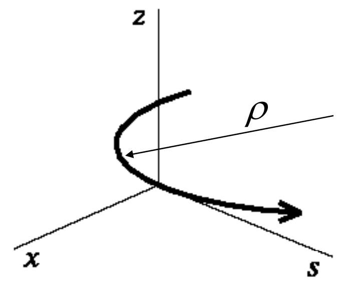
Recap. of Transverse Dynamics
E. Wilson – 15th September 2003

- Transverse Coordinates
- Relativistic definitions
- Magnetic rigidity
- Bending Magnet
- Fields and force in a quadrupole
- Alternating gradients
- Equation of motion in transverse co-ordinates
- Transverse ellipse
- Physical meaning of Q and β
- Twiss Matrix
- Effect of a drift length and a quadrupole
- Focusing in a sector magnet
- Calculating the Twiss parameters
- FODO Cell
- Stability
- Dispersion
- Chromaticity

Transverse Coordinates



- Magnetic force on a moving particle
-

$$\mathbf{F} = e\ \mathbf{v} \times \mathbf{B}$$

- Coordinates
  » z - vertical
  » x - horizontal
  » y means either vertical or horizontal
  » s - direction of the beam

- Quadrupoles act on x and z like lenses

- RF Cavities accelerate in the s direction

Relativistic definitions

Energy of a particle at rest

$$E_0 = m_0 c^2$$

Total energy of a moving particle (definition of γ)

$$E = \gamma E_0 = m_0 c^2 \gamma \qquad \gamma = \frac{E}{E_0}$$

Another relativistic variable is defined:

$$\beta = \frac{\text{momentum} \times c}{\text{energy}} = \frac{pc}{E} = \frac{v}{c}$$

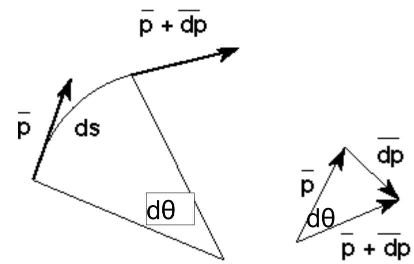
Alternative axioms you may have learned

$$E = \frac{m_0 c^2}{\sqrt{1-\beta^2}} \qquad p = mv = \frac{m_0 v}{\sqrt{1-\beta^2}} = \frac{m_0 c \beta}{\sqrt{1-\beta^2}} \qquad \gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

You can prove:

$$pc = \beta E = m_0 c^2 (\beta\gamma)$$

Magnetic rigidity



$$\frac{1}{\rho} = \frac{d\theta}{ds}$$

$$\frac{d\mathbf{p}}{dt} = |\mathbf{p}|\frac{d\theta}{dt} = |\mathbf{p}|\frac{d\theta}{ds}\frac{ds}{dt} = \frac{|\mathbf{p}|}{\rho}\frac{ds}{dt}$$

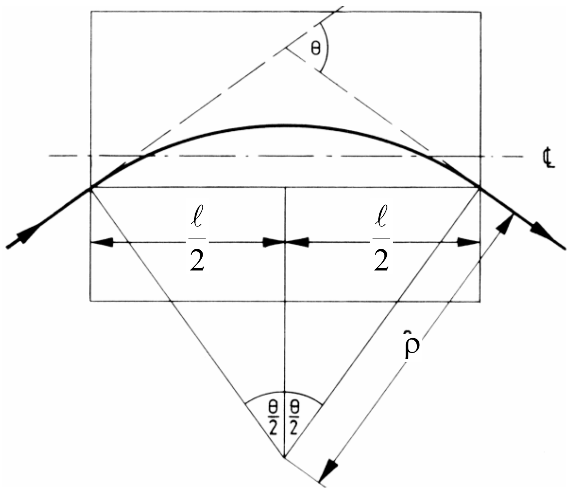
$$= e\mathbf{v} \times \mathbf{B} = e\frac{ds}{dt}B$$

$$(B\rho) = \frac{p}{e} = \frac{pc}{ec} = \frac{\beta E}{ec} = \frac{\beta\gamma E_0}{ec} = \frac{m_0 c}{e}(\beta\gamma)$$

$$(B\rho)\ [\mathrm{T.m}] = \frac{pc}{ec} = \frac{pc[eV]}{c[m.s^{-1}]} = 3.3356\ (\mathrm{pc})\ [\mathrm{GeV}]$$

