LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

Centre for Advanced Laser Applications

CALA

LASER-DRIVEN WAKEFIELDS

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PART I: WAKEFIELD GENERATION AND DETECTION

Laser pulses

An electromagnetic wave has to fulfil Maxwell's equations

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times B = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j}$$

It is convenient to introduce a scalar (Φ) and vector (A) potential, which allows the fields to be expressed as:

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

(check by plugging electric field into Maxwell Eqn. and using ∇×∇Φ = 0)

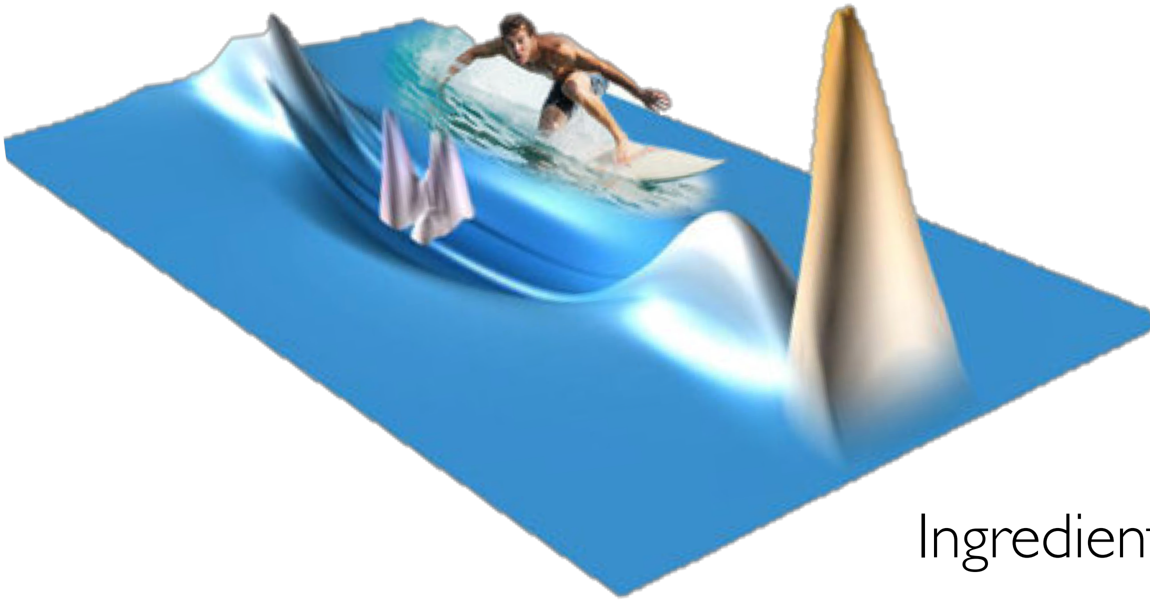
$$\vec{\nabla} \times \left(\vec{\nabla}\Phi + \frac{\partial \vec{A}}{\partial t} \right) = \frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \frac{\partial \vec{A}}{\partial t} \qquad \Rightarrow \ \vec{B} = \vec{\nabla} \times \vec{A}$$

A plane wave ansatz for A (Φ =0 *in vacuo*) yields the fields:

$$\vec{A} = A_0 \, \mathrm{e}^{i\left(\omega t - \vec{k}\vec{x}\right)}$$

$$\Rightarrow \vec{E} = i\omega \vec{A}_0 \, \mathrm{e}^{i\left(\omega t - \vec{k}\vec{x}\right)} \qquad \vec{B} = i\vec{k}\vec{A}_0 \, \mathrm{e}^{i\left(\omega t - \vec{k}\vec{x}\right)} \cdot \vec{e}_{\perp k, A} \Rightarrow \left|B\right| = \frac{\left|k\right|}{\omega}\left|E\right| = \frac{\left|E\right|}{c}$$

Laser-driven wakefields



Ingredients:

- intense laser pulse
- ponderomotive force
- wake generation
- surfing electrons

Solving the rel. equation of motion for an electron in an e-m wave $d\vec{p}/dt = e\left(E + \vec{v}\times\vec{B}\right)$

- monochromatic plane wave:
  E.S. Sarachik, G.T. Schappert; Phys. Rev. D 1, 2738 (1970)

$$x = \frac{ca_0}{\omega}\left(\cos\Phi - 1\right)$$

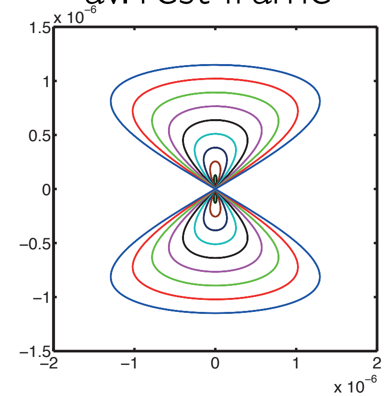
$$z = \frac{ca_0^2}{2\omega}\left(\frac{\Phi}{2} - \frac{1}{4}\sin 2\Phi\right)$$

$$\tilde{p}_x = a_0 \sin\Phi$$

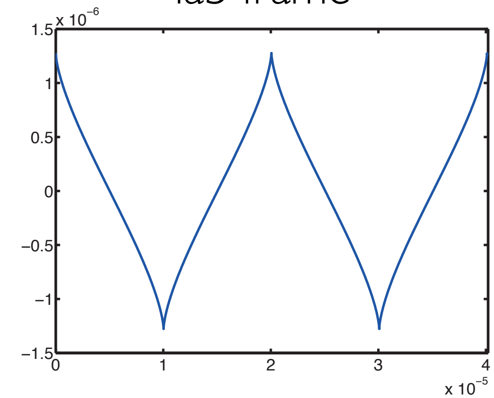
$$\tilde{p}_z = -\frac{1}{2}a_0^2\sin^2\Phi$$

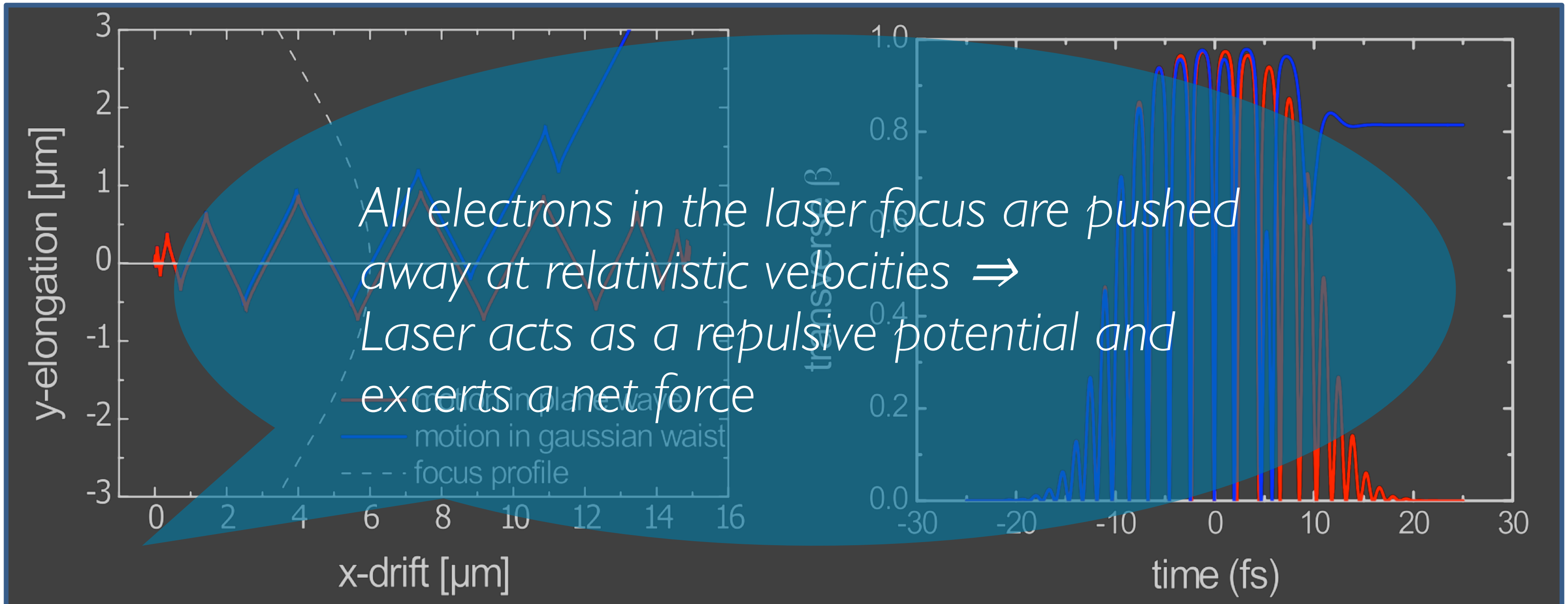
- pulsed plane wave and laser focus (numerical)

av. rest frame



lab frame





All electrons in the laser focus are pushed away at relativistic velocities ⇒ Laser acts as a repulsive potential and excerts a net force

Ponderomotive force

While the direct derivation of the relativistic ponderomotive force is quite involving, we can use identification of the ponderomotive potential as the mean kinetic energy of the quivering electrons as a short-cut:

$$\bar{E}_{kin} = \Phi_{pond} = -m_e c^2 \left\langle \gamma - 1 \right\rangle \overset{\gamma=\sqrt{1+\frac{a_0^2}{2}}}{\propto} \sqrt{I}$$

This yields the relativistic ponderomotive force as:

$$\vec{F}_{pond} = -m_e c^2 \nabla \left\langle \gamma \right\rangle = -\frac{m_e c^2}{e} \nabla \sqrt{\frac{a_0^2}{2}}$$

	non-relativistic	relativistic
F_{pond}	$-\frac{e^2}{4m_e\omega_L^2}\nabla(E_L^2)$	$-\frac{mc^2}{e}\nabla\sqrt{a_0^2/2}$
Φ_{pond}	$\frac{e^2}{4m_e\omega_L^2}E_L^2$	$\frac{mc^2}{e}\langle\gamma-1\rangle$
proportionality	$I, \nabla I$	$\sqrt{I}, \nabla\sqrt{I}$

The **ponderomotive force** pushes electrons aside, causing the laser pulse to **snowplow** through the plasma and excite a plasma wave in its wake.

