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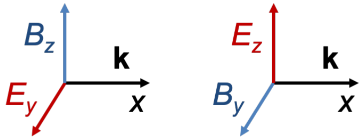
Introduction to Plasma Physics

Part II: Electron dynamics and wave propagation

Sesimbra, Portugal, 11-22 March 2019 | Paul Gibbon

Lecture 2: Electron dynamics and wave propagation

Electromagnetic plane waves



Transverse EM wave can be described by general, elliptically polarized vector potential $\boldsymbol{A}(\omega, \boldsymbol{k})$ travelling in the positive x-direction:

$$\boldsymbol{A} = A_0(0, \delta \cos\phi, (1-\delta^2)^{\frac{1}{2}} \sin\phi), \quad (16)$$

where $\phi = \omega t - kx$ is the phase of the wave; A_0 its amplitude ($v_{os}/c = eA_0/mc$) and δ the polarization parameter :

- $\delta = \pm 1, 0 \rightarrow$ linear pol.: $\boldsymbol{A} = \pm \hat{\boldsymbol{y}} A_0 \cos\phi$; $\boldsymbol{A} = \hat{\boldsymbol{z}} A_0 \sin\phi$
- $\delta = \pm \frac{1}{\sqrt{2}} \rightarrow$ circular pol.: $\boldsymbol{A} = \frac{A_0}{\sqrt{2}}(\pm \hat{\boldsymbol{y}} \cos\phi + \hat{\boldsymbol{z}} \sin\phi)$

Single electron motion in EM plane wave

Electron momentum in electromagnetic wave with fields $\boldsymbol{E}$ and $\boldsymbol{B}$ given by Lorentz equation (SI units):

$$\frac{d\boldsymbol{p}}{dt} = -e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}), \qquad (17)$$

with $\boldsymbol{p} = \gamma m \boldsymbol{v}$, and relativistic factor $\gamma = (1 + p^2/m^2c^2)^{\frac{1}{2}}$.

This has an associated energy equation, after taking dot product of $\boldsymbol{v}$ with Eq. (17):

$$\frac{d}{dt}\left(\gamma mc^2\right) = -e(\boldsymbol{v} \cdot \boldsymbol{E}), \qquad (18)$$

Solution recipe

Bardsley et al., Phys. Rev. A 40, 3823 (1989)
Hartemann et al., Phys. Rev. E 51, 4833 (1995)

1. Laser fields $\boldsymbol{E} = -\partial_t \boldsymbol{A},\ \boldsymbol{B} = \nabla \times \boldsymbol{A}$
2. Use dimensionless variables such that
   $\omega = k = c = e = m = 1$
   (eg: $\boldsymbol{p} \rightarrow \boldsymbol{p}/mc,\ \boldsymbol{E} \rightarrow e\boldsymbol{E}/m\omega c$ etc.)
3. First integrals give conservation relations:
   $\boldsymbol{p}_\perp = \boldsymbol{A},\ \ \gamma - p_x = \alpha$, where $\gamma^2 - p_x^2 - p_\perp^2 = 1$; $\alpha = \text{const.}$
4. Change of variable to wave phase $\phi = t - x$
5. Solve for $\boldsymbol{p}(\phi)$ and $\boldsymbol{r}(\phi)$

Solution: *laboratory* frame

Lab frame: the electron initially at rest before the EM wave arrives, so that at $t = 0$, $p_x = p_y = 0$ and $\gamma = \alpha = 1$.

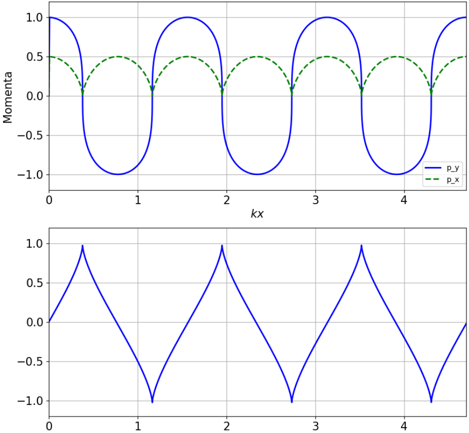
$$\begin{aligned} p_x &= \frac{a_0^2}{4}\left[1 + (2\delta^2 - 1)\cos 2\phi\right], \\ p_y &= \delta a_0 \cos\phi, \\ p_z &= (1-\delta^2)^{1/2} a_0 \sin\phi. \end{aligned} \tag{19}$$

