms}}{\beta_x}$$

We get:

$$k_B T_{beam,x} = \gamma m_o \left\langle v_x^2 \right\rangle = \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_x^2} = \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}}{\beta_x}$$

$$P_{beam,x} = n k_B T_{beam,x} = n \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_x^2} = N_T \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_L \sigma_x^2}$$

$$k_B T_{beam,x} = \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}}{\beta_x}$$

Property	Hot beam	Cold beam
ion mass (m_o)	heavy ion	light ion
ion energy ($\beta\gamma$)	high energy	low energy
beam emittance (ε)	large emittance	small emittance
lattice properties ($\gamma_{x,y} \approx 1/\beta_{x,y}$)	strong focus (low β)	high β
phase space portrait		

Electron Cooling: Temperature relaxation by mixing a hot ion beam with co-moving cold (light) electron beam.

Particle Accelerators
1973, Vol. 5, pp. 61–65

EMITTANCE, ENTROPY AND INFORMATION

J. D. LAWSON
Rutherford Laboratory, Chilton, Berkshire, England

P. M. LAPOSTOLLE
Centre National d'Études des Télécommunications, Issy-les-Moulineaux, France

and

R. L. GLUCKSTERN
Department of Physics and Astronomy, University of Massachusetts, Amherst, Mass. USA

$$S = kN \log(\pi\varepsilon)$$

Lets now consider for example the simple case with $\langle xx'' \rangle = 0$ describing a beam drifting in the free space.

The envelope equation reduces to:

$$\sigma_x^3 \sigma_x'' = \varepsilon_{rms}^2$$

With initial conditions σ_o, σ_o' at z_o, depending on the upstream transport channel, the equation has a hyperbolic solution:

$$\sigma(z) = \sqrt{\left(\sigma_o + \sigma_o'(z - z_o)\right)^2 + \frac{\varepsilon_{rms}^2}{\sigma_o^2}(z - z_o)^2}$$

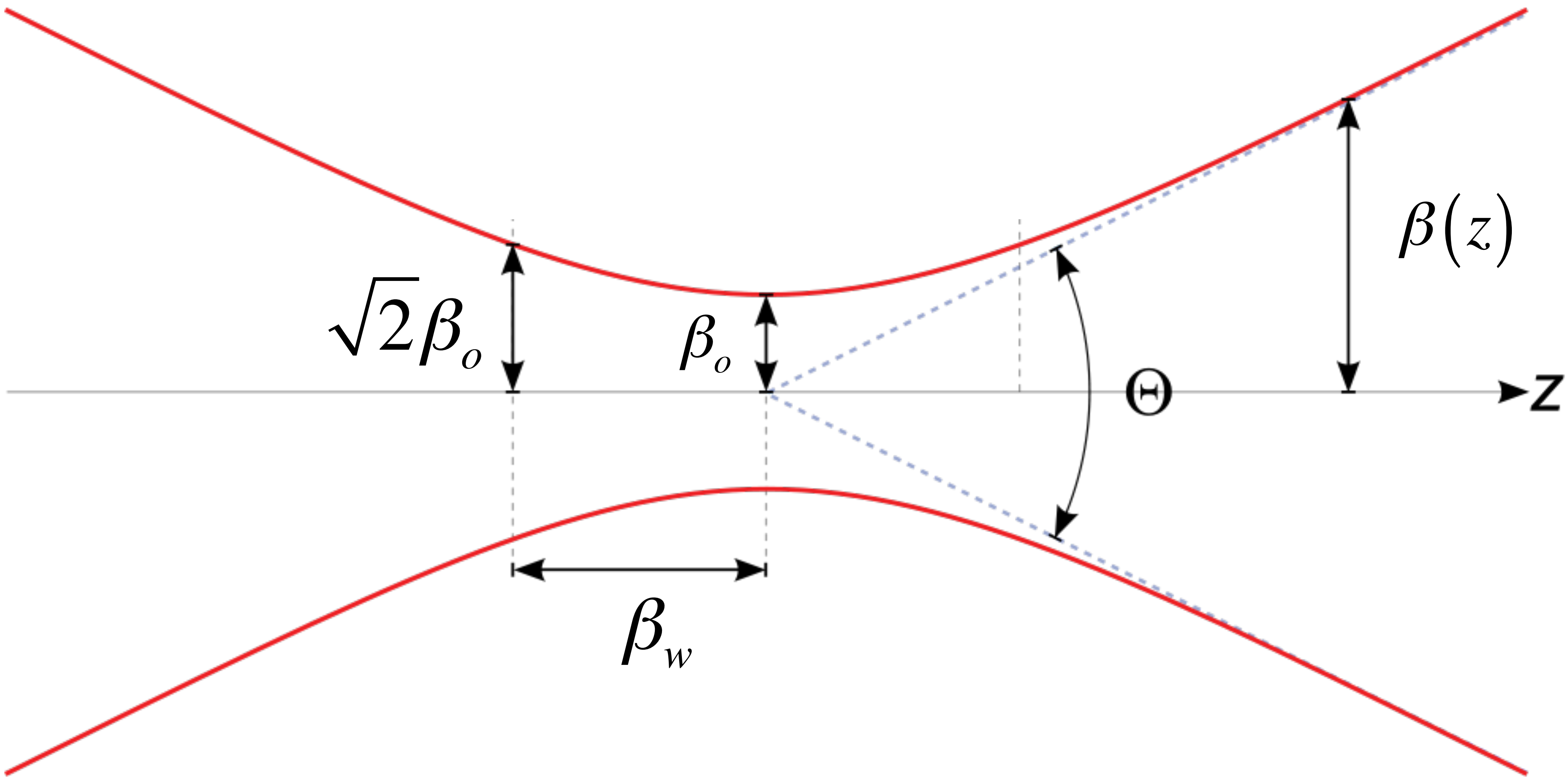
Considering the case $\sigma'_o = 0$ (beam at waist)