The **excitation** (which determines the **phase velocity** of the plasma wave) propagates with the **laser pulse group velocity**:

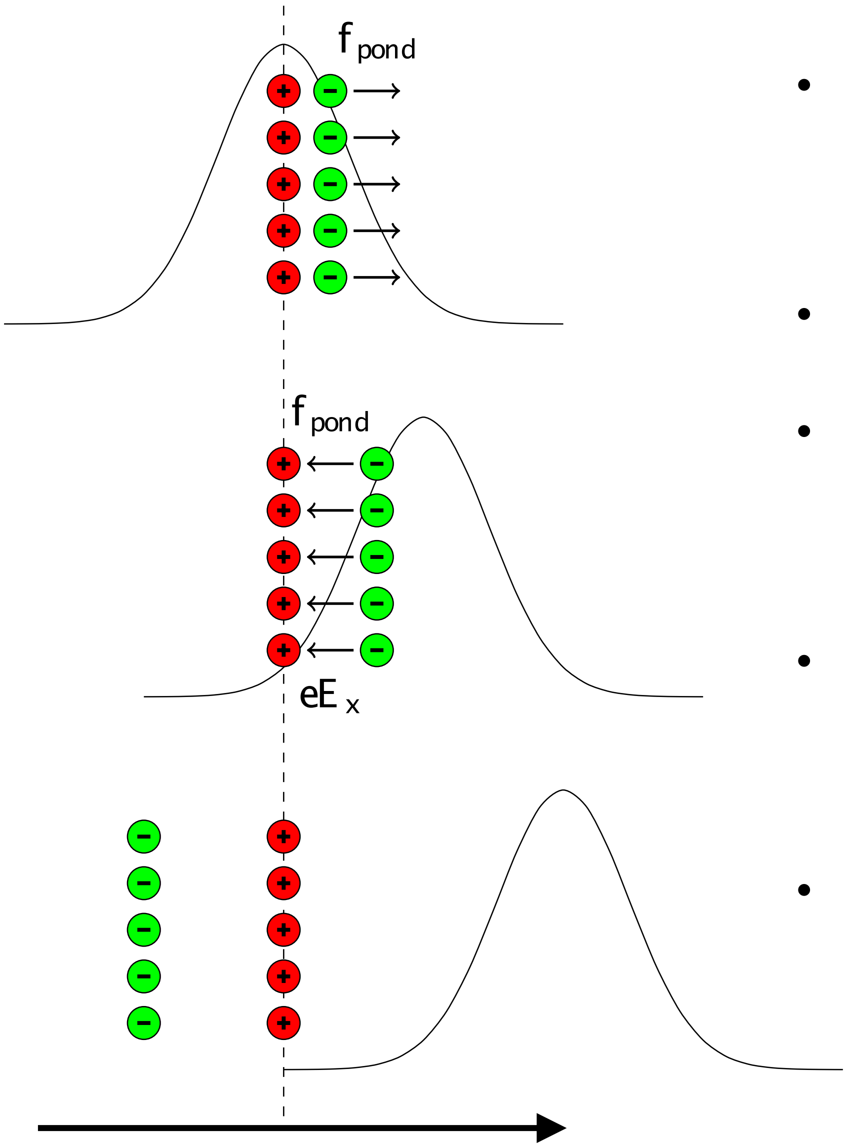
$$v_{gr,\,laser} = \frac{\partial \omega_l}{\partial k} = c\sqrt{1-\left(\frac{\omega_p}{\omega_p}\right)^2} = c\sqrt{1-\frac{n_e}{n_{cr}}} = v_{ph,wake}$$

We see that $\omega_l > \omega_p$ has to be fulfilled for a non-evanescent wave

Typical density range for LWFA: $n_e = n_{cr}/1000$

$\Rightarrow$ $v_{ph,wake} = 0.9995\ c$, $\lambda_p = 10\text{-}30\mu m$

Double ponderomotive push



- Two kicks by the ponderomotive force, corresponding to the rising and the falling edge of the laser pulse.
- Optimum pulse duration τ_{FWHM}=0.37 λ_p/c.
- Wake excitation is dominated by the rising edge kick due to longer interaction between co-moving electrons and driver.
- Resulting charge separation separation causes electric fields to exhibit a strong longitudinal component.
- The wave structure travels with v_{ph} = cη, and hence can constantly accelerate a co-moving electron.

Wake generation

Since in plasma the laser pulse interacts with many particles at the same time, it is impractical to treat each particle individually. Instead, the motion of electrons driven by an electromagnetic wave an a plasma can be derived from a set of fluid equations:

rel. Lorentz force
$$\frac{d\vec{p}}{dt} = \left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\vec{p} = -e\left[\vec{E} + \vec{v}\times\vec{B}\right]$$

continuity equation
$$0 = \frac{\partial n_e}{\partial t} + \vec{\nabla}\left(n_e\vec{v}\right)$$

Poisson equation
$$\vec{\nabla}^2\Phi = -\frac{\rho}{\varepsilon_0} = e\frac{\delta n_e}{\varepsilon_0}$$

$p = m\vec{v},$ momentum

$\delta n_e = n_e - n_{e,0},$ density modulation

$\vec{j} = -en_e\vec{v},$ current density

$\rho = -e\delta n_e$ charge density

For further derivations, we will also transform to a co-moving frame with speed v_{gr}

$$\xi = z - v_{gr}t, \qquad \tau = t, \qquad \frac{\partial}{\partial z} = \frac{\partial\xi}{\partial z}\frac{\partial}{\partial\xi} = \frac{\partial}{\partial\xi}, \qquad \frac{\partial}{\partial t} = \frac{\partial\xi}{\partial t}\frac{\partial}{\partial\xi} = -v_{gr}\frac{\partial}{\partial\xi} \overset{v_{gr}\to c}{\approx} -c\frac{\partial}{\partial\xi}$$

Linear wakefields

For small laser intensities ($a_0 << 1$), the plasma density is only weakly perturbed $\delta n_e \ll n_{e,0}$ and the continuity equation can be written as:

$$\frac{\partial \delta n_e}{\partial t} + n_{e,0} \nabla \vec{v} = 0$$

The above expression and Poisson's equation can be now inserted into the derivative of the Lorentz force. Keeping in mind $\nabla A = 0$ (Coulomb gauge) and $\mathbf{p} = m_e \mathbf{v}$ yields for initially resting electrons at low intensities, i.e., $\gamma = 1 + a^2/2$:

$$\left(\frac{\partial^2}{\partial t^2} + \omega_p^2 \right) \frac{\delta n_e}{n_{e,0}} = c^2 \nabla^2 \frac{a^2}{2}$$

The RHS represents the driving term of a forced oscilllator, and is proportional to the ponderomotive force $F_{pond} = m_e c^2 \nabla^2 a^2/2$. With Poisson's equation we express the charge imbalance with the scalar wake potential in the moving frame coordinates ($\xi = z - v_g t$, $\tau = t$)

$$\left(\frac{\partial^2}{\partial \xi^2} + k_p^2 \right) \phi = k_p^2 \frac{a^2}{2}$$

Assuming a radial symmetry, an analytical solution of the inhomogeneous wave equation can be found in 3D. It is given by

$$\phi(r,\xi) = -\frac{k_p}{4} \int_{\xi}^{\infty} a^2(r,\xi') \sin\left(k_p(\xi - \xi')\right) d\xi'$$

Linear wakefields II

For a Gaussian laser envelope a = $a_0 \exp(-\xi^2/(c\tau_0)^2)\exp(-r^2/w_0^2)$, the solution of the integral for $\xi \rightarrow -\infty$, i.e. after the laser transit is given by:

$$\phi(r,\xi) = -a_0^2\sqrt{\frac{\pi}{2}}\frac{k_p}{4}c\tau_0 e^{-\left(2r^2/w_0^2\right)} e^{-\left(k_p c\tau_0\right)^2/8}\sin\left(k_p\xi\right)$$

