Recap. of Transverse Dynamics
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- Transverse Coordinates
- Relativistic definitions
- Magnetic rigidity
- Bending Magnet
- Fields and force in a quadrupole
- Alternating gradients
- Equation of motion in transverse co-ordinates
- Transverse ellipse
- Physical meaning of Q and β
- Twiss Matrix
- Effect of a drift length and a quadrupole
- Focusing in a sector magnet
- Calculating the Twiss parameters
- FODO Cell
- Stability
- Dispersion
- Chromaticity

Transverse Coordinates

- Magnetic force on a moving particle
  \[ F = e \times v \times B \]
- Coordinates
  - z - vertical
  - x - horizontal
  - y means either vertical or horizontal
  - s - direction of the beam
- Quadrupoles act on x and z like lenses
- RF Cavities accelerate in the s direction
Relativistic definitions

Energy of a particle at rest
\[ E_0 = m_0 c^2 \]

Total energy of a moving particle
(definition of \( \gamma \))
\[ E = \gamma E_0 = m_0 c^2 \gamma \quad \gamma = \frac{E}{E_0} \]

Another relativistic variable is defined:
\[ \beta = \frac{\text{momentum} \times c}{\text{energy}} = \frac{pc}{E} = \frac{v}{c} \]

Alternative axioms you may have learned
\[ E = \frac{m_0 c^2}{\sqrt{1-\beta^2}} \]
\[ p = mv = \frac{m_0 c \beta}{\sqrt{1-\beta^2}} \]
\[ \gamma = \frac{1}{\sqrt{1-(\beta v)^2}} = \frac{1}{\sqrt{1-\beta^2}} \]

You can prove:
\[ pc = \beta E = m_0 c^2 (\beta \gamma) \]

Magnetic rigidity

\[ \frac{dp}{dt} = \left| \frac{d\theta}{dt} \right| = \left| \frac{d\theta}{dt} \right| \frac{ds}{dt} = \frac{|p| ds}{\rho dt} \]
\[ = ev \times B = e \frac{ds}{dt} B \]

\( (B \rho) = \frac{P}{e} = \frac{pc}{ec} = \frac{\beta E}{ec} = \frac{\beta \gamma E_0}{ec} = \frac{m_0 c}{e} (\beta \gamma) \)

\( (B \rho) \ [\text{T.m}] = \frac{pc}{ec} = \frac{pc[ev]}{c[m.s^{-1}]} = 3.3356 \ (pc) \ [\text{GeV}] \)
Bending Magnet

- Effect of a uniform bending (dipole) field
  \[ \sin(\theta/2) = \frac{\ell}{2\rho} = \frac{EB}{2(\rho B)} \]
- If \( \theta \ll \pi/2 \) then \( \theta \approx \frac{EB}{(\rho B)} \)
- Sagitta
  \[ \rho \approx \frac{\ell^2}{2} \left[ 1 - \cos(\theta/2) \right] \approx \frac{\rho^2}{16} \approx \pm \frac{\theta}{16} \]

Fields and force in a quadrupole

- No field on the axis
- Field strongest here
  \[ B \propto x \]
  (hence is linear)
- Force restores
- Gradient
  \[ \frac{dB}{dx} \]
- Normalised:
  \[ k = -\frac{1}{(B\rho)} \frac{dB}{dx} \]

Defocuses in vertical plane

SOLUTION IS TO ALTERNATE THE GRADIENTS OF A SERIES OF QUADS
Alternating gradients

Equation of motion in transverse coordinates

- Hill’s equation (linear-periodic coefficients)
  \[ \frac{d^2 y}{ds^2} + k(s)y = 0 \]
  where 
  \[ k = -\frac{1}{B\beta} \frac{dB_z}{dx} \] at quadrupoles
  like restoring constant in harmonic motion
- Solution (e.g. Horizontal plane)
  \[ \dot{y} = \sqrt{\beta(s)} \sqrt{\varepsilon} \sin[\phi(s) + \phi_0] \]
  \[ \varphi = \int \frac{ds}{\beta(s)} \]
- Condition
- Property of machine \( \sqrt{\beta(s)} \)
- Property of the particle (beam) \( \varepsilon \)
- Physical meaning (H or V planes)
  Envelope \( \sqrt{\varepsilon \beta(s)} \)

Maximum excursions
  \[ \dot{y} = \sqrt{\varepsilon \beta(s)} \]
  \[ \dot{y}' = \sqrt{\varepsilon / \beta(s)} \]
Transverse ellipse

- Definition of displacement and divergence

Physical meaning of Q and β

Signal = No. of electrons collected

Beam profile

Scanning voltage = beam displacement

Period of revolution (1 turn)

Envelope = $X_0 \sin 2 \pi f t$
Twiss Matrix

- All such linear motion from points 1 to 2 can be described by a matrix like:

\[
\begin{pmatrix}
y(s_2) \\
y'(s_2)
\end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y(s_1) \\
y'(s_1)\end{pmatrix} = M_{12} \begin{pmatrix} y(s_1) \\
y'(s_1)\end{pmatrix}.
\]

- Can be simplified if we define the “Twiss” parameters:

\[
\beta = w^2, \quad \alpha = -\frac{1}{2} \beta', \quad \gamma = \frac{1 + \alpha^2}{\beta}
\]