Integrate again to get trajectories:

$$\begin{aligned} x &= \frac{1}{4}a_0^2\left[\phi + \frac{2\delta^2 - 1}{2}\sin 2\phi\right], \\ y &= \delta a_0 \sin\phi, \\ z &= -(1-\delta^2)^{1/2} a_0 \cos\phi. \end{aligned} \tag{20}$$

NB: solution is *self-similar* in the variables $(x/a_0^2, y/a_0, z/a_0)$

Linearly polarized wave ($\delta = 1$)



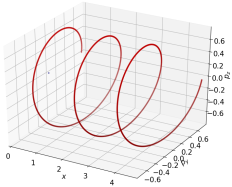
Electron *drifts* with average momentum

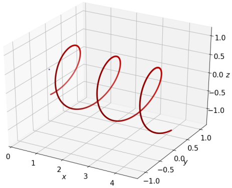
$$p_D \equiv \overline{p_x} = \frac{a_0^2}{4},$$

or velocity

$$\frac{v_D}{c} = \overline{v_x} = \frac{\overline{p_x}}{\overline{\gamma}} = \frac{a_0^2}{4+a_0^2}$$

Circularly polarized wave ($\delta = \pm 1/\sqrt{2}$)



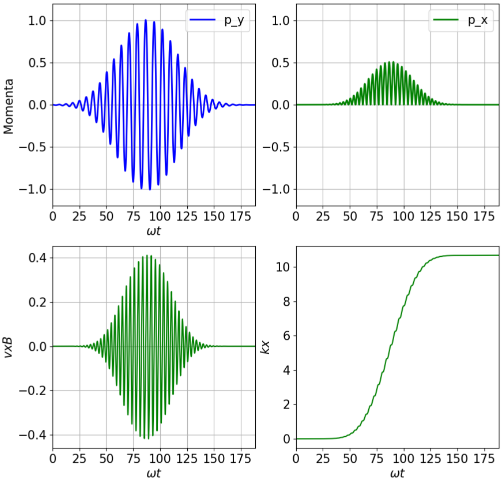


Oscillating p_x component at 2ϕ vanishes, but drift p_D remains.

Orbit is *Helix* with:

- radius $kr_\perp = a_0/\sqrt{2}$
- momentum $p_\perp/mc = a_0/\sqrt{2}$
- pitch angle $\theta_p = p_\perp/p_D = \sqrt{8}a_0^{-1}$

Finite pulse duration - LP

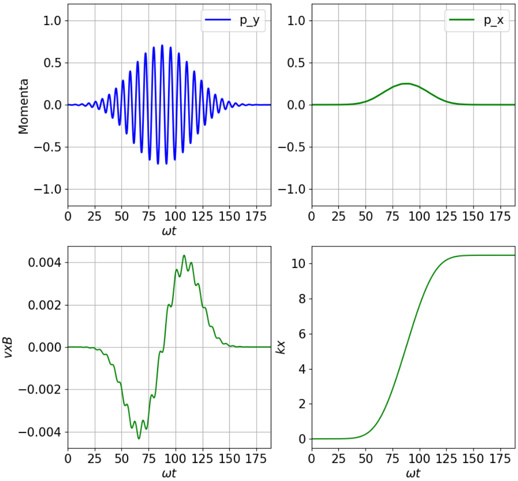


Pulse with *temporal envelope* in the wave vector Eq. (16).

$$\boldsymbol{A}(x,t) = f(t)a_0\cos\phi,$$

No net energy gain! Lawson-Woodward theorem

Finite pulse duration - CP



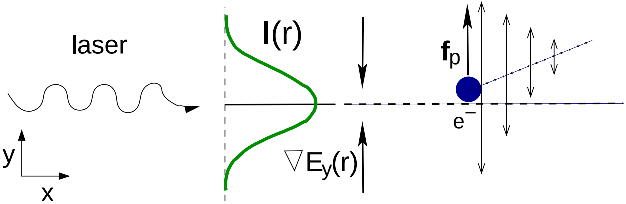
No oscillations in p_x, but drift still there.