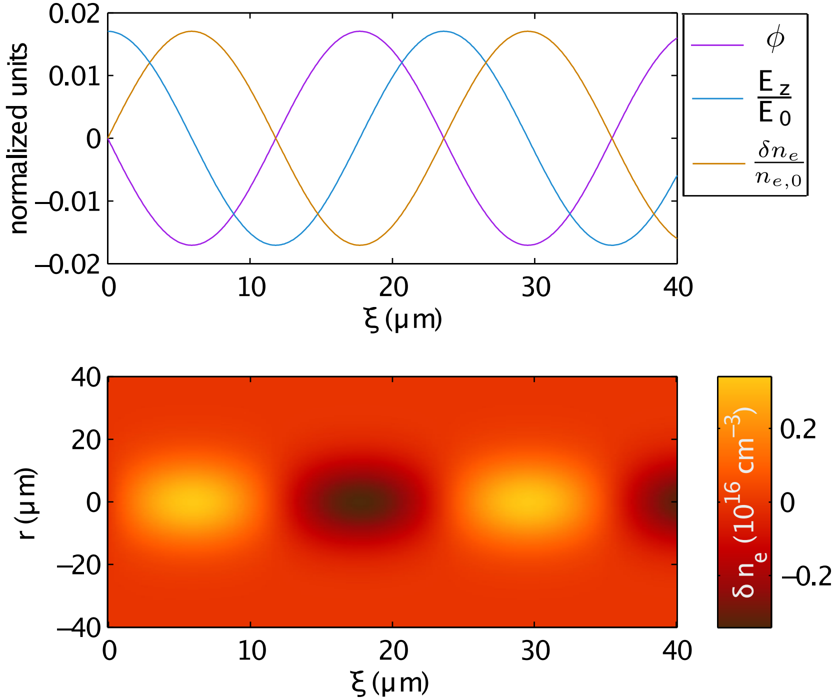
From this scalar potential ϕ the electric field and the electron density can be derived as:

$$\frac{E_z}{E_{p,0}} = -\frac{1}{k_p}\frac{\partial\phi}{\partial\xi}, \qquad \frac{E_r}{E_{p,0}} = -\frac{1}{k_p}\frac{\partial\phi}{\partial r}, \qquad \frac{\delta n_e}{n_{e,0}} = -\frac{1}{k_p^2}\frac{\partial^2\phi}{\partial\xi^2},$$

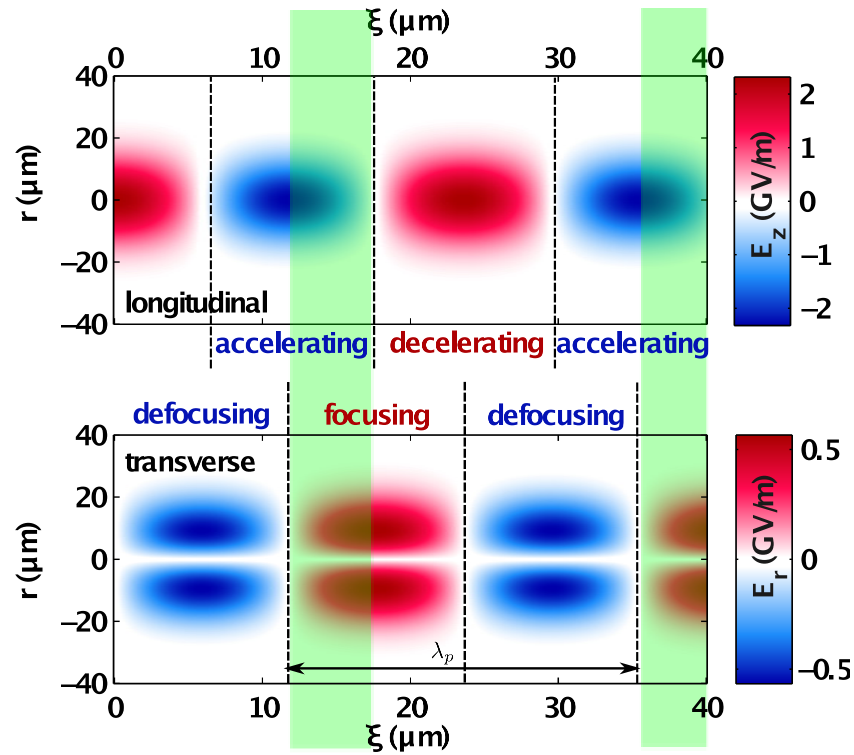
$E_{p,0}$ corresponds to the maximal electric field of the plasma wave in the linear regime, known as the cold fluid wavebreaking field:

$$E_{p,0} = \frac{m_e c\omega_p}{e}, \qquad E_{p,0}\left[\mathrm{GV/m}\right] = 96\sqrt{n_{e,0}\left[10^{18}\,\mathrm{cm}^{-3}\right]}$$

Linear wakefields III



Top: Normalized plasma potential ϕ, longitudinal electric field E_z/E_0 and density perturbation $\delta n_e/n_{e,0}$ on axis ($r = 0$). Bottom: color coded plasma density perturbation $\delta n_e(r,\xi)/n_{e,0}$ generated by the ponderomotive force in the vicinity of a Gaussian laser focus.



top: Spatial extent of the longitudinal E_z (r , ξ) and botton: the radial electric field E_r (r , ξ). The green area marks a λ_p / 4-phase region of the wakefield with an accelerating and transverse focusing field.

3D linear wakefield quantities in the co-moving frame created by a laser pulse with $a_0 = 0.2$, $t_{FWHM} = 28$fs and $d_{FWHM} = 22\mu$m in a plasma density of 2×10^{18}cm^{-3}

Nonlinear regime

No analytic 3-D solution exists for high intensities (a0 > 1), since $(\delta n_e/n_{e,0}) \sim 1$ and the response of the plasma becomes highly nonlinear. However, a solution can be found for the 1D nonlinear regime (Esarey et al., 1997a; Sprangle et al., 1990).

To proceed, we introduce the normalized quantities:

$$\vec{\beta} = \frac{\vec{v}}{c}, \quad \vec{a}_0 = \frac{e\vec{A}}{m_0 c}, \quad \phi = \frac{e\Phi}{m_e c^2}, \quad \gamma = \frac{E}{m_e c^2}, \quad \vec{u} = \frac{\vec{p}}{m_e c}$$

It is convenient to separate the plasma electron motion into parallel and perpendicular components relative to the laser propagation direction. The perpendicular equation of motion reads:

$$\frac{d\vec{p}_\perp}{dt} = e\left(\vec{E}_\perp + v_z \times \vec{B}_\perp\right) = e\frac{d\vec{A}_\perp}{dt}$$

The longitudinal motion can be found by subtracting the longitudinal components of

$$\underbrace{\frac{d\vec{p}}{dt} = e\left(\vec{E} + v_e \times \vec{B}\right)}_{\text{Lorentz force}}, \qquad \underbrace{\frac{d}{dt}\left(\gamma m c^2\right) = -e\left(\vec{v} \cdot \vec{E}\right)}_{\text{energy equation}}$$

Nonlinear regime II

and keeping in mind that the wake is a function of (t-z):

$$\frac{dE}{dt} - c\frac{d\vec{p}_{\parallel}}{dt} = 0$$

Integration for zero initial velocty yields:

$$\vec{p}_{\perp} = e\vec{A}_{\perp} \quad \Leftrightarrow \quad \vec{u}_{\perp} = \gamma\,\vec{\beta}_{\perp} = \vec{a}$$

$$E - cp_{\parallel} = m_e c^2 \quad \Leftrightarrow \quad \gamma - 1 = u_{\parallel}$$

Likewise, it is convenient to split the relativistic gamma factor $\gamma = \sqrt{1+u_{\parallel}^2+u_{\perp}^2} = 1\big/\sqrt{1-\beta^2}$ into transverse and longitudinal parts and express them in terms of *a:*

$$\gamma^2 = 1 + u_{\perp}^2 + (\gamma - 1)^2 \quad \Rightarrow \quad \gamma = 1 + a^2/2$$

$$\gamma^2 = \frac{1+\gamma^2\beta_{\perp}^2}{1-\beta_{\parallel}^2} \equiv \gamma_{\perp}^2\gamma_{\parallel}^2 \quad \Rightarrow \quad \gamma_{\perp} = \sqrt{1+a^2}$$

Nonlinear Regime III

Express the Lorentz force in terms of the potential-dependent E and B-fields:

$$\left(\frac{\partial}{\partial t}+\vec{v}\cdot\vec{\nabla}\right)\vec{p}=e\left[\frac{\partial}{\partial t}\vec{A}+\vec{\nabla}\Phi-\vec{v}\times\vec{\nabla}\times\vec{A}\right]$$