Fig.Brho 4.8

Bending Magnet



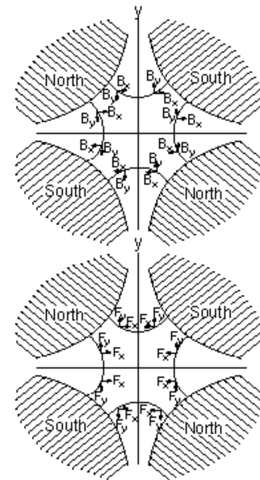
- **Effect of a uniform bending (dipole) field**

$$\sin(\theta/2) = \frac{\ell}{2\rho} = \frac{\ell B}{2(B\rho)}$$

- **If** $\theta << \pi/2$ **then** $\theta \approx \frac{\ell B}{(B\rho)}$

- **Sagitta** $\pm\frac{\rho}{2}\left[1-\cos(\theta/2)\right] \approx \pm\frac{\rho\theta^2}{16} \approx \pm\frac{\ell\theta}{16}$

Fields and force in a quadrupole



No field on the axis

Field strongest here

$B \propto x$

(hence is linear)

Force restores

Gradient $\frac{dB_z}{dx}$

Normalised:

$$k = -\frac{1}{(B\rho)}\frac{dB_z}{dx}$$

Defocuses in vertical plane

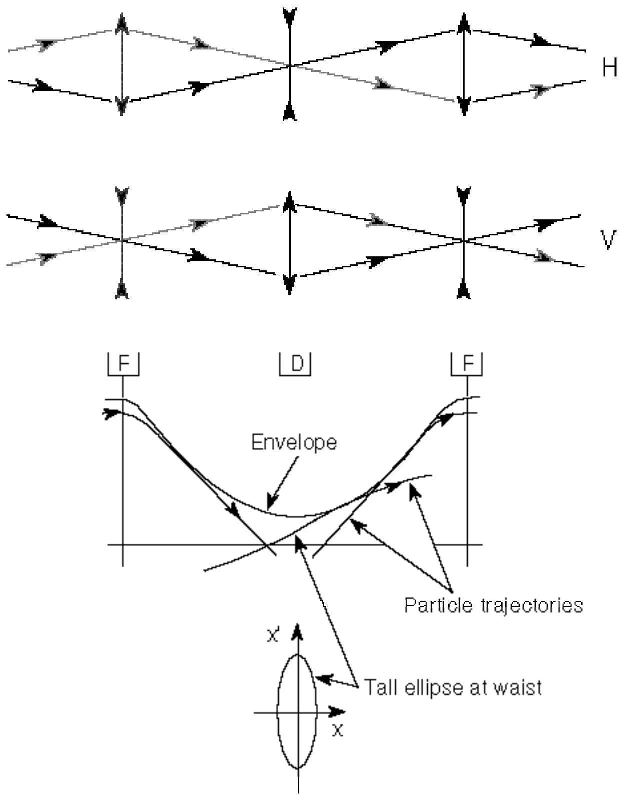
POWER OF LENS

$$\ell k = -\frac{1}{(B\rho)}.\frac{\partial B_z}{\partial x} = \frac{1}{f}$$

SOLUTION IS TO ALTERNATE THE GRADIENTS OF A SERIES OF QUADS

Fig. cas 10.8

Alternating gradients



Equation of motion in transverse co-ordinates

- **Hill's equation (linear-periodic coefficients)**

$$\frac{d^2 y}{ds^2} + k(s)y = 0$$

where $k = -\frac{1}{(B\rho)}\frac{dB_z}{dx}$ **at quadrupoles**

like restoring constant in harmonic motion

- **Solution (e.g. Horizontal plane)**

$$y = \sqrt{\beta(s)}\sqrt{\varepsilon}\sin\left[\phi(s) + \phi_0\right]$$

- **Condition**

$$\varphi = \int \frac{ds}{\beta(s)}$$

- **Property of machine** $\sqrt{\beta(s)}$
- **Property of the particle (beam)** ε
- **Physical meaning (H or V planes)**