and using the definition $\sigma_x = \sqrt{\beta \varepsilon_{rms}}$

the solution is often written in terms of the β function as:

$$\sigma(z) = \sigma_o \sqrt{1 + \left(\frac{z - z_o}{\beta_w}\right)^2}$$

This relation indicates that without any external focusing element the beam envelope increases from the beam waist by a factor $\sqrt{2}$ with a characteristic length $\beta_w = \dfrac{\sigma_o^2}{\varepsilon_{rms}}$



For an effective transport of a beam with finite emittance is mandatory to make use of some external force providing beam confinement in the transport or accelerating line.

$$\sigma(z)=\sqrt{\left(\sigma_o+\sigma_o'(z-z_o)\right)^2+\frac{\varepsilon_{rms}^2}{\sigma_o^2}(z-z_o)^2}$$

At waist holds also the relation: $\varepsilon_{rms}^2=\sigma_{o,x}^2\sigma_{o,x'}^2$ $\sigma_o'=0$

that leads to: $\sigma_x^2(z)\approx\sigma_{o,x'}^2(z-z_o)^2$

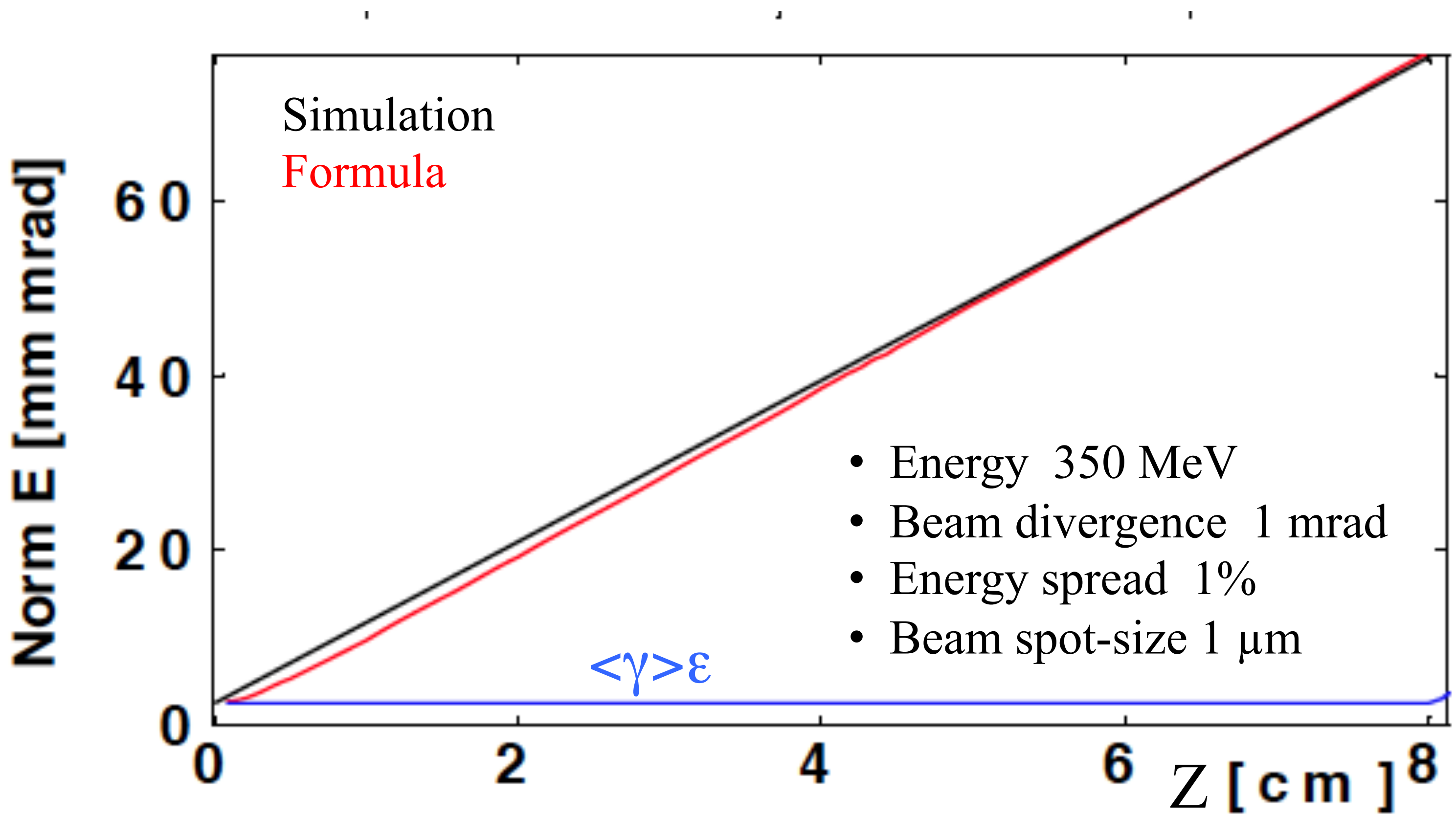
$$\varepsilon_{n,rms}^2=\langle\gamma^2\rangle\left(\sigma_\gamma^2\sigma_x^2\sigma_{x'}^2+\varepsilon_{rms}^2\right)=\langle\gamma^2\rangle\left(\sigma_\gamma^2\sigma_{o,x'}^4(z-z_o)^2+\varepsilon_{rms}^2\right)$$

showing that beams with large energy spread an divergence undergo a significant normalized emittance growth even in a drift

Migliorati et al., Phys. Rev. STAB 16, 011302 (2013)

$$\varepsilon_{n,rms}^2 = \langle\gamma^2\rangle\left(\sigma_\gamma^2\sigma_x^2\sigma_{x'}^2 + \varepsilon_{rms}^2\right) = \langle\gamma^2\rangle\left(\sigma_\gamma^2\sigma_{o,x'}^4\left(z - z_o\right)^2 + \varepsilon_{rms}^2\right)$$

showing that beams with large energy spread an divergence undergo a significant normalized emittance growth even in a drift



Envelope Equation with Linear Focusing

$$\sigma_x'' - \frac{\langle x x'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}$$