With $\vec{\nabla}\vec{p}^2=2\left[\left(\vec{p}\cdot\vec{\nabla}\right)\vec{p}+\vec{p}\times\left(\vec{\nabla}\times\vec{p}\right)\right]\overset{p=m_e\gamma c}{\overbrace{=}}m_e^2c^2 2\gamma\,\vec{\nabla}\gamma$ we obtain:

$$m_ec^2\;\vec{\nabla}\gamma=\left(\vec{v}\cdot\vec{\nabla}\right)\vec{p}+\vec{v}\times\left(\vec{\nabla}\times\vec{p}\right)$$

$\vec{p}=e\vec{A}$ (only ponderomotive driver) yields:

$$\frac{\partial\vec{p}}{\partial t}=e\vec{\nabla}\Phi+e\frac{\partial}{\partial t}\vec{A}-m_ec^2\vec{\nabla}\gamma$$

The longitudinal derivative dA/dt (dA/dx) can be neglected (quasistatic approximation) and the equation of motion is simplified in the laboratory and co-moving frame, respectively:

$$\frac{1}{c}\frac{\partial u_z}{\partial t}=\frac{\partial}{\partial z}\left(\phi-\gamma\right),\qquad\qquad \frac{1}{c}\frac{\partial u_z}{\partial\tau}=\frac{\partial}{\partial\xi}\left(\phi-\gamma\left(1-\beta_p\beta_z\right)\right)$$

where $\beta_p = v_g / c$ the normalized plasma wave velocity.

Nonlinear regime IV

Analogous, the continuity equation and Poisson's equation can be written in the co-moving frame:

$$\frac{\partial}{\partial\tau}\frac{n_e}{n_{e,0}} = c\frac{\partial}{\partial\xi}\left(\frac{n_e}{n_{e,0}}\left(\beta_p - \beta_z\right)\right), \qquad \frac{\partial^2\phi}{\partial\xi^2} = k_p^2\left(\frac{n_e}{n_{e,0}} - 1\right)$$

Applying the quasistatic approximation allows to neglect the partial derivative **∂/∂**τ relative to **∂/∂**ξ. All three equations can be integrated by finding the integration constant in the absence of the plasma wave, i.e., before the arrival of the pulse ξ ⟶ -**∞**

$$\frac{n_e}{n_{e,0}}\left(\beta_p - \beta_z\right) = const. \quad \overset{n(\xi\to-\infty)=1}{\Longrightarrow} \quad \frac{n_e}{n_{e,0}} = \frac{\beta_p}{\beta_p - \beta_z}$$

$$\phi - \gamma\left(1 - \beta_p\beta_z\right) = const. \quad \underset{\beta_z(\xi\to-\infty)=0}{\overset{\phi=0,\gamma=1}{\Longrightarrow}} \quad \phi + 1 = \gamma\left(1 - \beta_p\beta_z\right)$$

Nonlinear regime V

Rearranging these expressions to the following explicit form (Gibbon, 2005)

$$\gamma = \gamma_p^2 (1+\phi)(1-\beta_p \Psi), \qquad \beta_z = \frac{\beta_p - \Psi}{1-\beta_p \Psi}, \qquad \Psi = \left(1 - \frac{1+a^2}{\gamma_p^2 (1+\phi)^2}\right)^{1/2}$$

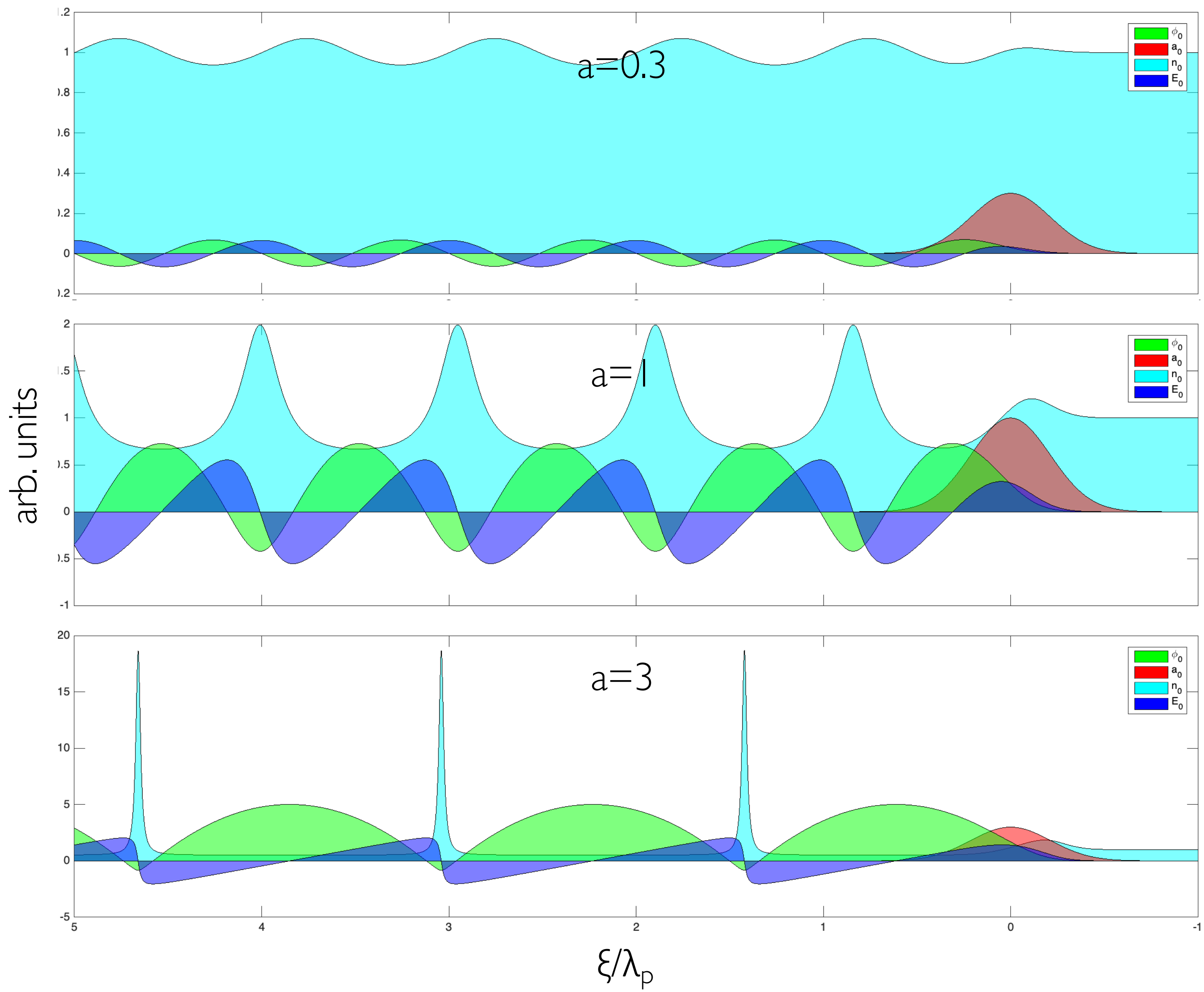
allows to eliminate β_z which results in:

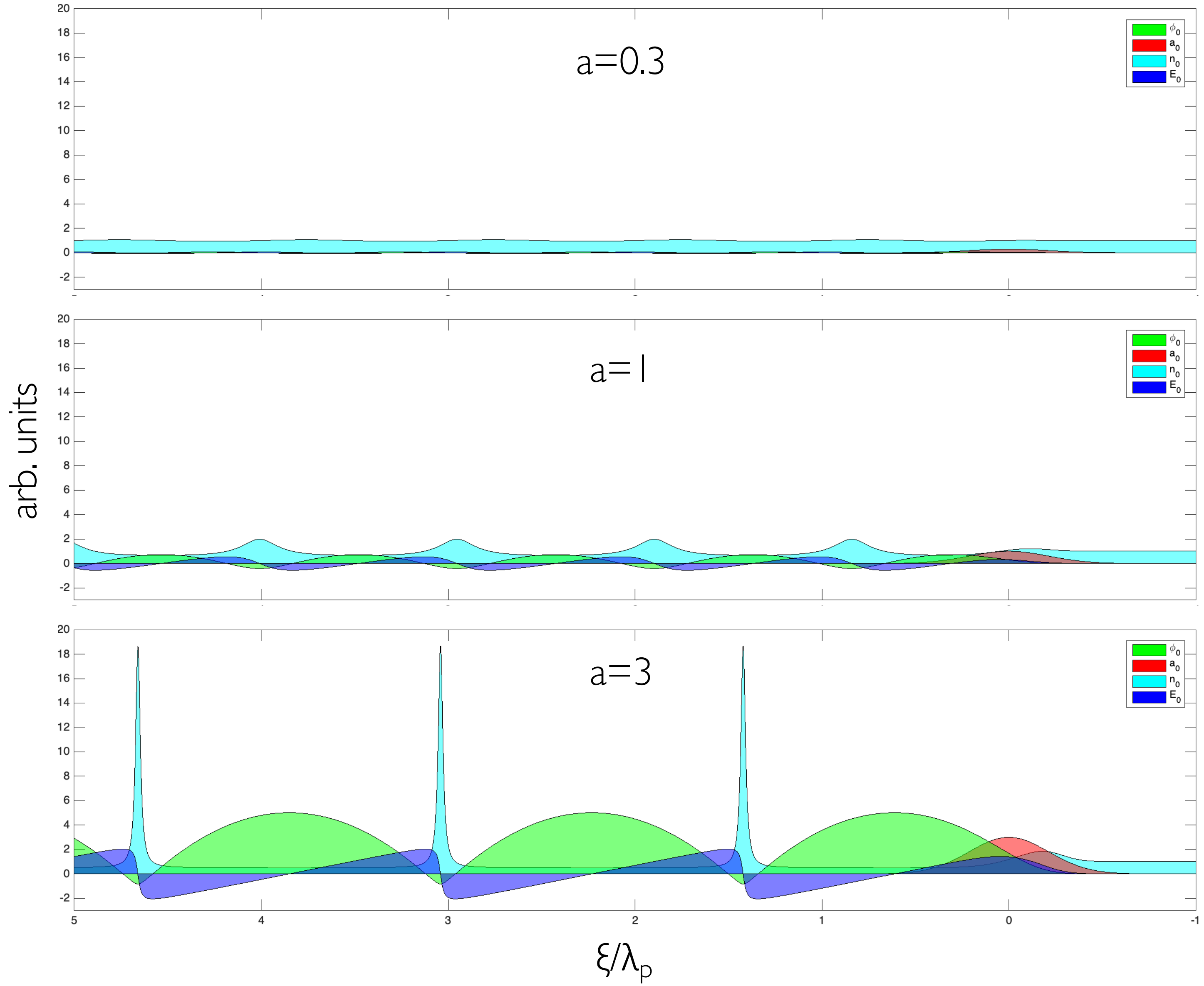
$$\frac{n_e}{n_{e,0}} = \gamma_p^2 \beta_p \left(\frac{1}{\Psi} - \beta_p\right)$$