$v \times B$ oscillations also nearly vanish, but 'DC' part retained:

longitudinal ponderomotive force!

Motion in laser focus

- Single electron oscillating slightly off-centre of focused laser beam:



- After 1st quarter-cycle, sees lower field
- Doesn't quite return to initial position

⇒ Accelerated away from axis

Ponderomotive force: transverse

In the limit $v/c \ll 1$, the equation of motion (25) for the electron becomes:

$$\frac{\partial v_y}{\partial t} = -\frac{e}{m} E_y(\boldsymbol{r}). \tag{21}$$

Taylor expanding electric field about the current electron position:

$$E_y(\boldsymbol{r}) \simeq E_0(y) \cos\phi + y \frac{\partial E_0(y)}{\partial y} \cos\phi + ...,$$

where $\phi = \omega t - kx$ as before.
To lowest order, we therefore have

$$v_y^{(1)} = -v_{\rm os} \sin\phi; \quad y^{(1)} = \frac{v_{\rm os}}{\omega} \cos\phi,$$

where $v_{\rm os} = eE_L/m\omega$.

Ponderomotive force: transverse (contd.)

Substituting back into Eq. (21) gives

$$\frac{\partial v_y^{(2)}}{\partial t} = -\frac{e^2}{m^2\omega^2} E_0 \frac{\partial E_0(y)}{\partial y} \cos^2 \phi.$$

Multiplying by m and taking the laser cycle-average,

$$\bar{f} = \int_0^{2\pi} f \, d\phi,$$

yields the transverse ponderomotive force on the electron:

$$f_{py} \equiv \overline{m \frac{\partial v_y^{(2)}}{\partial t}} = -\frac{e^2}{4m\omega^2} \frac{\partial E_0^2}{\partial y}. \tag{22}$$

General relativistic ponderomotive force

Rewrite Lorentz equation (17) in terms of the vector potential $\boldsymbol{A}$:

$$\frac{\partial \boldsymbol{p}}{\partial t} + (\boldsymbol{v}.\nabla)\boldsymbol{p} = \frac{e}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \frac{e}{c}\boldsymbol{v}\times\nabla\times\boldsymbol{A}. \tag{23}$$

Make use of identity:

$$\boldsymbol{v}\times(\nabla\times\boldsymbol{p}) \quad = \quad \frac{1}{m\gamma}\boldsymbol{p}\times\nabla\times\boldsymbol{p} = \frac{1}{2m\gamma}\nabla \mid p \mid^2 - \frac{1}{m\gamma}(\boldsymbol{p}.\nabla)\boldsymbol{p},$$

separate the timescales of the electron motion into slow and fast components $\boldsymbol{p} = \boldsymbol{p}^s + \boldsymbol{p}^f$ and average over a laser cycle, get $\boldsymbol{p}^f = \boldsymbol{A}$ and

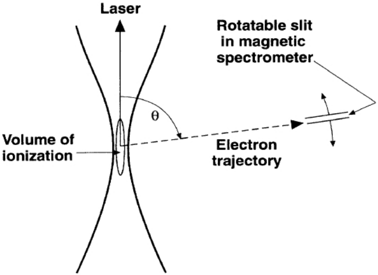
$$\boldsymbol{f}_p = \frac{d\boldsymbol{p}^s}{dt} = -mc^2\nabla\overline{\gamma}, \tag{24}$$

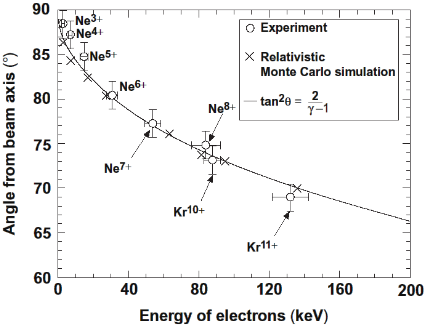
Exercise

where $\overline{\gamma} = \left(1 + \frac{p_s^2}{m^2c^2} + \overline{a_y^2}\right)^{1/2}, \;\; a_y = eA_y/mc.$

Ejection angle of electrons from laser focus

Moore, Meyerhofer *et al.* (1995)





Ionized gases: when is plasma response important?