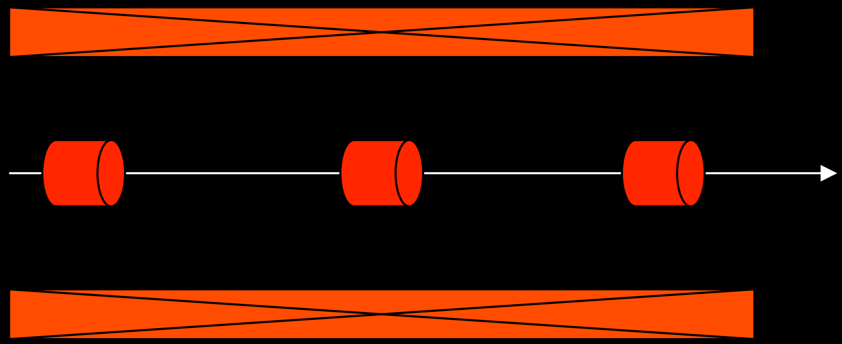
Assuming that each particle is subject only to a linear focusing force, without acceleration: $x'' + k_x^2 x = 0$

take the average over the entire particle ensemble $\langle x x'' \rangle = -k_x^2 \langle x^2 \rangle$

$$\sigma_x'' + k_x^2 \sigma_x = \frac{\varepsilon_{rms}^2}{\sigma_x^3}$$

We obtain the rms envelope equation with a linear focusing force in which, unlike in the single particle equation of motion, the rms emittance enters as defocusing pressure like term.

Matching Condition:



$$k_o = \frac{qB}{2mc\beta\gamma}$$

$$\sigma_r'' + k_o^2\sigma_r = \frac{\varepsilon_{rms}^2}{\sigma_r^3}$$

Equilibrium solution: $$\sigma_{eq} = \sqrt{\frac{\varepsilon_{rms}}{k_o}}$$ Matched beam

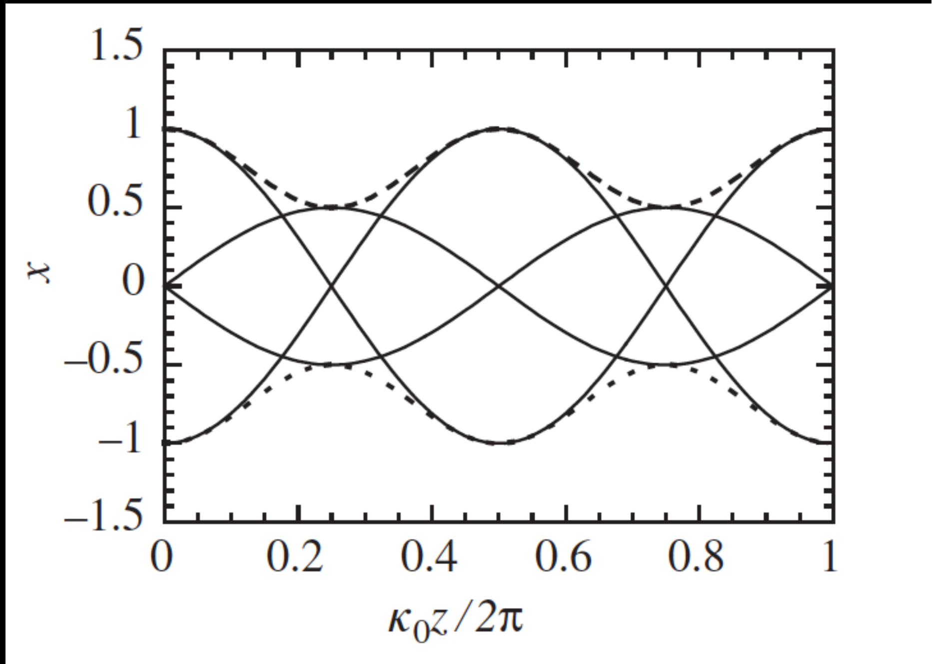
$$\sigma_r'' + k_o^2 \sigma_r = \frac{\varepsilon_{rms}^2}{\sigma_r^3}$$