Inserting the above expression into Poisson's equation yields the wake potential in the co-moving coordinates for arbitrary pump strength:

$$\frac{\partial^2 \phi}{\partial \xi^2} = k_p^2 \gamma_p^2 \left(\beta_p \left(1 - \frac{1+a^2(\xi)}{\gamma_p^2 (1+\phi)^2}\right)^{-1/2} - 1 \right)$$

This non-linear ordinary differential equation can now be solved for any desired laser pulse shapes a(ξ) numerically


a=0.3
a=1
a=3
arb. units
ξ/λp
φ0
a0
n0
E0
5
4
3
2
1
0
-1
20
15
10
5
0
-5


a=0.3
a=1
a=3
ϕ_0
a_0
n_0
E_0
arb. units
20
18
16
14
12
10
8
6
4
2
0
-2
5
4
3
2
1
0
-1
ξ/λ_p

WAKEFIELD DIAGNOSTICS

How can we detect and characterize the wakefield?

for a comprehensive review, consult:
M.C. Downer et al, Reviews of Modern Physics 90 035002 (2018)

Imaging methods

- Shadowgraphy
- Multi-plane shadowgraphy
- Electron radiography

require few-fs probe pulse

Frequency domain methods

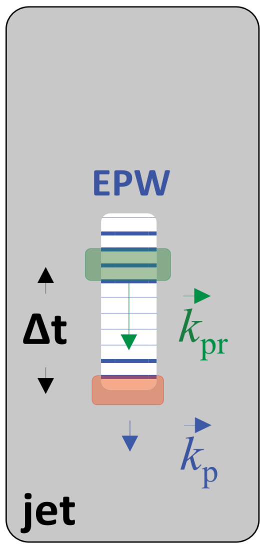
- Forward Thomson scattering
- Frequency-domain holography
- electron witness bunches

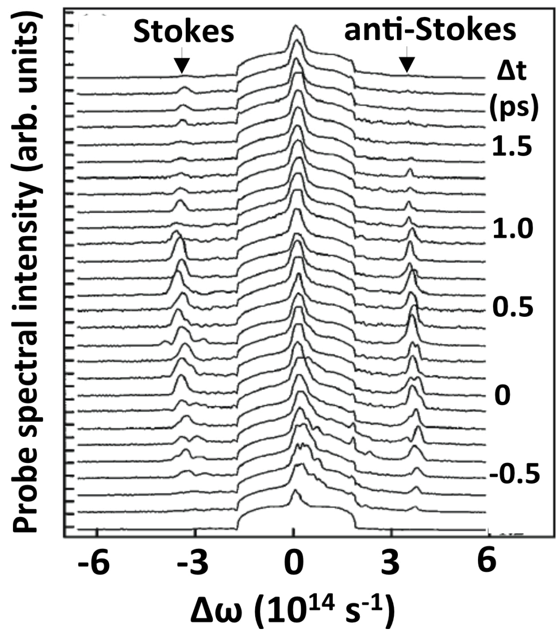
typically require reference pulse

Collective Thomson (probe pulse) forward scattering

The photon momentum of co-propagating probe pulse ($\tau_{probe} >> \lambda_p/c$) is shifted by the average wake electron momentum, leading to the creation of Thomson side-bands. Alternatively, the probe pulse is modulated by the average wake density perturbation (Raman scattering).

$$\omega_{sc} = \omega_{probe} \pm \omega_p$$

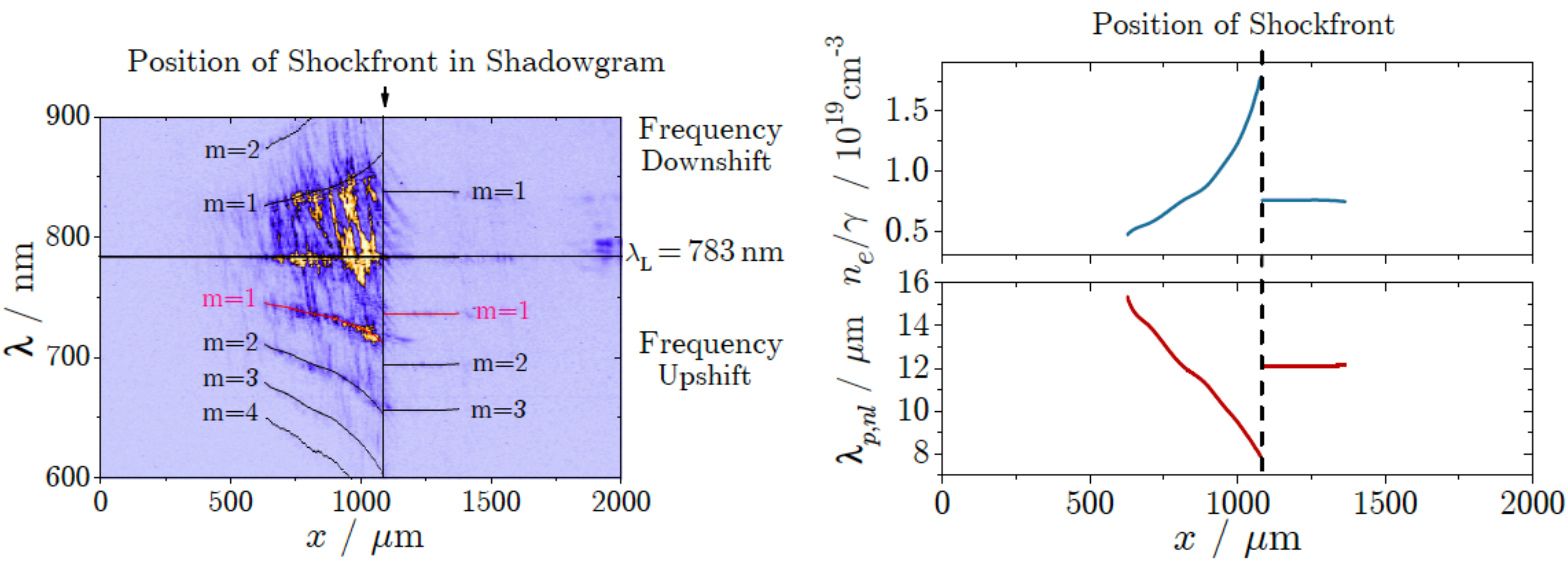




→ presence of wake, average wavelength

Le Blanc, S. P. et al, Phys. Rev. Lett. 77, 5381 (1996)
M.C. Downer et al, Reviews of Modern Physics 90 035002 (2018)