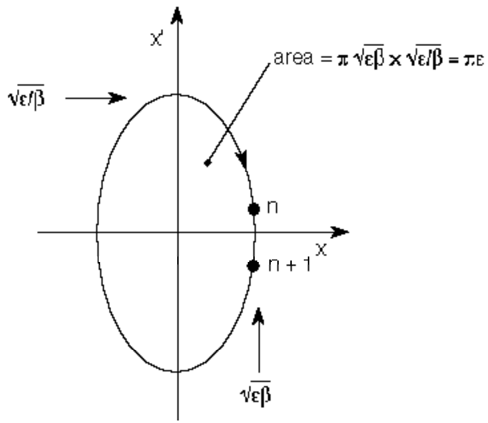
Envelope

$$\sqrt{\varepsilon\beta(s)}$$

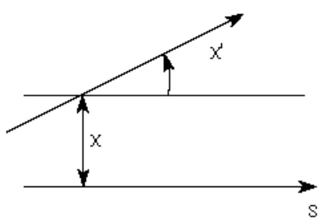
Maximum excursions

$$\hat{y} = \sqrt{\varepsilon\beta(s)} \qquad \hat{y}' = \sqrt{\varepsilon / \beta(s)}$$

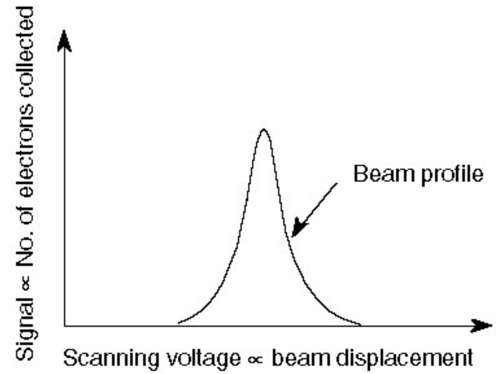
Transverse ellipse

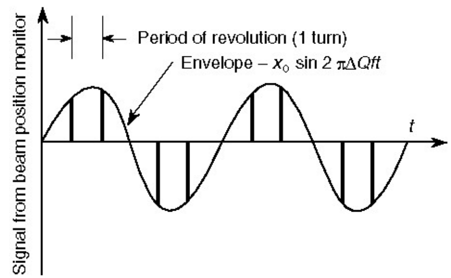


◆ **Definition of displacement and divergence**



Physical meaning of Q and β





Twiss Matrix

- All such linear motion from points 1 to 2 can be described by a matrix like:

$$\begin{pmatrix} y(s_2) \\ y'(s_2) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y(s_1) \\ y'(s_1) \end{pmatrix} = \mathbf{M}_{12} \begin{pmatrix} y(s_1) \\ y'(s_1) \end{pmatrix} .$$

- Can be simplified if we define the "Twiss" parameters:

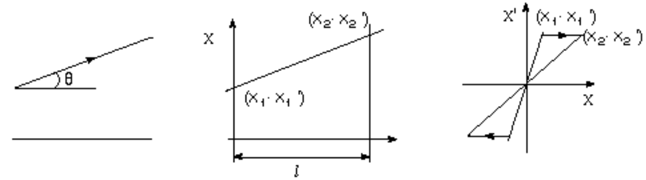
$$\beta = w^2 \ , \ \alpha = -\frac{1}{2}\beta' \ , \ \gamma = \frac{1+\alpha^2}{\beta}$$

- Giving the matrix for a ring (or period)

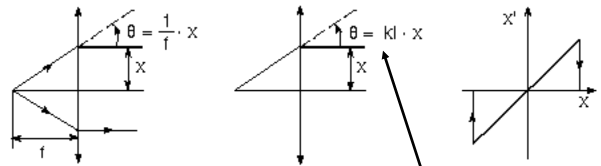
$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu \ , & \beta\sin\mu \\ -\gamma\ \sin\mu, & \cos\mu - \alpha\ \sin\mu \end{pmatrix}$$

Effect of a drift length and a quadrupole

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$



(a)