Simultaneous field ionization of many atoms produces a plasma with electron density n_e, temperature $T_e \sim 1 - 10$ eV.
Collective effects important if

$$\omega_p \tau_{\mathrm{inter}} > 1$$

Example (Gas jet)

$\tau_{\mathrm{inter}} = 100$ fs, $n_e = 10^{19}$ cm^{-3} $\rightarrow \omega_p \tau_{\mathrm{inter}} = 18$
Typical gas jets: $P \sim 1$bar; $n_e = 10^{18} - 10^{19}$ cm^{-3} so collective response important and cannot ignore charge separation.

Exploit plasma effects for: nonlinear refractive properties and high electric & magnetic fields, namely: for **particle acceleration**, or source of **short-wavelength radiation**.

Nonlinear wave propagation in plasmas

Starting point for most analyses is the Lorentz equation of motion for the electrons in a *cold* ($T_e = 0$), unmagnetized plasma, together with Maxwell's equations.

Two additional assumptions:

1. The ions are initially assumed to be singly charged ($Z = 1$) and are treated as a immobile ($v_i = 0$), homogeneous background with $n_0 = Zn_i$.
2. Thermal motion is neglected – OK because the temperature remains small compared to the typical oscillation energy in the laser field: $k_B T_e \ll m_e v_{\mathrm{os}}^2$.

Lorentz-Maxwell equations

Starting equations (SI units) are as follows:

$$\frac{\partial \boldsymbol{p}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{p} = -e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}), \tag{25}$$

$$\nabla \cdot \boldsymbol{E} = \frac{e}{\varepsilon_0}(n_0 - n_e), \tag{26}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{27}$$

$$c^2 \nabla \times \boldsymbol{B} = -\frac{e}{\varepsilon_0} n_e \boldsymbol{v} + \frac{\partial \boldsymbol{E}}{\partial t}, \tag{28}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{29}$$

where $\boldsymbol{p} = \gamma m_e \boldsymbol{v}$ and $\gamma = (1 + p^2/m_e^2 c^2)^{1/2}$.

The EM wave equation I

Substitute $\boldsymbol{E} = -\nabla\phi - \partial\boldsymbol{A}/\partial t$; $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ into Ampère Eq.(28):

$$c^2\nabla \times (\nabla \times \boldsymbol{A}) + \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = \frac{\boldsymbol{J}}{\varepsilon_0} - \nabla\frac{\partial\phi}{\partial t},$$

where the current $\boldsymbol{J} = -en_e\boldsymbol{v}$.
Now we use a bit of vectorial magic, splitting the current into rotational (solenoidal) and irrotational (longitudinal) parts:

$$\boldsymbol{J} = \boldsymbol{J}_\perp + \boldsymbol{J}_{||} = \nabla \times \boldsymbol{\Pi} + \nabla\Psi$$

from which we can deduce (see Jackson!):

$$\boldsymbol{J}_{||} - \frac{1}{c^2}\nabla\frac{\partial\phi}{\partial t} = 0.$$

The EM wave equation II

Now apply Coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$ and transverse part of Lorentz eqn $v_y = eA_y/\gamma$, to finally get:

$$\frac{\partial^2 A_y}{\partial t^2} - c^2\nabla^2 A_y = \mu_0 J_y = -\frac{e^2 n_e}{\varepsilon_0 m_e \gamma} A_y. \tag{30}$$

The nonlinear source term on the RHS contains two important bits of physics:

$n_e = n_0 + \delta n \quad \rightarrow \quad$ Coupling to plasma waves

$\gamma = \sqrt{1 + \boldsymbol{p}^2/m_e^2 c^2} \quad \rightarrow \quad$ Relativistic effects

Dispersion properties: EM waves

For the moment we switch the plasma oscillations off ($n_e = n_0$) in Eq.(30) and look for plane wave solutions $A = A_0 e^{i(\omega t - kx)}$. The derivative operators become: $\frac{\partial}{\partial t} \rightarrow i\omega$; $\frac{\partial}{\partial x} \rightarrow -ik$, yielding:

Dispersion relation

$$\omega^2 = \frac{\omega_p^2}{\gamma_0} + c^2 k^2 \tag{31}$$

with associated

Nonlinear refractive index

$$\eta = \sqrt{\frac{c^2 k^2}{\omega^2}} = \left(1 - \frac{\omega_p^2}{\gamma_0 \omega^2}\right)^{1/2} \tag{32}$$

where $\gamma_0 = (1 + a_0^2/2)^{1/2}$, and a_0 is the normalized oscillation amplitude as in (14).