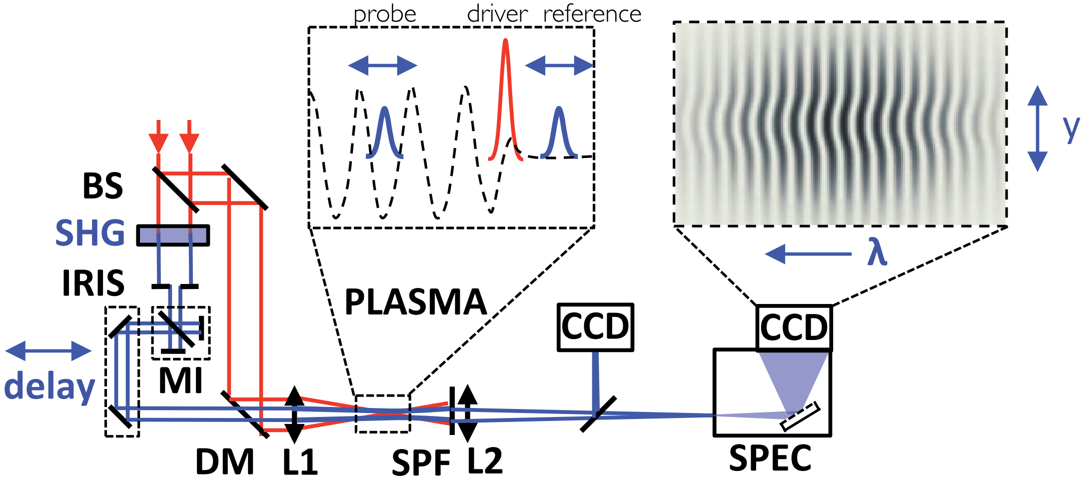
Raman (pump pulse) side scattering



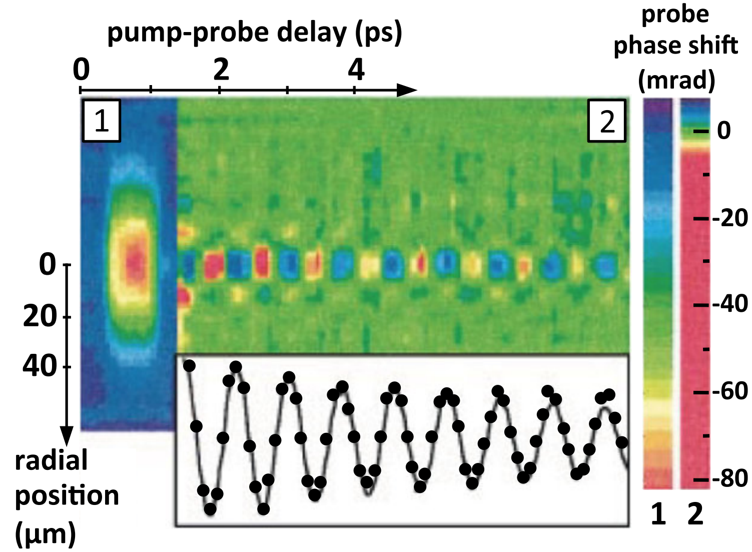
→Longitudinally resolved plasma wavelength

Maria Reuter, Dissertation, University of Jena, 2018

Frequency domain interferometry



Radially resolved interferograms yield 2-D information:



$$\tau_{probe} < \omega_p^{-1}:$$

$$\Delta\phi_{probe}(r,\Delta t) = (\omega_{probe}/c)\int_0^L \eta(r,\Delta t,z)\,dz$$

$$E_{probe}(t-T) = E_{0,pr}(t-T)\,e^{i\Delta\phi_{probe}(r,\Delta t)}$$

T: probe-reference timing

Δt: probe-driver timing

Δϕ is the integral over the longitudinal refractive index perturbation. Retrieving n(z) requires scanning the delay Δt

Marques, J. R et al, Phys. Rev. Lett. 76, 3566 (1996)

Marques, J. R et al, Phys. Rev. Lett. 78, 3463 (1997)

Single-shot: Frequency-domain holography

idea:

use many simultaneous probe pulses with variable delay

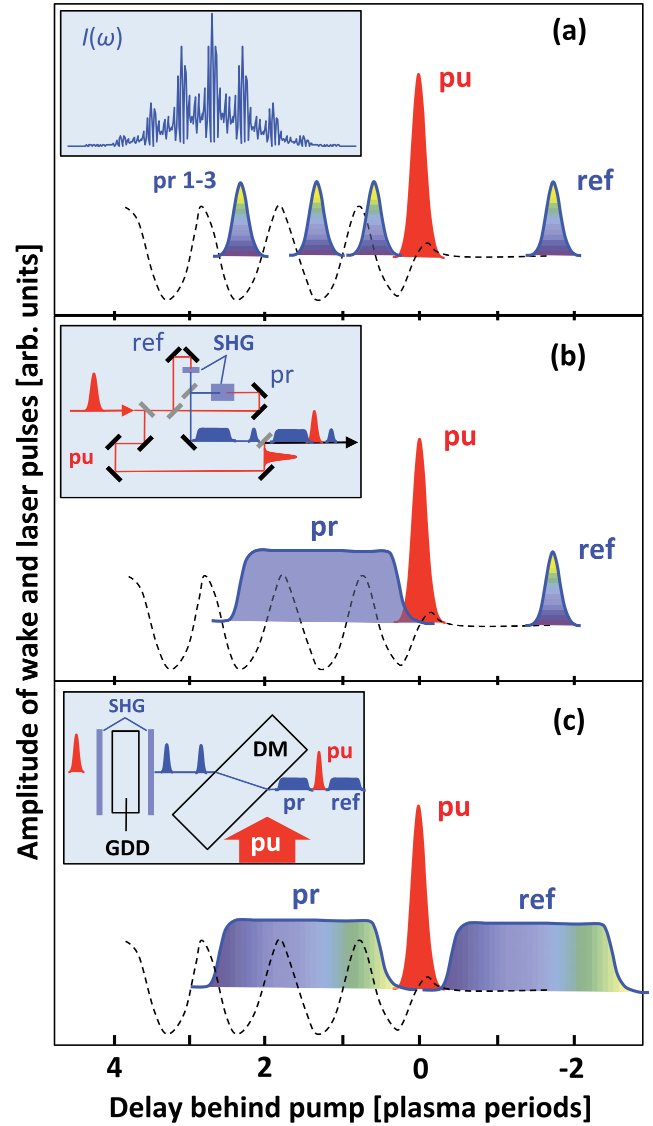
or →

TL short reference pulse and TL long probe pulse

or →

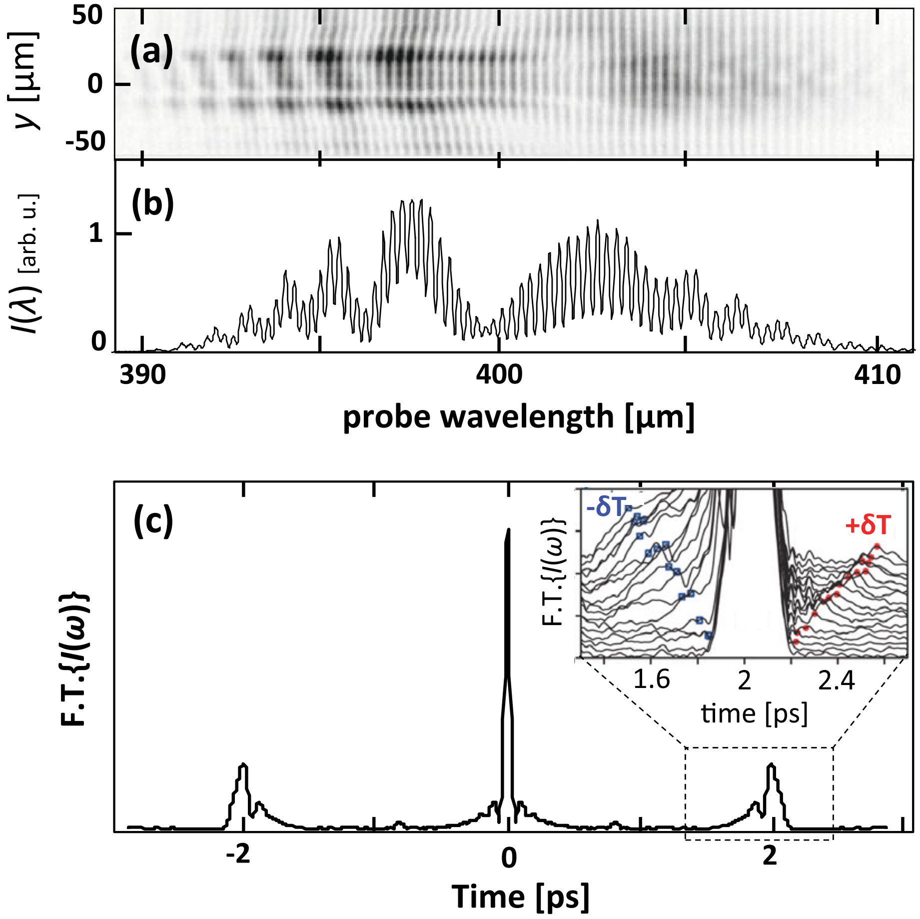
chirped probe and reference

longitudinal information is now encoded in probe wavelength!



