Averaged Lorentz Dynamics and application in Fluid Dynamics

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Using a geometric averaging procedure applied to a non-affine linear connection, we prove that for narrow one particle probability distribution functions and in the ultra-relativistic limit, a bunch of point charged particles can be described by a Charged Cold Fluid Model, without additional hypothesis on the momentum moments.

1. Introduction

For a beam of particles in Accelerators, it has been used continuous models, in particular the Cold Charged Fluid Model. However, how this continuous model comes from a more detailed description like the Relativistic Kinetic Theory that requires additional hypothesis, for instance on the moments.

We provide a justification of the use of continuous models and in particular of the Cold Charged Fluid Model, without using any special condition on the mometum moments. The method requires the geometrization of the Lorentz law and an averaging procedure applied to an associated non-affine linear connection.

2. The Lorentz Connetion

The Lorentz's law describing the interaction is

$$\frac{d^2\sigma^i}{dt^2}+{}^{\eta}\Gamma^i_{jk}\frac{d\sigma^j}{dt}\frac{d\sigma^k}{dt}+\eta^{ij}(dA)_{jk}\frac{d\sigma^k}{dt}\sqrt{\eta_x(\frac{d\sigma}{dt},\frac{d\sigma}{dt})}=0,\quad i,j,k=0,1,2,3,$$

where η is the Minkowski metric and ${}^{\eta}\Gamma^i_{jk}$ the associated connections coefficients; A is the electromagnetic potential

This equation can be written as the auto-parallel condition of a non-affine linear connection:

$${}^L\nabla_{\dot\sigma}\dot\sigma=0.$$

The connection coefficents are given by the formula:

$${}^L\Gamma^i_{jk}(x,y)={}^{\eta}\Gamma^i_{jk}+\frac{1}{2\eta(y,y)}(\mathbf{F}^i_j y^m\eta_{mk}+\mathbf{F}^i_k y^m\eta_{mj})+$$

$$+\mathbf{F}^i_m\frac{y^m}{\sqrt{\eta(y,y)}}(\eta_{jk}-\frac{1}{\eta(y,y)}\eta_{js}\eta_{kl}y^s y^l),\quad dA=\mathbf{F}.$$

3. The averaged Connection

The Lorentz's connection is not an affine connection on the tangent bundle of the spacetime. One way to obtain an affine connection is to integrate on the support of the distribution function the connection coefficients of the Lorentz's connection, for pdf with compact support:

$$<\Gamma^i_{jk}>:=\frac{1}{vol(\mathbf{\Sigma}_x)}\int_{\mathbf{\Sigma}_x}\Gamma^i_{jk}\,f(x,y)\,dvol(y),\quad vol(\mathbf{\Sigma}_x):=\int_{\mathbf{\Sigma}_x}f(x,y)\,dvol(y).$$

$\mathbf{\Sigma}_x$ is the mass-shell hyperboloid and $dvol(y)$ is the is the standard volume form induced from η on $\mathbf{\Sigma}_x$.

The results that we have obtained consist of: 1. comparing the Lorentz's connection and its averaged connection, 2. comparing the associated Vlasov's models and 3. performing an asymptotic analysis of the value of the derivative of the main velocity field.

Notation: the averaged quantites will be written between brackets.

4.Results

Theorem 1

Let $\mathbf{M}$ be a compact space-time manifold with boundary. Let us consider the connections ${}^L\nabla$ and $<{}^L\nabla>$ such that:

1. The auto-parallel curves of the connections ${}^L\nabla$ and $<{}^L\nabla>$ are defined for values of the parameter $t\in[0,t_0[$ and
2. The dynamics occurs in the ultra-relativistic regime, which means that $E>>\tilde\alpha>>1$, E is the minimal value of y^0 (measured in the laboratory frame) on the support of $\tilde f$ which is $\tilde\alpha$.

Then,given the same initial conditions, the solutions of the auto-parallel equations

$${}^L\nabla_{\dot x}\dot x=0,\qquad <{}^L\nabla>_{\dot{\tilde x}}\dot{\tilde x}=0$$

are such that

$$|\tilde x^i(t)-x^i(t)|\le C(\mathbf{M},\mathbf{F})\alpha^2E^{-2}t^2;$$

$C(\mathbf{M},\mathbf{F})$ is a finite constant.

Theorem 2

Let f and $\tilde f$ be solutions of Liouville's equations ${}^L\chi(f)=0$ and $<^L\chi>(\tilde f)=0$ such that $\tilde\alpha$, the diameter of $\tilde f$ is small; ${}^L\chi$ and $<{}^L\chi>$ are the spray vector fields obtained from the connections ${}^L\nabla$ and $<{}^L\nabla>$. Assume that $\tilde\alpha<<E=<y^0>_{\tilde f}$ in the laboratory coordinate system and the averaged is performed using $\tilde f$. Then, for the solutions of the Vlasov's and *averaged Vlasov's equation*, with the same initial conditions, the following estimate holds:

$$|f(x(t),y(t))-\tilde f(x(t),y(t))|<C_2(\mathbf{M},\mathbf{F})\tilde\alpha^2\,E^{-2}\,t^2.$$

Theorem 3

With the same notation than before,

$$<{}^L\chi>\tilde f=0\;\Rightarrow\;<{}^L\nabla>_{\tilde V}\tilde V\sim O(\tilde\alpha^3),$$

where $\tilde V^i=<y^i>_{\tilde f}$.

Theorem 4 (Cold Charged Fluid Model from Kinetic Theory)

The integral curves of the velocity field associated with the Lorentz equation ${}^L\nabla_{\dot x}\dot x=\iota_{\dot x}\mathbf{F}$ can be approximated by the integral curves of the normalized mean velocity field u of the distribution function f solution of the Vlasov's equation ${}^L\chi f=0$ and the difference is controlled by α.

5. Conclusion

We have obtained methods which allow to describe the dynamics of a bunch of particles using averaged quantities. We have shown that in the ultra-relativistic limit, the approximation is understood in terms of the diameter of f and the energy E.