- Giving the matrix for a ring (or period)

\[
M = \begin{pmatrix} \cos \mu + \alpha \sin \mu, \beta \sin \mu \\ -\gamma \sin \mu, \cos \mu - \alpha \sin \mu \end{pmatrix}
\]

Effect of a drift length and a quadrupole

\[
\begin{pmatrix} x_2 \\
x'_2 \end{pmatrix} = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\
x'_1 \end{pmatrix}
\]

\[
\theta = \frac{1}{f} x
\]

\[
\begin{pmatrix} x_2 \\
x'_2 \end{pmatrix} = \begin{pmatrix} 1, 0 \\ -k f, 1 \end{pmatrix} \begin{pmatrix} x_1 \\
x'_1 \end{pmatrix}
\]
### Focusing in a sector magnet

\[
M_x = \begin{pmatrix} \cos \theta & \rho \sin \theta \\ -\frac{1}{\rho} \sin \theta & \cos \theta \end{pmatrix}
\]

### Calculating the Twiss parameters

#### THEORY

\[
M = \begin{pmatrix} \cos \mu + \alpha \sin \mu & \beta \sin \mu \\ -\gamma \sin \mu & \cos \mu - \alpha \sin \mu \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

#### COMPUTATION

(multiply elements)

Real hard numbers

Solve to get Twiss parameters:

\[
\begin{align*}
\mu &= \cos^{-1}\left(\frac{\text{Tr} \ M}{2}\right) = \cos^{-1}\left(\frac{a+d}{2}\right) \\
\beta &= \frac{b}{\sin \mu} \\
\alpha &= \frac{a-d}{2\sin \mu} \\
\gamma &= \frac{-c}{\sin \mu}
\end{align*}
\]
FODO Cell

Write down matrices from mid-F to mid-F

\[
M = \begin{pmatrix}
\frac{1}{l} & 0 & 0 \\
0 & 1 & l \\
0 & 0 & \frac{1}{l}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{l} & 0 & 0 \\
0 & 1 & l \\
0 & 0 & \frac{1}{l}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{l} & 0 & 0 \\
0 & 1 & l \\
0 & 0 & \frac{1}{l}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{l} & 0 & 0 \\
0 & 1 & l \\
0 & 0 & \frac{1}{l}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{l} & 0 & 0 \\
0 & 1 & l \\
0 & 0 & \frac{1}{l}
\end{pmatrix}
\]

\[
M = \begin{pmatrix}
1 - L^2 f^2 & 2L f & L^2 f^2 \\
2L f & -1 & L f \\
L^2 f^2 & L f & 1 - L^2 f^2
\end{pmatrix}
\begin{pmatrix}
\cos \mu + \alpha \sin \mu & \beta \sin \mu \\
-\gamma \sin \mu & \cos \mu - \alpha \sin \mu
\end{pmatrix}
\]

Since they must equal the Twiss matrix:

\[
\begin{align*}
\cos \mu &= 1 - L^2 f^2 \\
\sin \left(\frac{\mu}{2}\right) &= 4l^2 f \\
\beta &= 2L \left[ \frac{1 + \sin \left(\frac{\mu}{2}\right)}{\sin \mu} \right] \\
\alpha_{yz} &= 0 \\
\beta &= \frac{1 + \sin \left(\frac{\mu}{2}\right)}{\sin \left(\frac{\mu}{2}\right)} \\
\end{align*}
\]

Stability diagram for FODO

Conditions

\[
\det |M| = 1 \\
\prod_{N_k} (M(s))^{N_k}
\]

bounded

If motion is bounded then \( \mu \) must be real

HENCE

\[
\cos \mu = \frac{\text{Tr} M}{2} \leq 1
\]
**Dispersion**

- Low momentum particle is bent more
- It should spiral inwards but:
- There is a displaced (inwards) closed orbit
- Closer to axis in the D’s
- Extra (outward) force balances extra bends

\[ x = D(s) \frac{\Delta p}{p} \]

**Dispersed beam cross sections**

- These are real cross-section of beam
- The central and extreme momenta are shown
- There is of course a continuum between
- The vacuum chamber width must accommodate the full spread
- Half height and half width are:

\[ a_V = \sqrt{\beta_V \epsilon_V}, \quad a_H = \sqrt{\beta_H \epsilon_H + D(s) \frac{\Delta p}{p}} \]
The Q is determined by the lattice quadrupoles whose strength is:

\[ k = \frac{1}{(B_p)} \frac{dB}{dx} \propto \frac{1}{p} \]

Differentiating:

Remember from gradient error analysis

\[ \frac{\Delta k}{k} = -\frac{\Delta p}{p} . \]

Giving by substitution

\[ \Delta Q = \frac{1}{4\pi} \int \beta(s) \delta k(s) \, ds . \]

\[ Q' \text{ is the chromaticity} \]

“Natural” chromaticity

\[ \Delta Q = \frac{1}{4\pi} \int \gamma(s) \delta \beta(s) \, ds = \frac{1}{4\pi} \int \beta(s) \delta \gamma(s) \, ds \Delta p \]

\[ \Delta Q = \frac{\Delta p}{p} \]

\[ Q' = -\frac{1}{4\pi} \int \delta \beta(s) \gamma(s) \, ds \approx -1.3Q \]

N.B. Old books say 

\[ \xi = \frac{p}{Q} \frac{dQ}{dp} = \frac{Q'}{Q} \]

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