M.C. Downer et al, Reviews of Modern Physics 90 035002 (2018)
Siders et al, IEEE Trans. Plasma Sci. 24, 301 (1996)
Chien, C.Y. at al., Opt. Lett. 25, 578 (2000)
Geindre, J. P. et al., Opt. Lett. 26, 1612 (2001)
Kim, K.Y. et al., Appl. Phys. Lett. 81, 4124 (2002)
Matlis, N. H., et al., Nat. Phys. 2, 749 (2006)

FDH reconstruction



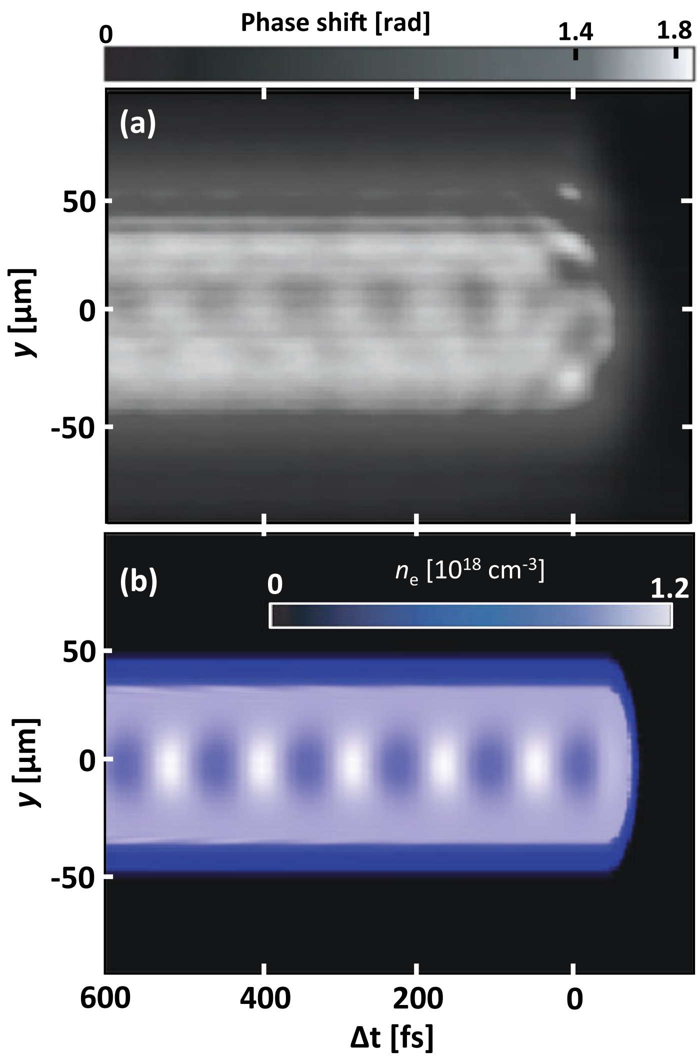
raw FDH interferogram

lineout

FT of lineout. Inset: Wakefield information is encoded in 2-ps sideband for different densities

Matlis, N. H., et al., Nat. Phys. 2, 749 (2006)
Dong P et al., New J. Phys. 12, 045016 (2010)
Matlis, N. H., et al., Opt. Lett. 41, 5503 (2016)

FDH reconstruction II



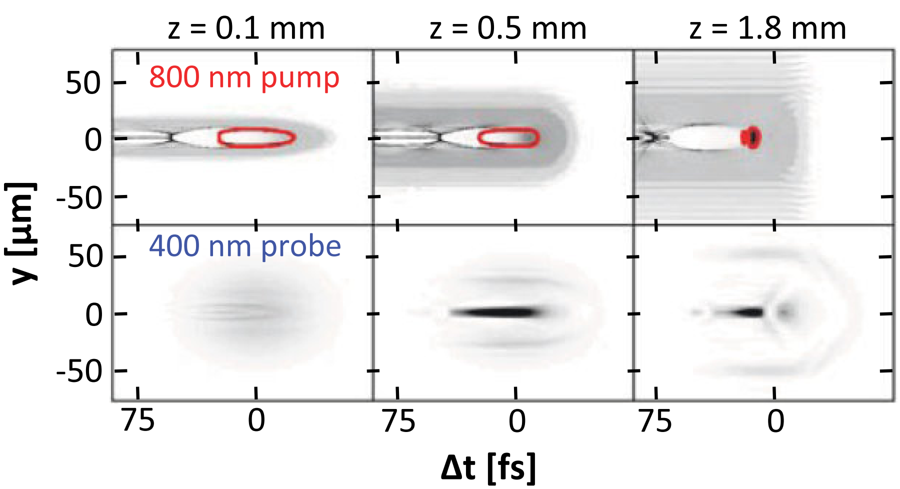
Reconstruction of wakefield from FT^{-1} of 2-ps sideband

Simulation of wakefield for conditions similar to the experiment

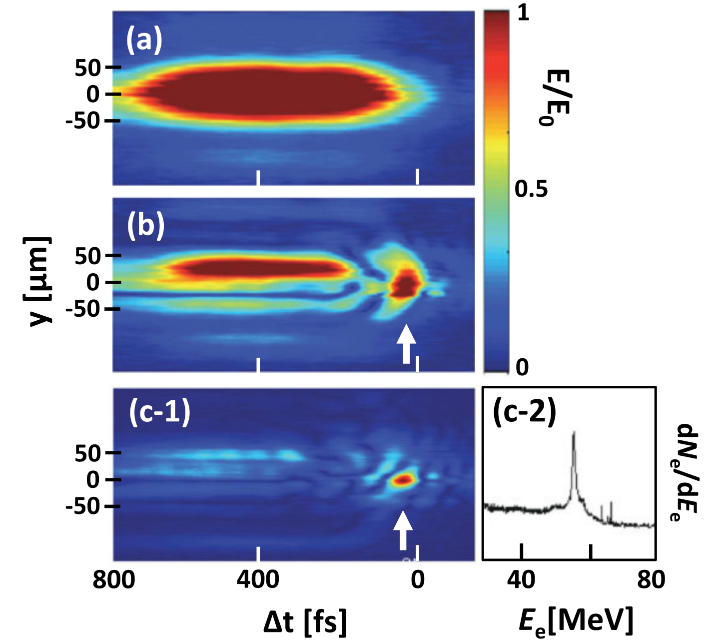
Matlis, N. H., et al., Nat. Phys. 2, 749 (2006)

Longitudinal shadowgraphy:

The strong index gradients inside a bubble can focus a FDH probe:



Simulation of wakefield with 800 nm driver evolution (top) and 400 nm FDH probe evolution (bottom)

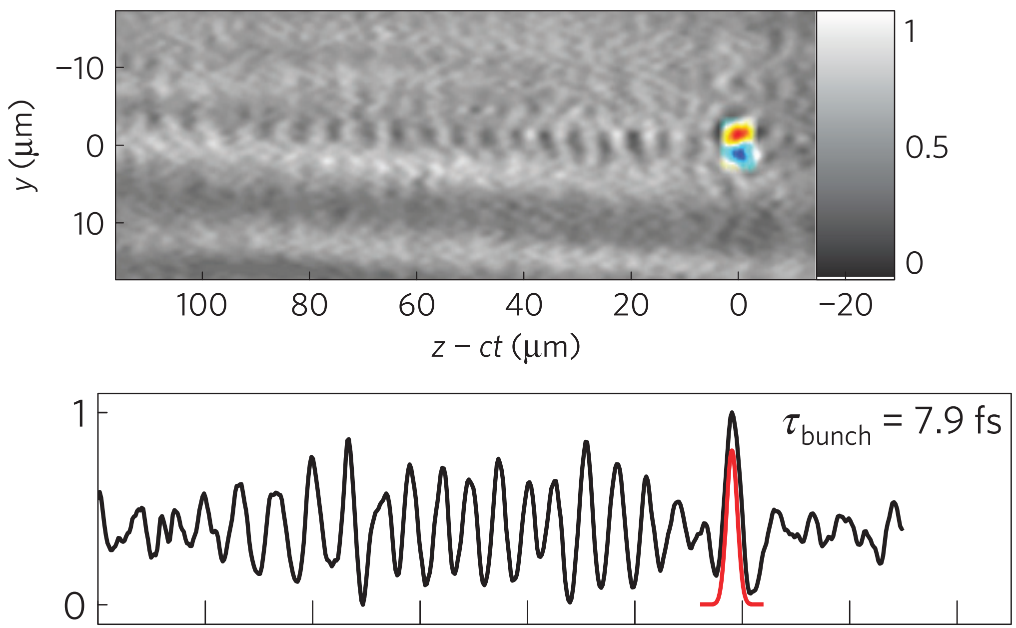


Experiment showing the formation of a 400 nm FDH probe " light bullet"
top: no plasma
middle & bottom: bullet trapped inside bubble.