Linear propagation characteristics ($a_0 \ll 1; \gamma_0 \to 1$)

Underdense plasmas

From the dispersion relation (31) a number of important features of EM wave propagation in plasmas can be deduced.

For *underdense* plasmas ($n_e \ll n_c$):

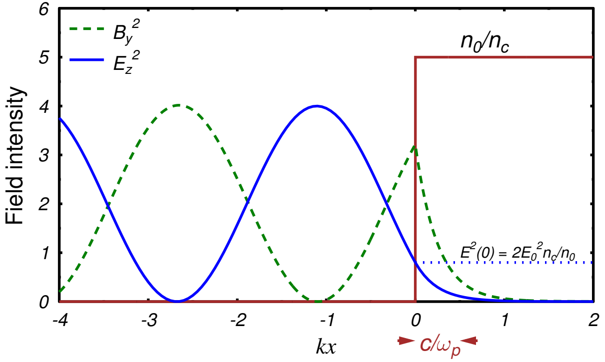
$$\text{Phase velocity} \quad v_p = \frac{\omega}{k} \simeq c\left(1 + \frac{\omega_p^2}{2\omega^2}\right) > c$$

$$\text{Group velocity} \quad v_g = \frac{\partial\omega}{\partial k} \simeq c\left(1 - \frac{\omega_p^2}{2\omega^2}\right) < c$$

Propagation characteristics (2)

Overdense plasmas

In the opposite case, $n_e > n_c$, or $\omega < \omega_p$, the refractive index η becomes imaginary. The wave can no longer propagate, and is instead attenuated with a decay length determined by the **collisionless skin depth** c/ω_p.



Underdense plasmas: nonlinear refraction effects

Real laser pulses are created with focusing optics & are subject to:

1. diffraction due to finite focal spot σ_L: $Z_R = 2\pi\sigma_L^2/\lambda$
2. ionization effects $dn_e/dt \Rightarrow$ refraction due to radial density gradients
3. relativistic self-focusing and self-modulation
   $\Rightarrow \eta(r) = \sqrt{\left(1 - \frac{\omega_p^2(r)}{\gamma_0(r)\omega^2}\right)}$
4. ponderomotive channelling $\Rightarrow \nabla_r n_e$
5. scattering by plasma waves $\Rightarrow k_0 \rightarrow k_1 + k_p$

All nonlinear effects important for laser powers $P_L > 1\,TW$

Focussing threshold – practical units

Litvak, 1970; Max *et al.*1974, Sprangle *et al.*1988

Relation between laser power and critical power:

$$
\begin{aligned}
P_L &= \left(\frac{m\omega c}{e}\right)^2 \left(\frac{c}{\omega_p}\right)^2 \frac{c\epsilon_0}{2} \int_0^\infty 2\pi r a^2(r) dr \\
&= \frac{1}{2}\left(\frac{m}{e}\right)^2 c^5 \epsilon_0 \left(\frac{\omega}{\omega_p}\right)^2 \tilde{P}, \\
&\simeq 0.35 \left(\frac{\omega}{\omega_p}\right)^2 \tilde{P}\,\text{GW}, \quad \text{where} \quad \tilde{P} \equiv \pi a_0^2 (\omega_p^2 \sigma_L^2 / c^2)
\end{aligned}
$$

The critical power $\tilde{P}_c = 16\pi$ thus corresponds to:

Power threshold for relativistic self-focussing

$$
P_c \simeq 17.5 \left(\frac{\omega}{\omega_p}\right)^2 \text{GW}. \tag{33}
$$

Focussing threshold – example

Critical power

$$P_c \simeq 17.5 \left(\frac{\omega}{\omega_p}\right)^2 \text{GW}, \qquad (34)$$