Mismatched Beam

$$\sigma_{eq} = \sqrt{\frac{\varepsilon_{rms}}{k_o}}$$

Small perturbation:

$$\sigma_r = \sigma_{eq} + \delta\sigma_r$$



$$\delta\sigma_r'' + k_o^2 \delta\sigma_r = -\frac{3\varepsilon_{rms}^2}{\sigma_r^4} \cong -3k_o^2 \delta\sigma_r$$

$$\delta\sigma_r'' + 4k_o^2 \delta\sigma_r = 0$$

$$\delta\sigma_r = \delta\sigma_o \cos\left(2k_o z + \varphi_o\right)$$

the envelope frequency of oscillations about the equilibrium is twice that of the single particle betatron frequency

From Envelope to Betatron function

(27) $$\sigma_x'' + k_{ext}^2 \sigma_x = \frac{\varepsilon_{rms}^2}{\sigma_x^3}$$

By substituting $\sigma_x = \sqrt{\beta_x \varepsilon_{rms}}$ in (27) one obtains an equation for the "betatron function" $\beta_x(z)$ that is independent on the emittance term:

(28) $$\beta_x'' + 2k_{ext}^2 \beta_x = \frac{2}{\beta_x} + \frac{\beta_x'^2}{2\beta_x}$$

Equation (28) containing only the transport channel focusing strength and being independent on the beam parameters, suggests that the meaning of the betatron function is to describe the transport line characteristic by itself. The betatron function reflects exterior forces from focusing magnets and is highly dependent on the particular arrangement of quadrupole magnets. The equilibrium, or matched, solution of eq. (28) is given by $\beta_{eq} = \frac{1}{k_{ext}} = \frac{\lambda_\beta}{2\pi}$ as one can easily verify. This result shows that the matched β_x function is simply the inverse of the focusing wave number, or equivalently is proportional to the "betatron wavelength" λ_β

Envelope Equation with Acceleration

$$\frac{dp_x}{dt} = \frac{d}{dt}(px') = \beta c \frac{d}{dz}(px') = 0$$

$$p = \beta\gamma m_o c$$

$$x'' + \frac{p'}{p}x' = 0$$

$$x'' = -\frac{(\beta\gamma)'}{\beta\gamma}x'$$

$$\sigma_x'' = \frac{\varepsilon_{rms}^2}{\sigma_x^3} + \frac{\langle xx''\rangle}{\sigma_x}$$

$$\langle xx''\rangle = -\frac{(\beta\gamma)'}{\beta\gamma}\langle xx'\rangle = -\frac{(\beta\gamma)'}{\beta\gamma}\sigma_{xx'} = -\frac{(\beta\gamma)'}{\beta\gamma}\sigma_x\sigma_x'$$

Space Charge De-focusing Force

$$\sigma_x'' + \frac{(\beta\gamma)'}{\beta\gamma}\sigma_x' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}$$

Adiabatic Damping

Emittance Pressure

Other External Focusing Forces

$$\varepsilon_n = \beta\gamma\varepsilon_{rms}$$

References:

[1] T. Shintake, Proc. of the 22nd Particle Accelerator Conference, June 25-29, 2007, Albuquerque, NM (IEEE, New York, 2007), p. 89.

[2] L. Serafini, J. B. Rosenzweig, PR E55 (1997) 7565

[3] M. Reiser, “Theory and Design of Charged Particle Beams” , Wiley, New York, 1994

[4] J. B. Rosenzweig, “Fundamentals of beam physics”, Oxford University Press, New York, 2003

[5] T. Wangler, “Principles of RF linear accelerators”, Wiley, New York, 1998

[6] S. Humphries, “Charged particle beams”, Wiley, New York, 2002

[7] F. J. Sacherer, F. J., IEEE Trans. Nucl. Sci. NS-18, 1105 (1971).

[8] M. Ferrario et al., Int. Journal of Modern Physics A, Vol 22, No. 23, 4214 (2007)

[9] J. Buon, “Beam phase space and emittance”, in CERN 94-01