Dong P et al., New J. Phys. 12, 045016 (2010)

Few-cycle shadowgraphy:

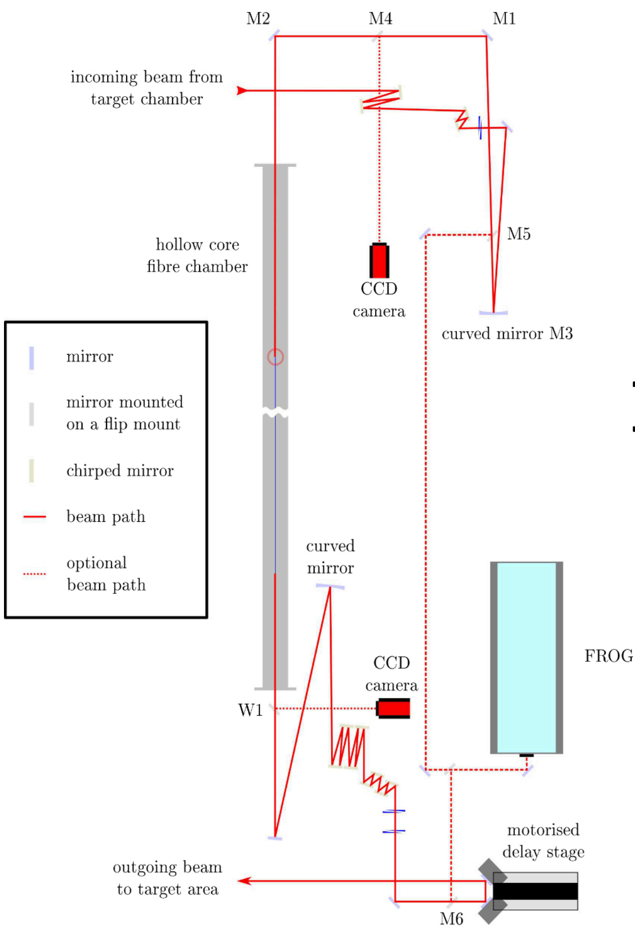
90° probe geometry requires few-fs probe duration in order to freeze motion blur. First successful transverse wake imaging experiment with a 7-fs driver laser:



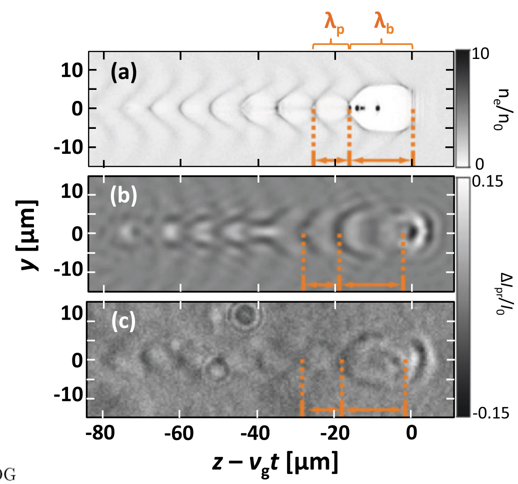
A. Buck et al., Nature Physics 7 543 (2011)

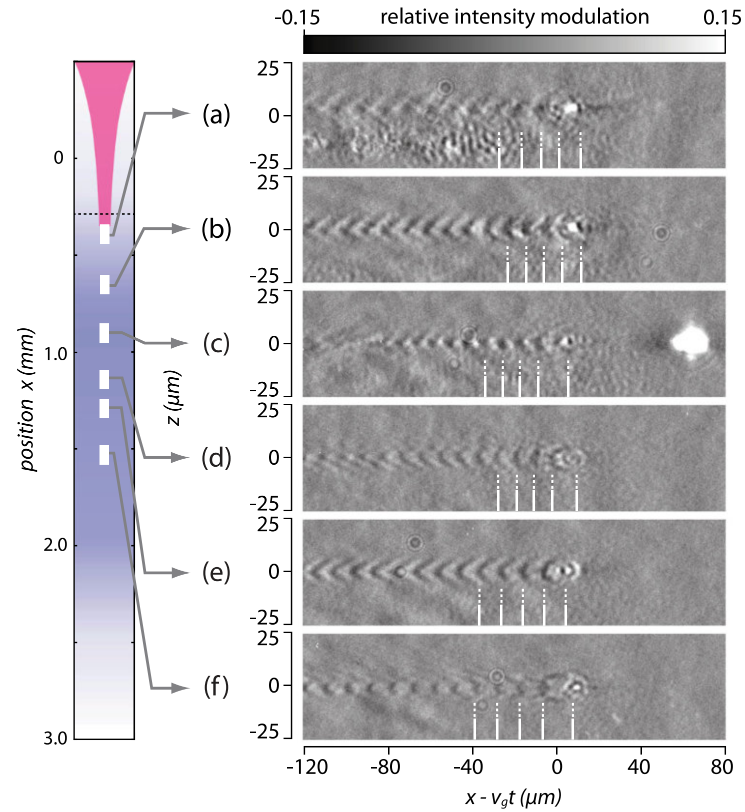
Few-cycle shadowgraphy II:

at "standard" Ti:Sa lasers, a few-cycle probe has to be generated



- PIC
- simulated shadowgram
- experiment

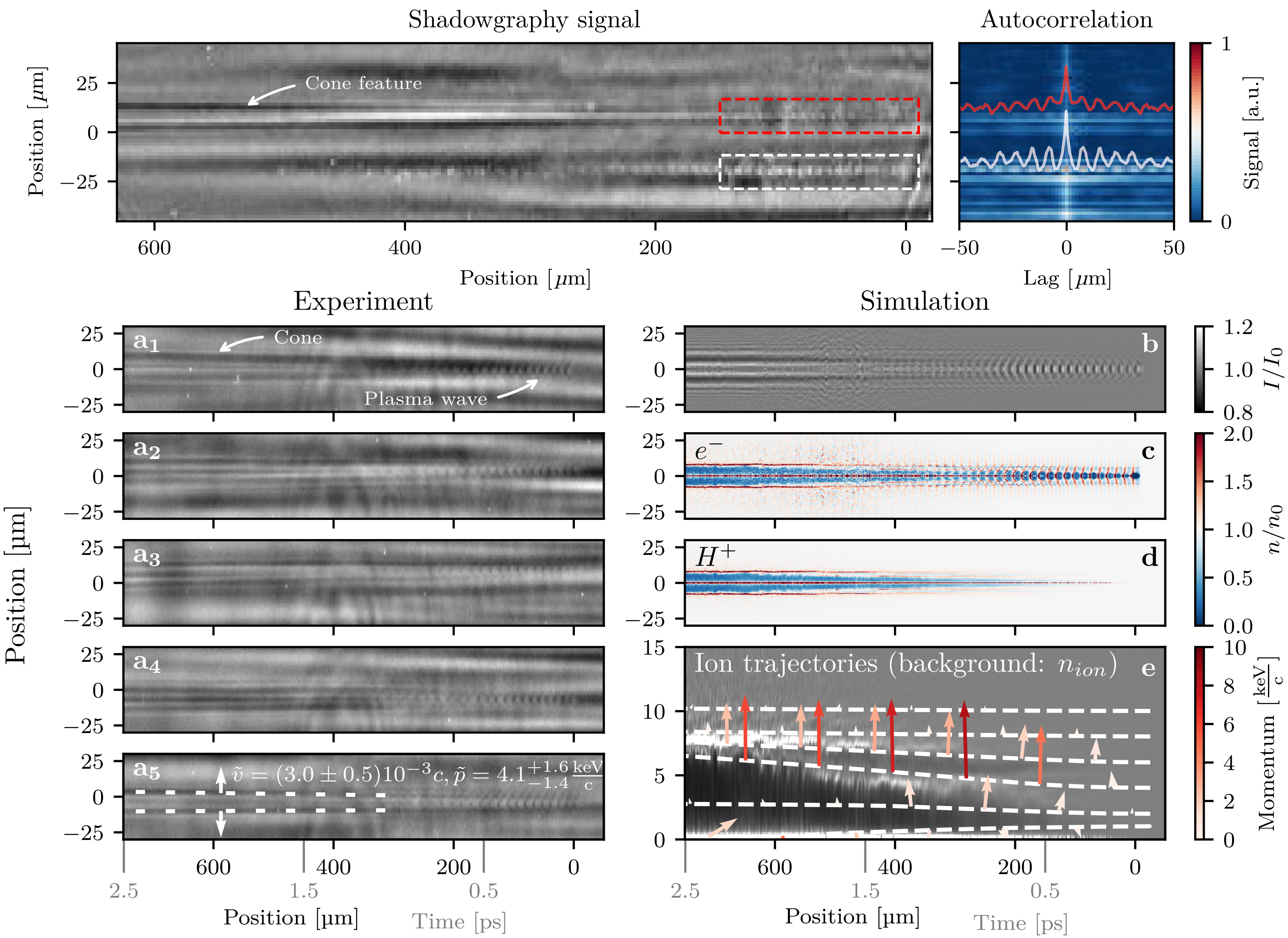




good morphology information, difficult to retrieve wake amplitude (phase object)

Sävert A, et al., Phys. Rev. Lett. 115 055002 (2015)
Siminos, E., et al.., Plasma Phys. Controlled Fusion 58, 065004 (2016)

More plasma waves – laser driven vs. beam driven



Gilljohann et al., Phys. Rev. X 9, 011046 (2019)