Example

$$\lambda_L = 0.8\mu m, \quad n_e = 1.6 \times 10^{20} \;\; \text{cm}^{-3}$$

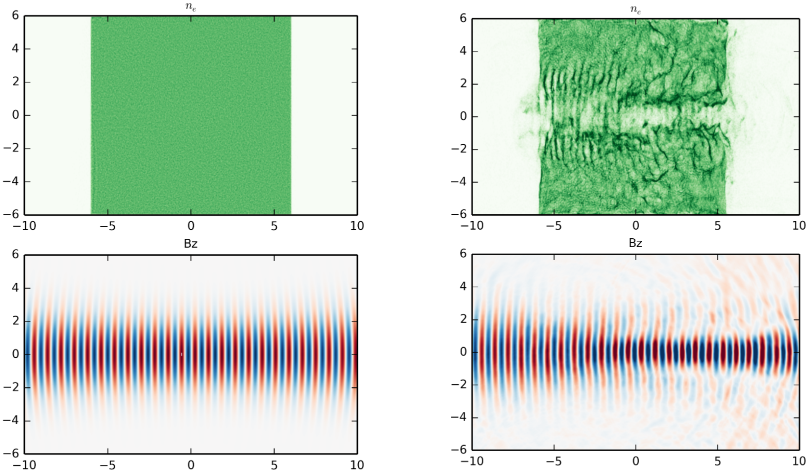
$$\Rightarrow \frac{n_e}{n_c} = \left(\frac{\omega_p}{\omega}\right)^2 = 0.1$$

$$\Rightarrow P_c = 0.175\text{TW}$$

Propagation examples: long pulses

i) $P_L/P_c \ll 1$

ii) $P_L = 2P_c$



Plasma waves

Recap Lorentz-Maxwell equations (25-29)

$$\begin{aligned}
\frac{\partial \boldsymbol{p}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{p} &= -e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}), \\
\nabla \cdot \boldsymbol{E} &= \frac{e}{\varepsilon_0}(n_0 - n_e), \\
\nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}, \\
c^2 \nabla \times \boldsymbol{B} &= -\frac{e}{\varepsilon_0} n_e \boldsymbol{v} + \frac{\partial \boldsymbol{E}}{\partial t}, \\
\nabla \cdot \boldsymbol{B} &= 0
\end{aligned}$$

Electrostatic (Langmuir) waves I

Taking the *longitudinal* (x)-component of the momentum equation (25) and noting that $|\boldsymbol{v} \times \boldsymbol{B}|_x = v_y B_z = v_y \frac{\partial A_y}{\partial x}$ from (**??**) gives:

$$\frac{d}{dt}(\gamma m_e v_x) = -eE_x - \frac{e^2}{2m_e\gamma}\frac{\partial A_y^2}{\partial x}$$

We can eliminate v_x using Ampère's law $(28)_x$:

$$0 = -\frac{e}{\varepsilon_0} n_e v_x + \frac{\partial E_x}{\partial t},$$

while the electron density can be determined via Poisson's equation (26):

$$n_e = n_0 - \frac{\varepsilon_0}{e}\frac{\partial E_x}{\partial x}.$$

Electrostatic (Langmuir) waves II

The above (closed) set of equations can in principle be solved numerically for arbitrary pump strengths. For the moment, we simplify things by *linearizing the plasma fluid* quantities:

$$
\begin{aligned}
n_e &\simeq n_0 + n_1 + \ldots \\
v_x &\simeq v_1 + v_2 + \ldots
\end{aligned}
$$

and neglect products like $n_1 v_1$ etc. This finally leads to:

Driven plasma wave

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\omega_p^2}{\gamma_0}\right) E_x = -\frac{\omega_p^2 e}{2 m_e \gamma_0^2} \frac{\partial}{\partial x} A_y^2 \qquad (35)$$

The driving term on the RHS is the *relativistic ponderomotive force*, with $\gamma_0 = (1 + a_0^2/2)^{1/2}$.