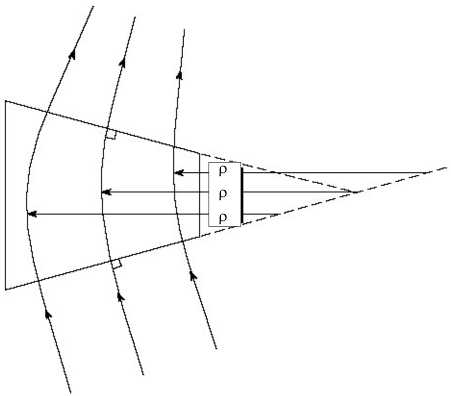
(b)

$$\theta = \frac{1}{f} \cdot x$$

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 \ , & 0 \\ -1/f \ , & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 \ , & 0 \\ -kl \ , & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

Focusing in a sector magnet



$$M_x = \begin{pmatrix} \cos\theta\ , & \rho\sin\theta \\ -\frac{1}{\rho}\ \sin\theta, & \cos\theta \end{pmatrix}$$

Calculating the Twiss parameters

THEORY **COMPUTATION (multiply elements)**

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu\ , & \beta\sin\mu \\ -\gamma\ \sin\mu, & \cos\mu - \alpha\ \sin\mu \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Real hard numbers

Solve to get Twiss parameters:

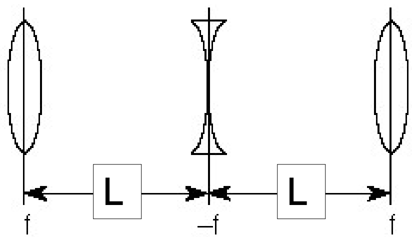
$$\mu = \cos^{-1}\left(\frac{Tr\ M}{2}\right) = \cos^{-1}\left(\frac{a+d}{2}\right)$$

$$\beta = b / \sin\ \mu$$

$$\alpha = \frac{a-d}{2\sin\ \mu}$$

$$\gamma = -c / \sin\ \mu$$

FODO Cell



Write down matrices from mid-F to mid-F

$$M=\begin{pmatrix}1 & 0\\ \mp 1/2f & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \pm 1/f & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \mp 1/2f & 1\end{pmatrix}$$

$$=\begin{pmatrix}1-L^2/2f^2 \ , & 2L(1\pm L/2f)\\ -L/2f^2(1\mp L/2f)\ , & 1-L^2/2f^2\end{pmatrix}=\begin{pmatrix}\cos\mu+\alpha\ \sin\mu\ , & \beta\ \sin\mu\\ -\gamma\ \sin\mu & , \cos\mu-\alpha\ \sin\mu\end{pmatrix}$$

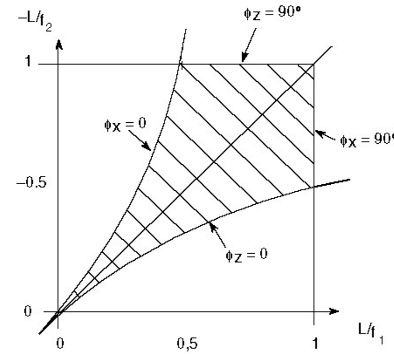
Since they must equal the Twiss matrix:

$$\left.\begin{aligned}\cos\mu &= 1-L^2/2f^2\\ \sin(\mu/2) &= L/2f\end{aligned}\right\}$$

$$\left.\begin{aligned}\beta &= 2L\frac{[1\pm\sin(\mu/2)]}{\sin\mu}\\ \alpha_{x,z} &= 0\ .\end{aligned}\right\}$$

$$\frac{\hat{\beta}}{\breve{\beta}}=\frac{1+\sin(\mu/2)}{1-\sin(\mu/2)}\ ,$$

Stability diagram for FODO



Conditions

$\boxed{\det|M|=1}$

$\boxed{\prod_{Nk}\{\mathrm{M}(s)\}^{Nk}}$

bounded

$$M=\begin{pmatrix}\cos\mu+\alpha\ \sin\mu\ , & \beta\ \sin\mu\\ -\gamma\ \sin\mu, & \cos\mu-\alpha\ \sin\mu\end{pmatrix}$$

$$=I\ \cos\mu+J\ \sin\mu$$

$$\mathbf{I}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\mathbf{J}=\begin{pmatrix}\alpha & \beta\\ -\gamma & -\alpha\end{pmatrix}$$