Plasma (Langmuir) wave propagation

Without the laser driving term ($A_y = 0$), Eq.(35) describes linear plasma oscillations with solutions

$$E_x = E_{x0} \sin(\omega t),$$

giving the dispersion relation:

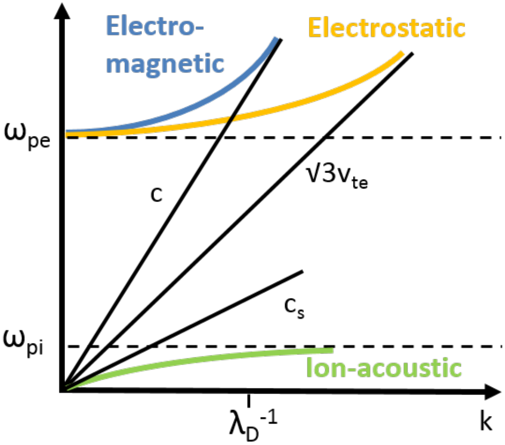
$$-\omega^2 + \omega_p^2 = 0. \quad (36)$$

The linear eigenmode of a plasma has $\omega = \omega_p$.

To account for a finite temperature $T_e > 0$, we would reintroduce a pressure term ∇P_e in the momentum equation (25), which finally yields the Bohm-Gross relation:

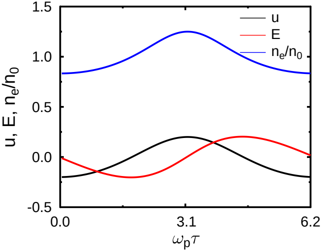
$$\omega^2 = \omega_p^2 + 3 v_t^2 k^2. \quad (37)$$

Summary: dispersion curves



Numerical solutions – linear Langmuir wave

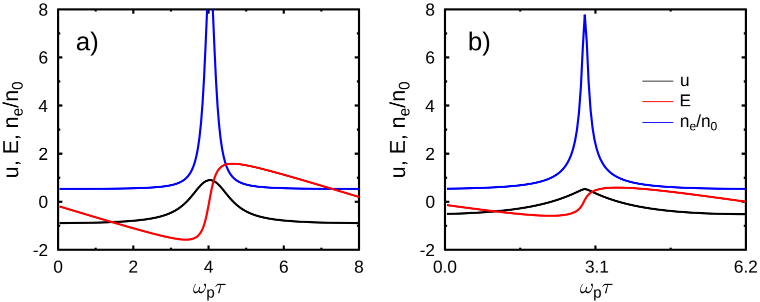
Numerical integration of the electrostatic wave equation on slide 59 for $v_{\text{max}}/c = 0.2$



NB: electric field and density 90^o out of phase

Numerical solutions – nonlinear Langmuir wave

Solution of fully nonlinear electrostatic wave equation (see Akhiezer & Polovin, 1956)



Typical features : i) sawtooth electric field; ii) spiked density; iii) lengthening of the oscillation period by factor γ

Maximum field amplitude - wave-breaking limit

For relativistic phase velocities, find

$$E_{\mathrm{max}} \sim m\omega_p c/e$$

– wave-breaking limit – Dawson (1962), Katsouleas (1988).

Example

$$\begin{aligned}
m_e &= 9.1 \times 10^{-28}\mathrm{g} \\
c &= 3 \times 10^{10}\mathrm{cms}^{-1} \\
\omega_p &= 5.6 \times 10^{4}(n_e/cm^{-3})^{1/2} \\
e &= 4.8 \times 10^{-10}\mathrm{statcoulomb} \\
E_p &\sim 4 \times 10^{8}\left(\frac{n_e}{10^{18}\ \mathrm{cm}^{-3}}\right)^{1/2}\ \mathrm{V\ m}^{-1}
\end{aligned}$$

Summary

- Electrons in EM wave oscillate transversely and longitudinally
- Net energy gain only when symmetry of plane-wave broken, eg: finite focus/duration $\rightarrow$ ponderomotive force
- Waves in plasmas described by fluid equations
- EM waves: propagation determined by ω_0/ω_p and γ; coupling to plasma via nonlinear current
- Longitudinal plasma waves (Langmuir waves) can be driven by laser ponderomotive force: field strengths $O(10^9)$ Vm^{-1} possible

Further reading (I)

1. J. Boyd and J. J. Sanderson, *The Physics of Plasmas*
2. W. Kruer, *The Physics of Laser Plasma Interactions*, Addison-Wesley, 1988
3. J. D. Jackson, *Classical Electrodynamics*, Wiley 1975/1998
4. J. P. Dougherty in Chapter 3 of R. Dendy *Plasma Physics*, 1993
5. P. Gibbon, *Short Pulse Interactions with Matter*, IC Press, London, 2005