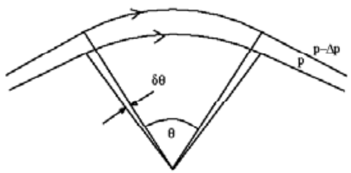
$$M^K=(I\ \cos\mu+J\ \sin\ \mu)^K=I\ \cos K\mu+J\ \sin\ K\mu$$

Just like: $\left(e^{i\mu}\right)^K=e^{iK\mu}$.

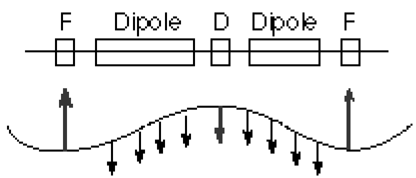
If motion is bounded then μ must be real

HENCE $\boxed{\cos\mu=\ \mathrm{Tr}\,M/2\le 1}$

Dispersion



- Low momentum particle is bent more
- It should spiral inwards but:
- There is a displaced (inwards) closed orbit
- Closer to axis in the D's
- Extra (outward) force balances extra bends

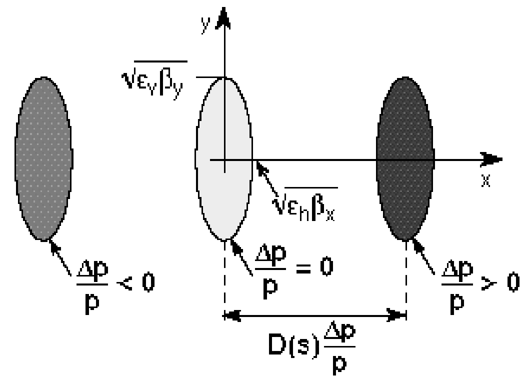


- *D(s)* is the "dispersion function"

$$x = D(s)\frac{\Delta p}{p}$$

Fig. cas 1.7-7.1C

Dispersed beam cross sections



- These are real cross-section of beam
- The central and extreme momenta are shown
- There is of course a continuum between
- The vacuum chamber width must accommodate the full spread
- Half height and half width are:

$$a_V = \sqrt{\beta_V \varepsilon_V} \quad , \quad a_H = \sqrt{\beta_H \varepsilon_H} + D(s)\frac{\Delta p}{p} \ .$$

Chromaticity

- The Q is determined by the lattice quadrupoles whose strength is:

$$k = \frac{1}{(B\rho)} \frac{dB_z}{dx} \propto \frac{1}{p}$$

- Differentiating:
- Remember from gradient error analysis

$$\frac{\Delta k}{k} = -\frac{\Delta p}{p} \ .$$

- Giving by substitution

$$\Delta Q = \frac{1}{4\pi} \int \beta(s) \delta k(s) \ ds \ .$$

$\boxed{Q'}$ is the chromaticity

- "Natural" chromaticity

$$\Delta Q = \frac{1}{4\pi} \int \beta(s) \Delta k(s) ds = \left[\boxed{\frac{-1}{4\pi} \int \beta(s) k(s) ds} \right] \frac{\Delta p}{p} \ .$$

$$\Delta Q = \boxed{Q} ' \frac{\Delta p}{p}$$

$$Q' = -\frac{1}{4\pi} \oint \beta(s) k(s) ds \approx -1.3Q$$

N.B. Old books say $\xi = \frac{p}{Q} \frac{dQ}{dp} = \frac{Q'}{Q}$

Summary

- Transverse Coordinates
- Relativistic definitions
- Magnetic rigidity
- Bending Magnet
- Fields and force in a quadrupole
- Alternating gradients
- Equation of motion in transverse co-ordinates
- Transverse ellipse
- Physical meaning of Q and β
- Twiss Matrix
- Effect of a drift length and a quadrupole
- Focusing in a sector magnet
- Calculating the Twiss parameters
- FODO Cell
- Stability
- Dispersion
- Chromaticity